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Econometrics

Center for Economic Studies
Discussions Paper Series (DPS) 02.19
http://www.econ.kuleuven.be/ces/discussionpapers/default.htm

December 2002

DISCUSSION
PAPER

THE COALITION STRUCTURE CORE IS ACCESSIBLE

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Abstract. For each outcome (i.e. a payoff vector augmented with a coalition structure) of a TU-game with a non-empty coalition structure core there exists a finite sequence of successively dominating outcomes that terminates in the coalition structure core. In order to obtain this result a restrictive dominance relation - which we call enforceable dominance - is employed.

1. Introduction

For a TU-game in coalitional form, there are two fundamental and strongly linked problems: (*i*) what coalitions will form, and (*ii*) how will the members of these coalitions distribute their total worth. We attempt to answer these questions for a certain class of games. We presuppose some bargaining process and show that the coalition structure core, provided it is non-empty, comes forward as a natural candidate for a solution.

We consider a TU-game and some initial individually rational payoff configuration - i.e. a coalition structure augmented with an individual rational and group rational payoff vector (Owen 1982, p236). In case some coalition could gain by acting for themselves, it can reject this initial outcome and propose a second outcome. We impose an *enforceability*-condition upon such a counter-proposal: first, the deviating coalition is a member of the new coalition structure; second, none of the players in the deviating coalition loose when moving towards the new outcome; third, the new coalition structure also contains those coalitions in the initial configuration that do not shelter deviating players; and fourth, these unaffected coalitions obtain the very same payoffs. Then again, another coalition may reject this counter-proposal in favour of a third outcome, and so forth. Apparently, this bargaining process turns the coalition structure core, if non-empty, into an accessible set of outcomes:

> *For each outcome of a TU-game with a non-empty coalition structure core, there exists a finite sequence of successive 'enforceable' counter-proposals that terminates in the coalition structure core.*

Date: October 3, 2001.

Key words and phrases. Coalition structure, core-extension, non-emptiness, dominance.

This paper benefited from presentations at the LGS2 Conference (St Petersburg, June 2001) and at the Warwick Summer Research Conference (July 2001). Discussions with Hideo Konishi and Licun Xue have led to a considerable improvement of the paper. The first author gratefully acknowledges the joint support (SOR-H/99/23 and SOR-H/00/01) of the Katholieke Universiteit Leuven and the Soros Foundation.

In the search for a dynamic foundation of the core, already Green (1974) made an important contribution. He established a finite process of successive counter-proposals that almost surely reaches the core. Later on, Wu (1977) showed the existence of a bargaining scheme that converges to the core and rephrased this result as the core is globally stable.

Finally, our result is a continuation of the work by Sengupta and Sengupta (1996). Formulated in the language of von Neumann and Morgenstern, they proved the *indirect stability* of the core: no payoff allocation dominates a core outcome, and each outcome is indirectly dominated by a core outcome. We refine this stability property in two dimensions.

First, Sengupta and Sengupta (and also Green and Wu) concentrate on the core. Hence they do not tackle problem (i). They take the coalition structure to be exogenously given and assume that the grand coalition forms. We also consider the coalition formation process and extend the stability result to the coalition structure core.

Second, we employ a more restrictive dominance relation based upon an enforceability condition. Sengupta and Sengupta allow a deviating coalition to affect the payoffs of *all* the players, and thus ignore the behaviour and the motivation of the outsiders. We impose, as already indicated, strong restrictions upon counter-proposals: the deviating players cannot intervene in the structure and in the payoffs of those coalitions that are not involved when they separate to form a coalition.

The next section collects preliminaries, introduces the coalition structure core, and defines enforceable dominance. Section 3 studies dominating chains and proves our result. The coalition structure core is characterised as the smallest set of outcomes that satisfies this accessibility property.

2. Preliminaries

We introduce the notation and define games, outcomes, dominance, and a core concept. As we do not assume that the grand coalition forms our notion of outcome slightly differs from the usual notion of imputation.

Let $N = \{1, 2, \ldots, n\}$ be a set of n players. Non-empty subsets of N are called coalitions. A *partition* is a set of pairwise disjoint coalitions so that their union is N and represents the breaking up of the grand coalition N. For a partition $\mathcal{P} = \{C_1, C_2, \ldots, C_m\}$ and a coalition C, the *partners' set* $P(C, \mathcal{P})$ of C in $\mathcal{P}$ is defined as the union of those coalitions in $\mathcal{P}$ that have a non-empty intersection with C:

$$P(C,\mathcal{P}) \;=\; \{i \,|\, i \in C_j \text{ with } j \text{ such that } C_j \cap C \neq \varnothing\} \;=\; \bigcup_{C_j \cap C \neq \emptyset} C_j.$$

A characteristic function $v : 2^N \setminus \{\varnothing\} \longrightarrow \mathbb{R}$ assigns a real value to each coalition. The pair (N, v) is said to be a transferable utility game in characteristic function form, in short, a game. An *outcome* of a game (N, v) is a pair $(x, \mathcal{P})$ with x in $\mathbb{R}^n$ and $\mathcal{P}$ a partition of N. The vector $x = (x_1, x_2, \ldots, x_n)$ lists the payoffs of each player and satisfies

$$\forall i \in N : x_i \geq v(\{i\}) \qquad \text{and} \qquad \forall C \in \mathcal{P} : x(C) = v(C),$$

with $x(C) = \sum_{j \in C} x_j$. The first condition is known as individual rationality: player i will cooperate to form a coalition only if his payoff x_i exceeds the amount he would get on his own. The second condition combines feasibility and the myopic behaviour of the players, it states that each coalition in the partition $\mathcal{P}$ allocates its value among its members. Outcomes with the same payoff vector are said to be *payoff equivalent*. The set of all outcomes is denoted by $\Omega(N, v)$.

In case the grand coalition forms, then an outcome is a pair $(x, \mathcal{P})$ with $\mathcal{P} = \{N\}$, $x_i \geq v(\{i\})$, and $x(N) = \sum_{i \in N} x_i = v(N)$. As such, outcomes generalise imputations. Also note that $\Omega(N, v)$ is non-empty: it contains the outcome in which the grand coalition is split up in singletons.

Now, we define an *enforceable* dominance relation. An interpretation and a discussion follows. Later on, we will drop the adjective 'enforceable', this shall not lead to confusion.

Definition 1. Let $x, y \in \mathbb{R}^n$ and let C be a coalition. Then, vector x dominates y by C, denoted by $x >_C y$, if

- for each player i in C we have $x_i \geq y_i$, and
- for at least one player i in C we have $x_i > y_i$.

Let (N, v) be a game and let $a = (x, \mathcal{P})$ and $b = (y, \mathcal{Q})$ be two outcomes. Then, outcome *a enforceably dominates b by C* if

- $\mathcal{P}$ contains C,
- $\mathcal{P}$ contains all coalitions in $\mathcal{Q}$ that do not intersect C,
- the payoff vector x dominates y by C, and
- the restrictions of x and y to the set of players outside $P(C, \mathcal{Q})$ coincide.

Outcome *a enforceably dominates b* if $\mathcal{P}$ contains a coalition C such that a dominates b by C.

This relation should be interpreted in a dynamic way. Let $a = (x, \mathcal{P})$ dominate $b = (y, \mathcal{Q})$ by C. Then, if b is considered as the initial outcome, one can say that coalition C deviates and enforces the new outcome a. Indeed, in order to obtain a higher total payoff, coalition C separates from its partners (and at least one member of C gets strictly better off). The players in $P(C, \mathcal{Q}) \setminus C$ become ex-partners of C. They may reorganise themselves and their payoffs might decrease when moving from b to a. In the worst case, these ex-partners fall apart to singletons. Finally, the outsiders, i.e. the players not in $P(C, \mathcal{Q})$, are left untouched.

Definition 1 also models a merger or a breaking up. In the former case, the deviating coalition is the union of some of the coalitions in the initial partition. In the latter case, an initial coalition is split up into two or more subcoalitions; each subcoalition that is better off in the new outcome can be considered as the deviating coalition.

Sengupta and Sengupta (1996) restrict their attention to the core, i.e. they assume that the grand coalition forms. As a consequence, they employ a dominance relation at the level of payoff *vectors*. However, if one is also concerned with the coalition formation process, enforceable dominance seems to be a natural and a straightforward extension. On the one hand, the idea of enforceability strongly limits the outlook of a dominating sequence, on the other, if outcome b is dominated by a at the level of payoff vectors, then there exists an outcome a' that enforceably dominates b. Therefore, the enforceability condition does not affect the set of undominated outcomes.

Definition 2. Let (N, v) be a game. The *coalition structure core* $C(N, v)$ is the set of outcomes no coalition can improve upon.

Equivalently, the pair $(x, \mathcal{P})$ is in the coalition structure core if and only if it satisfies feasibility and coalitional rationality:

- for each coalition C in $\mathcal{P}$ we have $x(C) \leq v(C)$, and
- for each coalition S we have $x(S) \geq v(S)$.

Note that the coalition structure core might contain payoff equivalent outcomes. Also, in case the grand coalition forms, the coalition structure core includes the core. Moreover, the linear programming problem to obtain the coalition structure core is very similar to the one behind the core. As a consequence the well-known Shapley-Bondareva conditions that guarantee a non-empty core extend to the present framework. We will not use these conditions. Therefore we state the result without explaining the notion of balanced collection (Owen 1982, Chapter 8).

Let (N, v) be a game. Let v^* be the maximum of $v(\mathcal{P}) = \sum_{C \in \mathcal{P}} v(C)$ where $\mathcal{P}$ runs over all partitions of N. The number v^* is called the value of the game (N, v). The value of a superadditive game is equal to the value of the grand coalition.

Proposition. *The coalition structure core $C(N, v)$ of a game (N, v) with value v^* is non-empty if and only if for each balanced collection $\mathcal{S}$ we have*

$$\sum_{S \in \mathcal{S}} \delta_S \, v(S) \;\leq\; v^*,$$

where the real numbers δ_S are the balancing weights. Replace the value v^* in the above inequalities with the value $v(N)$ to obtain the (stronger) conditions for a non-empty core.

3. The coalition structure core is accessible

Consider an initial outcome. If a coalition can obtain a higher payoff, it is allowed to deviate and to propose a second outcome, and so forth. This bargaining process gives rise to a 'dominating' sequence. We show that for each outcome there exists a dominating sequence that terminates in the coalition structure core. Let (N, v) be a game and let $\Omega = \Omega(N, v)$ be the set of all outcomes.

Definition 3. Let $a, b \in \Omega$. Outcome a is said to be *accessible* from b, and we write $a \leftarrow b$ (or $b \rightarrow a$), if one of the following conditions holds

- a and b are payoff equivalent, or
- a *sequentially dominates* b, i.e. there exists a positive integer k and a sequence of outcomes

$$a_0 = b,\; a_1,\; \ldots,\; a_{k-1},\; a_k = a$$

such that a_i dominates a_{i-1} for $i = 1, 2, \ldots, k$. The integer k is said to be the length of (or the number of steps in) the dominating sequence.

The relation '$\leftarrow$' describes a possible succession of transitions from one outcome to another. We are interested in the outcomes that appear at the *end* of these sequences.

Definition 4. Let Δ be a set of outcomes. Then, Δ is *accessible* from Ω if for each b in Ω there exists an a in Δ such that $a \leftarrow b$.

Lemma. *Let (N, v) be a game with a non-empty coalition structure core. Then, the coalition structure core is accessible.*

Proof. Let $b_0 = (y_0, \mathcal{Q}_0)$ be an outcome that is not in $C(N, v)$. In case b_0 is dominated by an outcome in $C(N, v)$, the proof is done. In case no outcome in $C(N, v)$ dominates b_0, we look for a dominating sequence that terminates in the *core*.[1] This sequence will be denoted by $b_0 \to b_1 \to b_2 \to \ldots$. As a consequence, coalitions and individual payoffs have a double subscript the first one of which refers to the position in this dominating sequence.

The proof is divided into several steps. First, we indicate how to detect those players that can be blamed for not being able to go to the *core* in one step. Then, we define a sequence of suitable deviations and show that the *core* is reached after a finite number of steps. As a matter of fact, we rule out the case that this dominating sequence has an infinite length.

Step 1. Defining the set of overpaid players.
Interpret $b_0 = (y_0, \mathcal{Q}_0)$ as the initial outcome. Let $a = (x, \mathcal{P})$ be a *core* outcome. A player i for which $y_{0i} > x_i$ is said to be *overpaid* relative to a. Let $O(b_0, a)$ collect these overpaid players. Since b_0 is not dominated by a, the set $O(b_0, a)$ is non-empty.
Now, we consider the collection of *core* outcomes that minimise the number of overpaid players. Within this collection, we look for an outcome $a^* = (x^*, \mathcal{P}^*)$ that minimises the amount overpaid $y_0(O_0) - x^*(O_0)$, where $O_0 = O(b_0, a^*)$. We consider a^* as a *core* outcome *close* to b_0.

Step 2. Selecting a deviating coalition.
Since the outcome $b_0 = (y_0, \mathcal{Q}_0)$ is not in the *core* there exists at least one blocking coalition, i.e. a coalition D for which $v(D) > y_0(D)$. We select a deviating coalition D as follows. First, we inspect the coalitions in the partition $\mathcal{P}^*$ and we look for a blocking coalition D among $\mathcal{P}^*$. Next, if the partition $\mathcal{P}^*$ does not contain a blocking coalition, then the outcome b_0 satisfies $y_0(N) = v^*$ and is said to be efficient (with respect to $\mathcal{P}^*$), v^* is the value of the game. In that case we select a minimal (for inclusion) blocking coalition.

Step 3. Defining a deviating outcome.
In order to define the payoff vector in the deviating outcome $b_1 = (y_1, \mathcal{Q}_1)$ we consider the different types of players separately.

First, we deal with the *deviating* players. Since D blocks b_0 and a^* is a *core* outcome, we

[1] In this proof we use the term '*core*' as a shorthand for 'coalition structure core'.

know that $y_0(D) < v(D) \leq x^*(D)$. Let $i \in D$. The payoff y_{1i} depends upon whether or not D contains overpaid players.

(1) If D does not contain overpaid players, then we define

$$y_{1i} = y_{0i} + \delta_i(D) \leq x_i^*,$$

with $\delta_i(D)$ non-negative and adding up to $\delta(D) = v(D) - y_0(D)$.

(2) If D does contain overpaid players, then we define

$$y_{1i} = \begin{cases} y_{0i} + \frac{1}{|D \cap O_0|}\,[v(D) - y(D)] & \text{in case } i \text{ is overpaid,} \\ y_{0i} & \text{in case } i \text{ is not overpaid.} \end{cases}$$

In words, the deviating coalition divides the surplus $v(D) - y_0(D)$ among its members. The overpaid players are served first and consume the whole surplus. The non-overpaid players experience either a status quo or an improvement.

Secondly, the *ex-partners* of D are assumed to split up into singletons. Hence, each player i in $P(D, \mathcal{Q}_0) \backslash D$ receives his value $v(\{i\})$ as payoff.[2]

Thirdly, the *outsiders* remain untouched: if $i \notin P(D, \mathcal{Q}_0)$, then $y_{1i} = y_{0i}$.

In conclusion: when moving from b_0 to b_1, the overpaid ex-partners of D become non-overpaid. In case b_1 is either a *core* outcome or dominated by a *core* outcome, the proof finishes. Otherwise, execute the next steps.

Step 4. An iteration.
We denote the set $O(b_1, a^*)$ of overpaid players in the outcome b_1 by O_1. This set O_1 is a subset of O_0. We repeat Steps 2 and 3 and we generate a dominating sequence $b_0 \to b_1 \to b_2 \to \ldots$ of outcomes and a corresponding sequence $O_0 \supseteq O_1 \supseteq O_2 \supseteq \ldots$ of sets of overpaid players. If the *core* is reached after a finite number of steps, then the iteration stops and the proof is done. Otherwise, we end up with an infinite dominating sequence $\{\, b_k \,\}_{k=0,1,2,\ldots}$ each outcome b_k being outside the *core* and not dominated by the *core*.

Step 5. Inspecting the infinite sequence.
As the *core* is not reached the set of overpaid players finds its non-empty minimal form after a finite number of steps. Denote this set by O. The iteration procedure is unable to reduce O. Due to the selection criteria for the deviating coalition (Step 2) there is somewhere in the sequence an outcome $b_t = (y_t, \mathcal{P}_t)$ that is efficient with respect to $\mathcal{P}^*$ and satisfies $O_t = O$. Let us replace b_t with $b = (y = y_t\,,\, \mathcal{P}^*)$.

We *claim* that the outcome b is in the *core*. Denote the partners' set of the overpaid players in $\mathcal{P}^*$ by B, i.e. $B = P(O, \mathcal{P}^*)$, and the complement of B by A. Since A does not contain overpaid players and b is efficient with respect to $\mathcal{P}^*$, the payoff vectors y and x^* restricted to A coincide: $y|_A = x^*|_A$.

We complete the proof of the claim by contradiction. Let D be a blocking coalition. Obviously, D is not a subset of A. Hence D intersects B. Let $\bar{D} = P(D, \mathcal{P}^*)$ be the partners' set of D. The efficiency of b implies that $y(\bar{D}) = v(\bar{D})$. Since O can not be

[2]This assumption can be relaxed. The ex-partners are allowed to reorganise themselves provided none of them is overpaid in the new outcome.

reduced, the coalition D contains all the overpaid players in $\bar{D}$. Therefore, $\bar{D} \setminus D$ only contains non-overpaid players and satisfies $y(\bar{D} \setminus D) \leq x^*(\bar{D} \setminus D)$. Use the efficiency of the outcome b together with the fact that a^* is a *core* outcome to conclude that $y(D) \geq x^*(D) \geq v(D)$. Hence, D is not blocking. A contradiction. □

In order to stress the impact of the particular construction in the above proof we give an example of a bargaining scheme that does not enter the coalition structure core.

Example. Let $\Gamma_\alpha = (\{1, 2, 3\}, v)$ denote the three-player game, where each singleton has value 0, each pair has value 2, and $v(N) = \alpha$. Let $\alpha = 6$. The core is non-empty. Nevertheless, the next three outcomes generate a cycle of dominating outcomes:

$$((1, 1, 0), \{1, 2\}, \{3\}), \quad ((1, 0, 1), \{1, 3\}, \{2\}), \quad \text{and} \quad ((0, 1, 1), \{2, 3\}, \{1\}).$$

We close this section with a characterisation of the coalition structure core.

Theorem. *The coalition structure core of a game, if non-empty, is the smallest (for inclusion) set of outcomes that satisfies accessibility.*

Proof. Accessibility follows from the previous lemma. Furthermore, each outcome in the coalition structure core is undominated. This implies minimality. □

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