A lost sales inventory model with a compound poisson demand pattern

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Abstract

In this paper, we study the decision problem of a retailer, who wants to optimize the amount of shelf inventory of a particular product, given that the demand for the product is stochastic and replenishment lead times (from the store's stockroom to the shelf) are negligible. The shelf inventory is managed according to a $(0, B^*)$-inventory policy: when the shelf inventory is sold out, the retailer gets a fixed amount of B^* units from the central stockroom to replenish the shelf inventory.

To adequately reflect the shopping behavior of retail customers, the demand process is modeled as a compound Poisson process, with Poisson distributed purchase quantities. When the purchase quantity of a customer exceeds the amount of shelf inventory still available, the unsatisfied demand is considered to be lost sales.

As the demand process is stochastic, the runout time of the shelf inventory will be stochastic too. The costs per cycle related to keeping inventory on the shelf can be split up into three components: average holding costs (which may be related to the scarcity of shelf space), a fixed handling cost (per replenishment trip), and an average lost sales cost. The purpose of the model is to determine the value of B^* that minimizes the average total cost per time unit.

Keywords: Discrete inventory models, compound Poisson process, lost sales, Jonquière's function

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1 Introduction

In this paper, we look at an inventory setting where demand does not arrive in units, but in batches following a given discrete probability distribution.
Our research is inspired by a retail store setting, where part of the inventory is kept on the shelf, and replenishments can be supplied from the warehouse in negligible time. Every movement of material entails a fixed cost, and the amount of inventory that can be stored on the shelf is fixed and limited. The demand pattern of customers in this type of setting will typically be stochastic: both the time between consecutive store visits and the amount of product purchased at each visit are random variables. In the literature, this type of demand process is modeled by means of a compound Poisson process [3]: the number of visits during a given timeframe is assumed to be Poisson distributed, while the purchase quantities follow an arbitrary discrete distribution of which the first two moments are given (see e.g. [8; 10; 17]).
As the demand is discrete, and the shelf inventory is fixed, it may happen that a customer does not find the desired purchase quantity during his visit; in this case, sales are lost. A replenishment order is triggered as soon as the shelf inventory drops to zero; in which case it is replenished with a fixed quantity, determined by the amount of shelf space reserved for that product type.
The amount of inventory kept on the shelf will determine the average number of replenishment trips to be made over a specified horizon (and, hence, the ordering costs), the average inventory holding costs related to the shelf inventory, and the average cost of lost sales. Obviously, a trade-off exists between these three cost components. The purpose of our model is to determine the amount of shelf inventory that minimizes the total costs.
Although the presence of stochastic batch-sized demand is common in real life, the literature on the impact of this demand pattern in an inventory setting is rather scarce. Most commonly used inventory management models indeed assume unit sized demand, with total demand following a normal distribution during replenishment leadtime (see e.g. [4; 22]). Other papers do consider batch-sized demand (see e.g. [1; 2; 8; 10; 11; 17]) but treat this problem in a very general manner, without making explicit assumptions about the probability distribution of the demand. While this approach is certainly useful from a theoretical point of view, the downside is that the resulting expressions are not directly usable (e.g. in an optimization scheme), and fail to give insight into the behaviour of the different cost components in relation to the decision parameters.
Our work differs from the previous literature in the sense that we explicitly assume the purchase quantities to be Poisson distributed. Though this assumption introduces an additional restriction in the model, it allows to derive some rather remarkable analytical insights, more precisely with respect to the behaviour of the lost sales cost in this setting.
In the next section, we summarize the assumptions of the model and introduce the notation. Section 3 describes the model, while section 4 takes a closer look at the optimization problem. Finally, section 5 summarizes the main insights and results.

2 Notations and assumptions

We will consider a store setting where for a given product type, an inventory of B^* units is kept on the shelf. Customers buy quantities from this shelf inventory according to a compound Poisson process: more specifically, we assume that the number of customer visits to the store during a time interval $[0, T]$ is Poisson distributed with average λT:

$$\eta_T \sim \text{Poisson}(\lambda T) \tag{1}$$

This assumption is appropriate, as we can safely assume that the customer population is large, and that customers act independently. The purchase quantities at arbitrary visits i (denoted by β_i) are assumed to be independent and Poisson distributed with average μ:

$$\beta_j \sim \text{Poisson}(\mu) \quad \forall j \tag{2}$$

Note that the assumption of a Poisson distribution for purchase quantities takes into account the possibility that the customer does not buy the product on a given shopping trip ($P[\beta_j = 0] > 0$).
It is assumed that the shelf inventory is replenished according to an order-up-to inventory policy: as soon as the shelf inventory is sold out, a replenishment order of B^* units is fetched from the store's central stockroom. The assumption of a zero reorder point is actually quite realistic in our setting, as it gives a clearly visible signal to storeroom personnel. Moreover, as the stockroom in a retail setting is commonly adjacent to the store, replenishment lead times can be assumed to be negligible. Hence, replenishment is quasi instantaneous, eliminating the need for a positive reorder point. Units that cannot be delivered from the shelf inventory are considered to be lost sales.
As both the time between shopping trips and the purchase quantity per trip are stochastic, the time between successive shelf replenishments (referred to as the replenishment cycle or the *runout time* τ_{B^*}) will also be stochastic. Consequently, the value of B^* will influence the number of replenishment orders issued over a given horizon, the average number of units in inventory, and the average cost of lost sales.
In this paper, we develop closed-form analytical expressions for the average ordering costs, average inventory holding costs and average lost sales costs, in terms of the system's characteristics. As a result, an optimization model for the global cost function is proposed.

3 Model development

3.1 Average runout time

In general, the runout time τ_{B^*} is a stochastic variable, which can be written as the sum of individual customer intervisit times (Y_j):

$$\tau_{B^*} = \sum_{j=1}^{N} Y_j \tag{3}$$

In this expression, N itself is a random variable, referring to the number of customers whose individual purchase quantities add up to a quantity larger than or equal to B^*.

Considering τ_{B^*} as a random sum of random variables, we then know that the average value can be written as (see e.g. [9]):

$$E\left[\tau_{B^*}\right] = E\left[N\right] E\left[Y\right] \tag{4}$$

As customer visits are Poisson distributed with rate λ, the average time between two customer visits is:

$$E\left[Y\right] = \frac{1}{\lambda} \tag{5}$$

The average of N can in general be determined as follows:

$$E\left[N\right] = \sum_{n=1}^{\infty} nP[N = n] \tag{6}$$

The probability mass function of N depends on the probability mass function of the purchase quantities:

$$P[N = 1] = P[\beta_1 \geqslant B^*] \tag{7}$$

$$P[N = n] = \sum_{a=0}^{B^*-1} \left\{ P\left[\sum_{j=1}^{n-1} \beta_j = a\right] P[\beta_n \geqslant B^* - a] \right\},\ n = 2, \ldots, \infty \tag{8}$$

Assuming independent Poisson distributed purchase quantities with average value μ, this yields:

$$E[N] = 1 - e^{-\mu} \sum_{j=0}^{B^*-1} \frac{\mu^j}{j!} + \sum_{n=2}^{\infty} n \left[\sum_{a=0}^{B^*-1} e^{-(n-1)\mu} \ \frac{((n-1)\mu)^a}{a!} \times \right. \\ \left. \times \left(1 - e^{-\mu} \sum_{j=0}^{B^*-1-a} \frac{\mu^j}{j!}\right)\right] \tag{9}$$

and consequently:

$$E\left[\tau_{B^*}\right] = \frac{1}{\lambda} \left[1 - e^{-\mu} \sum_{j=0}^{B^*-1} \frac{\mu^j}{j!} + \sum_{n=2}^{\infty} n \left[\sum_{a=0}^{B^*-1} e^{-(n-1)\mu} \times \right.\right. \\ \left.\left. \times \frac{((n-1)\mu)^a}{a!} \left(1 - e^{-\mu} \sum_{j=0}^{B^*-1-a} \frac{\mu^j}{j!}\right)\right]\right] \tag{10}$$

It can be proven by considering equal powers of μ, that this expression is nothing but

$$E\left[\tau_{B^*}\right] = \frac{1}{\lambda} \left[1 + \sum_{j=0}^{B^*-1} \frac{\mu^j}{j!} Li_{-j}\left(e^{-\mu}\right) \right] \tag{11}$$

where $Li_n(z)$ stands for Jonquière's function [16], which is defined in the following manner, for integer values of n (see e.g. [15]):

$$Li_n(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^n} \text{ with } |z| < 1. \tag{12}$$

An alternative derivation for expression (11) is given in Appendix A.
In case of a negative integer value of $n = -j$, Jonquière's function can also be written as [18]:

$$Li_{-j}(z) = \left(z\frac{d}{dz}\right)^j \frac{1}{1-z} \tag{13}$$

with

$$Li_0(z) = \frac{z}{1-z} \tag{14}$$

showing its strong relation with the geometric series. As a result one may easily derive, using an inductive procedure, the following explicit expression:

$$Li_{-j}(z) = \sum_{k=1}^{j+1} (k-1)! S(a+1,k) \frac{z^k}{(1-z)^k} \tag{15}$$

where $S(n,k)$ stands for a Stirling number of the second kind (see e.g. [5]), defined as

$$S(n,k) \equiv \frac{1}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} (k-l)^n = \frac{1}{k!} \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} l^n \tag{16}$$

3.2 Average number of units in inventory at an arbitrary time

Obviously, we can write the average number of units in inventory $E[I_{B^*}]$ at an arbitrary time as follows:

$$E[I_{B^*}] = B^* - E[L_{B^*}] \tag{17}$$

where $E[L_{B^*}]$ denotes the average number of units purchased at an arbitrary time. To determine $E[L_{B^*}]$, we can rely upon previous research results on the so-called shuttle dispatch problem, a subject which, in spite of its totally different setting, strongly resembles our research problem. Figure 1 below illustrates the similarity.
In the shuttle dispatch problem, passengers arriving according to a simple or compound Poisson process need to wait until a minimum number of passengers B^* (the control limit) is reached before the shuttle is dispatched. The shuttle may have infinite capacity, which implies that all passengers are transported at the moment of dispatch (see e.g. [7; 12; 14; 23]), or finite capacity, implying that the shuttle is loaded up to its capacity (see e.g. [6; 19; 20; 21]).
The inventory problem that we consider is similar to the shuttle dispatch problem with compound poisson arrivals, an infinite capacity shuttle (as we assume that sales which cannot be delivered from inventory are lost), and a zero travel time for the shuttle (as we assume immediate replenishment).

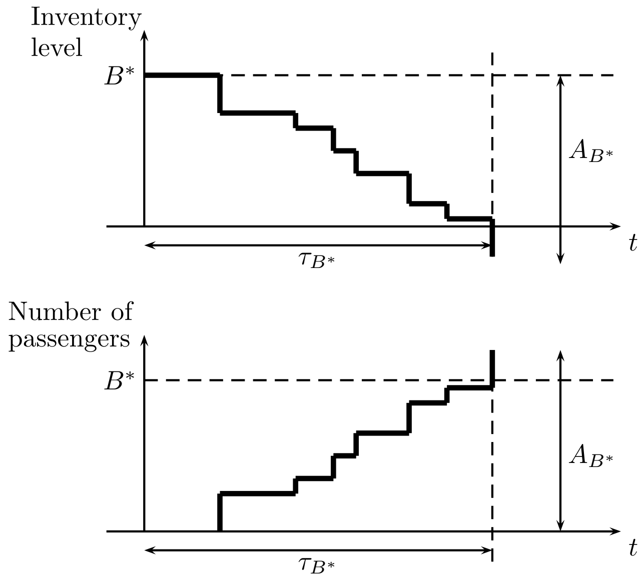

Figure 1: Similarities between the inventory problem in our setting (top), and the shuttle dispatch problem (bottom)

From Figure 1, it is clear that the average number of units purchased at an arbitrary time $E[L_{B^*}]$, is analogous to the average length of the queue in the shuttle dispatch problem. Hence, it can be written as (see e.g. [12; 13; 14]):

$$E\left[L_{B^*}\right] = \frac{1}{2}\left[\frac{A_{B^*}^{(2)}}{E[A_{B^*}]} - \frac{\beta^{(2)}}{E[\beta]}\right] \tag{18}$$

where the random variable A_{B^*} refers to the total number of units demanded during the runout time, including lost sales (or, in the shuttle dispatch problem, the total number of passengers on the infinite capacity shuttle at the moment that it leaves the terminal). In the expression for $E[L_{B^*}]$, $A_{B^*}^{(2)}$ and $\beta^{(2)}$ represent, respectively, the second factorial moments of A_{B^*} and β:

$$A_{B^*}^{(2)} = E\left[A_{B^*}(A_{B^*} - 1)\right] \tag{19}$$

$$\beta^{(2)} = E\left[\beta(\beta - 1)\right] = \mu^2 \tag{20}$$

The expression for $A_{B^*}^{(2)}$ can be efficiently determined using the following result from [12]:

$$\frac{A_{B^*}^{(2)} - A_{B^*-1}^{(2)}}{E\left[A_{B^*}\right] - E\left[A_{B^*-1}\right]} = \frac{\beta^{(2)}}{E\left[\beta\right]} + 2(B^* - 1) \quad \text{for } B^* \geqslant 1 \tag{21}$$

which yields (the inductive proof is given in Appendix B):

$$A_{B^*}^{(2)} = \frac{\beta^{(2)}}{E\left[\beta\right]}\left[E\left[A_{B^*}\right] - E\left[A_1\right]\right] + 2B^* E\left[A_{B^*}\right] - 2\sum_{r=1}^{B^*} E\left[A_r\right] \tag{22}$$

In this expression, $E\left[A_r\right]$ denotes the expected number of passengers on the shuttle when a control limit r is used. It is determined as (see e.g. [12]):

$$E\left[A_r\right] = \lambda\mu E\left[\tau_r\right] \tag{23}$$

Using expression (11), relation (23) can be rewritten as

$$E\left[A_r\right] = \mu \left[1 + \sum_{j=0}^{r-1} \frac{\mu^j}{j!} Li_{-j}(e^{-\mu})\right] \tag{24}$$

On account of eq. (22) and (23), the expression for $E\left[L_{B^*}\right]$ can be casted into the following form

$$E\left[L_{B^*}\right] = \frac{1}{2}\left[2B^* - \mu\frac{E\left[\tau_1\right]}{E\left[\tau_{B^*}\right]} - \frac{2}{E\left[\tau_{B^*}\right]}\sum_{r=1}^{B^*} E\left[\tau_r\right]\right] \tag{25}$$

Hence, the average number of units in inventory reduces to

$$E\left[I_{B^*}\right] = B^* - E\left[L_{B^*}\right] = \frac{\mu}{2}\frac{E\left[\tau_1\right]}{E\left[\tau_{B^*}\right]} + \frac{1}{E\left[\tau_{B^*}\right]}\sum_{r=1}^{B^*} E\left[\tau_r\right] \tag{26}$$

3.3 Average number of units lost

As mentioned before, the expected total number of units demanded during the runout time is given by $E\left[A_r\right]$ when the initial stock level is r, which can easily be determined by means of formula (24). The average number of sales units that are lost is then given by

$$E\left[Z_{B^*}\right] = E\left[A_{B^*}\right] - B^* = \mu\left[1 + \sum_{j=0}^{B^*-1} \frac{\mu^j}{j!} Li_{-j}(e^{-\mu})\right] - B^* \tag{27}$$

when the initial inventory level is B^*.
For large values of the replenishment order B^*, this average number of lost sales tends to $\frac{\mu}{2}$. In order to prove this, let us first remark that

$$Li_0(e^{-\mu}) = \frac{1}{2}\coth\left(\frac{\mu}{2}\right) - \frac{1}{2} \tag{28}$$

and that (see Appendix C)

$$Li_j(e^{-\mu}) = \frac{(-1)^j}{2}\frac{d^j}{d\mu^j}\left[\coth\left(\frac{\mu}{2}\right)\right] \tag{29}$$

so that expression (27) is rewritten as

$$E\left[Z_{B^*}\right] = \frac{\mu}{2} + \frac{\mu}{2}\sum_{j=0}^{B^*-1}\frac{(-\mu)^j}{j!}\frac{d^j}{d\mu^j}\left[\coth\left(\frac{\mu}{2}\right)\right] - B^* \tag{30}$$

Using the Laurent series for $\coth\left(\frac{\mu}{2}\right)$ and expanding the obtained rational expressions as Taylor series in an appropriate manner (see Appendix C), relation (30) takes the following form:

$$E\left[Z_{B^*}\right] = \frac{\mu}{2} + \sum_{\substack{n \\ \text{with } 1 \leqslant n \\ \text{and } (2\pi n)^2 \leqslant \mu^2}} \frac{\mu}{n\pi} \underbrace{\left[\frac{\mu^2}{\mu^2+(2\pi n)^2}\right]^{\frac{B^*}{2}}}_{\downarrow\, B^* \to \infty \atop 0} \sin\left[B^* \arctan\left(\frac{2n\pi}{\mu}\right)\right]$$

$$+ \sum_{\substack{n \\ \text{with } (2\pi n)^2 > \mu^2 \geqslant 1}} \underbrace{\frac{(-1)^{B^*}\mu^2 \sin(\pi B^*)}{2(B^*-1)n^2\pi^2}}_{\downarrow\, B^* \to \infty \atop 0} {}_3F_2\left(\tfrac{1}{2}, 1, 1 \,\middle|\, 1-\tfrac{B^*}{2}, \tfrac{3-B^*}{2} \,\middle|\, -\tfrac{\mu^2}{4n^2\pi^2}\right) \tag{31}$$

with ${}_3F_2(a_1, a_2, a_3|b_1, b_2|x)$ a so-called hypergeometric function.
Considering the limit $B^* \to \infty$, both sums vanish, resulting in the conclusion

$$\lim_{B^* \to \infty} \left[\mu\left(1 + \sum_{j=0}^{B^*-1} \frac{\mu^j}{j!} Li_{-j}(e^{-\mu})\right) - B^*\right] = \frac{\mu}{2} \tag{32}$$

This is a rather remarkable result. It states that, when the initial shelf quantity B^* is sufficiently large, the average lost sales for the last customer equals half of his average demand, which is independent of the actual value of B^*.

4 Optimization of the order quantity

In this section we first study the behaviour of the different cost components. Next the total cost function is derived. The findings will be illustrated by means of an example, for which the parameters are given in table 1.

C_f	C_h	C_l	μ	λ
1	1	7	30	4

Table 1: Parameters of the cost function

4.1 Cost components

Let us assume that a fixed cost C_f is incurred per replenishment cycle and that a holding cost (per time unit) C_h has to be taken into account for each product unit. Moreover, a cost C_l per unit of lost sales must be considered.
The expected fixed cost per time unit FC_m is given by

$$FC_m = \frac{C_f}{E\left[\tau_m\right]} \tag{33}$$

which appears to be a nonincreasing function of the order quantity m. This is a direct consequence of the positivity of Jonquière's function $Li_{-n}(z)$, $\forall z \in [0, 1]$.

Furthermore, it can be stated that FC_m is a monotonically decreasing function of m. This follows from the difference

$$FC_{m+1} - FC_m = \frac{C_f(E[\tau_m] - E[\tau_{m+1}])}{E[\tau_m]\,E[\tau_{m+1}]} = \frac{-C_f \mu^j Li_{-j}(e^{-\mu})}{j!\lambda E[\tau_m]\,E[\tau_{m+1}]} \tag{34}$$

and is illustrated by the example in Figure 2 for the parameters of Table 1.

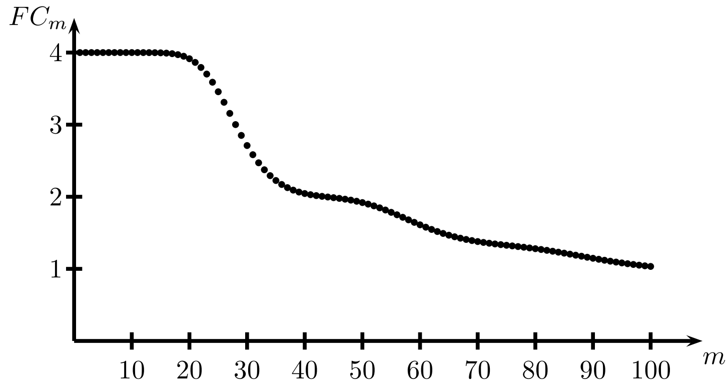

Figure 2: The expected fixed cost per time unit for the parameters of Table 1.

The expected holding cost per time unit HC_m can be written as

$$HC_m = C_h\,(m - E[L_m]) \tag{35}$$

where $E[L_m]$ is given by expression (25) with B^* replaced by m.
From the analysis of the difference $HC_{m+1} - HC_m$, one can conclude that the expected holding cost per time unit as a function of m may exhibit oscillatory behaviour. Indeed, let us consider HC_m for the example. As shown in Figure 3, the resulting function oscillates, but clearly shows an upward trend. It seems to converge to a straight line with positive slope for large values of m, however the proof is beyond the scope of this paper.

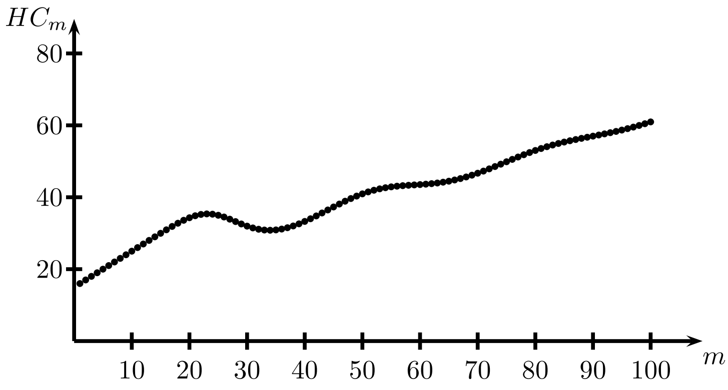

Figure 3: The expected holding cost per time unit for the parameter setting in Table 1.

Finally, the lost sales cost per time unit LC_m is given by

$$LC_m = \frac{E[Z_m]}{E[\tau_m]} \tag{36}$$

where $E[Z_m]$ is determined by eq.(27) in which we have replaced B^* by m. As m increases, LC_m will show a downward trend. Like HC_m, it exhibits oscillatory behaviour: Figure 4 depicts LC_m in terms of m for the example.

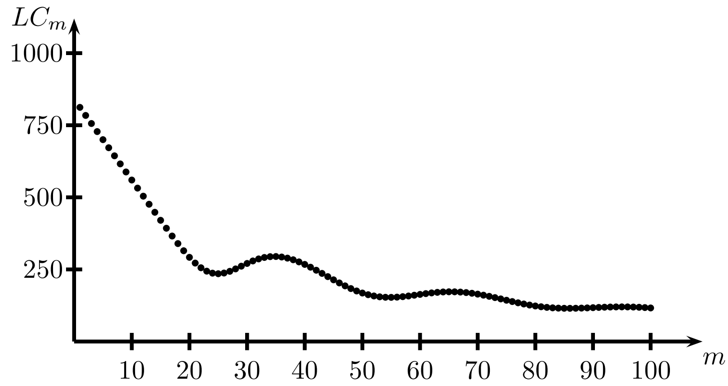

Figure 4: The expected lost sales cost per time unit for the example.

Remarkably, the expected number of units lost sales ($E[Z_m]$) will be a perfectly oscillating function of m, as shown in Figure 5. The figure also illustrates that $E[Z_m]$ converges to $\mu/2$ for large values of m, as discussed above (see eq.(32)).

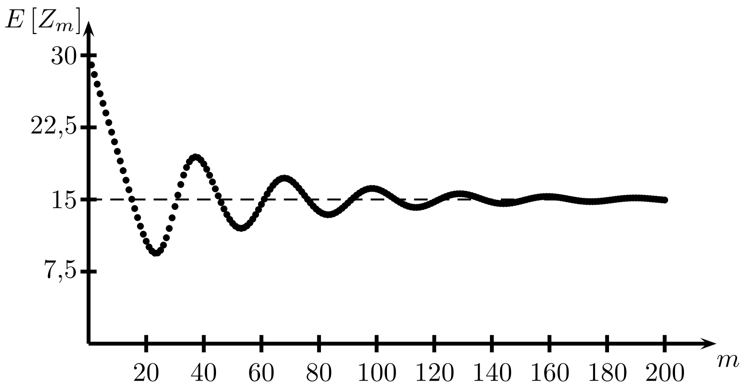

Figure 5: The expected lost sales for the example.

4.2 Minimization of the total cost

As a result, the total expected cost per time unit, for an arbitrary m, can be written as

$$TK_m = \frac{C_f}{E[\tau_m]} + C_h [m - E[L_m]] + \frac{C_l E[Z_m]}{E[\tau_m]} \tag{37}$$

which is nothing but the sum of the three cost components discussed above. On account of eqs. (23), (25) and (27) this total expected cost per time unit reduces to

$$TK_m = C_l \lambda \mu + \frac{1}{E[\tau_m]} \left[C_f + \frac{C_h \mu E[\tau_1]}{2} + C_h \sum_{r=1}^{m} E[\tau_r] - m C_l \right] \tag{38}$$

where $E[\tau_j]$ is given by expression (11).
This function appears to be nonconvex in m, for all possible instances of the parameters. Figure 6 shows TK_m in terms of m, for the example. As discussed in the previous section, HC_m is the only cost component with an upward trend in terms of m; hence, it is obvious that for large values of m, TK_m will be dominated by the almost linearly increasing behaviour of HC_m, which implies that the relevant local minima for an optimization scheme will be finite in number.

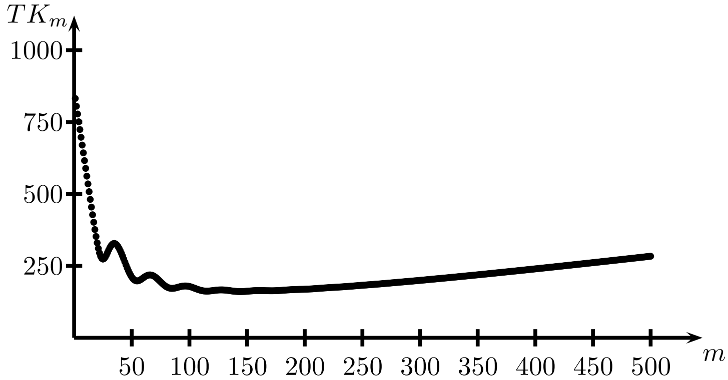

Figure 6: The expected total cost per time unit for the example.

In order to optimize this function TK_m of the integer variable m, we consider the difference

$$\triangle TK_m \equiv TK_{m+1} - TK_m \tag{39}$$

which, by using the definition (38) and expression (11), becomes the following

$$\begin{aligned}\triangle TK_m \equiv \frac{-\mu^m Li_{-m}(e^{-\mu})}{\lambda\, m! E[\tau_m] E[\tau_{m+1}]} \Bigg[& \frac{C_h}{\lambda} \sum_{j=0}^{m-1} \frac{\mu^j}{j!}(m-j) Li_{-j}(e^{-\mu}) + \frac{mC_h}{\lambda} \\ & + \frac{\mu}{2} C_h E[\tau_1] + C_f - C_l m \Bigg] + C_h - \frac{C_l}{E[\tau_{m+1}]}\end{aligned} \tag{40}$$

This expression can for example be used in a steepest descent based algorithm, which would be the local search part in a metaheuristic in order to find the global minimum of TK_m for the set of parameters given in Table 1.

m	25	55	85	115	144
TK_m	273.560	197.835	171.836	162.310	160.707

Table 2: Results of the optimization of the expected total cost per time unit for the example.

If we consider the case $C_f = C_h = 1$, $C_l = 7$, $\lambda = 4$ and $\mu = 30$, we obtain the results presented in table 2 for the consecutive best values of the cost function. From this table one may conclude that the optimal order quantity will be $B^* = 144$, for which we obtain the following contributions to the cost function:

$$FC_{B^*} = 0.7568,\ HC_{B^*} = 82.7731,\ LC_{B^*} = 77.1767 \tag{41}$$

leading to the value $TK_{B^*} = 160.7066$. One may also verify that the average number of lost sales in this case becomes $E[Z_{B^*}] = 14.5688$, which approaches $\frac{\mu}{2} = 15$ as expected due to relation (32).

5 Conclusion

In this paper, we have derived a closed-form formula for the average runout time of a shelf inventory in a retail setting, assuming that customers arrive according to a Poisson process and purchase quantities are Poisson distributed. It is revealed that this average runout time can be written by means of Jonquiére's functions. Using the analogy of this problem with the shuttle dispatch problem, closed-form expressions can also be found for the average number of lost sales and the average number of units in inventory, in terms of these functions.
These results can be embedded in a total cost function, which in general turns out to be nonconvex. Consequently, the optimal amount of shelf inventory can only be traced through the use of a common metaheuristic.
As the shape of the total cost function depends on the specific parameters, settings might be derived for which the convexity of this function is guaranteed. Hence, we plan to focus our future work on the analysis of these settings, in order to derive convexity conditions on the parameters. When these conditions are fulfilled, the application of a simple steepest descent algorithm, using expression (40), suffices to determine the globally optimal shelf inventory.

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A Appendix

In this section, it will be shown that for the type of process considered, it is possible to derive a closed-form expression for the average runout time $E[\tau_{B^*}]$. In order to do this we first derive the cumulative probability distribution and the frequency distribution of τ_{B^*}.
The cumulative probability distribution is given by:

$$\begin{aligned}
P\left[\tau_{B^*} \leqslant T\right] &= 1 - P\left[\text{number of units purchased in } [0,T] < B^*\right] \\
&= 1 - \sum_{j=0}^{\infty} P\left[\eta_T = j\right] P\left[\sum_{k=1}^{j} \beta_k < B^*\right] \\
&= 1 - \sum_{j=0}^{\infty} P\left[\eta_T = j\right] P\left[\sum_{k=1}^{j} \beta_k \leqslant B^* - 1\right] \\
&= 1 - \sum_{j=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^j}{j!} \left[\sum_{k=0}^{B^*-1} e^{-j\mu} \frac{(j\mu)^k}{k!}\right] \\
&= F_{\tau_{B^*}}(T) \text{ for } T \geqslant 0
\end{aligned}$$

The frequency distribution is then given by:

$$\begin{aligned}
f_{\tau_{B^*}}(T) &= \frac{dF_{\tau_{B^*}}(T)}{dT} = 0 - \sum_{j=0}^{\infty} \frac{d}{dT}\left[\frac{e^{-\lambda T}(\lambda T)^j}{j!}\left[\sum_{k=0}^{B^*-1} e^{-j\mu}\frac{(j\mu)^k}{k!}\right]\right] \\
&= -\frac{d}{dT}\left(e^{-\lambda T}\right) - \sum_{j=1}^{\infty} \frac{d}{dT}\left[\frac{e^{-\lambda T}(\lambda T)^j}{j!}\left[\sum_{k=0}^{B^*-1} e^{-j\mu}\frac{(j\mu)^k}{k!}\right]\right] \\
&= \lambda e^{-\lambda T} - \sum_{j=1}^{\infty}\left[\frac{-\lambda e^{-\lambda T}(\lambda T)^j + j\lambda e^{-\lambda T}(\lambda T)^{j-1}}{j!}\right]\left[\sum_{k=0}^{B^*-1} e^{-j\mu}\frac{(j\mu)^k}{k!}\right] \\
&= \lambda e^{-\lambda T} + \sum_{j=1}^{\infty}\left[\frac{\lambda\left(e^{-\lambda T}(\lambda T)^j - j e^{-\lambda T}(\lambda T)^{j-1}\right)}{j!}\right]\left[\sum_{k=0}^{B^*-1} e^{-j\mu}\frac{(j\mu)^k}{k!}\right]
\end{aligned}$$

The average runout time can now be determined as follows:

$$\begin{aligned}
E\left[\tau_{B^*}\right] &= \int_0^{\infty} T f_{\tau_{B^*}}(T) dT \\
&= \int_0^{\infty} \lambda T e^{-\lambda T} dT \\
&\quad + \int_0^{\infty} \sum_{j=1}^{\infty}\left[\frac{e^{-\lambda T}(\lambda T)^{j+1} - j e^{-\lambda T}(\lambda T)^j}{j!}\right]\left[\sum_{k=0}^{B^*-1} e^{-j\mu}\frac{(j\mu)^k}{k!}\right] dT
\end{aligned}$$

Provided that the convergence conditions are satisfied, one may rewrite this expression for $E\left[\tau_{B^*}\right]$ as:

$$E\left[\tau_{B^*}\right] = \frac{1}{\lambda} + \sum_{j=1}^{\infty} \frac{1}{j!}\left[\sum_{k=0}^{B^*-1} e^{-j\mu}\frac{(j\mu)^k}{k!}\right] \int_0^{\infty} \left[e^{-\lambda T}(\lambda T)^{j+1} - j e^{-\lambda T}(\lambda T)^j\right] dT$$

Since we have that

$$\int_0^{\infty} \left[e^{-\lambda T}(\lambda T)^{j+1} - j e^{-\lambda T}(\lambda T)^j\right] dT = \frac{j!}{\lambda}$$

the expression for $E\left[\tau_{B^*}\right]$ reduces to

$$
\begin{aligned}
E\left[\tau_{B^*}\right] &= \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{j=1}^{\infty} \sum_{k=0}^{B^*-1} e^{-j\mu} \frac{(j\mu)^k}{k!} = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{k=0}^{B^*-1} \frac{\mu^k}{k!} \sum_{j=1}^{\infty} j^k e^{-j\mu} \\
&= \frac{1}{\lambda} \left[1 + \sum_{k=0}^{B^*-1} \frac{\mu^k}{k!} Li_{-k}(e^{-\mu}) \right]
\end{aligned}
$$

B Appendix

For an arbitrary control limit m, we can write:

$$
\frac{A_m^{(2)} - A_{m-1}^{(2)}}{E\left[A_m\right] - E\left[A_{m-1}\right]} = \frac{\beta^{(2)}}{E\left[\beta\right]} + 2(m-1) \quad \text{for } m \geqslant 1
$$

Hence, $A_m^{(2)}$ is given by the following relation:

$$
A_m^{(2)} = \left[\frac{\beta^{(2)}}{E\left[\beta\right]} + 2(m-1) \right] \left[E\left[A_m\right] - E\left[A_{m-1}\right] \right] + A_{m-1}^{(2)} \quad \text{for } m \geqslant 1
$$

Working recursively one may derive the following expression for $A_m^{(2)}$:

$$
A_m^{(2)} = \frac{\beta^{(2)}}{E\left[\beta\right]} \left[E\left[A_m\right] - E\left[A_1\right] \right] + 2mE\left[A_m\right] - 2 \left[\sum_{j=1}^{m} E\left[A_j\right] \right]
$$

In order to prove this, we suppose the above expression is valid for $m = M$ and prove it for $m = M+1$:

$$
\begin{aligned}
A_{M+1}^{(2)} &= \left[\frac{\beta^{(2)}}{E\left[\beta\right]} + 2M \right] \left[E\left[A_{M+1}\right] - E\left[A_M\right] \right] + A_M^{(2)} \\
&= \left[\frac{\beta^{(2)}}{E\left[\beta\right]} + 2M \right] \left[E\left[A_{M+1}\right] - E\left[A_M\right] \right] + \frac{\beta^{(2)}}{E\left[\beta\right]} \left[E\left[A_M\right] - E\left[A_1\right] \right] \\
&\quad +2ME\left[A_M\right] - 2 \left[\sum_{j=1}^{M} E\left[A_j\right] \right] \\
&= \frac{\beta^{(2)}}{E\left[\beta\right]} \left[E\left[A_{M+1}\right] - E\left[A_1\right] \right] + 2ME\left[A_{M+1}\right] - 2 \left[\sum_{j=1}^{M} E\left[A_j\right] \right]
\end{aligned}
$$

Adding and substracting the term $2E\left[A_{M+1}\right]$ leads to the expected result.

C Appendix

In this section we prove relation (32), through the use of the theory of special functions for which we used extensively [16]. First we reconsider the expression to prove

$$
\lim_{B^* \to \infty} \left[\sum_{j=0}^{B^*-1} \frac{\mu^j}{j!} Li_{-j}(e^{-\mu}) - \frac{B^*}{\mu} \right] = -\frac{1}{2} \tag{42}
$$

In the following we analyse the l.h.s. of this expression between brackets and rewrite it in a more appropriate form, taking into account the number of terms in the summation over j:

$$\sum_{j=0}^{B^*-1} \left[\frac{\mu^j}{j!} Li_{-j}(e^{-\mu}) - \frac{1}{\mu} \right]. \tag{43}$$

We first remark that Jonquière's function $Li_\nu(z)$ is related to Lerch's transcendent $\Phi(z, \nu, \alpha)$ for integer values of its parameter ν:

$$Li_{-j}(z) = z\Phi(z, -j, 1), \text{ for } |z| < 1. \tag{44}$$

For this transcendental function, there exists a special relationship, called the functional equation by Lerch:

$$\begin{aligned} \Phi(e^{-\mu}, -j, \alpha) & - \Gamma(j+1)e^{\mu\alpha}\mu^{-(j+1)} = -i(2\pi)^{-(j+1)}\Gamma(j+1)e^{\mu\alpha} \times \\ & \times \left[e^{i(2\pi\alpha - \frac{j\pi}{2})}\Phi\left(e^{2i\pi\alpha}, j+1, 1+\frac{\mu}{2\pi i}\right) \right. \\ & \left. \quad - e^{-i(2\pi\alpha - \frac{j\pi}{2})}\Phi\left(e^{-2i\pi\alpha}, j+1, 1-\frac{\mu}{2\pi i}\right) \right] \end{aligned} \tag{45}$$

with $i^2 = -1$ and $\Gamma(x)$ the eulerian gamma-function. This functional equation appears to be very useful since it expresses the relation between a Lerch-function, in terms of an exponential of an argument, and Lerch-functions in terms of that same argument.
On account of relation (45) and the fact that j is integer, we may write

$$\begin{aligned} Li_{-j}(e^{-\mu}) &= e^{-\mu}\Phi(e^{-\mu}, -j, 1) \\ &= \frac{j!}{\mu^{j+1}} + \frac{j!}{i(2\pi)^{j+1}} \left[e^{2i\pi - \frac{i\pi j}{2}}\Phi\left(e^{2i\pi}, j+1, 1+\frac{\mu}{2i\pi}\right) \right. \\ & \qquad \left. - e^{-2i\pi + \frac{i\pi j}{2}}\Phi\left(e^{-2i\pi}, j+1, 1-\frac{\mu}{2i\pi}\right) \right] \end{aligned} \tag{46}$$

Using $e^{2i\pi} = 1$, $e^{\frac{i\pi}{2}} = i$ and $e^{\frac{-i\pi}{2}} = -i$, this expression reduces to

$$\begin{aligned} Li_{-j}(e^{-\mu}) = \frac{j!}{\mu^{j+1}} + \frac{j!}{i(2\pi)^{j+1}} & \left[(-i)^j\Phi\left(e^{2i\pi}, j+1, 1+\frac{\mu}{2i\pi}\right) \right. \\ & \left. - i^j\Phi\left(e^{-2i\pi}, j+1, 1-\frac{\mu}{2i\pi}\right) \right] \end{aligned} \tag{47}$$

Notice that we did not apply these relations in the argument of the Lerch function, since it admits a branch in the complex plane on the real axis from 1 to $+\infty$. However, one may use the limiting case:

$$\lim_{z \to e^{\pm 2i\pi}} \Phi(z, n, \alpha) = \lim_{z \to e^{\pm 2i\pi}} \Gamma(1-n)(-\ln z)^{n-1}z^{-\alpha} + \zeta(n, \alpha) \tag{48}$$

with $\zeta(z, \alpha)$ Hurwitz' zeta-function.
Using relation (48) one may rewrite expression (47) for $j \neq 0$ in the following manner:

$$\begin{aligned} Li_{-j}(e^{-\mu}) = \frac{j!}{\mu^{j+1}} + \frac{j!}{(2i\pi)^{j+1}} & \left[\zeta\left(j+1, 1+\frac{\mu}{2i\pi}\right) \right. \\ & \left. + (-1)^{j+1}\zeta\left(j+1, 1-\frac{\mu}{2i\pi}\right) \right] \end{aligned} \tag{49}$$

Since j is a nonnegative integer the Hurwitz zeta-function, occurring in (49), can also be written as

$$\zeta(j+1,z) = \frac{(-1)^{j+1}}{j!}\psi_j(z) \tag{50}$$

with $\psi_j(z) \equiv \frac{d^j\psi_0(z)}{dz^j}$ the polygamma function. For this polygamma function there exists a recurrence relation

$$\psi_n(z+1) = \psi_n(z) + (-1)^n n! z^{-n-1} \tag{51}$$

and a reflection formula

$$\psi_n(1-z) = (-1)^n\psi_n(z) + (-1)^n\pi\frac{d^n\cot(\pi z)}{dz^n} \tag{52}$$

On account of the expressions (50),(51),(52) one may rewrite (49) as

$$Li_{-j}(e^{-\mu}) = \frac{(-1)^j}{2}\frac{d^j}{d\mu^j}\coth\left(\frac{\mu}{2}\right), \qquad j \neq 0 \tag{53}$$

From the explicit definition of $Li_0(z)$, one may easily derive that

$$Li_0(e^{-\mu}) = \frac{1}{2}\coth\left(\frac{\mu}{2}\right) - \frac{1}{2} \tag{54}$$

Let us also remark that

$$\frac{1}{\mu} = \frac{(-1)^j\mu^j}{j!}\frac{d^j}{d\mu^j}\frac{1}{\mu} \tag{55}$$

As a result it suffices to show that

$$\lim_{B^*\to\infty}\sum_{j=0}^{B^*-1}\frac{(-\mu)^j}{2j!}\frac{d^j}{d\mu^j}\left[\coth\left(\frac{\mu}{2}\right) - \frac{2}{\mu}\right] = 0 \tag{56}$$

In order to do this, we use the Laurent series for $\coth(x)$:

$$\coth\left(\frac{\mu}{2}\right) = \frac{2}{\mu} + \mu\sum_{n=1}^{\infty}\frac{1}{(n\pi)^2 + \frac{\mu^2}{4}} \tag{57}$$

Hence, the limit (56) may be rewritten as follows:

$$2\lim_{B^*\to\infty}\sum_{j=0}^{B^*-1}\frac{(-\mu)^j}{j!}\frac{d^j}{d\mu^j}\left[\mu\sum_{n=1}^{\infty}\frac{1}{(2n\pi)^2+\mu^2}\right] = 0 \tag{58}$$

or equivalently, when interchanging the sums over n and j and dropping the common factor 2:

$$\lim_{B^*\to\infty}\sum_{n=1}^{\infty}\sum_{j=0}^{B^*-1}\frac{(-\mu)^j}{j!}\frac{d^j}{d\mu^j}\left[\frac{\mu}{(2n\pi)^2+\mu^2}\right] = 0 \tag{59}$$

To prove this limit, we use the Taylor expansion of the fraction. However, one should be very careful with respect to the convergence of the series. Therefore we must split the sum over n into two parts:

$$\begin{aligned} &\sum_{n:(2n\pi)^2\leqslant\mu^2} \frac{1}{2n\pi} \sum_{j=0}^{B^*-1} \frac{(-\mu)^j}{j!} \frac{d^j}{d\mu^j} \left[\sum_{k=0}^{\infty} (-1)^k \left(\frac{2n\pi}{\mu}\right)^{2k+1} \right] \\ &\qquad + \sum_{n:(2n\pi)^2>\mu^2} \frac{1}{2n\pi} \sum_{j=0}^{B^*-1} \frac{(-\mu)^j}{j!} \frac{d^j}{d\mu^j} \left[\sum_{k=0}^{\infty} (-1)^k \left(\frac{\mu}{2n\pi}\right)^{2k+1} \right] \end{aligned} \tag{60}$$

It can be shown by induction on the parameter j that

$$\frac{(-\mu)^j}{j!} \frac{d^j}{d\mu^j} \left[\sum_{k=0}^{\infty} (-1)^k \left(\frac{2n\pi}{\mu}\right)^{2k+1} \right] = \sum_{k=0}^{\infty} (-1)^k \binom{2k+j}{j} \left(\frac{2n\pi}{\mu}\right)^{2k+1} \tag{61}$$

and that

$$\frac{(-\mu)^j}{j!} \frac{d^j}{d\mu^j} \left[\sum_{k=0}^{\infty} (-1)^k \left(\frac{\mu}{2n\pi}\right)^{2k+1} \right] = \sum_{k=0}^{\infty} (-1)^{k+j} \binom{2k+1}{j} \left(\frac{\mu}{2n\pi}\right)^{2k+1} \tag{62}$$

Now using the fact that

$$\sum_{j=0}^{B^*-1} \binom{2k+j}{j} = \binom{2k+B^*}{B^*-1} \tag{63}$$

and

$$\sum_{j=0}^{B^*-1} (-1)^j \binom{2k+1}{j} = (-1)^{B^*-1} \binom{2k}{B^*-1} \tag{64}$$

one may rewrite expression (60), on account of relations (61) and (62), as

$$\begin{aligned} &\sum_{n:(2n\pi)^2\leqslant\mu^2} \frac{1}{2n\pi} \sum_{k=0}^{\infty} (-1)^k \binom{2k+B^*}{B^*-1} \left(\frac{2n\pi}{\mu}\right)^{2k+1} \\ &\qquad + \sum_{n:(2n\pi)^2>\mu^2} \frac{1}{2n\pi} \sum_{k=0}^{\infty} (-1)^{k+B^*-1} \binom{2k}{B^*-1} \left(\frac{\mu}{2n\pi}\right)^{2k+1} \end{aligned} \tag{65}$$

It can also be shown that

$$\begin{aligned} &\sum_{k=0}^{\infty} (-1)^k \binom{2k+B^*}{B^*-1} \left(\frac{2n\pi}{\mu}\right)^{2k+1} \\ &\qquad = \sqrt{\left(\frac{\mu^2}{\mu^2+(2\pi n)^2}\right)^{B^*}} \sin\left(B^* \arctan\left[\frac{2\pi n}{\mu}\right]\right) \end{aligned} \tag{66}$$

and that

$$\begin{aligned} &\sum_{k=0}^{\infty} (-1)^{k+B^*-1} \binom{2k}{B^*-1} \left(\frac{\mu}{2n\pi}\right)^{2k+1} \\ &\qquad = \frac{(-1)^{B^*} \mu \sin(\pi B^*)}{2(B^*-1)n\pi^2} \, {}_3F_2\left(\frac{1}{2}, 1, 1 \,\middle|\, 1-\frac{B^*}{2}, \frac{3-B^*}{2} \,\middle|\, -\frac{\mu^2}{4n^2\pi^2}\right) \end{aligned} \tag{67}$$

with ${}_3F_2(a_1, a_2, a_3|b_1, b_2|x)$ a so-called hypergeometric function.
Since $n \geqslant 1$ it is clear that in the limit $B^* \to \infty$ the r.h.s. of the expressions (66) and (67) vanish, which proves relation (42)