The uniform distributions puzzle

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This note deepens a problem proposed and discussed by Kadane and O'Hagan (JASA, 1995). Kadane and O'Hagan discuss the existence of a uniform probability on the set of natural numbers (they provide a sufficient and necessary condition for the existence of such a uniform probability). I question the practical relevance of their solution. I show that a purely finitely additive measure on the set of natural numbers cannot be constructed (its existence needs some form of the Axiom of Choice, the prototype of a nonconstructive axiom).

KEY WORDS: Finitely additive probabilities; Charges; Axiom of choice; Constructivism; Natural numbers.

1 Introduction

Professors Joseph B. Kadane and Anthony O'Hagan (1995) formulate the uniform distributions puzzle as follows:

> Suppose that a computer scientist asks you for help with finding the probability that a random integer has some property, ... She says that by "random" she means something akin to a uniform distribution, in the sense that every integer, no matter how large, will have the same probability. How can such a question be answered in terms of modern probability theory?

This problem of a suitable definition of "pick a natural number at random" has caused considerable difficulty at the foundational level. Kadane and O'Hagan use relative frequencies and residue classes to make precise notions of uniformity, and they provide a necessary and sufficient condition for the "existence" of a finitely additive probability defined on all sets of natural numbers that agrees with these interpretations.

For a study of finitely additive measures, see Bhaskara Rao and Bhaskara Rao (1983). They indicate that a lot of the standard theory of measure and integration can be carried through assuming only finite additivity. For discussions on whether countable additivity should be dispensed with, see, for example, Savage (1954) and Dubins and Savage (1976). Wakker (1993), Stinchcombe (1997) and Kadane, Schervish, and Seidenfeld (1986) discuss some implications of the failure of countable additivity.

More than fourty years ago, Deal et al (1963) discuss the existence of a finitely additive probability on the set of natural numbers which attaches zero probability to all finite sets. They use a free ultrafilter[1] to define a finitely additive 0-1-probability measure (sets that belong to the ultrafilter obtain probability one, sets that do not belong to the ultrafilter obtain probability zero). A result of Sierpiński (1938), however, implies that each 0-1-probability (attaching zero probability to finite sets) generates a non-measurable set of real numbers. Since the existence of a non-measurable set hinges on the Axiom of Choice (Solovay, 1970), such a 0-1-probability cannot be constructed.

In view of this observation and—similar to Kadane and O'Hagen—having the computer scientist in mind, I want to deepen the uniform distributions puzzle: Is it possible to "construct" a finitely additive measure on the set of natural numbers which attaches zero probability to all finite sets? This note provides the answer: No!

I show that the uniform distributions puzzle cannot be solved in a constructive way. The result of Sierpiński extends to arbitrary finitely additive probabilities which attach zero probability to all finite sets. More precisely and in terms of set theory: the existence of such a measure is independent of the Zermelo-Fraenkel axioms of set theory together with the Axiom of Dependent Choice. Although finitely additive probabilities provide the right approach to the uniform distributions puzzle, each computer scientist is unable to

[1] A free ultrafilter on the set $\mathbb{N}$ of natural numbers is a collection $\mathcal{U}$ of subsets of $\mathbb{N}$ such that (i) $\mathbb{N} \in \mathcal{U}$ and $\varnothing \notin \mathcal{U}$, (ii) if $A \subseteq B$ and $A \in \mathcal{U}$, then $B \in \mathcal{U}$, (iii) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$, and (iv) for each $A \subseteq \mathbb{N}$, either A or $\mathbb{N} - A$ belongs to $\mathcal{U}$. The existence of a free ultrafilter follows from Zorn's Lemma.

"implement" such a uniform distribution. Results on the "existence" of a finitely additive probability do not provide any help.

The next section touches the notion of constructivism and recalls the Axiom of Choice and the Axiom of Dependent choice. Section 3 states and proves the main result.

2 Constructivism

In constructive mathematics, a problem is considered as solved only if its solution can be explicitly produced. Let me provide a typical example.

Claim. There exist two irrational real numbers a and b such that a^b is rational.

Proof. The real number $c = \sqrt{2}^{\sqrt{2}}$ *either* is rational, and then we put $a = b = \sqrt{2}$; *or* is not rational, and then we put $a = c$ and $b = \sqrt{2}$.

Classical mathematics accepts this proof. On the other hand, this proof uses the law of the excluded middle which states the truth of 'P or not-P' for each proposition P (i.c. the real number c is rational) and does not provide an explicit way to indicate which pair of real numbers satisfies the claim. Therefore, within constructivism this proof is rejected. In constructivist logic the classical law of the excluded middle is not assumed. Of course, constructive mathematics also rejects principles that are stronger than the law of the excluded middle. The Axiom of Choice (AC) is such a principle.

AC postulates for each nonempty family $\mathcal{D}$ of nonempty sets the existence of a function f such that $f(S) \in S$ for each set S in the family $\mathcal{D}$. The function f is referred to as a choice function. AC does not provide an explicit way to construct such a choice function and provoked considerable criticism in the aftermath of Zermelo's formulation in 1904.

AC is (i) consistent: AC can be added to the Zermelo-Fraenkel axioms of set theory (ZF) without yielding a contradiction; and (ii) independent: AC is not a theorem of ZF (Fraenkel et al, 1973, Section II.4.2). Among the applications of AC, we mention Zorn's Lemma, the theorem of Hahn-Banach, and the existence of free ultrafilters. AC also implies a number of paradoxes such as the decomposition of a sphere into a sphere of smaller size, and the existence of a non-measurable set of real numbers.

Constructive mathematics rejects AC. On the other hand, the Axiom of Dependent Choice (DC) is generally accepted by constructivists (Beeson, 1988, p. 42). Let S be a nonempty set and let R be a binary relation in S such that for each a in S there is a b in S with $(a, b) \in R$. Then, DC postulates the existence of a sequence $(a_1, a_2, \ldots, a_n, \ldots)$ of elements in S such that $(a_k, a_{k+1}) \in R$ for each $k = 1, 2, \ldots$.

The nonconstructive object used in this note is known as a non-Ramsey set. Let I be an infinite set and let n be a positive integer. Let $[I]^n$ collect all the subsets of I with exactly n elements. Ramsey (1928) shows that for each subset S of $[I]^n$, there exists an infinite set $J \subset I$ such that either $[J]^n \subset S$ or $[J]^n \cap S = \varnothing$. When n is replaced by countable infinity, then Ramsey's theorem fails. There exists a subset S of $[I]^\infty$ such that for each infinite subset J of I the class $[J]^\infty$ intersects S and its complement $[I]^\infty - S$ as well. Such a set S is said to be non-Ramsey. Mathias (1977) showed that the existence of non-Ramsey sets does not follow from ZF (without AC).[2]

3 Finitely additive probabilities

Let $\mathbb{N}$ be the set of natural numbers. Let $\mathcal{F}$ be the field of all subsets of $\mathbb{N}$. A finitely additive probability is a map $\mu : \mathcal{F} \to \mathbb{R}^+$ that satisfies $\mu(\mathbb{N}) = 1$ and the condition

$$\mu\Big(A_1 \cup A_2 \cup \ldots \cup A_n\Big) \;=\; \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n),$$

where the sets $A_1, A_2, \ldots, A_n$ are disjoint, for all finite n. We now state the main result of this note.

Theorem. Let μ be a finitely additive probability on $\mathbb{N}$ that attaches zero probability to each natural number. Then, μ generates a non-Ramsey set.

Proof. We start with some additional notation. For two natural numbers $i > j$, let $[\,i, j[$ denote the set $\{i, i+1, \ldots, j-1\}$. Furthermore, to each infinite set $A \subseteq \mathbb{N}$ we connect a set, denoted by A_0, as follows. Let $A = \{n_0, n_1, \ldots, n_k, \ldots\}$ with $n_k < n_{k+1}$ for each k, then $A_0 = [\,n_0, n_1[\, \cup [\,n_2, n_3[\, \cup \ldots \cup [\,n_{2k}, n_{2k+1}[\, \cup \ldots$.

Now, let μ satisfy the requirements listed in the theorem. We show that

$$S = \{A \subseteq \mathbb{N} \,|\, \mu(A_0) > 0.5\}$$

is a non-Ramsey set. It is sufficient to show that each infinite set $A = \{n_0, n_1, \ldots, n_k, \ldots\}$ includes an infinite subset B such either A or B belongs to S (the 'either-or' being exclusive). We distinguish three cases.

Case 1. $A \notin S$, in particular $\mu(A_0) < 0.5$. Let $B = A - \{n_0\}$. Then, $[\,0, n_0[\, \cup A_0 \cup B_0 = \mathbb{N}$. Since $\mu([\,0, n_0[) = 0$, $\mu(A_0) < 0.5$, and $\mu(\mathbb{N}) = 1$; we obtain that $\mu(B_0) > 0.5$. Therefore, $B \subseteq A$ and $B \in S$.

Case 2. $A \notin S$, in particular $\mu(A_0) = 0.5$. Let $B = \{n_0, n_3, n_4, n_7, \ldots, n_{4k}, n_{4k+3}, \ldots\}$ and let $B' = \{n_0, n_1, n_2, n_5, n_6, n_9, \ldots, n_{4k+2}, n_{4k+5}, \ldots\}$. Then, $A_0 = B_0 \cap B'_0$. Hence, we have $\mu(B_0) \geq 0.5$ and $\mu(B'_0) \geq 0.5$. Furthermore,

$$[\,0, n_0[\, \cup (B_0 \,\Delta\, B'_0) \;\cup\; A_0 \;=\; \mathbb{N}.$$

Conclude that the symmetric difference $B_0 \,\Delta\, B'_0$ has a measure equal to 0.5. Hence, at least one of the sets B_0 or B'_0 has a measure strictly larger than 0.5. Select the subset B or B' of A for which the corresponding set B_0 or B'_0 has the highest measure. The selected subset of A belongs to S.

[2]More precisely, Solovay (1970) proposed a model in which ZF and DC are true and in which AC fails. Mathias showed that in this Solovay-model a non-Ramsey set does not exist. Hence, the existence of a non-Ramsey set is independent of ZF + DC.

Case 3. $A \in S$. Similar to Case 1, we put $B = A - \{n_0\}$. Conclude that $B \subseteq A$ and that $B \notin S$. $\square$

The next summary closes the paper. If each natural number has the same probability, then this probability is zero and the resulting probability distribution μ is finitely additive and not countably additive (observe the inequality $\mu(\cup_0^\infty\{k\}) > \sum_0^\infty \mu(\{k\}))$. Whether one uses the limiting relative frequency, the residue sets, or translation invariance as an interpretation of uniformity (Kadane and O'Hagan, 1995), the above Theorem applies: uniformity of the natural numbers cannot be constructed.

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