# DERIVATIVES WITH STOCHASTIC VOLATILITY MODELS

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*In our paper we build a reccurence from generalized Garman equation and discretization of 3-dimensional domain. From reccurence we build an algorithm for computing values of an option based on time, momentan volatility of support and value of support on a spot market of an exchange.*

*Keywords: financial derivatives, Black-Scholes PDE, Garman PDE, reccurence, algorithm*

*JEL classification: G13, C69*

## 1. Garman functions attached on generalized model with stochastic volatility

In Black-Scholes model[623] we supose that volatility is a constant value. Unfortunately, volatility is not constant, and more, can not be predicted for periods greater than several months or even can not be observed directly. For these reasons, it seems natural to present itself as the volatility of a random variable that satisfies a stochastic process. Assume that the variation rate asset support satisfy a stochastic differential equation (SDE) of the form:

$$dS(t) = A(t,S(t),V(t))dt+B(t,S(t),V(t))dW_1(t) \qquad (1)$$

but in addition we assume that the square of volatility (v), follows the stochastic process:

$$dv(t) = C(t,S(t),V(t))dt+D(t,S(t),V(t))dW_2(t), \qquad (2)$$

where the two *Wiener processes* $W_1(t)_{t\geq 0}$ şi $W_2(t)_{t\geq 0}$ are correlated:

$$(dW_1(t))(dW_2(t)) = \rho dt.$$

In this case the value of an option is a function of three variables, V (t, S, v) and because we have two sources of risk need to build a portfolio of coverage to include the option in question and two other components. Assume the existence of a derivative financial unca on the same support, so that the value of this option is a function of three variables, V1 (t, s, v) to check with a law counterparts of V (t, S, v). After applying extended lemma of Ito[624], we obtain the existance of a special value β named as *the market price of volatility risk*[625], where:

$$\beta = (-rV+V_S rS+V_t+AV_S+CV_v+\tfrac{1}{2}B^2V_{SS}+\tfrac{1}{2}D^2V_{vv}+\rho BDV_{Sv}-AV_S)/(DV_v) \qquad (3)$$

If β is known, (3) rewrite as:

$$V_t+rSV_S+(C-D\beta)V_v+\tfrac{1}{2}B^2V_{SS}+\tfrac{1}{2}D^2V_{vv}+\rho BDV_{Sv}-rV = 0 \qquad (4)$$

623 see [Black1973].
624 see [Socaciu2009a].
625 see [Moodley2005], p. 8.

is known as *Garman equation*. Note that (4) is a partial differential equation (PDE), like *Black-Scholes* equation, obtained from SDE of model.

## 2. Particular cases of Garman functions

1. *Black-Scholes*; in conditions:

$$B^2 = vS^2$$
$$C = D = 0 \Leftrightarrow dv = 0 \text{ (constant volatility)}$$
$$v = \sigma^2$$

we re-obtain Black-Scholes equation[626]:

$$V_t + rSV_S + ½\sigma^2 S^2 V_{SS} - rV = 0$$

2. *Heston*; in conditions:

$$A = \mu S$$
$$B = v^{½} S$$
$$C = k(\theta - v)$$
$$D = \xi v^{½}$$
$$\beta = \lambda v^{½}$$

we re-obtain[627]:

$$V_t + rSV_S + (k(\theta - v) - \xi\lambda v)V_v + ½vS^2 V_{SS} + ½\xi^2 v V_{vv} + \rho S\xi v V_{Sv} - rV = 0 \quad (5)$$

3. *other cases* for particular models can be obtained via particular A, B, C, D and β. An incomplete list of models[628] can be retrieved from [Sundaresan2000][629].

## 3. Garman equation on domains' frontier

Equation (5) has on frontiers[630]:

$$V(t, 0, v) = 0, \text{ with } 0 \le t \le T, \text{ all } v. \quad (6)$$

and:

$$V(T, S, v) = \text{payoff}(S), \text{ with } 0 \le S, \text{ all } v. \quad (7)$$

We need add a new condition on v axis. Simplest condition is based on zero-volatility. In this case, (2) is vanished, and (1) will be rewrite as a regullar BS equation with a specific solutions like[631]:

$$V(t, S, 0) = BS\ (t, S) \quad (8)$$

## 4. Domain's discretization and reccurence obtained from Garman equation

We will apply a *Finite Element Method* (FEM[632]) for resolve (5). We know that FEMs are main method for resolve PDE[633]. First step is to reduce initial domain $D_i$:

$$D_i = [0,T] \times R_+ \times R_+$$

626 see [Black1973].
627 see [Lazăr2006], [Mikhailov2003].
628 with specification of A and B.
629 an other incomplete list can be retrieved in [Lazăr2006]).
630 like in [Socaciu2009b].
631 Numerical solutions will be presented in [Socaciu2009a], [Socaciu2009b] and [Socaciu2009c].
632 see [Gârbea1990].
633 see [Coman1994].

to a *limited domain* D:

$$D = [0,T] \times [0,MaxS] \times [0,MaxV],$$

where MaxS is a *big number* that cover all values of potential quotations, and MaxV is an big number that cover all potential volatilities. After discretization with $\Delta t$, $\Delta S$ and $\Delta v$ *steps* we will obtain a grid of points. We denote:

$$V[i,j,k] = V(i(\Delta t),j(\Delta S),k(\Delta v)), \text{ with } 0 \le i \le M,\ 0 \le j \le N,\ 0 \le k \le P$$

where number of points on axis:

$$M := MaxS/(\Delta S)$$
$$N := T/(\Delta t)$$
$$P := MaxV/(\Delta v)$$

than (4) will be rewrite as:

$$\begin{aligned}
&(V[i,j,k]-V[i-1,j,k])/(\Delta t)+rj(\Delta S)(V[i,j+1,k]-V[i,j-1,k])/(2(\Delta S))\\
&+(C-D\beta)\ (V[i,j,k+1]-V[i,j,k-1])/(2(\Delta v))\\
&+\tfrac{1}{2}B^2(V[i,j+1,k]-2V[i,j,k]+V[i,j-1,k])/(\Delta S)^2\\
&+\tfrac{1}{2}D^2(V[i,j,k+1]-2V[i,j,k]+V[i,j,k-1])/(\Delta v)^2\\
&+\rho BD(V[i,j+1,k+1]-V[i,j+1,k-1]-V[i,j-1,k+1]+V[i,j+1,k+1])/((\Delta S)(\Delta v))\\
&-rV[i,j,k] = 0
\end{aligned}$$

That can be rewrite as:

$$\begin{aligned}
&a(V[i,j,k]-V[i-1,j,k])+b(V[i,j+1,k]-V[i,j-1,k])+c(V[i,j,k+1]-V[i,j,k-1])\\
&+d(V[i,j+1,k]-2V[i,j,k]+V[i,j-1,k])+e(V[i,j,k+1]-2V[i,j,k]+V[i,j,k-1])\\
&+f(V[i,j+1,k+1]-V[i,j+1,k-1]-V[i,j-1,k+1]+V[i,j+1,k+1])-gV[i,j,k] = 0
\end{aligned} \tag{9}$$

where:

$$\begin{aligned}
a &= (\Delta t)(\Delta S)^2(\Delta v)^2\\
b &= \tfrac{1}{2}rj(\Delta t)(\Delta S)^2(\Delta v)^2\\
c &= \tfrac{1}{2}(C-D\beta)(\Delta t)(\Delta S)^2(\Delta v)\\
d &= \tfrac{1}{2}B^2(\Delta t)(\Delta v)^2\\
e &= \tfrac{1}{2}B^2(\Delta t)(\Delta v)^2\\
f &= \rho BD(\Delta t)(\Delta v)(\Delta S)\\
g &= r(\Delta t)(\Delta v)^2(\Delta S)^2
\end{aligned}$$

But (9) can be rewrite in explicit form:

$$\begin{aligned}
V[i-1,j,k] &= pV[i,j,k]+qV[i,j+1,k]+rV[i,j-1,k]+sV[i,j,k+1]+tV[i,j,k-1]\\
&+uV[i,j+1,k+1]-uV[i,j+1,k-1]-uV[i,j-1,k+1]+uV[i,j+1,k+1]
\end{aligned} \tag{10}$$

where:

$$\begin{aligned}
p &= (a-2d-2e-g)/a\\
q &= (b+d)/a\\
r &= (d-b)/a\\
s &= (c+e)\\
t &= (e-c)\\
u &= f/a
\end{aligned}$$

Note that coefficients p-u are based on t, S and v, and we must rewrite all of them in spirit of (i,j,k).

**5. Numerical algorithm based for Garman equation based on triple reccurence**

Numerical algorithm is based on (10) reccurence with supplementally (6)-(8) frontier condition is:

```
For i:=MaxM downto 0 do
  For j:=0 to MaxN+MaxM do
    For k:=0 to MaxP+MaxM do
      If i=MaxM then
        V[i,j,k]:=payoff(j(ΔS))
      ElseIf j=0 then
        V[i,j,k]:=0
      ElseIf k=0 then
        V[i,j,k]:=BS(i,j)
      Else
        V[i,j,k]:=pV[i+1,j,k]+qV[i+1,j+1,k]
        +rV[i+1,j-1,k]+sV[i+1,j,k+1]+tV[i+1,j,k-1]
        +uV[i+1,j+1,k+1]-uV[i+1,j+1,k-1]-uV[i+1,j-1,k+1]
        +uV[i+1,j+1,k+1]
      End If
    End For
  End For
End For
```

where BS is a function that call a numerical solution of Black-Scholes like equation on zero-volatility condition. Note that iteration will be computed for j to MaxN+MaxM and for k to MaxP+MaxM because we have some "out of range" problems if we compute V[0,MaxN,MaxP] and cheap resolvation is to compute a supradomain of D.

**6. Further works**

We work on obtaining parallel version of this algorithm with slicing of D like in [Socaciu2009a], [Socaciu2009b] and [Socaciu2009c]. We try to obtain similar formulae for a multi-SDE model, like Chen[634]. We want to start a parallel project for obtain on symbolic way an extendended Ito type lemma[635] and from this lemma to build an Garman-type equation.

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634 See [Socaciu2009a], [Chen1996].
635 See [Socaciu2009a], [Iftimie2008].

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