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Nash Networks with Heterogeneous Agents[1]

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Abstract

A non-cooperative model of network formation is developed. Agents form links with others based on the cost of the link and its assessed benefit. Link formation is one-sided, i.e., agents can initiate links with other agents without their consent, provided the agent forming the link makes the appropriate investment. Information flow is two-way. The model builds on the work of Bala and Goyal, but allows for agent heterogeneity. Whereas they permit links to fail with a certain common probability, in our model the probability of failure can be different for different links. We investigate Nash networks that exhibit connectedness and super-connectedness. We provide an explicit characterization of certain star networks. Efficiency and Pareto-optimality issues are discussed through examples. We explore alternative model specifications to address potential shortcomings.

JEL Classification: D82, D83

1 Introduction

The *Internet* provides ample testimony to the fact that information dissemination affects all aspects of economic activity. It is creating globalization that has hitherto been unprecedented in human history. Nowadays fashions and fads emerging in one country are easily communicated across the world with almost no time lag. Financial troubles in one country now have devastating consequences for other economies as the contagion moves across boundaries with relative ease. Yet, the East Asian financial crisis also demonstrated that economies where information networks were relatively primitive remained largely insulated from the crisis. This indicates that both the structure and the technology of information dispersion are important determinants of its consequences.

In the present paper, we look at the formation of social networks which serve as a mechanism for information transmission. The structural aspects of information dissemination are modelled by means of a social network. The role of technology is studied by examining the reliability of the network. Social networks have played a vital role in the diffusion of information across society in settings as diverse as referral networks for jobs (Granovetter (1974) and Loury (1977)) and in assessing quality of products ranging from cars to computers (Rogers and Kincaid (1981)). Information in most societies can either be obtained in the market-place or through a non-market environment like a social network. For instance, in developed countries credit agencies provide credit ratings for borrowers, while in many developing countries credit worthiness is assessed through a social network organized along ethnic lines.

Agents in our model are endowed with some information which can be accessed by other agents forming links with them. Link formation is costly and links transmit information randomly. More precisely, agents in our model can form links and participate in a network by incurring a cost for each link, which may be interpreted in terms of time, money or effort. The cost of establishing a link is incurred only by the agent who initiates it, and the initiating agent has access to the other agent's information with a certain probability. In addition, he has access to the information from all the links of the other agent. Thus each link can generate substantial positive externalities of a non-rival nature in the network. Moreover, the flow of benefits through a link occurs both ways. It can differ across agents, since the strength of ties varies across agents (although all links cost the same) and links fail with possibly different probabilities. This reflects the fact that in reality, communication often embodies a degree of costly uncertainty. We frequently have to ask someone to reiterate what they tell us, explain it again and even seek second opinions.

Foreign immigrants often form such networks. When an immigrant lands on the shores of a foreign country he usually has a list of people from the home country to get in touch with. Once contacted, some compatriots are more helpful than others. Often a substantial information exchange takes place in this process, where the new arrival learns about the foreign country, while providing the established immigrants current information about the home country and an opportunity to indulge in nostalgia.

Bala and Goyal (2000a, b) suggest telephone calls as an example of such networks. Another example (especially of the star networks considered here) of this kind is a LISTSERVE or an e-mail system. Costs have to be incurred in setting up and joining such electronic networks, but being a part of the network does not automatically ensure access to the information of other agents. Member participation rates in an electronic network often vary, and messages may get lost as in the celebrated "e-mail game" (Rubinstein, (1989)).[1]

Motivated by these examples, we, like Bala and Goyal (2000a, b) develop a non-cooperative model of network formation which generalizes theirs. The non-cooperative game formulation of network formation models typifies one of three strands of literature of major concern to us. The two other strands that have recently emerged in the context of economics and game theory are differentiated by their use of cooperative game theory and the notion of pairwise stability, repectively. The early cooperative literature treats costs as a set of constraints on coalition formation (see for example, Myerson (1977), Kalai *et al.* (1978) and Gilles *et al.* (1994)). An excellent survey of that literature can be found in van den Nouweland (1993), and Borm, van den Nouweland and Tijs (1994). Aumann and Myerson (1988) were the first to incorporate both costs and benefits of coalition formation. This line of research has been extended by Slikker and van den Nouweland (2000).

[1]Like most of the networks literature we shall preclude the possibility of harmful information, like nuisance phone calls. The same is assumed for intermediate agents or indirect links in a network who function as purveyors of information between other agents without incurring any disutility.

Jackson and Wolinsky (1996) introduced the concept of pairwise stability (known from the matching literature) as an equilibrium concept in models of network formation. This gave rise to a completely new strand of the literature focusing on the tension between stability and efficiency. Pairwise stability requires mutual consent of a pair of agents for link formation whereas links can be deleted unilaterally. Dutta and Muttuswami (1997) and Watts (1997) refine the Jackson-Wolinksy framework further by introducing other stability concepts and derive implementation results for these concepts. Johnson and Gilles (2000) introduce a spatial dimension to the Jackson-Wolinsky model through spatial costs of link formation.

Several dynamic models using pairwise stability have been investigated as well, starting with Jackson and Watts (1998). Jackson and Watts (1999) and Goyal and Vega-Rodondo (1999) consider coordination games played on a network. The choice of partners in the game is endogenous and players are periodically allowed to add or sever links. Droste *et al.* (2000) also analyze coordination games played on a network — with spatial costs of link formation.

The non-cooperative version of network formation has first been developed in two papers by Bala and Goyal (2000a, b). In all cases, agents choose to form links on the basis of costs and a (deterministic or stochastic) flow of benefits that accrue from links. Bala and Goyal assume that a player can create a one-sided link with another player by making the appropriate investment. Their assumption differs fundamentally from the concept of pairwise stability since mutual consent of both players is no longer required for link formation. They further investigate the reliability issue in networks by allowing links to fail independently of each other with a certain probability. Links are deterministic in Bala and Goyal (2000a). They are random, with identical probabilities of failure for all established links, in Bala and Goyal (2000b). Thus both their models deal with homogeneous agents. The corresponding static equilibrium outcomes are called Nash networks.[2]

Our model also belongs to the non-cooperative tradition and is a generalization of Bala and Goyal (2000b). We introduce agent heterogeneity by allowing for the probability of link failure (or success) to differ across links. This distinctive feature reflects the nature of the transmission tech-

[2]They also identify strict Nash networks and study the formation of Nash networks in a modified version of best-response dynamics.

nology or the quality of information. The generalization provides a richer model in terms of answering theoretical as well as practical questions: connectivity and super-connectivity, selection of central agents in star networks, efficiency, and Pareto-optimality. Besides imparting greater realism to the model, the introduction of heterogeneous agents allows us to check the robustness of the conclusions obtained in Bala and Goyal (2000b). Whereas their findings still hold under certain conditions, heterogeneity gives rise to a greater variety of equilibrium outcomes, tends to alter results significantly and even generates some of the results of Bala and Goyal's (2000a) deterministic model.

Bala and Goyal show for both their models that Nash networks must be either connected or empty. With heterogeneous agents, this proves true only when the probabilities of success are not very different from each other. The range in which the probabilities must lie depends on the cost of links and the cardinality of the player set. Another central finding of Bala and Goyal is that compared to information decay imperfect reliability has very different effects on network formation. With information decay, minimally connected networks (notably the star) are Nash for a wide range of cost and decay parameters, independently of the size of society. In contrast, with imperfect reliability and small link formation costs, minimally connected networks tend to be replaced by super-connected networks (connected networks with redundant links) as the player set increases. However, with agent heterogeneity neither connectedness nor super-connectedness need arise asymptotically. Furthermore, in order for star networks to be Nash, probabilities must lie in a certain range and exceed costs as in Bala and Goyal, but we find that as a rule, they have to satisfy additional conditions. In particular, it never pays in the Bala and Goyal framework to connect to the center of the star indirectly. In our context, however, such a connection might be beneficial and further conditions on probabilities are required to prevent these connections. Interestingly enough, heterogeneity helps resolve a particular ambiguity associated with the homogeneous model: Owing to the additional equilibrium conditions, the coordination problem inherent in selecting the central agent of a star is mitigated to a certain degree.

We also investigate efficiency issues and find that Nash networks may be nested and Pareto-ranked. We demonstrate by example that inefficient Nash networks can be Pareto-optimal. Criticisms of the non-cooperative approach to network formation are addressed as well. We extend the model to allow for duplication of links and to analyze Nash networks with incomplete information. We find that redundant links will be established when the agents beliefs about the probabilities of the indirect links are lower than the actual probabilities. Thus network failure can arise just like market failures. Finally, the implications of mutual consent, negative externalities, and endogenous probabilities for Nash networks are discussed.

The network literature almost completely lacks models with heterogeneous agents, with the notable exception of Johnson and Gilles (2000) and Droste *et al.* (2000) who introduce spatial heterogeneity of agents and obtain results substantially different from both static and dynamic versions of homogeneous-agent pairwise stability models. Their model and ours differ in two respects: the kind of agent heterogeneity and the equilibrium concept. They follow Jackson and Wolinsky (1996) and use *pairwise stability* as the equilibrium concept. We analyze Nash networks.

In Section 2, we introduce the basic notation and terminology used throughout the paper. In Section 3, we present some general results on Nash networks. Alternative formulations of the model are considered in Section 4. Section 5 concludes. Section 6 contains proofs and derivations.

2 The Model

Let $N = \{1, \ldots, n\}$ denote the set of agents, with generic members i and j. For ordered pairs $(i, j) \in N \times N$, the shorthand notation ij is used. The symbol $\subset$ for set inclusion permits equality. We assume throughout that $n \geq 3$. Each agent has some information of value to the other agents. An agent can get access to more information by forming links with other agents. Agents form their links simultaneously. The formation of links is costly. Each link denotes a connection between a pair of agents which is not fully reliable. It may fail to transmit information with a positive probability that can differ across links.

Each agent's strategy is a vector $g_i = (g_{i1}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in})$ where $i \in N$ and $g_{ij} \in \{0, 1\}$ for each $j \in N \backslash \{i\}$. The value $g_{ij} = 1$ means that agents i and j have a link initiated by i whereas $g_{ij} = 0$ means that agent i does not initiate the link. This does not preclude the possibility of agent j initiating a link with i. A link between agents i and j potentially allows for **two-way (symmetric) flow of information**. The set of all pure strategies of agent i is denoted by $\mathcal{G}_i$. We focus only on pure strategies in this paper.

Given that agent i has the option of forming or not forming a link with each of the remaining $n-1$ agents, the number of strategies available to agent i is $|\mathcal{G}_i| = 2^{n-1}$. The strategy space of all agents is given by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. A strategy profile $g = (g_1, \ldots, g_n)$ can be represented as a **directed graph** or **network**. Notice that there is a one-to-one correspondence between the set of all directed networks with n vertices or nodes and the set of strategies $\mathcal{G}$. The link g_{ij} will be represented pictorially by an edge starting at j with the arrowhead pointing towards i to indicate that agent i has initiated the link. The reader may refer to Figure 2 shown below where agents 1 and 2 establish the links with agent 6 and agents 3 and 4 establish the links with agent 7. Consequently, the cost of forming these links are borne by agents 1, 2, 3 and 4 and the arrowhead always points towards the agent who pays for the link.

To describe information flows in the network, let for $i \in N$ and $g \in \mathcal{G}$, $\mu_i^d(g_i) = |\{k \in N : g_{ik} = 1\}|$ denote the number of links in g initiated by i which is used in the determination of i's costs. Next we define the closure of g which is instrumental for computing benefits, since we are concerned with the symmetric, two-way flow of benefits. Pictorially the closure of a network is equivalent to replacing each directed edge of g by a non-directed one.

Definition 1 *The **closure** of g is a non-directed network denoted by $h = cl(g)$ and defined as $cl(g) = \{ij \in N \times N : i \neq j \text{ and } g_{ij} = 1 \text{ or } g_{ji} = 1\}$.*

Benefits. The benefits from network g are derived from its closure $h = cl(g)$. Each link $h_{ij} = 1$ succeeds with probability $p_{ij} \in (0, 1)$ and fails with probability $1 - p_{ij}$ where p_{ij} is not necessarily equal to p_{ik} for $j \neq k$. It is assumed, however, that $p_{ij} = p_{ji}$. Furthermore, the success or failure of different links are assumed to be independent events. Thus, h may be regarded as a random network with possibly different probabilities of realization for different edges. We define h' as a realization of h (denoted by $h' \subset h$) if for all i, j with $i \neq j$ we have $h'_{ij} \leq h_{ij}$.

At this point the concept of a path (in h') between two agents proves useful.

Definition 2 *For $h' \subset h$, a **path** of length m from an agent i to a different agent j is a finite sequence $i_0, i_1, \ldots, i_m$ of pairwise distinct agents such that $i_0 = i$, $i_m = j$, and $h'_{i_k i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. We say that player i **observes** player j in the realization h', if there exists a path from i to j in h'.*

Invoking the assumption of independence, the probability of the network h' being realized given h is

$$\lambda(h' \mid h) = \prod_{ij \in h'} p_{ij} \prod_{ij \in h \setminus h'} (1 - p_{ij}).$$

Let $\mu_i(h')$ be the number of players that agent i observes in the realization h', i.e. the number of players to whom i is directly or indirectly linked in h'. Each observed agent in a realization yields a benefit $V > 0$ to agent i. Without loss of generality assume that $V = 1$.[3]

Given the strategy tuple g agent i's expected benefit from the random network h is given by the following benefit function $B_i(h)$:

$$B_i(h) = \sum_{h' \subset h} \lambda(h' \mid h) \mu_i(h')$$

where $h = cl(g)$. The probability that network h' is realized is $\lambda(h' \mid h)$, in which case agent i gets access to the information of $\mu_i(h')$ agents in total. Note that the benefit function is clearly non-decreasing in the number of links for all the agents.

Payoffs. We assume that each link formed by agent i costs $c > 0$. Agent i's expected payoff from the strategy tuple g is

$$\Pi_i(g) = B_i(cl(g)) - \mu_i^d(g_i)c. \tag{1}$$

Given a network $g \in \mathcal{G}$, let g_{-i} denote the network that remains when all of agent i's links have been removed. Clearly $g = g_i \oplus g_{-i}$ where the symbol $\oplus$ indicates that g is formed by the union of links in g_i and g_{-i}.

[3] Another formulation could be used to obtain agent heterogeneity. Under this formulation, the value of agent i's information would be given by V_i which differs across agents, while p, the probability of the link success, is identical for all agents $i \in N$. The direct expected benefit from a link g_{ij} to agent i would now be given by pV_j which would then differ across links. This amounts to assuming that $p_{ij} \neq p_{ji}$ and would add another layer of heterogeneity. In contrast, Johnson and Gilles (2000) assume $p = 1$ and $V = 1$, with differing costs based on a spatial distribution of agents.

Definition 3 *A strategy g_i is said to be a **best response** of agent i to g_{-i} if*

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g_i' \oplus g_{-i}) \text{ for all } g_i' \in \mathcal{G}_i.$$

Let $BR_i(g_{-i})$ denote the set of agent i's best response to g_{-i}. A network $g = (g_1, \ldots, g_n)$ is said to be a **Nash network** if $g_i \in BR_i(g_{-i})$ for each i, i.e., agents are playing a Nash equilibrium. A strict Nash network is one where agents are playing strict best responses.

Agent i's benefit from the direct link ij to agent j is at most $p_{ij}(n-1)$. Set $p_0 = p_0(c, n) = c \cdot (n-1)^{-1}$. If $p_{ij} < p_0$, it never benefits agent i to initiate a link from i to j, no matter how reliably agent j is linked to other agents and, therefore, $g_{ij} = 0$ in any Nash equilibrium g.

We now introduce some additional definitions which are of a more graph-theoretic nature. A network g is said to be **connected** if there is a path in $h = cl(g)$, between any two agents i and j. A connected network g is said to be **super-connected**, if there exist links after whose deletion the network is still connected. A connected network g is **minimally connected**, if it is no longer connected after the deletion of *any* link. A network g is called **complete**, if all links exist in $cl(g)$. A network with no links is called an **empty network**.

Definition 4 *A set $C \subset N$ is called a **component** of g if there is a path in $cl(g)$ between any two agents i and j in C and there is no strict superset C' of C for which this is true.*

The commonly used welfare measure is defined as the sum of the payoffs of all the agents. Formally, let $W : \mathcal{G} \to \mathcal{R}$ be defined as

$$W(g) = \sum_{i=1}^{n} \Pi_i(g) \text{ for } g \in \mathcal{G} \ .$$

Definition 5 *A network g is **efficient** if $W(g) \geq W(g')$ for all $g' \in \mathcal{G}$.*

An efficient network is one that maximizes the total value of information available to all agents net of the aggregate costs of forming the links. The definition of (strong) Pareto-optimality is the usual one: A network g is

Pareto-optimal, if there does not exist another network g' such that $\Pi_i(g') \geq \Pi_i(g)$ for all i and $\Pi_i(g') > \Pi_i(g)$ for some i. Obviously, every efficient network is Pareto-optimal. However, we will show that not every Pareto-optimal network is efficient. In fact, we provide an example of a Pareto-optimal Nash network which is inefficient, while the unique efficient network is not Nash.

We finally introduce the notion of an essential network. A network $g \in \mathcal{G}$ is **essential** if $g_{ij} = 1$ implies $g_{ji} = 0$. Note that if $c > 0$ and $g \in \mathcal{G}$ is a Nash network or an efficient network, then it must be essential. This follows from the fact that the benefits from a link are given by the closure of the link $h_{ij} = \max\{g_{ij}, g_{ji}\}$ (making the probability of failure independent of whether it is a single link or a double link) and from the fact that the information flow is symmetric and independent of which agent invests in forming the link. If $g_{ij} = 1$, then by the definition of Π_j agent j pays an additional cost c for setting $g_{ji} = 1$, while neither he nor anyone else gets any benefit from it. Hence if g is not essential it cannot be Nash or efficient.

3 Nash Networks

In this section we look at Nash networks. We begin with an analysis of connectedness and redundancy in Nash networks. Then we identify conditions under which the complete network and the empty network, respectively, will be Nash. This is followed by a subsection that covers the popular star networks. We also discuss efficiency issues by means of examples.

3.1 Connectivity and Super-Connectivity

With homogeneous agents, Nash networks are either connected or empty (Bala and Goyal (2000b)). With heterogeneous agents, this dichotomy does not always hold. The proposition below identifies conditions under which Nash networks will show this property.

PROPOSITION 1: *If* $p_{ij} \geq \dfrac{1}{1 + c/n^2} p_{mk}$ *for any* $i \neq j$ *and* $m \neq k$, *then every Nash network is either empty or connected.*

Proof: Consider a Nash network g. Suppose g is neither empty nor connected. Then there exist three agents i, j, and k such that i and j belong to one connected component of $cl(g)$, C_1 and k belongs to a different connected component of $cl(g)$, C_2. Then $g_{ij} = 1$ or $g_{ji} = 1$, whereas $g_{mk} =$

$g_{km} = 0$ for all $m \in C_1$. Without loss of generality assume $g_{ij} = 1$. Then the incremental benefit to i of having the link from i to j is $b_1 \geq c$. Let g' denote the network which one obtains, if in g all direct links with i as a vertex are severed. The incremental expected benefit to i of having the link ij in g' is $b_2 \geq b_1 \geq c$ and can be written as $b_2 = p_{ij}(1 + V_j)$ where V_j is j's expected benefit from all the links j has in addition to ij.

Now consider a link from k to j, given $g' \oplus g_{ij}$. This link is worth $b_3 = p_{kj}(p_{ij} + 1 + V_j)$ to k. A link from k to j, given g, is worth $b_4 \geq b_3$ to k. We claim that $b_3 > b_2$, i.e.,

$$p_{kj} > p_{ij} \frac{1 + V_j}{1 + V_j + p_{ij}}$$

Since g is Nash and $g_{ij} = 1$, we know $p_{ij} \geq p_0 > c/n$. By assumption, $p_{kj} \geq \dfrac{1}{1 + c/n^2} p_{ij}$. Therefore,

$$p_{kj} > \frac{1}{1 + p_{ij}/n} p_{ij} = p_{ij} \frac{1 + n - 1}{1 + n - 1 + p_{ij}} \geq p_{ij} \frac{1 + V_j}{1 + V_j + p_{ij}}$$

where we use the fact that V_j is bounded above by $n - 1$. This shows the claim that $b_4 \geq b_3 > b_2 \geq b_1 \geq c$. Initiating the link kj is better for k than not initiating it, contradicting that g is Nash. Hence to the contrary, g has to be either empty or connected. ■

This result means that if the probabilities are not too widely dispersed, then the empty versus connected dichotomy still holds. If, however, the probabilities are widely dispersed, then a host of possibilities can arise and a single dichotomous characterization is no longer adequate. Bala and Goyal (2000b) further show that with homogeneous agents and imperfect reliability, Nash networks become super-connected as the size of the society increases. This result warrants several qualifications.

The first one concerns an obvious trade-off even in the case of homogeneous agents. While it is correct that for any given probability of success $p > 0$, super-connectivity obtains asymptotically, the minimum number of players it takes to get super-connectivity goes to infinity as p goes to zero. Let n^* be any number of agents. If $p < p_0(c, n^*)$, then it takes at least $n^* + 1$ agents to obtain even a connected Nash network.

Secondly, in our model with heterogeneous agents, asymptotic connectivity need no longer obtain, eliminating any scope for super-connectivity.

Consider an infinite sequence of agents $i = 1, 2, \ldots, n, \ldots$ and a sequence of probabilities $p_2, p_3, \ldots$ such that $p_{ij} = p_{ji} = p_j$ for $i < j$. Then the sequence $p_k, k \geq 2$, can be constructed in such a way that the empty network is the only Nash network for any agent set $I_n = \{1, \ldots, n\}$, $n \geq 2$. Of course, with heterogeneous agents, asymptotic super-connectivity obtains, if there exists a $q_0 > 0$ such that $p_{ij} \geq q_0$ for all ij. The argument for homogeneous agents can easily be adapted to this case.

Finally, the lack of a common positive lower bound for the success probabilities does not necessarily rule out asymptotic super-connectivity, provided the probabilities do not drop too fast. A positive example is given by $c = 1$ and $p_{ij} = p_{ji} = p_j = j^{-1/2}$ for $i < j$. Basically, the argument developed for homogeneous agents can be applied here, too. This follows from the fact that for $1 < m < n$,

$$\sum_{i=m}^{n} p_{1i} > \int_{m}^{n+1} s^{-1/2} ds = [2s^{1/2}]_{m}^{n+1} = 2((n+1)^{1/2} - m^{1/2}).$$

Furthermore, with heterogeneous agents, other possibilities exist. For instance, super-connectivity may be established at some point, but connectivity may break down when further agents are added and reemerge later, etc. Or several connected components can persist with super-connectivity within each component. Thus the Bala and Goyal result is altered significantly in our model.

3.2 The Polar Cases

The next proposition identifies conditions under which the complete network and the empty network are Nash. Let $P = [p_{ij}]$ denote the matrix of link success probabilities for all agents $(i, j) \in N \times N$, where $p_{ij} \in (0, 1)$.

Proposition 2*: For any P, there exists $c(P) > 0$ such that each essential complete network is (strict) Nash for all $c \in (0, c(P))$. The empty network is strict Nash for $c > \max\{p_{ij}\}$.*

Proof: Let $g = g_i \oplus g_{-i}$ be any essential complete network. Consider an arbitrary agent i with one or more links in his strategy g_i. Let $\mathcal{G}'_i = \{g'_i \in \mathcal{G}_i : g'_{ij} \leq g_{ij} \text{ for all } j \neq i\}$. Clearly, if $c = 0$ then for agent i, g_i is a strict best response in $\mathcal{G}'_i$ against g_{-i}. By continuity, there exists $c_i(P, g_{-i}) > 0$ so that g_i is a strict best response in $\mathcal{G}'_i$ against g_{-i} for all $c \in (0, c_i(P, g_{-i}))$. Suppose $c \in (0, c_i(P, g_{-i}))$. If $g^*_i \in \mathcal{G}_i \backslash \mathcal{G}'_i$, then $g^*_{ij} = g_{ji} = 1$ for some $j \neq i$

and there exists a better response $g_i' \in \mathcal{G}_i'$ without redundant costly links. Since g_i is a better response than g_i', it is also a better response than g_i^*. Hence for $c \in (0, c_i(P, g_{-i}))$, g_i is a strict best response in $\mathcal{G}_i$ against g_{-i}. Now let $c(P)$ be the minimum of $c_i(P, g_{-i})$ over all conceivable combinations of i and g_{-i}. The first part of the claim follows from this.

For the second part, if $c > \max\{p_{ij}\}$ and no other agent forms a link, then it will not be worthwhile for agent i to form a link. Hence the empty network is strict Nash as asserted. ■

3.3 Star Networks

Star networks are among the most widely studied network architectures. They are characterized by one agent who is at the center of the network and the property that the other players can only access each other through the central agent. There are three possible types of star networks. The inward pointing (center-sponsored) star has one central agent who establishes links to all other agents and incurs the cost of the entire network. An outward pointing (periphery-sponsored) star has a central agent with whom all the other $n-1$ players form links. A mixed star is a combination of the inward and outward pointing stars. Here we will focus on the periphery-sponsored star and the proofs provided below can be easily adapted to the other types of stars.

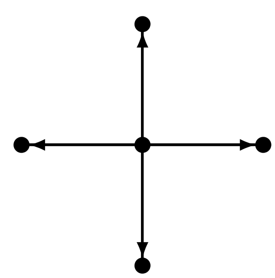

FIGURE 1: Outward Pointing (Periphery-Sposored) Star Network

While the method of computing Nash networks does not change with the introduction of heterogeneous agents, the process of identifying the different parameter ranges for Nash networks is now considerably complicated. We will next establish two claims about Nash networks to illustrate the complex nature of the situation with agent heterogeneity. Without loss of generality we will assume that player n is the central agent in the star. Define M to be the set of all the agents except n or $M = N \backslash \{n\}$ and let $K_m = M \backslash \{m\}$ be the set M without agent m. Also let $J_k = K_m \backslash \{k\}$ denote a set K_m without agent k and $\Sigma_k = \sum_{j \in J_k} p_{jn}$.

Proposition 3: *Given $c \in (0,1)$, there exists a threshold probability $\delta \in (0,1)$ such that the outward pointing star is Nash if , the outward pointing star is Nash if :*

1. $p_{ij} \in (\delta, 1)$ for all pairs ij;
2. *for all $m \in M$, $k \in K_m$:* either $p_{mn} > p_{mk}$,
   or $p_{mn} < p_{mk}$, $p_{mn} > p_{mk}p_{kn}$
   and $(p_{mn} - p_{mk}p_{kn})\Sigma_k > (p_{mk} - p_{mn}) + p_{kn}(p_{mk} - p_{mn})$.

Proof: Consider the outward pointing star with agent n as the central agent. Choose the threshold probability $\delta \in (c, 1)$ to satisfy the inequality

$$\max_{m \in M} \left[(1 - p_{nm}) + \left(1 - p_{nm} \sum_{k \in K_m} p_{nk} \right) \right] < c \tag{2}$$

if $p_{ij} \in (\delta, 1)$ for all ij. Next we identify the conditions under which no player wants to deviate. We know that n has no links to sever, and does not want to add a link since $g_{mn} = 1$ for all $m \in M$ and the flow of benefits is two-way. Now consider an agent $m \neq n$ who might wish to sever the link with n and instead link with some other $k \in K_m$. Player m's payoff from the outward pointing star is $\Pi_m(g^{ot}) = p_{mn} + p_{mn} \sum_{k \in K_m} p_{kn} - c$. His payoff from deviating and forming the new link is $\Pi_m(g^{ot} - g_{mn} + g_{mk}) = p_{mk} + p_{mk}p_{kn} + p_{mk}p_{kn}\Sigma_k p_{jn} - c$. We get $\Pi_i(g^{ot}) - \Pi_i(g^{ot} - g_{in} + g_{ik}) = (p_{mn} - p_{mk}) + p_{kn}(p_{mn} - p_{mk}) + (p_{mn} - p_{mk}p_{kn})\Sigma_k$.
This is clearly positive when $p_{mn} > p_{mk}$ for all $m \in M$, i.e., when every non-central agent ($i \neq n$) has her best link with the central agent .

However, when the inequality is reversed, we need $p_{mn} > p_{mk}p_{kn}$ and $(p_{mn} - p_{mk}p_{kn})\Sigma_k > (p_{mk} - p_{mn}) + p_{kn}(p_{mk} - p_{mn})$, i.e., agent k's link with n is so weak that it is not worthwhile for m to form this link. Essentially, the difference between the benefits from accessing agents $j \in J_k$ through n

instead of the indirect link through k in this case should exceed net benefits from agents n and k when agent m establishes a link with k instead of the central agent. Note that player m can only sever one link in an outward pointing star and hence we need not consider any more instances of link substitution by player m. Next we need to check that no agent wants to add an extra link. This means that no $m \in M$ wants to form a link with any $k \in K_m$. Note that payoffs with this additional link are bounded above by $(n-1) - 2c$. Taking the difference between $\Pi_m(g^{ot} + g_{mk})$ and $\Pi_m(g^{ot})$ we get $[(1-p_{mn}) + (1-p_{1n}p_{mn}) + \cdots + (1-p_{m-1n}p_{mn}) + (1-p_{m+1n}p_{mn}) + \cdots + (1-p_{n-1n}p_{mn})] < c$ as the condition that the additional link is lowering m's payoff. Verifying that this is satisfied for all $m \in M$, gives us $\max_{m \in M} [(1-p_{mn}) + (1-p_{1n}p_{mn}) + \cdots + (1-p_{m-1n}p_{mn}) + (1-p_{m+1n}p_{mn}) + \cdots + (1-p_{n-1n}p_{mn})] < c$, which is equivalent to (**2**). Since we use the upper bound on the payoffs to show that it is not worthwhile to add even one extra link by any player $m \in M$, this obviates the need to check that a player may want to add more than one link. ■

Compared to the Bala and Goyal framework, the introduction of heterogeneous agents alters the situation significantly. While part of the difference involves more complex conditions for establishing any star network, heterogeneity comes with its own reward. A different probability for the success of each link resolves the coordination problem implicit in the Bala and Goyal framework. With a constant probability of success, once we identify conditions under which a given star network will be Nash, the role of the central agent can be assigned to any player. With heterogeneous agents, however, there are some natural candidates for the central agent. The agent who has the least benefit net of costs from a single link, is the natural choice for the central agent in the outward pointing star. There are also some other differences from the Bala and Goyal framework. Notice that the determination of δ involves probabilities of all other links, making it quite complicated. Further, the benefits from deviation are also altered now. In the Bala and Goyal framework, no agent in the outward pointing star will ever deviate by severing a link with the central agent. In our model, links to the central agent will be severed unless the probabilities in the relevant range satisfy some additional conditions.

Note that in our framework the inward pointing star is Nash in the above specified range of costs if the central agent's worst link yields higher benefits than c and (**2**) is satisfied. Clearly, the role of the central agent for this star can be assigned to the agent whose payoff net of costs from forming the $(n-1)$ links is the highest. The mixed star can be supported as Nash when conditions required by the inward and the outward pointing star are satisfied for the relevant agents.

We next consider the case where $c > 1$. Here $c > p_{ij}$ for all links g_{ij}. We provide conditions under which the outward pointing star is Nash.

PROPOSITION 4: *Given $c \in (1, n-1)$ there exists a threshold probability $\delta < 1$ such that for $p_{ij} \in (\delta, 1)$ the outward pointing star is Nash.*

Proof: Let agent n be the center with whom all the other players establish links. Since $c \in (1, n-1)$ we can choose $\delta \in (0,1)$ such that if $p_{ij} \in (\delta, 1)$ for all ij, then (**2**) holds and $\min_{m \in M} [p_{mn}(1+ \sum_{k \in K_m} p_{kn})] > c$. Then no $m \in M$ wants to sever his link with n. The remainder of the proof is similar to the proof of Proposition 3. ∎

Once again it is possible to identify a natural candidate for the role of the central player. Also, note that the inward pointing and mixed star will never be Nash in this range of costs.

3.4 Efficiency Issues

Efficiency is a key issue in Jackson and Wolinsky (1996), Bala and Goyal (2000a,b), and Johnson and Gilles (2000). When costs are very high or very low, or when links are highly reliable, there is virtually no conflict between Nash networks and efficiency in the Bala and Goyal (2000b) framework. This observation still holds in our context. However, there is a conflict between Nash networks and efficiency for intermediate ranges of costs and link reliability, even with the same probability of link failure for all links. In particular, Nash networks may be under-connected relative to the social optimum as the subsequent example shows.

Let us add two important observations not made before. First, it is possible that Nash networks are nested and Pareto-ranked. Second, at least in our context, the following can coexist: a Nash network which is not efficient, but Pareto-optimal and a unique efficient network which is not Nash and does not weakly Pareto-dominate the Nash network. The first observation is supported by the following example: $c = 1$, $n = 4$ and $p_{ij} = 0.51$ for all ij. In this case, both the empty network and the outward pointing star with center 4, are Nash networks. The "outward pointing star" consisting of the links 14, 24 and 34 contains and strictly Pareto-dominates the empty network. Moreover, the empty network is under-connected. Our second observation is based on the following example.

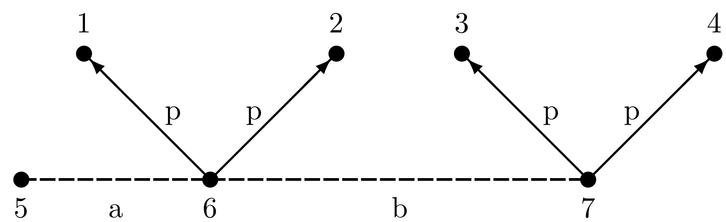

FIGURE 2: Inefficient and Pareto Nash Network

Example 1: $c = 1$, $n = 7$. $p_{16} = p_{26} = p_{37} = p_{47} = p = 0.6181$, $p_{56} = a = 0.2$, $p_{67} = b = 0.3$, and corresponding probabilities for the symmetric links. All other links have probabilities $p_{ij} < p_0$. Now g given by $g_{16} = g_{26} = g_{37} = g_{47} = 1$ and $g_{ij} = 0$ otherwise is a Nash network. Indeed, p is barely large enough to make this a Nash network. The critical value for p satisfies $p(1+p) = 1$ with solution 0.6180.... But g is not efficient. Linking also 5 with 6 and 6 with 7 provides the following added benefits where we use $2p = 1.2362$ and $1 + 2p = 2.2362$:

For 1+2:	$1.2362 \cdot (a + b \cdot 2.2362)$	$=$	1.07656
For 3+4:	$1.2362 \cdot b \cdot (a + 2.2362)$	$=$	0.90349
For 5:	$a \cdot 2.2362 + ab \cdot 2.2362$	$=$	0.58141
For 6:	$a + b \cdot 2.2362$	$=$	0.87086
For 7:	$b \cdot (a + 2.2362)$	$=$	0.73086
Total:			4.16318

Hence the total added benefit exceeds the cost of establishing these two additional links. The network thus created would be efficient. But neither 6 nor 7 benefits enough from the additional link between them to cover the cost of the link. Thus, the enlarged efficient network is not Nash. Since the rules of the game stipulate that one of the two agents assumes the entire cost of the new link, the enlarged efficient network cannot weakly Pareto-dominate g. In fact, g is Pareto-optimal while inefficient. Reconciling efficiency and Pareto-optimality would require the possibility of cost sharing and side payments. ■

4 Alternative Model Specifications

In this section we will consider three alternative specifications of our current model. The first variation introduces more realism in the formation of networks by allowing agents to duplicate existing links. The second specification considers network formation under incomplete information. Here, each agent $i \in N$ is aware of the success probabilities $p_{ij}, i \neq j$ of her own links, but is ignorant of the probabilities of link successes of the other agents. Further, we discuss the implications for Nash networks in a model where pairwise link formation requires the consent of the other agent and when links can impose a cost on the other agent. Finally, we present an example with endogenous probabilities.

4.1 An Alternative Formulation of Network Reliability

The payoff function in the previous section is based on the closure of the network implying that the links $g_{ij} = 1$ and $g_{ji} = 1$ are perfectly correlated. Thus, no agent will ever duplicate a link if it already exists. A more accurate way of modelling information flows would be to assume that the event $g_{ij} = 1$ and $g_{ji} = 1$ are independent. Then, the link $h_{ij} = \max\{g_{ij}, g_{ji}\}$ exists with probability $r_{ij} = 1 - (1 - p_{ij})^2$ providing an incentive for two-way connection between agents i and j. This never occurs in the previous model since duplicating a link can only increase costs. We retain the assumption that $p_{ij} = p_{ji}$. Also, the flow of benefits is still both ways.

The consequences of the new formulation are now explored by reexamining Proposition 3. The incentives for modifying links by deviating do not change under this formulation, i.e., the conditions of Proposition 3 are assumed to hold. The main impact is on the threshold probability value δ, altering the range of costs and probabilities under which the outward star can be supported as Nash. Note that the payoff function used earlier for determining the payoff from an additional link gets around this issue by assuming that payoffs have an upper bound of $(n-1) - \alpha c$ where α denotes the number of links formed. In order to see how this new formulation will affect reliability we need to compute the precise value of the payoffs from additional links instead of using the upper bound. We denote the resulting new threshold value by $\widetilde{\delta}$. We find that a threshold of the form $\widetilde{\delta} = \max_{m \in M} \max_{i \neq m} \delta_i^m$ will suffice. Each δ_i^m is a threshold value related to the specific link mi.

Proposition 5: *Suppose that the links $g_{ij} = 1$ and $g_{ji} = 1$ are independent, and $c \in (0, 1)$. Then the outward pointing star can be supported as Nash under the threshold probability value $\widetilde{\delta}$, if Proposition 3 holds.*

Proof: See Appendix. ■

In our previous formulation, δ can also be obtained as the maximum of link-specific thresholds. The latter tends to be lower if duplication has no benefits. Thus, in general $\widetilde{\delta} > \delta$. Proposition 1 holds under this alternative formulation since the players are still endowed with information which is mutually beneficial and non-rival. The proof relies on showing that the net benefits to an agent k in the connected subgraph C_2 exceeds to net benefits to an agent i in the connected subgraph C_1 from a link with some agent $j \in C_1$. This is clearly independent of the number of double links in either of the two connected components. Hence the proof of Proposition 1 can be easily adapted for this alternative model of reliability. It can also be shown that Proposition 2 holds under this formulation since it relies on a continuity argument. Finally, this formulation can lead to super-connected networks of a different sort − one where agents may reinforce existing higher probability links instead of forming new links with other players.

4.2 Nash Networks under Incomplete Information

The previous sections have assumed that the agents are fully aware of all link success probabilities. However, this is not always a very realistic assumption. As an alternative, we introduce incomplete information in the game. Each agent $i \in N$ has knowledge of the probability of success of all her direct links. However, she is not aware of the probability of success of indirect links, i.e., agent i knows the value of p_{ij}, but is unaware of the value of p_{jk}, where $i \neq j, k$. The assumption that $p_{ij} = p_{ji}$ is still retained. We re-examine Proposition 3 for this specification.

In order to solve for equilibria, each agent i must now have some beliefs about the indirect links. We assume that each agent postulates that, on average, every other agent's world is identical to her own. She assigns the average success value of all her own direct links to the indirect links, imparting a symmetry to the problem of indirect links. Thus, agent i assigns a value of $p_i = \frac{1}{n-1} \sum_{i \neq m} p_{im}$, to all indirect links p_{jk} for $i \neq j, k$. This has some immediate consequences for the payoff function. Consider some agent $m \in M$. This agent now believes that her payoff from the outward pointing star is given by $\Pi_m(g^{ot}) = p_{nm} + |K_m| \, p_{nm} p_m - c = p_{nm} + (n-2) p_{nm} p_m - c$, which is clearly different from her actual payoff.

Proposition 6*: Given each agent's beliefs about her indirect links, the outward pointing star is Nash if every non-central agent ($i \neq n$) has her best link with the central agent or when $p_{mn} < p_{mk}$, then $p_{mn} > p_m p_{mk}$ and $(n-3)(p_{mn} - p_m p_{mk})p_m > p_{mn} - p_{mk} + p_m(p_{mn} - p_{mk})$ and for each agent $m \in M$, the inequality $(n-2)(1 - p_{nm}p_m) < (1 - p_{nm}\sum_{k \in K_m} p_{nk})$ holds.*

Proof: See Appendix. ■

This formulation provides us with some interesting insights about the role of the indirect links and the vulnerability of Nash networks. First note that the set of conditions for the outward pointing star to be Nash are very similar to those obtained in Proposition 3. However there are some differences. For the purpose of comparison, let us assume that the actual probabilities satisfy the conditions of Proposition 3. Now it is possible that $(n-2)(1 - p_{nm}p_m) > c > (1 - p_{nm}\sum_{k \in K_m} p_{nk})$, in which case agents will create new links destroying the star architecture. Consequently, the realized network yields lower payoffs than the star network. This is an instance when the outward star is Nash under complete information, but due to incorrect beliefs about indirect links, agents create additional links under incomplete information. Thus, the introduction of incomplete information can easily lead to *network failure*.

We now examine the consequences of this formulation through an explicit example.

Example 2: Consider a network with $n = 6$. Suppose agents 1 to 4 are linked in a star formation with agent 4 being the central agent, i.e., $g_{14} = g_{24} = g_{34} = 1$. Further $g_{56} = 1$ and we will examine what happens to the link g_{45} under complete and incomplete information. Let $c = 1/12$, $p_{14} = p_{24} = p_{34} = p = 4/10$, $p_{56} = r = 1/2$ and $p_{45} = q = 1/24$. The probabilities of all other links are assumed to be zero.

Under these objective probabilities it is easy to verify that $q(1 + r) < c$ and hence agent 4 will never initiate the link with agent 5. However, agent 5 will initiate this link since $q(1 + 3p) > c$. The resulting connected network is Nash since all other links yield no benefits.

Note that for our current formulation with incomplete information, $p_5 = \overline{r} = \frac{1}{5}(r + q)$ and $p_4 = \overline{p} = \frac{1}{5}(3p + q)$. Under these beliefs about the probabilities of the indirect links, agent 4 will never establish the link since $q(1+\overline{r}) < c$. Similarly, agent 5 will not establish the link since $q(1+3\overline{p}) < c$. With incomplete information the above disconnected network with $g_{45} = 0$ is a Nash network. ■

Thus incomplete information may destroy a crucial link and give rise to two connected components.

4.3 Further Ramifications

In this subsection, we touch upon two further ramifications, mutual consent requirements and certain negative network externalities. In our setting and in much of the literature, it is assumed that agent i does not need the consent of agent j to initiate a link from i to j. All it takes is that agent i incurs the cost c. This may be construed as a drawback of the non-cooperative formulation. Though one might argue that when asked agent j might give her permission anyway, since she would only benefit from an additional link that does not cost her anything.[4] Thus it appears that introducing an implicit consent requirement is inconsequential, a descriptive improvement at best, a notational burden at worst. Yet Nash networks have another more serious weakness. Namely, it seems somewhat preposterous that agent j should divulge all the information from her other links without her consent. For this reason, we now discuss the implications of a consent game. Formally, such a requirement can be accommodated by replacing each player's strategy set $\mathcal{G}_i$ by $\mathcal{G}_i \times \mathcal{G}_i$ with generic elements $(g_i, a_i) = (g_{i1}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in}; a_{i1}, \ldots, a_{ii-1}, a_{ii+1}, \ldots, a_{in})$ where the second component, a_i, stands for i's consent decisions. A link from i to j is initiated by mutual consent if and only if $g_{ij} = 1$ and $a_{ji} = 1$. Agents incur only the cost of links that are permitted. Denied links are absolutely costless.

Every graph g that was a Nash network before is still a Nash network. But now there is room for mutual obstruction: $g_{ij} = 0$ is always a best response to $a_{ji} = 0$ and vice versa. Therefore, the empty network is always Nash under a mutual consent requirement. More generally, take any set of potential edges $E \subseteq N \times N$ and replace p_{ij} by $q_{ij} < p_0$ for all $ij \in E$ in the original model. Then any Nash network of the thus defined hypothetical game constitutes a Nash network of the network formation game requiring mutual consent. In particular, for any $N' \subseteq N$, the Nash networks with reduced player set N' form Nash networks (as long as the architecture of

[4] This argument is less compelling in the case of one-way information flow.

the network is preserved) of the network formation game requiring mutual consent with player set N, if one adds the agents in $N \setminus N'$ as isolated nodes. One could modify the mutual consent game by requiring that agents must incur the cost of all links they initiate, irrespective of consent. Since agents are rational and have complete information, links that will be denied will never be initiated in equilibrium. The Nash networks will be identical under this specification.

All the new equilibria from the mutual consent game will be eliminated, if one imposes 2-player coalition-proofness or introduces conjectural variations of the kind that a player interested in initiating a link presumes the other's consent. A more serious issue is why two agents cannot split the cost in a Pareto-improving way when both would benefit from an additional link. Addressing endogenous cost sharing in a satisfactory way necessitates a radically different approach which is beyond the scope of the present generation of models.

The Jackson-Wolinsky "connections model" assumes exogenous cost sharing. In such a case, agent i can have an incentive to reject a link from j to i. More generally, one can consider a modification of the payoff function (**1**) that yields the expanded form

$$\Pi_i(g) = B_i(cl(g)) - \sum_j g_{ij} a_{ji} c - \sum_j g_{ji} a_{ij} c' \tag{3}$$

where i incurs the cost or disutility c', if he agrees to a link initiated by agent j. The quantity c' can be interpreted as a composite cost which includes an explicit cost contribution towards the creation of a link ji as well as a certain disutility (negative externality) that i experiences when others contact him through this link.[5] The special case $c = c'$ is tantamount to equal cost sharing. While we leave the in-depth analysis of this model variant for future research, two observations can be made without further scrutiny. First, if giving one's consent is costly, Nash networks tend to be smaller. This would still be true, if we allowed for the additional possibility that at an extra cost, agent j can impose the link ji against i's objection. Second, the possibility of mutual obstruction persists under costly consent.

[5] In a more refined version, one could differentiate the cost structure further and make the cost of accepting a specific link ji depend on whether or not there is duplication, that is the reverse link ij is initiated by i and accepted by j.

On a more speculative note, one can also think of the possibility that the addition of a link renders all adjacent links less reliable. For conceivably, any given node might become less effective in responding to information requests via its direct links, if it gets accessed through one more direct link. In other words, the additional link causes a negative externality on the other links competing for access to the same node. Incorporating this particular feature into a model of network formation would lead to endogenous failure probabilities. One of the consequences is that a complete network need no longer be Nash, even if links are costless. To illustrate this and other interesting possibilities, let us consider

Example 3: Let $c = 0$. For $i \in N$ and $g \in \mathcal{G}$, set

$$n_i(g) = |\{k \in N \setminus \{i\} : g_{ik} = 1 \text{ or } g_{ki} = 1\}|,$$

the number of agents to whom i has direct links in g. For any two agents i and j and any network g, let the endogenous probability of success of link ij be given as

$$p_{ij}(g) = \begin{cases} \frac{1}{n_i(g)} \cdot \frac{1}{n_j(g)}, & \text{if } g_{ij} + g_{ji} > 0; \\ 0, & \text{if } g_{ij} + g_{ji} = 0. \end{cases}$$

First consider the case $n = 3$ and the wheel or circle g with links 12, 23, and 31, an essential complete network where each link has success probability 1/4. Each player i receives payoff $\Pi_i(g) = 19/32$. After severance of the link initiated by him, the two remaining links have each success probability 1/2 and i's payoff becomes 3/4 or 24/32. This shows our claim that with endogenous success probabilities and zero or negligible costs, complete networks need no longer be Nash — in stark contrast to Proposition 2. Moreover, for $n \geq 4$, wheels with simple links, line networks with simple links, and stars are not Nash under the current assumptions. Regarding stars, a peripheral agent gains from initiating links to all other peripheral agents in addition to the existing link to the central agent. Finally, the example exhibits non-empty Nash networks with very small connected components. It turns out that a network g is Nash if each component C either satisfies $|C| = 3$ and is incomplete (is not a wheel) or satisfies $|C| = 2$. ■

This somewhat extreme example clearly shows that the negative externality caused by additional links can affect the outcomes significantly.

5 Concluding Remarks

The model developed here as well as a substantial part of the network literature is concerned with information flows. Such models may be interpreted as a reduced form where all costs and benefits have been attributed to information flows. Under perfect reliability, the primary focus lies on the size and efficiency of networks. With imperfect reliability the strength of social ties, or the nature and quality of information can be discussed as well. In our model for instance, one could argue that the information exchange between i and j is valuable with probability p_{ij} and is of a dubious nature with the complementary probability. Thus, imperfect reliability raises questions about a possible quantity-quality trade-off as well as the related efficiency issues.

The assumption of agent heterogeneity in the form of imperfect reliability in social networks provides a richer set of results than the homogeneous setting. In conjunction with our adopted solution concept, Nash equilibrium, it accentuates open questions that also arise − though perhaps to a lesser degree − in the context of pairwise stability. An example is the issue of cost sharing and side payments. Twice in the course of our current investigation we came across this issue: First, in the discussion of efficiency and Pareto-optimality. For a second time in the context of the mutual consent model. The issue of cost sharing and bargaining over the costs of link formation is especially crucial when benefits mainly accrue from indirect links. It indicates an important direction for future research. Currarini and Morelli (2000) take a first step in this direction. They introduce a noncooperative game of sequential network formation in which players propose links and demand payoffs. They show that for networks which satisfy size monotonicity, there is no conflict between efficiency and stability.

Bala and Goyal's work on Nash networks shows that results under imperfect reliability are quite different from those in a deterministic setting. With the introduction of heterogeneity this clear distinction no longer prevails. Our findings encompass results of both types of models. For example, with perfect reliability and information decay, Nash networks are always minimally connected, irrespective of the size of society (Bala and Goyal (2000a)). In contrast, with homogeneous imperfect reliability and no information decay, redundant links between agents always arise asymptotically (Bala and Goyal (2000b)). In our model, with heterogeneous imperfect reliability and no information decay, both types of outcomes can be generated through appropriate choice of the p_{ij}'s. For instance, decay models (with perfect reliability) compute benefits by considering only the shortest path between agents. Extra indirect links do not contribute to benefits. Given a resulting minimally connected Nash network g of such a model, there exists a parameter specification of our model that also gives rise to g as a Nash network. In our framework this requires lowering the p_{ij} to zero or below p_0 for all ij with $g_{ij} = 0$ and $g_{ji} = 0$ and choosing sufficiently high probabilities p_{ij} for all other ij so that all benefits accrue from the direct links only. On the other hand, as discussed in subsection 4.1, choosing the p_{ij}'s appropriately leads to super-connected networks as well.

Finally, to end on a cautionary note, we have indicated the possibility of network failure in the discussion following *Proposition 6.* It is only appropriate to mention Greif's (1994) tale of two historical societies $-$ the Maghribi traders, with an Islamic culture who shared trading information widely, and the Genoese traders exemplifying the Latin world, who operated individually and did not share information amongst each other, relying more on legal contracts. He argues that the culture and social organization of these two communities ultimately determined their long-run survival. The Genoese kept business secrets from each other, improved their contract law and operated through the market. Consequently they ended up with an efficient society. The Maghribis on the other hand operated through an informal network where behavior of a single pair of agents affected everyone in the network. As opposed to the Genoese traders the Maghribis invested considerable time and effort to gather information about their network. Since one bad link could adversely affect the entire network, the Maghribis often had to engage in superfluous links as well without adequate concern for efficiency. Efficiency became a critical issue once new business opportunities arose in far away lands, where operating through an ethnically based network became very expensive. In the end these organizational differences created by the cultural beliefs of the two societies led to the survival of the more efficient of the two. Thus social networks may be good substitutes for anonymous markets in certain societies, but the market paired with the proper infrastructure may be a more efficient institution. In fact for trade in standardized commodities, a frictionless and informationally efficient anonymous market, if feasible, would be best. Some of the trade-offs between networks and anonymous markets are addressed by Kranton (1996) who investigates the persistence and coexistence of personalized exchange arrangements when anonymous market channels are available and would be more efficient.

6 Appendix: Proofs

1. *Proof of Proposition 5*:

Consider the outward pointing star. All agents $m \in M$ have a link with the central agent, and the conditions for not deviating from the Nash strategy identified in *Proposition 3* remain unchanged. However, we must also verify that neither agent n nor any $m \in M$ will gain by adding a link. For all $m \in M$ and $k \in K_m$ we need to compute $\Pi_m(g^{ot} + g_{mk})$ which is the sum of payoffs from three different terms: the payoff related to player n, the payoff related to player k, and the payoff from links to all others players $j \in J_k$.

• The payoff related to player n is given by $r'_{nm} \equiv p_{nm}(1 - p_{mk}p_{nk}) +$
$(1 - p_{nm})p_{mk}p_{nk} + p_{nm}p_{mk}p_{nk}$.
• The payoff related to player k is given by $r'_{mk} \equiv p_{mk}(1 - p_{nm}p_{nk}) +$
$(1 - p_{mk})p_{nm}p_{nk} + p_{nm}p_{mk}p_{nk}$.
• Finally, the payoff from all other players is given by $r'_{nm}\Sigma_k$. Adding all these up yields

$$\Pi_m(g^{ot} \oplus g_{mk}) = r'_{nm} + r'_{mk} + r'_{nm}\Sigma_k.$$

The link mk will not be formed when $\Pi_m(g^{ot} \oplus g_{mk}) - \Pi_m(g^{ot}) < 0$, or $(1 - p_{nm})p_{mk}p_{nk} + p_{mk}(1 - p_{nm}p_{nk}) + (1 - p_{nm})p_{mk}p_{nk}\Sigma_k < c$. Choose the threshold probability δ_k^m as the smallest number such that this inequality holds, if $p_{ij} > \delta_k^m$ for all $i \neq j$. Regarding agent n, he does not want to form an additional link with $m \in M$, if

$$p_{mn}(1 - p_{mn}) < c.$$

But this condition follows from $p_{nm} > \delta_n^m$, where δ_n^m is the smallest number such that the inequality $p_{nm}(1 - p_{nm}) + p_{nm}(1 - p_{nm}) \sum\limits_{k \in K_m} p_{nk} < c$ holds. Finally, set $\widetilde{\delta} = \max\limits_{m \in M} \max\limits_{i \neq m} \delta_i^m$. Then if $p_{ij} \in (\widetilde{\delta}, 1)$ for all ij, we can support the outward pointing star as Nash. ■

2. *Proof of Proposition 6*:

Consider an outward pointing star. The central agent n plays no role in this case. Every agent m receives a perceived payoff of $\Pi_m(g^{ot}) = p_{mn} + (n-2)p_{mn}p_m - c$. Consider the possibility that agent m wants to deviate and form a link with some $k \in K_m$. Her payoffs from this are given by

$\Pi_m(g^{ot} + g_{mk} - g_{mn}) = p_{mk} + p_{mk}p_m + (n-3)p_{mk}(p_m)^2 - c$. Hence the condition for no deviation is given by

$$p_{mn} - p_{mk} + p_m(p_{mn} - p_{mk}) + (n-3)(p_{mn} - p_m p_{mk})p_m > 0$$

which is true when either $p_{mn} > p_{mk}$, or if $p_{mn} < p_{mk}$, then $p_{mn} > p_m p_{mk}$ and $(n-3)(p_{mn} - p_m p_{mk})p_m > p_{mn} - p_{mk} + p_m(p_{mn} - p_{mk})$. In order to rule out additional links, we require that just as before $(n-2)(1 - p_{nm}p_m) < (1 - p_{nm}\sum_{k \in K_m} p_{nk})$. This completes the proof. ■

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