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# Subsidies, Knapsack Auctions and Dantzig's Greedy Heuristic [1]

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**Abstract**

A budget-constrained buyer wants to purchase items from a shortlisted set. Items are differentiated by quality and sellers have private reserve prices for their items. Sellers quote prices strategically, inducing a knapsack game. The buyer's problem is to select a subset of maximal quality. We propose a buying mechanism which can be viewed as a game theoretic extension of Dantzig's greedy heuristic for the classic knapsack problem. We use Monte Carlo simulations to analyse the performance of our mechanism. Finally, we discuss how the mechanism can be applied to award R&D subsidies.

JEL classifications: D21, D43, D44, D45

Keywords: Auctions, Subsidies, Market Design, Knapsack Problem

# 1. INTRODUCTION

Consider a buyer who has a fixed budget to spend on items from a shortlisted set. The items differ in quality. The quality of a subset of items is the sum of the individual qualities of its elements. A subset of higher quality is preferred to one of lower quality. Subsets of the same quality are considered as perfect substitutes. Each seller has private information about his reserve price for his item. The buyer's problem is to select a subset of items of maximal quality subject to his budget constraint.

Under complete information, the buyer faces a binary knapsack problem with qualities corresponding to values and reserve prices corresponding to weights in the standard notation. In the realm of incomplete information, any buying mechanism induces a knapsack game where sellers choose the weight of their item (i.e. the price they quote) strategically. In this paper we extend Dantzig's greedy heuristic (Dantzig (1957)) for the knapsack problem to this game–theoretic setting. More precisely, we design an auction mechanism with the following property: In equilibrium (in dominant strategies) every item that is chosen has better quality relative to its price than every item that is not chosen.

For an important application of this problem, consider government funds to subsidize R&D activities by private businesses. Firms apply for subsidies by submitting a research proposal. Proposals vary both in quality and the stated level of required funding. A selection committee chooses an allocation that maximizes total quality. Since costs are private information, firms might overstate the amount they need, leading to suboptimal allocations.

The problem was first studied by Giebe, Grebe, and Wolfstetter (2006) who give an economic analysis of the R&D subsidies program for small and medium-sized companies in Germany (see Binks, Lockett, Siegel, and Wessner (2003) for an account of the US and UK programs). Aggarwal and Hartline (2006) study knapsack auctions with applications to advertisement and broadband bandwidth markets.

This paper proceeds as follows. Section 2 introduces the problem in a formal setting. In section 3, we present our mechanism. We discuss several properties and give examples. In particular, we outline its connection to Dantzig's algorithm. Section 4 employs a Monte Carlo simulation to compare the performance of the mechanism to theoretical benchmarks. Section 5 briefly debates some aspects of applying the mechanism to award R&D subsidies.

# 2. THE MODEL

## 2.1 *The formal setting*

Assume we have $n$ potential sellers numbered $1, \ldots, n$. We use $i$ to represent a typical seller and $N$ to represent the whole set of sellers, i.e. $N = \{1, \ldots, n\}$. Each seller wants to sell an indivisible item for which he has a private reserve price $z_i$. Items differ in quality. Let $q_i$ denote the quality of seller $i$'s item relative to an item of highest quality, so that $1 \geq q_i$ for all $i \in N$ (This requires the possibility to compare items' qualities quantitatively). Sellers are faced by a single buyer who has a fixed and finite budget $\mathcal{B}$ to spend on these items. The buyer's objective is to purchase a subset of items of maximal quality, where the quality of a set of items is the sum of qualities of its elements.

## 2.2 *Complete Information: The Knapsack Problem*

The **0-1 (binary) knapsack problem** is one of the most studied problems in combinatorial optimization (see e.g. Korte and Vygen (2005, ch. 17)). It reads as follows:

*Given a set of items each with a weight and a value, determine a subset of items so that the total weight is less than a given capacity and the total value is as large as possible.*

Note that for the special case of complete information, (i.e. if all $z_i$ are known) the buyer's problem is a knapsack problem, with qualities corresponding to values and reserve prices corresponding to weights. Indeed, the buyer merely has to solve the optimization problem and pay every seller in the chosen allocation his reserve price. More formally, the buyer's problem reduces to finding

$$\arg\max_x \sum_{i=1}^{n} q_i x_i \quad s.t. \sum_{i=1}^{n} z_i x_i \leq \mathcal{B}, \quad x \in \{0,1\}^n \tag{1}$$

The binary knapsack problem is $NP$-hard. Dantzig proposed the following simple heuristic algorithm:

***Dantzig's greedy heuristic.*** *Sort items in increasing order of weight–per–value. Insert items in this order until the cap binds.*

The heuristic can perform arbitrarily bad relative to the optimal solution. However, one can derive a relative bound:

Lemma 2.2.1 *(Dantzig (1957)) Suppose that the first $i-1$ items of the list fit into the knapsack. Then the absolute error is bounded by the value of the $i$th item (the critical item).*

## 2.3 *Incomplete information: A knapsack game*

In the presence of incomplete information (i.e. private reserve prices) any buying mechanism induces a knapsack game where sellers choose the weight of their item (the price they quote) strategically. The special case of complete information poses a natural benchmark for any buying mechanism for this general case. In the next section we develop a simple and very practicable auction mechanism that performs remarkably well (compared to this benchmark) in simulations. Our mechanism can be viewed as an extension of Dantzig's greedy heuristic to this game theoretic knapsack problem in the way that, in equilibrium, it chooses those items which offer the best price-per-quality ratio.

## 3. THE MECHANISM

In this section we present our main results. In 3.1 we introduce the Dantzig auction for the buyer's problem and discuss some properties. We give two examples in 3.2.

### 3.1 *The Dantzig auction*

Consider the following open descending clock auction for the buyer's problem introduced in section 2: Each potential seller makes an initial offer (or bid). Should the sum of all bids exceed the buyer's budget, the buyer tells each bidder to lower its bid by a certain decrement (this might be zero or a positive number). If a bidder refuses to lower his bid by the required amount, he is eliminated from the auction. If he accepts, he stays active for the next round (if there is one). This procedure continues until the budget exceeds the sum of all active bids, i.e. until there is no excess demand. Those bidders who are still active at the end of the last round receive their final bids in return for their items.

A **stopping rule** $\beta_i(z_i)$ is the lowest bid that bidder $i$ is willing to make depending on his private reserve price $z_i$.

LEMMA 3.1.1 *$\beta_i(z_i) = z_i$ is a weakly dominant strategy for all $i = 1, \dots, n$*

PROOF Assume bidder $i$ chooses $\beta_i > z_i$. If $i$ wins at price $\beta_i(z_i)$ or higher there is no difference. But if $i$ loses there is either no difference or $i$ would have won (and preferred to win) by playing $z_i$ instead. The argument is analogous for $z_i > \beta_i(z_i)$. □

Thus, $(\beta_1, \dots, \beta_n) = (z_1, \dots, z_n)$ is an equilibrium in (weakly) dominant strategies. Whenever we speak of an equilibrium of the auction game, we will implicitly refer to this particular equilibrium.

Next, we want to specify how the decrements are calculated. To this end, we propose the following **Dantzig clock rule**: Each bidder starts at his maximal bid. This might simply be the budget $\mathcal{B}$ for all bidders. Alternatively, there may be predefined individual maximal bids for the bidders. A price-per-quality bidding-clock starts at the highest initial bid-per-quality ratio. As long as there is excess demand, this price-per-quality clock is lowered by one unit every round and all active bidders are informed if and by how much they have to lower their bids to meet the new ratio. As an example, consider a bidder whose item has quality $q = \frac{1}{2}$ and assume that his current bid is $b = 100$, giving him a current price-per-quality ratio of 200. Now, suppose the price-per-quality clock is lowered to 199. The bidder has to lower his bid to 99.5 to stay active.

DEFINITION 3.1.1 *A descending clock auction as above with a Dantzig clock rule is a* ***Dantzig auction****.*

The next proposition shows that this name is indeed justified.

PROPOSITION 3.1.1 *In equilibrium, the Dantzig auction leads to an allocation* $\mathcal{A}$ *with the following Dantzig-like property: Let* $i, j \in N$*, with* $i \in \mathcal{A}$ *and* $j \in N \backslash \mathcal{A}$*. Let* $b_i^w$ *be* $i$*'s winning bid. Then* $\frac{b_i^w}{q_i} < \frac{z_j}{q_j}$*, i.e. every item in the final allocation has a better price–quality ratio than every item that is not in that allocation.*

PROOF As $j$ is not in the final allocation, there exists a round $r$ in which $j$ drops out of the auction. Suppose the price-per-quality clock shows $x$ after round $r-1$. Then, in equilibrium, $x \geq \frac{z_j^w}{q_j} > x - \varepsilon$, where $\varepsilon$ is the decrement. But $i$ is still active after round $r$, thus $x - \varepsilon \geq \frac{b_i^r}{q_i}$ ($b_i^r$ being $i$'s bid in round $r$). Since $q_i$ is constant and bids can only decrease from round to round, the result follows. □

In other words: Take the complete set of information generated during the auction, that is the set of winning bids and the final bids of all other bidders. Then the Dantzig auction yields the same allocation as the Dantzig heuristic applied to this information set.

Instead of allowing for a multiround open bidding procedure, the buyer could design an equivalent automated mechanism where bidders simply submit a stopping rule to an algorithm.[1]

Let $\mathcal{B}$ be the budget, $\mathbf{c} = (c_1, \ldots, c_n)$ the vector of individual maximal bids (again, $c_i = \mathcal{B}$ for all $i \in N$ is possible), $\mathbf{q} = (q_1, \ldots . q_n)$ the vector of quality ratios relative to the best project, $\varepsilon > 0$ the size of the decrement and $\beta = (\beta_1, \ldots, \beta_n)$ the vector of chosen stopping rules. Additionally, let $\mathbf{b}$ denote an arbitrary set of bids and $\mathcal{A} \subseteq N$ an arbitrary subset of items. With this notation, the following proxy algorithm is equivalent to the Dantzig auction.

ALGORITHM 3.1.1

*INPUT:* $\mathcal{B}, N, \varepsilon, \boldsymbol{c}, \boldsymbol{q}, \beta$

*INITIALIZE:* $\boldsymbol{b}=\boldsymbol{c}$, $\mathcal{A} = \{1, \ldots, n\}$

*WHILE:* $|\boldsymbol{b}| > \mathcal{B}$

- *Let* $r = max\left\{\frac{b_1}{q_1}, \ldots, \frac{b_{|\mathcal{A}|}}{q_{|\mathcal{A}|}}\right\}$
- *For all* $i$ *with* $\frac{b_i}{q_i} = r$

[1] Of course, for equivalence the algorithm must not exploit the collected information to the bidders' disadvantage.

- *IF* $b_i > \beta_i$
  - $b_i \leftarrow b_i - \varepsilon q_i$ *// lower the bid*
- *ELSE*
  - $\mathcal{A} \leftarrow \mathcal{A} \backslash \{i\}$, $\boldsymbol{b} \leftarrow \boldsymbol{b} \backslash \{b_i\}$ *// eliminate from the auction*
  - *rearrange indices appropriately*

*RETURN:* $\mathcal{A}$, $\boldsymbol{b}$

It should be clear that the algorithm terminates as the sum of active bids decreases every round.

### 3.2 *Examples*

Consider the following examples

1. Let $\mathcal{B} = 100$, $c_i = \mathcal{B}$ for all $i$ and suppose there are four items.

| Items | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $q_i$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |
| $z_i$ | 50 | 20 | 20 | 10 |
| $\frac{c_i}{q_i}$ | 100 | 200 | 200 | 400 |
| $\frac{z_i}{q_i}$ | 50 | 40 | 40 | 40 |

   Consider the Dantzig auction with this input. The price-per-value clock starts at 400 ($\frac{c_4}{q_4}$). The auction ends when the clock is at 49, once 1 dropped out. 2, 3 and 4 win at prices 24.5, 24.5 and 12.25 yielding a total quality $\frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4}$. Assuming sellers play their equilibrium strategies, the buyer learned that $z_1 = 50$, $z_2, z_3 \leq 24.5$ and $z_4 \leq 12.25$. Exploiting this information, the buyer would chose 1, 2 and 3 at a total cost of 99, yielding a quality of 2. Under complete information the buyer would select the *first best* allocation and purchase all the items at their reserve prices, giving him quality of $\frac{9}{4}$ at a total cost 100.

2. Let $\mathcal{B} = 100$, and suppose there are two projects, 1 and 2, with $q_1 = q_2$. Let $z_1, = z_2 = 51$. Then the algorithm chooses the empty set. This shows that the algorithm can perform arbitrarily bad relative to the optimal solution.

## 4. SIMULATION

In this section we analyse the performance of the Dantzig auction using a Monte Carlo simulation. For this, we assume that sellers play their equilibrium strategies. We compare the resulting allocation, $D$, to two natural benchmarks. The first benchmark, $D'$, is the optimal allocation that results if one considers all bids that are made in the auction (recall that the chosen allocation ignores players who drop out before the auction ends). The second benchmark, $CI$, is the optimal choice under complete information (see Section 2).

We chose the following simulation parameters:

- Each item has one of three quality values: $(q_A, q_B, q_C) = \left(1, \frac{1}{2}, 0.333\right)$
- budget: $\mathcal{B} = 10,000$
- $z_i$ uniformly distributed with support $[0, 1000]$
- $c_i$ uniformly distributed with support $[z_i, 1000]$
- number of projects: n=100
- number of simulated auctions: 500

Note that every project assumes one of only three possible quality values. Also, we included individual caps for the bidders. Both of these assumptions were introduced as they become relevant when applying our mechanism to award R&D subsidies (see section 5). In the following we present the results of this simulation.

Our theoretical results imply that the total quality improvement from $D$ to $D'$ cannot exceed the value of the most valuable item $q_A = 1$. This is confirmed by the simulation results. Figure 1 shows the empirical distribution of absolute improvements for all simulated auctions. Moreover, we see that for more than 60 per cent of cases an improvement was impossible. In about 80 per cent of all cases the improvement does not exceed the value of $q_B = 0.5$. Figure 2 shows the relative quality gain that one obtains if one

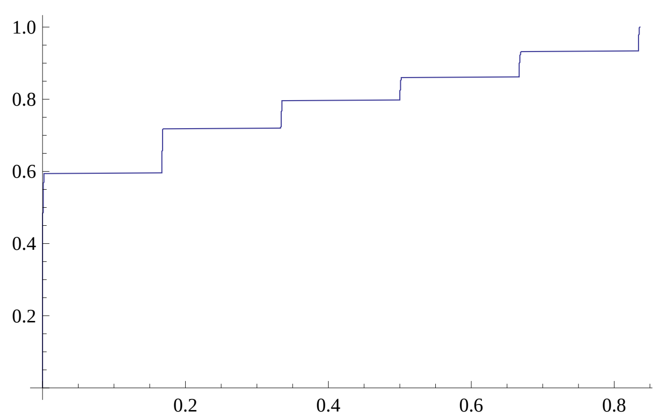


Figure 1: cdf of absolute welfare gain after move from $D$ to $D'$

moves from $D$ to $D'$. In only a few cases, the quality gain was more than four per cent. The above simulation results suggest that applying Dantzig's heuristic to the collected information during the auction is indeed a good

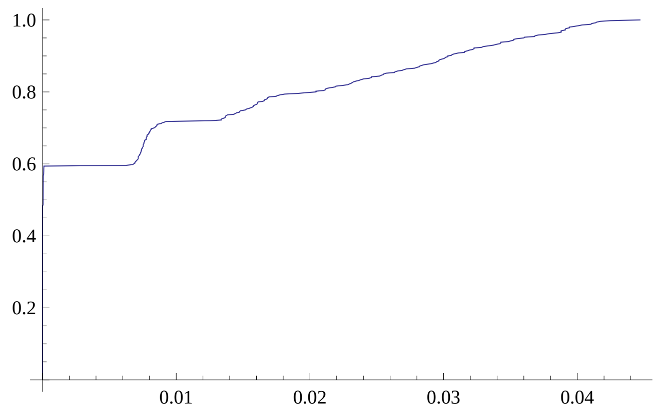


Figure 2: cdf of **relative** quality gain after move from $D$ to $D'$ (0.01 = 1 per cent)

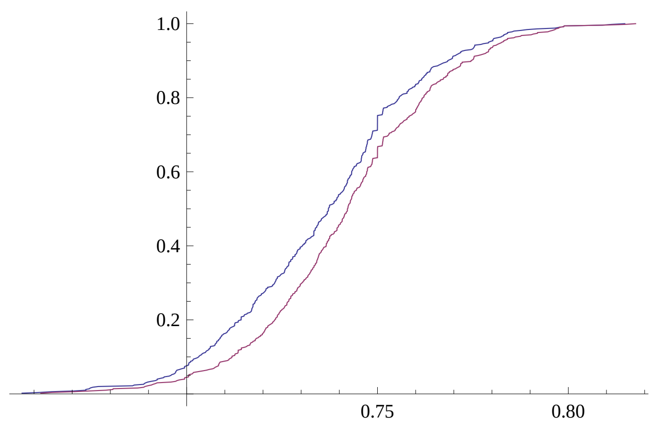


Figure 3: cdf of relative total quality of $D$ and $D'$ as compared to the first–best allocation (100%)

approximation to a maximization over all allocations that are feasible given that information.

Figure 3 compares the quality (welfare) of $D$ and $D'$ as compared to the complete–information benchmark, $CI$, by computing the empirical distributions of quality. For most of the observations, allocations D and D' achieve between 70 and 80 percent of the first best quality.

## 5. APPLICATION TO R&D SUBSIDIES

In this section we briefly sketch a possible application of the Dantzig mechanism.

As mentioned above, this work is motivated by Giebe, Grebe, and Wolfstetter (2006). The authors analyse the German R&D subsidy programs for small and medium-sized private businesses (which is similar in spirit to the SBIR program in the US). Firms from all sectors of the economy are encouraged to submit their research proposals. Proposals are evaluated by a panel of experts and ranked into one of three quality categories $A$, $B$ or $C$ (experts frequently point out that a finer grading is hard if not impossible to achieve). Successful applicants receive a subsidy of 50 per cent of their stated personnel cost. Current practice allocates subsidies as follows: As many $A$ projects as possible are funded. If there is money left, it is used to fund as many $B$ projects as possible etc. Giebe et al outline the two major flaws of this procedure. First, it puts relatively cheap $C$ and $B$ projects at disadvantage to expensive $A$ projects even though they might promise much better research results relative to their cost. Second it is highly likely that some applicants receive more money than they require to carry out the project (if they agree to do it for 50 per cent of their personnel cost it is a fair guess that some would do it for less). The proposed Dantzig mechanism addresses both these problems. Introducing a price-per-quality clock enforces competition across the quality classes and the open bidding procedure brings the subsidies closer to the firms' reserve prices. The 50 per cent personnel cost serve as the firms' individual maximal bids.

There is one design issue that we haven't addressed yet, namely the concept of quality ratios. In current practice quantitative ratios between the different quality classes exist only implicitly in the sense that $q_A >> q_B >> q_C$, i.e. any $A$ project is always preferred to any $B$ project is always preferred to any $C$ project independent of the costs. To apply the Dantzig mechanism one has to explicitly attach numerical equivalence ratios to the three categories (The ratios of $q_B = 0.5$ and $q_C = 0.33$ for our simulation were chosen arbitrarily).

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