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Target Shortfall Orderings and Indices*

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Abstract

Given any income distribution, to each income we associate a subgroup containing all persons whose incomes are not higher than this income and a person's target shortfall in a subgroup is the gap between the subgroup highest income and his own income. We then develop an absolute target shortfall ordering, which, under constancy of population size and total income, implies the Lorenz and Cowell-Ebert complaint orderings. Under the same restrictions, one distribution dominates the other by this ordering if and only if the dominated distribution can be obtained from the dominant one by a sequence of rank preserving progressive transfers, where each transfer is shared equally by all persons poorer than the donor of the transfer. The relationship of the ordering with the absolute deprivation and differential orderings, and its consistency with ranking of distributions by absolute target shortfall indices are explored. Well-known inequality indices like the absolute Gini index and the standard deviation are interpreted as absolute target shortfall indices. Finally, the possibility of a relative target shorfall ordering is also discussed.

JEL Classification Numbers: D31, D63.
Keywords: Target shortfall orderings, transfer, indices.

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1 Introduction

It is often argued that attitudes such as envy and deprivation are important components of individual judgments so far as distributive justice is concerned. There is also a view that social status of an individual - approximated by his position on the social hierarchy - plays an important role in the determination of his well-being (see, for example, Weiss and Fershtman, 1998).

The notion of individual deprivation originating in the works of Runciman (1966) precisely accommodates these views making the individual assessment of a given social state depend on his situation compared with those of persons more favourably treated than him. The deprivation profile, which indicates the level of deprivation felt by each individual, constitutes the basis of social judgement. Sen (1976), who introduced the idea of deprivation into the income distribution literature, posited that an individual's level of deprivation is an increasing function of the number of persons who are better off than him in the income scale. Most of the subsequent researchers assumed that individual deprivation of a person is simply the sum - possibly normalized in a suitable way - of the income gaps between all individuals richer than him and the individual himself (see, for example, Yitzhaki, 1979, Hey and Lambert, 1980, Kakwani, 1984, Chakravarty, 1997, Chakravarty and Mukherjee, 1999, and Chakravarty and Moyes, 2003).[1] Temkin (1986, 1997) argued that we may view inequality as an aggregate of complaints of different individuals located at disadvantaged positions in the income distribution. Temkin considered a number of possibilities of which a major case is that the highest income is the same reference point for all, and everyone, but the person with the highest income, has a legitimate complaint. Cowell and Ebert (2002) used this structure to derive a new class of inequality indices and an inequality ordering.

In this paper we adopt a view which is similar in spirit to the above notions of value judgements, but in a different structure. We assume that, in a n-person society, any person in subgroup i of persons with i lowest incomes regards the subgroup highest income as his target income. The difference between this target income and his own income is a measure of his absolute target shortfall in the subgroup. This notion of target shortfall formally resembles the individuals' poverty gap in the subgroup, where the poverty line is set equal to the targeted income. These target shortfalls of

[1] An axiomatic characterization of individual deprivation in such a framework was developed by Ebert and Moyes (2000).

different individuals in different subgroups form the basis of our distributive justice in the current context. Using this framework we develop an ordering which we refer to as the absolute target shortfall ordering, because it satisfies absolute scale invariance condition in the sense that it does not alter under equal absolute changes in all incomes.

Clearly, while the Cowell-Ebert ordering (Cowell-Ebert, 2002) is based on differences of individual incomes from the highest income, the target shortfall ordering relies on differences from subgroups' highest incomes.

For two distributions of a given total income over a given population size, this ordering implies but is not implied by the well-known Lorenz ordering and the Cowell-Ebert complaint ordering. It is also different from the absolute deprivation ordering which is based on individual absolute deprivations.

We further demonstrate, under constancy of total income and population size, that if one distribution dominates the other by the absolute target shortfall criterion, then the latter can be obtained from the former by a sequence of rank preserving progressive transfers, where the nature of a transfer is such that each person poorer than the donor of the transfer gets an equal share of it. The converse is also true. We also identify the class of absolute target shortfall indices that implies and is implied by the ordering. It is shown explicitly that well-known indices like the absolute Gini index and the standard deviation are coherent with the absolute target shortfall ordering.

The paper is organized as follows. The absolute target shortfall ordering, its relationship with the absolute differential, Lorenz, absolute deprivation and Cowell-Ebert complaint orderings, its equivalent redistributive mechanism, its function in ranking distributions by absolute target shortfall indices and examples of such indices are presented in Section 2. Section 3 explores the possibility of a relative target shorfall ordering, which remains unchanged under equi-proportionate variations in all incomes. Section 4 concludes.

2 The Absolute Target Shortfall Ordering and Its Implications

For a population of size n, the typical income distribution is given by a vector $x = (x_1, x_2, ..., x_n)$, where $x_i \geq 0$ is the income of person i. The distribution x is an element of D^n, the nonnegative orthant of the Euclidean n-space $\mathbf{R}^n$ with the origin deleted. By deleting origin from the domain we ensure that there is at least one person with positive income. The set of all possible income distributions is $D = \bigcup_{n \in \mathbf{N}} D^n$, where $\mathbf{N}$ is the set of natural numbers. Throughout the paper we will adopt the following notation. For any function $f : D \longrightarrow \mathbf{R}^1$ the restiction f on D^n is denoted by f^n. When we say that a function $f : D \longrightarrow \mathbf{R}^1$ satisfies some properties, we mean that all restrictions f^n of f satisfy them. For all $n \in \mathbf{N}$, $x \in D^n$, let $\lambda(x)$, or λ, be the mean of x. By our assumption $\lambda > 0$. All income distributions are assumed to be illfare ranked, that is, for all $n \in \mathbf{N}$, $x \in D^n$, $x_1 \leq x_2 \leq ... \leq x_n$. Since x_i's are ordered, any kind of change in x_i will have to be rank preserving. For all $n \in \mathbf{N}$, $x \in D^n$, the subgroup of the population with i lowest incomes, $\{x_1, x_2, ..., x_i\}$, in x is $S_i = \{1, 2, ..., i\}$. For $n = 1$, the concept of inequality is vacuous. We, therefore, assume that $n \geq 2$.

Any person j in subgroup S_i may suffer from depression on finding that his income is lower than x_i and regard x_i as his targeted income. Therefore, we can consider the normalized gap $(x_i - x_j)/n$ as a measure of j's target shortfall in S_i. We will see that this normalized difference has several advantages including poverty interpretation, satisfaction of population replication principle and convenient role in interpreting inequality indices as target shortfall indices.

Definition 1 *: For any $x \in D^n$, $t(x, S_i) = \sum_{j=1}^{i} \frac{(x_i - x_j)}{n}$ is the aggregate target shortfall of individuals in subgroup S_i.*

Clearly, some of the shortfalls $x_i - x_j$ may be zero. If x_i is taken as the poverty line in S_i, then $x_i - x_j$ is individual j's poverty gap and $\sum_{j=1}^{i} (x_i - x_j)$ gives us the total amount of money necessary to put the persons in S_i at the poverty line itself. Note that here $\frac{i}{n}$ is the headcount ratio, the proportion of persons whose incomes do not exceed the poverty line, and $\sum_{j=1}^{i} \frac{(x_i - x_j)}{i}$ is the average poverty gap of the poor, when x_i is the poverty line. Thus, $t(x, S_i)$ is the product of these two indicators of poverty.

The following are some of the properties possessed by the aggregate index $t(x, S_i)$.

1. It is independent of the incomes higher than x_i. This property reflects the importance of the reference group S_i. When $j \in S_i$, his targeted income is in the subvector of x that corresponds to S_i.

2. It is an absolute or translation invariant index, it remains unaltered under equal absolute changes in all incomes.

3. It is linear homogeneous.

4. A rank preserving reduction in x_i reduces $t(x, S_i)$.

5. A rank preserving progressive (regressive) transfer from the individual i, with income x_i, to anyone with a lower (higher) income reduces $t(x, S_i)$.

6. A rank preserving progressive transfer between two individuals, $i, k \in S_i$, where $x_j, x_k < x_i$, does not change $t(x, S_i)$. Similarly, a rank preserving progressive transfer between individuals $i, k \notin S_i$, does not change $t(x, S_i)$.

7. A rank preserving progressive transfer from anyone richer than i to someone poorer than i reduces $t(x, S_i)$. Likewise, a rank preserving regressive transfer from anyone poorer than i to someone richer than i increases $t(x, S_i)$.

8. A rank preserving reduction in the income of anybody poorer that i increases $t(x, S_i)$.

9. A rank preserving addition/reduction in any income higher than x_i does not affect the value of $t(x, S_i)$.

10. It remains unchanged under replications of the population.

11. It is bounded between zero and $\frac{(i-1)x_i}{n}$, where the lower bound is attained when there is no feeling of depression in the subgroup, that is, the income vector corresponding to S_i is equal. In contrast, the maximum value is obtained when for each $j < i$, the target shortfall is maximum, that is, everybody except person i has a zero income.

Later in the section we will see how the aggregate target shortfalls of different subgroups can be used in constructing a summary target shortfall index for the society as a whole. Now, instead of focusing on subgroup S_i, we can concentrate on the whole population and, as before, assume that for the pair (i, j), person j's depression is measured by $x_i - x_j$, where $x_i \geq x_j$. Then the average of all such depressions in all pairwise comparisons becomes proportional to the Gini index (Sen, 1973).

When we focus on the entire population (the subgroup S_n), instead of making pairwise comparisons of incomes, following Temkin (1986, 1993) we can argue that x_n is the reference point for all persons and $x_n - x_i$ is the size of complaint of person i. Cowell and Ebert (2002) characterized a general class of inequality indices $C = C_1 \bigcup C_2$ in terms of such complaints, where:

$$\begin{aligned} C_1 &= \left\{ C_\varepsilon^n(x) : \varepsilon > 1,\ \sum_{k=1}^{n-1} w_k = 1,\ w_k \geq w_{k+1} > 0 \right\}, \\ C_2 &= \left\{ C_\varepsilon^n(x) : \varepsilon = 1,\ \sum_{k=1}^{n-1} w_k = 1,\ w_k > w_{k+1} > 0 \right\}, \\ C_\varepsilon^n(x) &= \left[\sum_{k=1}^{n-1} w_k (x_n - x_k)^\varepsilon \right]^{\frac{1}{\varepsilon}}. \end{aligned} \tag{1}$$

Members of the class C decrease under a rank preserving income transfer from a person to anyone poorer. The relevance of the family C is shown later in detail.

It thus appears that absolute income differences can be used for interpreting several inequality indices from different perspectives. In the following we explain their roles more formally for ordering income distributions.

To show how the concept of target shortfall may be used in ranking alternative distributions of income we introduce:

Definition 2 *: Given $x, y \in D^n$, we say that x dominates y by the absolute target shortfall ordering, which we write $x \succeq_{ATS} y$, if*

$$t(x, S_i) \geq t(y, S_i), \tag{2}$$

for all $i = 1, 2, ..., n$ with $>$ for at least one i.

Thus, for $x \succeq_{ATS} y$ to hold we require that the aggregate target shortfall of each subgroup of persons with i lowest incomes in x will not be lower than that in y, and will be higher for at least one subgroup, where $i = 1, 2, ..., n$. Since $\succeq_{ATS}$ is translation invariant, it is an absolute ordering. We may also represent the ordering $\succeq_{ATS}$ in terms of the absolute target shortfall curve.

Definition 3 *: For any $x \in D^n$, the absolute target shortfall curve of x, $AT\left(x, \frac{i}{n}\right)$, is a plot of $t(y, S_i)$ against $\frac{i}{n}$, where $AT(x, 0) = 0$.*

That is, $AT\left(x, \frac{i}{n}\right)$ graphically shows the aggregate target shorfalls $t(x, S_i)$ of the subgroups S_i, as i goes from 0 to n, where $S_0 = 0$. Clearly, for a perfectly equal distribution the curve coincides with the horizontal axis. For an unequal distribution x, $AT\left(x, \frac{i}{n}\right)$ is nondecreasing but nonconstant. Then $x \succeq_{ATS} y$ means that the target shortfall curve of x dominates that of y, that is, $AT\left(x, \frac{i}{n}\right)$ lies nowhere below $AT\left(y, \frac{i}{n}\right)$ and at some places (at least) strictly above.

The ordering we consider next is more demanding, as it requires comparison of pairwise inequalities in two distributions.

Definition 4 *: Given $x, y \in D^n$, we say that x dominates y by absolute differentials, which we write $x \succeq_{AD} y$, if*

$$y_i - x_i \geq y_{i+1} - x_{i+1}, \tag{3}$$

for all $i = 1, 2, ..., n$ with $>$ for some i.

Since we can rewrite $y_i - x_i \geq y_{i+1} - x_{i+1}$ as $x_{i+1} - x_i \geq y_{i+1} - y_i$, (3) simply means that differences between any two consecutive incomes in nondecreasing order are not less in x than in y and will be higher in some case(s). It was first introduced by Marshall et al. (1967) and has been considered as a suitable inequality criterion by Preston (1990) and Moyes (1994, 1999).

The following result gives the relationship between $\succeq_{ATS}$ and $\succeq_{AD}$.

Theorem 1 *: For all $x, y \in D^n$, $x \succeq_{AD} y$ implies $x \succeq_{ATS} y$, but the converse is not true unless $n = 2$.*

Proof:
Suppose $x \succeq_{ATS} y$, which by definition is equivalent to:

$$\sum_{j=1}^{i} (x_i - x_j) \geq \sum_{j=1}^{i} (y_i - y_j), \tag{4}$$

for all $i = 1, 2, ..., n$ with $>$ for at least one i. Given i, a sufficient condition for (4) to hold is that every term within brackets on the left hand side is greater than or equal to the corresponding term on the right hand side. On decomposition this requirement becomes

$$(x_i - x_{i-1}) \geq (y_i - y_{i-1}), ..., (x_i - x_1) \geq (y_i - y_1).$$

We can rewrite these inequalities as:

$$(y_{i-1} - x_{i-1}) \geq (y_i - x_i), ..., (y_1 - x_1) \geq (y_i - x_i). \tag{5}$$

Now, a sufficient condition for (5) to hold is that:

$$(y_1 - x_1) \geq (y_2 - x_2) \geq ... \geq (y_i - x_i),$$

which is implied by $x \succeq_{AD} y$. Clearly, if a strict $>$ occurs for some i, say i_0 in (3), then there will be a strict inequality for i_0 in (2).

To see that the converse is false, whenever $n > 2$, consider the distributions $x = (10, 20, 30, 40)$ and $y = (15, 20, 25, 40)$. The we have that $x \succeq_{ATS} y$, but not $x \succeq_{AD} y$. ■

We now discuss the relationship between $x \succeq_{ATS} y$, and $\succeq_L$, the Lorenz ordering. Given $x, y \in D^n$, $x \succeq_L y$ holds if:

$$\frac{\sum_{j=1}^{i} x_j}{n\lambda(x)} \geq \frac{\sum_{j=1}^{i} y_j}{n\lambda(y)},$$

for all $i = 1, 2, ..., n$ with $>$ for at least one $i < n$. Thus, in contrast to the average subgroup shortfalls that we compare under $\succeq_{ATS}$, $\succeq_L$ compares the proportions of the total income enjoyed by the subgroups.

Note that $\succeq_L$ is a relative ordering, it remains unaltered when all incomes are multiplied by a positive scalar. Thus, while for an absolute ordering income differentials are a source of envy, for a relative ordering people's deprivations depend on income shares. However, if mean income is fixed, we do not differentiate between the two notions.

The following theorem shows that $\succeq_{ATS}$ is a sufficient but not a necessary condition for $\succeq_L$.

Theorem 2 *: Let $x, y \in D^n$, where $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$, be arbitrary. Then $x \succeq_{ATS} y$ implies $y \succeq_L x$, but the converse is not true unless $n = 2$.*

Proof:

Throughout the proof we refer to the assumption $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$ as condition A.

The n^{th} inequality in $x \succeq_{ATS} y$ is $nx_n - \sum_{j=1}^{n} x_j \geq ny_n - \sum_{j=1}^{n} y_j$, which in view of A, gives $x_n \geq y_n$. Therefore it follows that $\sum_{j=1}^{n-1} x_j \leq \sum_{j=1}^{n-1} y_j$. In $x \succeq_{ATS} y$ for $i = n-1$, we get:

$$(n-1)\, x_{n-1} - x_1 - ... - x_{n-2} - x_{n-1} \geq (n-1)\, y_{n-1} - y_1 - ... - y_{n-2} - y_{n-1}. \quad (6)$$

Given $x_n \geq y_n$ and $n > 2$, we can add $(n-1)\, x_n$ ($(n-1)\, y_n$) and subtract x_n (y_n) from the left hand (right hand) side of (6) to deduce:

$$(n-1)(x_{n-1} + x_n) - \sum_{j=1}^{n} x_j \geq (n-1)(y_{n-1} + y_n) - \sum_{j=1}^{n} y_j, \quad (7)$$

which because of A implies:

$$(n-1)(x_{n-1} + x_n) \geq (n-1)(y_{n-1} + y_n). \quad (8)$$

Hence $x_{n-1} + x_n \geq y_{n-1} + y_n$. Therefore, by A,

$$\sum_{j=1}^{n-2} x_j \leq \sum_{j=1}^{n-2} y_j. \tag{9}$$

Thus we have proved that:

$$\sum_{j=1}^{n-r+1} x_j \leq \sum_{j=1}^{n-r+1} y_j \tag{10}$$

holds for $r = 1, 2, 3$. (For $r = 1$ equality holds by the assumption). Assume that (10) is true for all $r \leq k$. We show that it is true for $r = k + 1$ also. By assumption:

$$\sum_{j=1}^{n-k+1} x_j \leq \sum_{j=1}^{n-k+1} y_j, \tag{11}$$

which by A gives:

$$\sum_{j=n-k+2}^{n} x_j \geq \sum_{j=n-k+2}^{n} y_j. \tag{12}$$

The $(n-k+1)^{th}$ inequality in $x \succeq_{ATS} y$ gives $(n-k)\, x_{n-k+1} - x_{n-k} - ... - x_1 \geq (n-k)\, y_{n-k+1} - y_{n-k} - ... - y_1$ from which we get:

$$\begin{aligned} &(n-k+1)\, x_{n-k+1} + \textstyle\sum_{j=n-k+2}^{n} x_j - \sum_{j=1}^{n} x_j \geq \\ &(n-k+1)\, y_{n-k+1} + \textstyle\sum_{j=n-k+2}^{n} y_j - \sum_{j=1}^{n} y_j. \end{aligned} \tag{13}$$

Given A, this implies that:

$$(n-k+1)\, x_{n-k+1} + \sum_{j=n-k+2}^{n} x_j \geq (n-k+1)\, y_{n-k+1} + \sum_{j=n-k+2}^{n} y_j. \tag{14}$$

Multiplying both sides of (12) by $(n-k)$ and adding the right hand (left hand) side of the resulting expression to the right hand (left hand) side of (14), we get:

$$(n-k+1) \sum_{j=n-k+1}^{n} x_j \geq (n-k+1) \sum_{j=n-k+1}^{n} y_j. \tag{15}$$

Cancelling $(n-k+1)$ from both sides of (15) and using A it follows that:

$$\sum_{j=1}^{n-k} x_j \leq \sum_{j=1}^{n-k} y_j. \tag{16}$$

Since $1 \leq k \leq n$ is arbitrary, it follows that $y \succeq_L x$. If there is at least one strict inequality in $\succeq_{ATS}$, there will be at least one strict inequality in $\succeq_L$. For instance, if the inequality in (6) is strict, then the inequality in (9) is strict as well.

To see that the reverse implication is not true, let $x = (10, 20, 30, 40)$ and $y = (10, 24, 26, 40)$. Then we have $y \succeq_L x$ but not $x \succeq_{ATS} y$. ■

The intuitive reasoning behind why the reverse implication in Theorem 2 is not true can be explained using the above example. We get y from x by transferring 4 units of income from the second richer person to the third richer person. In view of equivalence between inequality reduction due to rank preserving progressive transfer and the Lorenz domination, we have $y \succeq_L x$ (Kolm, 1969, Atkinson, 1970 and Dasgupta, Sen and Starrett, 1973). But the transfer while reduces the aggregate target shortfall in the third subgroup increases that in the second subgroup, and, consequently, the net effect becomes ambiguous. From the proof of Theorem 2 we can also conclude that under constancy of total income and population size of distributions x and y, if $x \succeq_{ATS} y$ holds, then the former becomes at least as good as the latter by the maximax criterion, that is, $\max_i \{x_i\} \geq \max_i \{y_i\}$.

The Lorenz domination does not seem to be compatible with attitudes such as envy and resentment, which according to experimental studies, seem to be important component of individual judgements. The concept of individual deprivation developed in the work of Runciman (1966) harmonizes these views under the assumption that the individual's feeling of deprivation arises out of the comparison of his situation with those of better off persons.

Given $x \in D^n$, we let:

$$ADP(i, x) = \frac{1}{n} \sum_{j=i}^{n} (x_j - x_i) \tag{17}$$

measure the absolute deprivation of person i. That is, the magnitude of absolute deprivation felt by a person is a normalized sum of the differences between the situation of better off persons and his own situation. By definition the most well-off person is never deprived and the poorest person suffers from maximum deprivation. $ADP(i, x)$ may be contrasted with the target shortfall measure of person i in a subgroup. The latter simply concentrates on the difference between the maximum income of the subgroup and the person's own income, whereas the former is based on the comparison of his income with those of the better off persons in the entire population.

Following Hey and Lambert (1980), Kakwani (1984), Chakravarty (1997) and Chakravarty and Moyes (2003) we now have the following.

Definition 5 *: For $x, y \in D^n$, we will say that x dominates y by the absolute deprivation criterion, which we write $x \succeq_{ADP} y$, if:*

$$ADP\,(i,x) \geq ADP\,(i,y)\,, \tag{18}$$

for all $i = 1, 2, ..., n$ with $>$ for at least one i.

The following theorem shows that $\succeq_{ATS}$ and $\succeq_{ADP}$ are different.

Theorem 3 *: The orderings $\succeq_{ATS}$ and $\succeq_{ADP}$ are logically independent.*

Proof:

To show that the two orderings are logically independent, we need to demonstrate existence of distributions such that (i) both $\succeq_{ATS}$ and $\succeq_{ADP}$ hold, (ii) one of them holds but not the other, and (iii) neither holds. The following table enlists five such distributions. The presence of '1' means that the ordering in the corresponding column holds and '0' means it does not.

Table 1

Distributions	$x \succeq_{ATS} y$	$x \succeq_{ADP} y$
$x = (10, 20, 30, 40)$, $y = (15, 20, 30, 35)$	1	1
$x = (10, 20, 30, 40)$, $y = (15, 15, 30, 40)$	1	0
$x = (10, 20, 30, 40)$, $y = (10, 20, 34, 36)$	0	1
$x = (10, 20, 30, 40)$, $y = (10, 24, 26, 40)$	0	0

■

Theorem 3 says explicitly that the distributive judgements embodied in the orderings $\succeq_{ATS}$ and $\succeq_{ADP}$ are not the same.

Following Cowell and Ebert (2002) we now have:

Definition 6 *: For $x, y \in D^n$, x exhibits more complaint inequality than y, which we write $x \succeq_C y$, if:*

$$\sum_{j=1}^{i} (x_n - x_j) \geq \sum_{j=1}^{i} (y_n - y_j), \tag{19}$$

for all $i = 1, 2, ..., n$, with $>$ for at least one i.

That is, $x \succeq_C y$ means that the cumulative complaints of the i poorest person in x are not lower than that in y and will be higher for some i, $i = 1, 2, ..., n$. Cowell and Ebert (2002) proved that $x \succeq_C y$ is related to the generalized Lorenz Ordering $\succeq_{GL}$.[2] Formally, Lemma 6 of Cowell and Ebert (2002) shows that $x \succeq_C y$ implies and is implied by $(y - y_n 1^n) \succeq_{GL} (x - x_n 1^n)$, where 1^n is the $n-$coordinated vector of ones. They also showed that $x \succeq_C y$ is equivalent to the condition that $C_\epsilon^n(x) > C_\epsilon^n(y)$ for all $C_\epsilon^n \in C$. In our next theorem we show that $\succeq_{ATS}$ subsumes $\succeq_C$.

Theorem 4 *: Let $x, y \in D^n$, where $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$, be arbitrary. Then $x \succeq_{ATS} y$ implies $x \succeq_C y$, but the converse is not true unless $n = 2$.*

Proof:

Suppose $x \succeq_C y$, which by Lemma 6 of Cowell and Ebert (2002) is same as the condition that $(y - y_n 1^n) \succeq_{GL} (x - x_n 1^n)$. We can write this latter relation explicitly as:

$$\frac{1}{n}\left(ix_n + \sum_{j=1}^{i} y_i\right) \geq \frac{1}{n}\left(iy_n + \sum_{j=1}^{i} x_i\right), \tag{20}$$

for all $i = 1, 2, ..., n$ with $>$ for at least one i. Given the equality of total incomes in x and y, two sufficient conditions for (20) to hold are $x_n \geq y_n$ and $y \succeq_L x$. From the proof of Theorem 2 we know that under the assumption $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, $x \succeq_{ATS} y$ implies $x_n \geq y_n$ and $y \succeq_L x$. Hence $x \succeq_{ATS} y$ implies $x \succeq_C y$.

To see that $x \succeq_C y$ does not imply $x \succeq_{ATS} y$, let $x = (10, 20, 30, 40)$ and $y = (10, 24, 26, 40)$. Then we have $x \succeq_C y$ but $x \succeq_{ATS} y$ does not hold. ■

[2] The generalized Lorenz curve of an income distribution is obtained by scaling up its Lorenz curve by the mean income, and $x \succeq_{GL} y$ means that the generalized Lorenz curve of x is nowhere below that of y, and at some places (at least) lies strictly inside, see Shorrocks (1983).

From Theorems 2 and 3 it emerges that redistributive principles involving egalitarian transfers that are compatible with $\succeq_L$ and $\succeq_{ADP}$ respectively will be inappropriate in the case of $\succeq_{ATS}$. Therefore, $\succeq_{ATS}$ needs a separate treatment in this context. For this we introduce:

Definition 7 *: Given $x \in D^n$ we say that y is obtained from x by an impartially favourable transfer if:*

$$\begin{array}{lr} y_j = x_j - \delta \geq y_{j-1} & \textit{for for some } j > 1, \textit{ where } \delta > 0, \\ y_k = x_k + \frac{\delta}{j-1} & \textit{for } 1 \leq k \leq j-1, \\ y_k = x_k & \textit{for } k > j. \end{array}$$

That is, an impartially favourable transfer is a rank preserving progressive transfer from some person (j) and it is equally shared by all persons poorer than him. Since the transfer is from someone to all persons poorer than him, it is favourable, and since it treats all recipients identically, we call it impartial as well. Note that in particular, if there are k persons with the minimum income and the second poorest person is the donor, then the transfer is shared equally by these k persons only.

The following theorem identifies the type of egalitarian transfer that can take us from a less equitable distribution to a more equitable one according to $\succeq_{ATS}$.

Theorem 5 *: Let $x, y \in D^n$, where $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$, be arbitrary. Then the following statements are equivalent:*

(a) $x \succeq_{ATS} y$,

(b) *y can be obtained from x through a sequence of impartially favourable transfers.*

Proof:

We verify the theorem in the case where the minimum income is unique, and there is only one impartially favourable transfer. The proof in the general case follows similarly. Note that $x \succeq_{ATS} y$ can be simplified as:

$$x_i - y_i \geq \frac{1}{(i-1)} \sum_{j=1}^{i-1} (x_j - y_j), \tag{21}$$

for all $i = 2, 3, ..., n$ with $>$ for some i.

Suppose that (b) holds, that is, y has been obtained from x by an impartially favourable transfer of δ units of income from person t (with income x_t). Then for all i, $2 \leq i \leq t-1$,

$$-\frac{\delta}{t-1} = x_i - y_i = \sum_{j=1}^{i-1} \frac{(x_j - y_j)}{i-1} = -\frac{\delta}{t-1}.$$

For $i = t$,

$$\delta = x_i - y_i > \sum_{j=1}^{t-1} \frac{(x_j - y_j)}{t-1} = -\frac{\delta}{t-1}.$$

For all $i \geq t+1$,

$$0 = x_i - y_i = \frac{1}{(i-1)} \sum_{j=1}^{i-1} (x_j - y_j) = 0.$$

Thus we have $x \succeq_{ATS} y$.

To prove the converse, suppose that x has been transformed into y through a rank preserving transfer of δ units of income from the person t, where δ is shared equally by all persons poorer than him except the poorest person. Then $x \succeq_{ATS} y$ requires $-\frac{\delta}{t-2} = x_2 - y_2 \geq x_1 - y_1 = 0$, a contradiction!

Thus, we have shown that (b) implies (a) and not (b) implies not (a). Hence (a) and (b) are equivalent. ■

From the proof of theorem 5 it emerges that if $x \succeq_{ATS} y$ holds under the conditions stated in the theorem, then the poorest person in y gets some positive amount of income from someone richer. Therefore, the following result drops out as an interesting corollary to Theorem 5.

Corollary 1 *: Let $x, y \in D^n$, where $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$, be arbitrary. Then $x \succeq_{ATS} y$ implies that* $\min_i \{y_i\} > \min_i \{x_i\}$.

That is, under the given conditions stated in Theorem 5, if one distribution dominates the other by the criterion $\succeq_{ATS}$, then an implied policy recommendation is similar to the Rawlsian maximin type (Rawls, 1971) that maximizes the welfare of the worst off individual.

Corollary 1 does not say anything about the ranking of depressions of the worst off individual in different subgroups in the concerned distributions.

In the following theorem we show that one implication of $\succeq_{ATS}$ is that the depression of the worst off person in the dominant profile is at least as large as that in the dominated one for all subgroups and will be higher for at least one subgroup. The result holds even if the total incomes in the two situations are different.

Theorem 6 *: Let $x, y \in D^n$ be arbitrary. Then $x \succeq_{ATS} y$ implies that $x_i - x_1 \geq y_i - y_1$ for all $i = 1, 2, ..., n$ with $>$ for at least one i. However, the converse is not true.*

Proof:
For $i = 2$, $x \succeq_{ATS} y$ gives:

$$x_2 - x_1 \geq y_2 - y_1. \tag{22}$$

For $i = 3$, we get:

$$2x_3 - x_2 - x_1 \geq 2y_3 - y_2 - y_1. \tag{23}$$

Adding the left hand (right hand) side of (22) with the corresponding side of (23), we have $2x_3 - 2x_2 \geq 2y_3 - 2y_2$, from which it follows that:

$$x_3 - x_1 \geq y_3 - y_1. \tag{24}$$

Next, for $i = 4$, we have:

$$3x_4 - x_3 - x_2 - x_1 \geq 3y_4 - y_3 - y_2 - y_1. \tag{25}$$

Adding the left hand (right hand) side of (22) with the corresponding side of (24) and (25), it can be deduced that:

$$x_4 - x_1 \geq y_4 - y_1. \tag{26}$$

Continuing this way we get $x_i - x_1 \geq y_i - y_1$ for all i. If for some i (say i_0) strict inequality occurs in $x \succeq_{ATS} y$, then $x_{i_0} - x_1 > y_{i_0} - y_1$.

To see that the opposite is not true, let $x = (20, 30, 40, 50, 60)$ and $y = (15, 15, 35, 35, 50)$. Then $x_i - x_1 \geq y_i - y_1$ for all i, with three inequalities being strict. But $x \succeq_{ATS} y$ does not hold. ■

Apart from the implications of the absolute target shortfall ordering and its equivalent redistributive mechanism, it is natural to consider how this ordering may be used in ranking distributions unanimously by target shortfall indices. Let $T^n : D^n \to \mathbf{R}^1$ be an arbitrary target shorfall index that meets symmetry and decreases under an impartially favourable transfer, where symmetry requires invariance of T^n under any reordering of the incomes. Thus, symmetry means that anything other than income is irrelevant to the measurement of target shortfall. An implication of symmetry is that we can define target shortfall indices directly on ordered distributions (as we have done).

We then have:

Theorem 7 *: Let $x, y \in D^n$ be arbitrary. Then the following conditions are equivalent:*

(a) $x \succeq_{ATS} y$

(b) $T^n(x) > T^n(y)$ *for all absolute symmetric target shortfall indices* $T^n : D^n \to \mathbf{R}^1$ *that reduce under an impartially favourable transfer.*

Proof:

$(a) \Rightarrow (b)$: Suppose $\lambda(x) \geq \lambda(y)$. Define $\overline{y} = y + (\lambda(x) - \lambda(y)) 1^n$. Given that $\succeq_{ATS}$ is an absolute ordering, $x \succeq_{ATS} y$ implies $x \succeq_{ATS} \overline{y}$. Since $\lambda(x) = \lambda(\overline{y})$, by theorem 5, we have $T^n(x) > T^n(\overline{y})$ for any index T^n that decreases whenever there is an impartially favourable transfer. Note that T^n is symmetric because it is defined on ordered distributions. By translation invariance of T^n, we get $T^n(x) > T^n(y)$.

A similar proof holds if $\lambda(x) < \lambda(y)$.

$(b) \Rightarrow (a)$: Consider the index:

$$T_i^n(x) = \frac{1}{n} \sum_{j=1}^{i} (x_i - x_j), \; 2 \leq i \leq n.$$

This index is translation invariant, symmetric and its value decreases under an impartially favourable transfer. Hence $T_i^n(x) > T_i^n(y)$ for all i, which in turn shows that $x \succeq_{ATS} y$ holds. ■

Theorem 7 shows that an unanimous ranking of income distributions by absolute target shortfall indices can be obtained through the comparison of the distributions by the ordering $\succeq_{ATS}$.

As an example of a target shortfall index of the type identified in part (b) of Theorem 7, we have:

$$T_r^n(x) = \left(\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{(x_i - x_j)^r}{n} \right)^{\frac{1}{r}}, \; r \geq 1. \tag{27}$$

For $r = 1$, T_r^n in (27) becomes the absolute Gini index (Blackorby and Donaldson, 1980):

$$G^n(x) = \lambda(x) - \frac{1}{n^2} \sum_{i=1}^{n} (2(n-i)+1) x_i. \tag{28}$$

When divided by the mean income it produces the well-known Gini coefficient. On the other hand, for $r = 2$, T_r^n becomes the standard deviation:

$$\sigma_n(x) = \left(\frac{1}{n} \sum_{i=1}^{n} (x_i - \lambda)^2 \right)^{\frac{1}{2}}, \tag{29}$$

which, when divided by the mean, becomes the coefficient of variation.

Thus, two well-known absolute inequality indices, namely the absolute Gini index and the standard deviation, can be regarded as absolute target shortfall indices.

The parameter r in T_r^n is a sensitivity parameter in the sense that an egalitarian transfer permissible under Theorem 5 decreases the value of T_r^n by a larger amount as r increases from 2 to plus infinity. For a given $r > 2$, the reduction in T_r^n due to the transfer will be higher the richer the donor is.

An alternative of interest arises from a normalization of the average of exponential transformation of $t(x, S_i)$ values:

$$E^n(x) = \log \frac{1}{n} \sum_{i=1}^{n} e^{t(x,S_i)}. \tag{30}$$

Both T_r^n and E^n achieve their minimum value, zero, if and only if the income distribution is perfectly equal. While T_r^n is linear homogeneous, E^n is not so. With respect to transfers, E^n has a similar behavior as T_r^n for a given $r > 2$.

Theorem 7 has limited applicability in the sense that it can be used for comparing target shortfalls of two distributions with the same population size. But often we may be interested in cross population comparisons of target shortfalls, for instance, of two different societies or of the same society in two periods. In order to do this, we assume that a target shortfall index $T : D \to \mathbf{R}^1$ satisfies the Principle of Population. Formally, for any $n \in \mathbf{N}$, $x \in D^n$, $T^n(x) = T^{nm}(y)$, where y is the m-fold replication of x, $m \geq 2$. Thus, Principle of Population demands invariance of the target shortfall index under replications of the population. In other words, this property says that target shortfall is an average concept.

Note that the target shortfall curve is population replication invariant, that is, for any $x \in D^n$, $AT\left(x, \frac{i}{n}\right)$ coincides with $AT\left(y, \frac{i}{nm}\right)$, where $y \in D^{nm}$ is the m-fold replication of x. Therefore, the following theorem can now be proved using theorem 7:

Theorem 8 *: For arbitrary $m, n \in \mathbf{N}$, $m, n \geq 2$, let $x \in D^m$, $y \in D^n$ be arbitrary. Then the following conditions are equivalent:*

(a) The target shorfall curve of x dominates that of y.

(b) $T^m(x) > T^n(y)$ for all absolute, symmetric, population replication invariant target shortfall indices $T : D \to \mathbf{R}^1$ that reduce under an impartially favourable transfer.

Theorem 8 thus shows how $\succeq_{ATS}$ can be used for comparing distributions over different population sizes using target shortfall indices that fulfil the Principle of Population. It may be noted that both T_r^n and E^n satisfy this principle.

3 Relative Target Shortfall Ordering

The results developed so far rely on the assumption that a person's feeling of depression depends on absolute income gaps. But often people may like to view depression as arising from relative losses. In such a case the extent of depression suffered by person j in subgroup S_i is given by $\frac{(x_i - x_j)}{n x_j}$. This is the relative target shortfall of j in S_i.[3] Consequently, the aggregate relative target shortfall in S_i is $\frac{1}{n}\sum_{j=1}^{i}\frac{(x_i - x_j)}{x_j}$.

Definition 8 *: Given any two situations $x, y \in D_+^n$, the positive part of D^n, we say that x dominates y by the relative target shortfall criterion, which we write $x \succeq_{RTS} y$, if*

$$\frac{1}{n}\sum_{j=1}^{i}\frac{x_i - x_j}{x_j} \geq \frac{1}{n}\sum_{j=1}^{i}\frac{y_i - y_j}{y_j}, \tag{31}$$

for all $i = 1, 2, ..., n$ with $>$ for at least one i.

[3] We assume in this section of the paper that all incomes are positive so that relative losses are well-defined.

The next ordering, which parallels (3), is based on relative income differentials.

Definition 9 *: For $x, y \in D_+^n$, we say that x dominates y by relative differentials, which we write $x \succeq_{RD} y$, if*

$$\frac{y_i}{x_i} \geq \frac{y_{i+1}}{x_{i+1}}, \tag{32}$$

for all $i = 1, 2, ..., n$ with $>$ for at least one i.

That is, ratios between any two consecutive incomes taken in non-decreasing order must not be higher in y than in x and will be lower in at least one case (see Marshall et al., 1967, Preston, 1990, and Moyes, 1994, 1997).

We demonstrate below that $\succeq_{RD}$ is stronger than $\succeq_{RTS}$.

Theorem 9 *: For all $x, y \in D_+^n$, $x \succeq_{RD} y$ implies $x \succeq_{RTS} y$, but the converse is not true unless $n = 2$.*

Proof:
Assume $x \succeq_{RTS} y$, which on simplification reduces to:

$$\sum_{j=1}^{i} \frac{x_i - x_j}{x_j} \geq \sum_{j=1}^{i} \frac{y_i - y_j}{y_j}, \tag{33}$$

for all $i = 1, 2, ..., n$, with $>$ for at least one i. Given i, a sufficient condition for (33) to hold is that $\frac{x_i}{x_j} \geq \frac{y_i}{y_j}$, for $j = 1, 2, ..., i$. That is, $\frac{x_i}{y_i} \geq \frac{x_j}{y_j}$, for $j = 1, 2, ..., i$. This requirement is satisfied if we assume that $\frac{x_i}{y_i} \geq \frac{x_{i-1}}{y_{i-1}} \geq \frac{x_{i-2}}{y_{i-2}} \geq ... \geq \frac{x_1}{y_1}$, which is implied by $x \succeq_{RD} y$. If a $>$ occurs in $x \succeq_{RD} y$, then there will be some $>$ in $\succeq_{RTS}$ as well.

To see that the converse is false, for $n > 2$, let $x = (10, 20, 30, 40)$ and $y = (15, 20, 25, 40)$. Then $x \succeq_{RTS} y$ holds but not $x \succeq_{RD} y$. ■

Chakravarty and Moyes (2003) argued that instead of focusing on absolute income differentials, one may rather conceive deprivation as resulting from relative income gaps. They defined the relative deprivation of individual i in situation $x \in D_+^n$ as:

$$RDP(i, x) = \frac{1}{n} \sum_{j=i}^{n} \frac{(x_j - x_i)}{x_i}. \tag{34}$$

Definition 10 *: For any two profiles $x, y \in D_+^n$, x is said to be unambiguously more relatively deprived than y, which we write $x \succeq_{RDP} y$, if:*

$$RDP(i, x) \geq RDP(i, y), \tag{35}$$

for all $i = 1, 2, ..., n$, with $>$ for at least one i.

A comparison of $\succeq_{RTS}$ can now be made with $\succeq_{RDP}$. Both concentrate on income ratios, but as we show in the following theorem, they are independent. Equivalently, we say that the egalitarian bias implicit in one is not related to that involved in the other.

Theorem 10 *: The orderings $\succeq_{RTS}$ and $\succeq_{RDP}$ are logically independent.*

Proof:

As in the case of proof of Theorem 3, the proof requires construction of examples showing the unambiguity/ambiguity of the two orderings in alternative comparable cases. The following table, which parallels Table 1, explicitly demonstrates independence of $\succeq_{RTS}$ and $\succeq_{RDP}$.

Table 2

Distributions	$x \succeq_{RTS} y$	$x \succeq_{RDP} y$
$x = (10, 20, 30, 40)$, $y = (15, 20, 30, 35)$	1	1
$x = (2, 4, 6, 8)$, $y = (6, 6, 12, 16)$	1	0
$x = (2, 4, 6, 8)$, $y = (4, 8, 14, 14)$	0	1
$x = (2, 4, 6, 8)$, $y = (2, 5, 5, 8)$	0	0

■

The first two distributions in Table 2 also show that $x \succeq_{ATS} y$ also holds (see first row of Table 1). From the two distributions x and y presented in the fourth row of Table 2, we can verify that $x \succeq_{ATS} y$ also does not hold. Next, from the situation in the second row of the above table, we have $x \succeq_{RTS} y$ but not $x \succeq_{ATS} y$. Finally, for $x = (2, 4, 6, 8)$ and $y = (1, 1, 4, 6)$, we have $x \succeq_{ATS} y$ but not $x \succeq_{RTS} y$. These observations lead us to the following theorem.

Theorem 11 *: The orderings $\succeq_{ATS}$ and $\succeq_{RTS}$ are logically independent.*

From Theorem 11, we can conclude that the redistributive operation underlying $\succeq_{RTS}$ will be different from that in $\succeq_{ATS}$. That is, an alternative notion of transfer mechanism which is compatible with $\succeq_{RTS}$ needs to be devised.

4 Conclusions

Given an income distribution x, to the i^{th} highest income x_i in x we associate subgroup S_i that consists of all persons whose incomes are less than or equal to x_i and the target shortfall of a person in S_i is the gap between x_i and his own income. Building on this, we consider an absolute target shortfall ordering, which, for a given population size and mean income, implies the Lorenz and Cowell-Ebert complaint orderings.

A redistributive principle that recommends rank preserving progressive transfer of income, where all persons poorer than the donor of the transfer share it equally, has been found to be compatible with this ordering. We also discussed the relationship of this ordering with absolute differential and deprivation orderings, and its role in ranking alternative income distributions by absolute target shortfall indices. Well-known inequality indices, like the absolute Gini index and the standard deviation, have been interpreted as absolute target shortfall indices. Finally, we briefly analyzed some properties and implications of a relative counterpart to this ordering.

It may be important to mention that our analysis has been of ordinal nature. It has been concerned with rankings of distributions from alternative perspectives. A worthwhile attempt will be to characterize target shortfall indices. Another line of extension can be the development of a redistributive principle compatible with the relative target shortfall ordering. We leave these for future research.

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