ISSN 1825-0211

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Quaderno n. 43 — Febbraio 2008

QUADERNI DEL DIPARTIMENTO DI ECONOMIA, FINANZA E STATISTICA

Hedging error in Lévy models with a Fast Fourier Transform approach

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Abstract

We measure, in terms of expectation and variance, the cost of hedging a contingent claim when the hedging portfolio is re-balanced at a discrete set of dates. The basic point of the methodology is to have an integral representation of the payoff of the claim, in other words to be able to write the payoff as an inverse Laplace transform. The models under consideration belong to the class of Lévy models, like NIG, VG and Merton models. The methodology is implemented through the popular FFT algorithm, used by many financial institutions for pricing and calibration purposes. As applications, we analyze the effect of increasing the number of tradings and we make some robustness tests.

JEL classification: G13 C63

Keywords: Hedging, Lévy models, Fast Fourier Transform

1 Introduction

The aim of this paper is the measurement of the cost of hedging a contingent claim when the hedging portfolio is re-balanced at a discrete set of dates. The basic point of the methodology is to have an integral representation of the payoff of the claim, in other words to be able to write the payoff as an inverse Laplace transform. This approach was proposed by Hubalek et al. [16] in order to efficiently compute the optimal strategy and its variance. It was then used by Angelini and Herzel [2] to valuate the variance of hedging error of a given strategy, satisfying a compatibility condition which is met by various important strategies, like the delta one. This approach provides a framework suitable for managing derivatives, namely for pricing, for computing hedging ratios and for measuring the cost of hedging. For pricing and calibration purposes, as well as for computation of Greeks, a popular method adopted by financial institution is the Fast Fourier Transform (FFT) algorithm, proposed by Carr and Madan [6]. We will show how the FFT algorithm may also be implemented to compute the expectation and the variance of the hedging error. One of the main contribution of the paper is to suggest that the FFT machinery may be an integrated tool to deal with different aspects of the derivative risk management. Notice that the different forms of the problem require in principle different probability measures, the martingale measure for pricing purposes and the "objective" measure for valuating the variance of the error. We will develop our results for a generic probability measure. However, as suggested in [1] and analogously to [8], in order to incorporate the market views of future scenarios and the risk premia attached to the prices, we will perform computations using a martingale measure for both problems.

Most of financial models for pricing and hedging derivatives assume that trading is possible in continuous time. Such an assumption does not hold in practice. For example, the widely used Black-Scholes delta hedging strategy produces an error even if all other assumptions of the Black-Scholes model are met. This type of error may be called "discretization error". The second font of error has to do with the incompleteness of the model. Here we will consider the class of Levy models, which have recently been extensively studied in finance. Cont et al. [8] study the incompleteness error in models with jumps, while Tankov and Voltchkova [23] show an asymptotic analysis of the discretization error in the same model setting.

The problem of hedging derivatives in incomplete markets has been studied by many authors [[21], [16], [8], [5], [20]], following the seminal papers of Föllmer and Sondermann [12]. The main approach to the problem is that of determining a strategy which minimizes the variance of the hedging error, which we will refer to as the optimal strategy. Generally more feasible to compute is the strategy that minimizes the variance of cost of local portfolio adjustments, which we will call the local optimal strategy. It is well known that, when the discounted price process is a martingale, the local optimal strategy is the same as the optimal one. However, the Black-Scholes delta hedging strategy is still very popular among practitioners. In a different model setting than the Black-Scholes one, this choice is not theoretically coherent, but it can nevertheless be followed by using the Black-Scholes implied volatility of the option to be hedged. Another feasible alternative is to compute the model-based delta simply as the derivative of the price of the derivative with respect to the underlying. As pointed out by Tankov [22], in a model with jumps this choice is not optimal, since it does not take into account the risk coming from the jumps, but only that coming from infinitesimal movements. Moreover, in some models, it may not even exist. A sensible solution is again to compute the optimal strategy.

In our model setting, for derivatives with integral representation, we are in the position of using results of Hubalek et al. [16] and Angelini and Herzel [2] to compute the expectation and the variance of the hedging error for all the strategies described above and to compare their performances. Moreover, we are able to analyze the effect on such quantities due to a model mispecification in the following sense. We fix a set of model parameters to compute the price of the claim and the hedging strategy; this set may be thought as obtained through a model calibration to option prices. Then we let the market evolve following the same model, but with different sets of parameters. This should give a measure of the robustness of the model for each given strategy and provide some insight on the influence of quantities such as the standard deviation, the skewness and the kurtosis of the underlying.

The rest of the paper is structured as follows. In the following section we set up the theoretical framework: first we give the concept of integral representation of payoffs and the general model setting; we also briefly describe the particular Lévy models under analysis, reporting, for convenience of the reader, their characteristic functions. Then we show how to perform pricing and computation of various relevant hedging ratios, like the model-delta, the Black-Scholes delta and the local optimal ratio. Finally, we define the hedging error and review results of [2] to be used in the sequel. Section 3 is devoted to the illustration of the FFT machinery. We start with a review of the pricing algorithm; we then describe how to adapt the algorithm to the computation of expected value and variance of the hedging error. In Section 4 we show some applications: after some remarks about numerical implementation, we give an asymptotic analysis for models like NIG and Merton models, as the number of trading dates increases; then we perform a robustness test. Section 6 draws some conclusions and gives hints for future research.

2 The Theoretical Framework

2.1 Integral representation of payoffs

In their works Hubalek et al. [16] and Angelini Herzel [2] consider European derivative securities on the stock S with maturity $T = N\Delta t$ and payoff $H = f(S_N)$, with the function $f : (0, \infty) \to \mathbb{R}$ of the form

$$f(S) = \int S^z \Pi(dz). \tag{2.1}$$

Π is a complex measure on a strip in the complex plane $\{z \in \mathbb{C} : R' \leq Re(z) \leq R\}$, where R' and R are real and are defined in such a way that $E[e^{2R'X_1}] < \infty$ and $E[e^{2RX_1}] < \infty$.

We remark that the form of the payoff function is that of an inverse Laplace transform of the measure Π. For example, taking $R = R'$ means to perform an inverse Laplace transform on a straight line parallel to the imaginary axis. Let us note that this transformation, apart from a scaling factor is nothing but the Fourier transform on the real line. For instance, the payoff of an European call option with strike price $K > 0$ and maturity T, is $(S - K)^+$ and it can be written as

$$(S - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} S^z \frac{K^{1-z}}{z(z-1)} dz, \tag{2.2}$$

for an arbitrary $R > 1$ and for each $S > 0$. Many payoffs can be represented as inverse Laplace (Fourier) transform: to cite the most important ones, we mention the put, the power call and the digital option [16]. In [15] Hubalek provides integral representation of many derivatives, even path-dependent or depending on multiple assets.

2.2 Model setting

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in(0,1,\ldots,N)}, P)$ denote a filtered probability space and let $X = (X_n)_{n=0,1,\ldots,N}$ be a real-valued process with independent and stationary increments satisfying:

1. X is adapted to the filtration $(\mathcal{F}_n)_{n\in(0,1,\ldots,N)}$,
2. $X_0 = 0$,
3. $\Delta X_n = X_n - X_{n-1}$ has the same distribution for $n = 1, \ldots, N$,
4. ΔX_n is independent of $\mathcal{F}_{n-1}$ for $n = 1, \ldots, N$.

We model the price process $S = (S_n)_{n=0,1,\ldots,N}$ of a non dividend paying stock at time $t = n\Delta t$, as follows

$$S_n = S_0 e^{X_n}. \tag{2.3}$$

We assume that $E[S_1^2] < \infty$ so that the moment generating function $m(z) = E[e^{zX_1}]$ is defined at least for complex z with $0 \le Re(z) \le 2$. Moreover, we exclude the case when S is a deterministic process.

Let X_t be a Lévy process (see for example [22] or [26]). It is known that the Lévy-Khintchine theorem provides an integral representation of the characteristic function of X_t of the form

$$\phi_{X_t}(u) = E[e^{iuX_t}] = e^{t\psi_X(u)}, \tag{2.4}$$

where $\psi_X(u)$ is the characteristic exponent:

$$\psi_X(u) = iu\mu - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}^0} (e^{iux} - 1 - iux\mathbf{1}_{|x|<1})\nu(dx). \tag{2.5}$$

Here μ is a constant drift, σ^2 describes the constant variance of the continuous component of the the Lévy process and $\nu(dx)$ is the Lévy density that represents the arrival rate for jumps of size x.

The characteristic function of the underlying process S_t can be written in terms of that of the Lévy process X_t in the following way:

$$\phi_{S_t}(u) = e^{iu\log(S_0)}\phi_{X_t}(u).$$

In a risk adjusted martingale measure, the discounted price is a martingale, while the drift $\mu = r - \log(E[e^{X_1}])$ can be obtained simply by setting the expected value of S_n equal to the forward value $E[S_n] = S_0 e^{rn\Delta t}$, where r

is the risk free interest rate. Notice that, differently from other related works [2, 16, 14], where r is set to zero, or equivalently the price process is intended discounted, in our work we prefer to make the dependence on the risk free interest rate explicit.

Now we describe in detail the Lévy models we analyze. The drift part is the same for all the models and we do not write it.

1. The Black-Scholes model for which $X_t = \sigma W_t$ is a pure diffusive continuous process with σ a constant deterministic volatility and W_t the Wiener process. The characteristic exponent is given by

$$\psi_X(u) = -\frac{1}{2}u^2\sigma^2.$$

2. The Merton's Jump diffusion model combines to the Brownian motion a compound Poisson process. Thus $X_t = \sigma W_t + \sum_{i=1}^{N_t} Y_i$, with N a Poisson process with mean arrival rate λ and Y_i a sequence of independent random variables normally distributed with mean μ_J and standard deviation σ_J. The characteristic exponent is

$$\psi_X(u) = -\frac{1}{2}u^2\sigma^2 + \lambda\left(e^{iu\mu_J - \frac{1}{2}u^2\sigma_J^2} - 1\right).$$

   The Merton's model exhibits a finite activity, meaning that the process generates a finite number of jumps within any finite time interval.

3. In the Variance Gamma (VG) model [18, 19], X_t is a pure jump process with an infinite activity and a finite variation. It can be obtained subordinating a Brownian motion of volatility σ and drift θ with an independent Gamma process Γ_t^ν [13] of variance rate ν: $X_t = \theta\Gamma_t^\nu + \sigma W_{\Gamma_t^\nu}$. The characteristic function is computed in [19] and its characteristic exponent is:

$$\psi_X(u) = -\frac{1}{\nu}\log\left(1 + \frac{u^2\sigma^2\nu}{2} - i\theta\nu u\right).$$

4. The Normal Inverse Gaussian (NIG) [4, 7] is obtained subordinating a Brownian motion of volatility σ and drift θ with an independent Inverse Gaussian process I_t^ν, of variance rate ν: $X_t = \theta I_t^\nu + \sigma W_{I_t^\nu}$. The NIG process is a pure jump process and its characteristic function is:

$$\psi_X(u) = \frac{1}{\nu} - \frac{1}{\nu}\sqrt{1 + u^2\sigma^2\nu - 2iu\theta\nu}.$$

This process has an infinite activity but differently from the VG one it is an infinite variation process.

A very attractive feature of VG and NIG models is that they depend only on three parameters (σ, θ, ν) and these parameters are related in a simple way to the variance, skewness and kurtosis of the log-return distribution implied by the process. And in fact while the variance for both the models is given by a linear combination of σ^2 and $\theta^2\nu$, the sign of skewness is given by the sign of θ, while the percentage excess of kurtosis is given by ν. So for example, if $\theta = 0$, such distribution is symmetric, as the skewness is zero, the annualized variance is simply σ^2 and the annualized percentage excess of kurtosis is just ν both for VG and NIG models. On the other hand, if θ is different from zero σ, θ and ν combine themselves in a non linear but still polynomial way to give the variance, skewness and kurtosis of the distribution, and thus they can no more be thought as three independent parameters in determining the desired moments.

For what concern the Merton model, the parameters are related to the moments of the log-return distribution in a way for which it is not easy to separate the effect of each single parameter on them, even if $\mu_J = 0$, for which the skewness of the distribution is zero.

2.3 Pricing and hedging

Starting from the integral representation of payoff (2.1), one can compute the price of the claim at time $t = n\Delta t$ just performing the expectation under a risk neutral probability measure, conditional to $\mathcal{F}_n$, and discounting at the risk free rate

$$C_n = e^{-r(N-n)\Delta t} E_n \left[\int S_N^z \Pi(dz) \right].$$

Exchanging the expected value integral with the Laplace integral by the Fubini's theorem and using the fact that the log-returns are i.i.d. variables, one gets the integral representation of the price value:

$$C_n = e^{-r(N-n)\Delta t} \int S_n^z m(z)^{N-n} \Pi(dz).$$

We remark that the previous price formula holds for any model satisfying the conditions of Section 2.2 and whose moment generating function $m(z)$ is known [1].

Now we can compute some interesting hedge ratios. From the above pricing formula, it is natural to take the derivative with respect to S_n

$$\Delta_{n+1} = \frac{\partial C_n}{\partial S_n} = e^{-r(N-n)\Delta t} \int z m(z)^{N-n} S_n^{z-1} \Pi(dz).$$

This is the delta within the model. As pointed out by Tankov [22], in a model with jumps this choice is not optimal, since it does not take into account the risk coming from the jumps, but only that coming from infinitesimal movements. Analogously, one can compute other Greeks within the model.

A better choice would be the local optimal hedging ratio, that is the strategy which minimizes the variance of the next period costs and whose formal definition can be found in [21]. Such a strategy is computed in Theorem 2.1 of [16] in the case of $r = 0$. Following [21], we performed the computation of the "local optimal" strategy also in the case of $r \neq 0$ obtaining:

$$\xi_{n+1} = e^{-r(N-n)\Delta t} \int f^{\xi}_{n+1}(z) S_n^{z-1} \Pi(dz), \tag{2.6}$$

where $f^{\xi}_{n+1}(z) = e^{r\Delta t} g(z) h(z)^{N-n-1}$, with

$$g(z) = \frac{m(z+1) - m(1)m(z)}{m(2) - m(1)^2},$$

$$h(z) = m(z) - (m(1) - e^{r\Delta t}) g(z).$$

The most used hedging strategy is the Black-Scholes delta. Given the market price of the claim to be hedged, or its implied volatility $\bar{\sigma}$, the delta of the position in the underlying from time $(n)\Delta t$ to time $(n+1)\Delta t$ is simply given by:

$$\Delta^{bs}_{n+1} = \frac{\partial C^{bs}_n}{\partial S_n} = e^{-r(N-n)\Delta t} \int z m^{bs}(z)^{N-n} S_n^{z-1} \Pi(dz),$$

where

$$m^{bs}(z) = e^{\left(\left(r - \frac{\bar{\sigma}^2}{2}\right) z + \frac{\bar{\sigma}^2}{2} z^2\right)\Delta t}$$

is the moment generating function for the Black-Scholes model.

[1] In fact the formula may be easily generalized to a wider class of models like Affine models and stochastic volatility models

Another interesting strategy is the "improved-delta" strategy provided by Wilmott in [25] in which the delta position is corrected by a term proportional to the gamma of the option:

$$\Delta^w_{n+1} = \Delta^{bs}_{n+1} + \Delta t(\mu - r + \frac{1}{2}\bar{\sigma}^2)\Gamma^{bs}_{n+1} S_n,$$

where the gamma function is the second derivative of C^{bs}_n with respect to S_n and both the Greeks delta and gamma are computed in the Black-Scholes framework, with volatility $\bar{\sigma}$. Note the presence of the drift μ of the evolution of the underlying price in the real world.

As for the delta strategy, one can give an integral representation of the improved-delta Wilmott's strategy, in terms of the Black-Scholes moment generating function and of the complex measure $\Pi(dz)$ relative to the claim:

$$\begin{aligned}\Delta^w_{n+1} &= e^{-r(N-n)\Delta t} \int S_n^{z-1} (z m^{bs}(z)^{N-n} + \\ &\quad \Delta t(\mu - r + \frac{1}{2}\bar{\sigma}^2) z(z-1) m^{bs}(z)^{N-n}) \Pi(dz),\end{aligned}$$

Let us remark that, in principle, the delta and the improved-delta strategies are conceived for a log-normal process. For a general dynamic of the underlying, our approach is that of setting the volatility to the market implied volatility $\bar{\sigma}$. Another approach could be to choose the parameter μ and the volatility $\bar{\sigma}$ in order to fit mean and variance of the log-returns at a given date.

Notice that the strategies we considered up to now are of the form:

$$\vartheta_n = e^{-r(N-n+1)\Delta t} \int \vartheta(z)_n \Pi(dz), \tag{2.7}$$

with $\vartheta(z)_n = f^{\vartheta}_n(z) S^{z-1}_{n-1}$, where $f^{\vartheta}_n(z)$ is a function of the complex variable z which does not contain S_k for any k. A hedging strategy which satisfies condition (2.7) is said to be compatible with a contingent claim whose payoff function satisfies condition (2.1). This definition is given adding a discount factor depending on r, in analogy with the case of a Δ^{bs}-strategy, to the definition (2.1) in [2].

Better than the local optimal strategy, one could try to minimize the expected square value of the total hedging error given a fixed initial endowment c. It is well known that such a strategy $\xi^{(c)}$ exists [21] and it is computed together with its variance in[16] using the same approach as here, namely using the approach of Laplace transform. Such an optimal strategy is the most important example of a strategy that is non-compatible. We recall that, if the discounted price process is a martingale, then this coincides with the local optimal strategy and it is therefore compatible.

2.4 Measurement of hedging error

Let $\vartheta = (\vartheta_n)$, for $n = 1, \ldots, N$, be a hedging strategy. The random variable ϑ_n is interpreted as the number of shares of the underlying asset held from time $(n-1)\Delta t$ up to time $n\Delta t$. Suppose moreover that the strategy is an admissible one, that is a predictable process such that the cumulative gains are square-integrable [16, 21].

The cumulative gains in the presence of a money market account can be simply obtained by capitalizing (or discounting) up to the same date at the risk free interest rate all the cash-flows deriving from the hedging strategy. We choose to capitalize the cash-flows up to the date of maturity of the claim to hedge. Thus the cumulative gains from the strategy at time $T = N\Delta t$ is:

$$G_N(\vartheta) = \sum_{k=1}^{N} \vartheta_k e^{r(N-k+1)\Delta t}(S_k e^{-r\Delta t} - S_{k-1}),$$

and the resulting final hedging error is:

$$\varepsilon(\vartheta, c) = H - e^{rT}c - G_N(\vartheta),$$

where c is the price at time $t = 0$ of the claim whose payoff at time T is H.

Let us remark that the strategy ϑ determines a unique self-financing portfolio. The hedging error $\varepsilon(\vartheta, c)$ is the net loss-gain one can get at maturity if one starts with the initial capital c and follows the strategy.

We suppose that the strategy $\vartheta = (\vartheta_n)$ is compatible with the given contingent claim. For such strategies, Angelini and Herzel in Theorem (3.1) of [2], computed the expected value $E[\varepsilon(\vartheta, c)]$ and the variance $var(\varepsilon(\vartheta, c))$ of the error. For convenience of the reader, here we give an extension of their results to the case of a risk free interest rate different from zero:

Theorem 2.1 *Let ϑ be a strategy which is compatible with a contingent claim*

H and let c be its initial value, then

$$E[\varepsilon(\vartheta,c)] = \int S_0^z \left[m(z)^N - (e^{-r\Delta t}m(1)-1)\sum_{k=1}^N f^\vartheta(z)_k m(z)^{k-1} \right] \Pi(dz) - e^{rT}c \tag{2.8}$$

and

$$E[\varepsilon(\vartheta,0)^2] = \int\int S_0^{y+z}(v_1(y,z)-v_2(y,z)-v_3(y,z)+v_4(y,z))\Pi(dz)\Pi(dy), \tag{2.9}$$

where

$$v_1(y,z) = m(y+z)^N,$$

$$v_2(y,z) = \sum_{n=1}^N f_n^\vartheta(y)m(y+z)^{n-1}m(z)^{N-n}(e^{-r\Delta t}m(z+1)-m(z)),$$

$$v_3(y,z) = \sum_{n=1}^N f_n^\vartheta(z)m(y+z)^{n-1}m(y)^{N-n}(e^{-r\Delta t}m(y+1)-m(y)),$$

$$\begin{aligned} v_4(y,z) &= (e^{-r\Delta t}m(2) - 2e^{-r\Delta t}m(1)+1)\sum_{n=1}^N f_n^\vartheta(y)f_n^\vartheta(z)m(y,z)^{n-1} + \\ &+ (e^{-r\Delta t}m(1)-1)\sum_{j<n}\sum_{n=2}^N f_j^\vartheta(y)f_n^\vartheta(z)m(y)^{n-1-j}m(y+z)^{j-1} \times \\ &\qquad \times(e^{-r\Delta t}m(y+1)-m(y)) + \\ &+ (e^{-r\Delta t}m(1)-1)\sum_{j<n}\sum_{n=2}^N f_n^\vartheta(y)f_j^\vartheta(z)m(z)^{n-1-j}m(y+z)^{j-1} \times \\ &\qquad \times(e^{-r\Delta t}m(z+1)-m(z)). \end{aligned}$$

Therefore, the variance of the hedging error is

$$\mathrm{var}(\varepsilon(\vartheta,c)) = \mathrm{var}(\varepsilon(\vartheta,0)) = E[\varepsilon(\vartheta,0)^2] - E[\varepsilon(\vartheta,0)]^2.$$

Let us remark that the expectations in the previous formulas are intended under a certain probability measure. In principle, the probability measure to consider depends on the problem at hand. For pricing purposes one would obviously need the martingale measure, while it would seem natural to compute expectations of hedging errors under the "objective" measure. Following [8], we will consider all the expectations computed under a risk neutral measure rather than under an objective measure. The first can be thought of as obtained from a calibration process and thus extracted from the quoted option prices. The "objective" measure can be retrieved from historical data matching the moments or the quantiles of distribution of log-returns. Under a martingale risk-adjusted measure, the expectation of the hedging error should reflect future uncertainty up to the maturity of the option and this is just the case for a risk adjusted measure drawn from the quoted option prices after a calibration procedure [1, 8, 9, 10]. In fact, such a measure is intended to discount all the market views of future scenarios. Moreover, as pointed out in [1], the risk-adjusted martingale measure should reflect not only the future probabilities of occurrence, but also the risk premia attached to asset and contingent claim prices from the market.

We shall apply the above results to Lévy models with pure jumps, as NIG and VG, and to the jump-diffusion Merton process, performing the computation with a Fast Fourier transform machinery.

3 Fast Fourier Transform (FFT)

In this section we are interested in performing a computation of the expectation and variance of hedging error for different strategies using a Fast Fourier Transform (FFT) approach. Carr and Madan in their work [6] showed how to determine the call and put option prices using the FFT algorithm. Their approach assumes that the characteristic function of the density in the risk neutral world is known analytically, and thus it can be applied to a large class of models, from the Lévy models to stochastic volatility ones. Moreover, the FFT approach to derivative pricing is fast and accurate and allows to perform a pricing in real time, even for a book with thousands of options. These features make the FFT technique a fundamental instrument today, especially for financial institutions, where the everyday needs of calibration on different financial instruments and of reevaluating books with many derivatives, require a fast and accurate pricing machinery. In this work our main motivation is to extend the FFT technique also to the analysis of hedging error.

3.1 A review of FFT applied to option pricing

Let us rewrite the Laplace representation for the call payoff (2.2) in terms of an inverse Fourier transform. Performing a translation followed by a rotation in the complex plane

$$z \to iv + R \tag{3.10}$$

and putting $\alpha = R - 1$, just to make uniform our notations with the ones in [6], one can rewrite the payoff of the call in terms of an inverse (in the variable k) Fourier transform

$$(S-K)^{+} = \frac{e^{-k\alpha}}{2\pi}\int_{-\infty}^{\infty} e^{-ikv}\frac{e^{is(v-i(\alpha+1))}}{(iv+\alpha)(iv+\alpha+1)}dv, \tag{3.11}$$

where $s = \log(S)$ and $k = \log(K)$ are respectively the log-value of S and the log-strike and where we consider the payoff as a function of the strike rather then of the stock price. Then, performing an expectation in a risk neutral world of the payoff (3.11) as in [2] and discounting with the risk free rate, one can obtain the price of the call at time $t = n\Delta t$:

$$C_n = \frac{e^{-\tilde{k}\alpha}}{\pi}\int_{0}^{\infty} e^{-i\tilde{k}v}\chi(v)dv, \tag{3.12}$$

where

$$\chi(v) = \frac{e^{-r(N-n)\Delta t}e^{s_n}\phi_{N-n}(v-i(\alpha+1))}{(iv+\alpha)(iv+\alpha+1)}, \tag{3.13}$$

in which one can recognize the valuation formula of Carr and Madan in [6] written for $\tilde{k} = k - s_n = \log(K/S_n)$. The characteristic function of the underlying process $\phi_{N-n}(u)$ is computed at time $(N-n)\Delta t$ and is intended under a risk neutral density.

The factor $e^{-\tilde{k}\alpha}$ can be viewed as a dampen to make the price call function of k square-integrable over the entire real line [6].

Just for review, let us sketch the fundamental steps to compute the integral (3.12) with an FFT technique. The FFT is an algorithm to compute efficiently sum of the form:

$$\Gamma(l) = \sum_{j=1}^{M} e^{-i\frac{2\pi}{M}(j-1)(l-1)}X(j) \tag{3.14}$$

for $l = 1, \ldots, M$. M has to be a power of 2. Performing a trapezoid rule in (3.12) one can obtain the desired sum simply setting $v_j = \eta(j-1)$, $\tilde{k}_l = -b + \lambda(l-1)$ and

$$\lambda\eta = \frac{2\pi}{M}. \tag{3.15}$$

where η is the lattice spacing of the integration variable and $M\eta$ becomes the effective upper limit of integration. $X(j)$ is the integrand (3.13) computed in v_j and multiplied by the grid size η. Note that the summation gives a vector of results for a vector of log-strikes equally spaced with lattice spacing λ, centered around $\tilde{k} = 0$ and running from $-b$ to b with $b = M\lambda/2$: $\tilde{k}_l = (l - 1 - M/2)\lambda$ for $l = 1, \ldots, M$. The prices are then obtained by scaling the vector of results, after the FFT summation has been performed, with a strike dependent factor

$$C(\tilde{k}_l) \approx \frac{e^{-\alpha\tilde{k}_l}}{\pi}\Gamma(l),$$

for each $l = 1, \ldots, M$.

The algorithm depends on the two parameters, M and η, and one has to find the right trade-off between a finer grid for integration and a good grid around the at the money for the strike dimension as the two lattice steps are related by formula (3.15). Introducing some Simpson's weights into the summation, one can obtain an accurate integration even with a larger value of η, thus leaving the possibility of a finer grid in the strike dimension. For more details see [6].

Recently, Chourdakis [11] adopted the fractional Fourier transform (FRFT) of Bailey and Swartztrauber [3] for computing option prices. This algorithm allows to make independent the integration grid from the log-strike one. An N-point FRFT can be implemented using three $2N$-point FFT. Thus this technique is more accurate but requires more computational time.

3.2 Expectation value and Variance of hedging error with FFT

In their work Angelini-Herzel [2] compute the expectation value and the variance of the hedging error, for any compatible strategy ϑ, and for a general Lévy model driving the underlying. Their results are reported in Theorem (2.1). Our aim here is to rewrite that formulas in terms of a Fourier representation and then to reduce the integrals to suitable summations that can be computed with an FFT algorithm.

To this goal, we perform the transformation given in (3.10). Moreover, we explicit the dependence on the risk free interest rate. Remember that s_0 and k are the log-values of the underlying spot price at time $t = 0$ and of the strike and $\tilde{k} = \log(K/S_0)$. Let ϑ be a strategy which is compatible with a contingent claim H as in (2.7) and let c be its initial value. Moreover let $\Pi(dz)$ be of the form $\Pi(dz) = K^{1-z}\widetilde{\Pi}(z)dz$, with $\widetilde{\Pi}(z)$ not depending on K. Indeed the following results are valid also if the dependence on K is of the form K^{n-z}, with n an integer (see [16]), but in that case the formulas must be multiplied by the suitable power of S_0.

The expected value of the hedging error at the maturity of the option can be written as

$$E[\varepsilon(\vartheta, c)] = \frac{e^{-\alpha\tilde{k}}}{\pi} \int_0^\infty e^{-iu\tilde{k}} \chi(iu + 1 + \alpha)du - e^{rT}c, \tag{3.16}$$

where the integrand is:

$$\chi(z) = e^{s_0} \left[m(z)^N - (e^{-r\Delta t}m(1) - 1) \sum_{n=1}^{N} f_n^\vartheta(z) m(z)^{n-1} \right] \widetilde{\Pi}(z). \tag{3.17}$$

Some remarks have to be made before computing the variance. First of all the expectation is a Fourier transform in the normalized log-strike variable $\tilde{k}$ of a kernel of integration χ. The integral is performed only on the positive real axis because the expectation of hedging error has to be a real number and this imposes the real part of the function χ to be symmetric, while the imaginary part is antisymmetric. The parameter α depends on the payoff studied and for a call is the usual dampen one can find in the integral representation of the price (see [6]).

The variance of the hedging error does not depend on the initial endowment c that is a deterministic constant, and thus we can compute it for $c = 0$:

$$var(\varepsilon(\vartheta, c)) = var(\varepsilon(\vartheta, 0)) = E[\varepsilon(\vartheta, 0)^2] - E[\varepsilon(\vartheta, 0)]^2,$$

where

$$E[\varepsilon(\vartheta, 0)^2] = \frac{e^{-\tilde{k}_1\alpha_1 - \tilde{k}_2\alpha_2}}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\tilde{k}_1 u_1 - i\tilde{k}_2 u_2} \mathcal{K}(iu_1 + \alpha_1 + 1, iu_2 + \alpha_2 + 1) du_1 du_2, \tag{3.18}$$

and the integration kernel is

$$\mathcal{K}(z_1, z_2) = e^{2s_0}\left[v_1(z_1, z_2) - v_2(z_1, z_2) - v_3(z_1, z_2) + v_4(z_1, z_2)\right]\widetilde{\Pi}(z_1)\widetilde{\Pi}(z_2), \tag{3.19}$$

with functions v_1, v_2, v_3, v_4, given in Theorem 2.1.

Invoking the trapezoid rule, we can approximate the integrals in (3.16) and (3.18) by sums. Let us start with the expectation value. The integral in (3.16) reduces to the following sum

$$E[\varepsilon(\vartheta, 0)] = \frac{e^{-\alpha\tilde{k}_l}}{\pi}\sum_{j=1}^{M} e^{-i\frac{2\pi}{M}(j-1)(l-1)}X(j) \tag{3.20}$$

for $j, l = 1, \ldots, M$. The vector $X(j)$ is the integrand (3.17) computed in $z_j = iu_j + 1 + \alpha$, with $u_j = \eta(j-1)$ and multiplied by the grid size η. The resulting sums are intended computed at the log-strikes $\tilde{k}_l = (l - 1 - M/2)\lambda$ where $\eta\lambda = 2\pi/M$. To get a better precision without reducing the grid size η, one can add the Simpson's weights. The vector to be transformed with FFT algorithm is

$$X(j) = \chi(iu_j + 1 + \alpha)\frac{\eta}{3}[3 + (-1)^j - \delta_{j-1}], \tag{3.21}$$

where δ_n is the Kronecker delta function that is unity for $n = 0$ and zero otherwise.

For what concern the integral (3.18), remember that a two-dimensional FFT computes for any two-dimensional complex input array $X(j_1, j_2)$, with $j_{1,2} = 1, \ldots, M$, the output array

$$\Gamma(l_1, l_2) = \sum_{j_1=1}^{M}\sum_{j_2=1}^{M} e^{-i\frac{2\pi}{M}((l_1-1)(j_1-1)+(l_2-1)(j_2-1))}X(j_1, j_2), \tag{3.22}$$

for $l_{1,2} = 1, \ldots, M$. The trapezoid rule in this case needs to define the following $M \times M$ grid $\{(u_{1,j_1}, u_{2,j_2}) : j_1, j_2 = 1 : \ldots, M\}$, with

$$u_{1,j_1} = (j_1 - 1 - M/2)\eta \qquad u_{2,j_2} = (j_2 - 1 - M/2)\eta,$$

while the arrival strike grid is $\{(\tilde{k}_{1,l_1}, \tilde{k}_{2,l_2}) : l_1, l_2 = 1 : \ldots, M\}$, with

$$\tilde{k}_{1,l_1} = (l_1 - 1 - M/2)\lambda \qquad \tilde{k}_{2,l_2} = (l_2 - 1 - M/2)\lambda.$$

As usual the strike grid and the integration grid are related by relation $\eta\lambda = 2\pi/M$.

The trapezoid rule with the above conditions gives for the integral (3.18) the following formula:

$$E[\varepsilon(\vartheta, 0)^2] \approx \frac{e^{-2\alpha\tilde{k}_l}}{(2\pi)^2}\Gamma(l, l), \tag{3.23}$$

where $\Gamma(l, l)$ is just the diagonal part of the two-dimensional array $\Gamma(l_1, l_2)$ that is computed performing a two-dimensional FFT algorithm like in (3.22), with the array $X(j_1, j_2)$ equal to

$$X(j_1, j_2) = (-1)^{(j_1-1)+(j_2-1)}\mathcal{K}(iu_{1,j_1} + 1 + \alpha_1, iu_{2,j_2} + 1 + \alpha_2)\eta^2,$$

and the kernel $\mathcal{K}$ defined in (3.19).

Notice that the computation for both the expectation and the variance obtained with the FFT machinery leads to results for a whole vector of strikes.

4 Applications

4.1 Numerical Implementation

We implemented the code in MATLAB, using the MATLAB functions "fft.m" and "fft2.m" to compute respectively the one dimensional and two dimensional Fourier transform. We tested our machinery for three different choice of the number of FFT points M: $512, 1024, 2048$, while keeping the grid integration size fixed to the value $\eta = 0.25$. Moreover we tested also the case $M = 2048$ with $\eta = 0.125$. The tests were performed for different choices of the model parameters and for all the models of our interest (BS, VG, NIG, Merton). Our choice of $M = 1024$ and $\eta = 0.25$ was the best trade-off between accuracy from one hand and no much computational time from the other hand. This choice corresponds to a log-strike spacing of $\lambda = 2\pi/(\eta M) = 0.0245$, that in turns gives a strike vector around the at the money with a spacing of about $2 \div 3$ percent points. The relative accuracy obtained is at least of order $10^{-4} \div 10^{-3}$ for the variance. For strikes out of the money ($K/S_0 \approx 1.15$) and pure jumps model like NIG or VG with extreme values of the parameters, the accuracy of the variance can be reduced and one should need a finer integration grid. The tests were performed comparing our results with the Laplace based machinery developed by Angelini and Herzel in [2], from one hand. On the other hand, we verified that the results do not change, ranging the value of M and η within the values given above.

Parameter α can be fixed to $\alpha = 1.5$. This is a good value both for pricing purpose and for the aim to compute variance and expectation of hedging error within a wide class of models including the ones considered in this work. We tried also to move α a bit around the value of 1.5 in order to see the sensibility of results to its value and we noted that the results are quite stable. Maybe one should try to span a wider range for α but our experience with pricing suggests that 1.5 is an appropriate value for α.

4.2 Asymptotic analysis

We want to compute the mean $E[\varepsilon]$ and the variance $var[\varepsilon]$ of the error produced to hedge an European call when trading in discrete time using the FFT machinery. For the reasons discussed in Section 2.4, in all the following applications, we have to think of the expectations in (3.16) and (3.18) as computed under a risk neutral measure.

As a first analysis we want to perform a computation to observe the behavior of the variance of the hedging error as the number of trading dates increases. In the case of the Black-Scholes model and using a standard delta strategy to hedge the call, some approximated formulas for the variance of the discretization error are known. In [24] Toft gives an approximation formula involving the option's gamma at time $t = 0$, valid as the number of trading dates N goes to infinity. Another approximation formula that is well known by practitioners is provided by Kamal and Derman [17]. In their formula, the variance of hedging error is proportional to the squared option's vega computed at time $t = 0$. In both formulas the variance goes to zero as $1/N$. In [2] it is given an appropriate analysis of the validity of such formulas comparing the approximated results with the exact results obtained in the general framework of Laplace representation developed in [16] and [2].

It is well known that in the presence of jumps in the stock prices, the market is no longer complete. In such a case, even if the hedging portfolio were re-balanced continuously, the hedging error would not be zero. Thus in the presence of jumps there are two kinds of hedging errors: the first is due to the discrete nature of hedging while the second is due to the incompleteness of the market. Tankov and Voltchkova show in [23] that while the first kind of error is dominated by the diffusion part of the price process, the second one is due to the jumps. In [23] the authors compute the asymptotic distribution of the hedging error due to its discrete nature for a general Lévy jump diffusion model.

In our work we analyze in the framework of FFT the asymptotic behavior of the variance of hedging error for the following models: Black-Scholes, VG, NIG, Merton. The issue of hedging in the Black-Scholes context was already extensively studied for example in [2]. Moreover the distribution properties of the VG process and the NIG process are very similar and in fact they lead to the same kind of results. For such reason we decide to show only the results for the NIG model, that is an infinite activity and infinite variation pure jump process, and for the Merton model that is a jump-diffusion process of finite activity.

For both Merton and NIG models we show the variance of error produced in hedging an at the money call as the number of trading dates increases. The call has a maturity $T = 0.25$ while the spot value of the underlying is $S_0 = 100$. Moreover the risk free interest rate is $r = 0.04$. We performed the computation for $N = [1, 2, 3, 4, 5, 6, 7, 9, 11, 13, 15, 17, 20, 25, 30, 40, 50]$.

The model parameters are calibrated to the call prices. Indeed at first we choose the model parameters and then we compute the call prices. In such a way the model results perfectly calibrated to the prices. Obviously we have different call prices for Merton and NIG models but that is not important for us as we are interested in comparing different hedging strategies rather than different models. The model parameters are $[\sigma, \theta, \nu] = [0.2, -0.1, 0.1]$ for the NIG model and $[\sigma, \lambda, \mu_{Jumps}, \sigma_{Jumps}] = [0.2, 1, -0.1, 0.05]$ for Merton model. Thus both the distributions have skewness and excess of kurtosis.

Let us remark that we show only the variance of error and not the expectation because the computation has been performed under a risk adjusted martingale measure and therefore the expectations are zero, regardless of the strategy.

In the first panel of Figures 1 and 2 we compare, both for Merton and NIG models, the asymptotic behavior of different hedging strategies when one hedges an at the money (ATM) call with maturity $T = 0.25$. We consider the Black-Scholes delta strategy computed at the ATM implied volatility. Then we show the Wilmott "improved delta" strategy, the "local optimal" one (2.6) and the model based delta strategy.

First of all one can note that for both models there is a residual hedging error due to the incompleteness of the market in the presence of jumps. The residual error for the "local optimal" strategy, in the case of NIG model and for $N = 50$, has a variance about equal to 4.24 that is of the same order of the call price $C \simeq 4.34$. In the Merton case such a residual error has a variance of $\simeq 1.94$ to be compared to the price $C \simeq 4.99$.

For both the NIG and the Merton models, the Wilmott "delta improved" strategy and the BS delta strategy are almost similar and not distinguishable on the Figures. In the Merton case, the "Local Optimal" strategy outperforms the Black-Scholes delta strategy with a variance smaller of about the 10 percent. On the contrary the model based Merton delta shows a variance slightly worst than the Black-Scholes delta one.

Also in the NIG case, the model based NIG delta is the worst strategy, showing a variance of about the 5 percent greater than the Black-Scholes's one and of about the 12 percent greater then the "local optimal" one.

Notice that for both models, the model based delta strategy is the one with the worst performance. On one hand, it could seem natural that if one calibrates the model on the market prices and then makes a pricing using that model, also the delta hedging has to be carried out in that model. Nevertheless our analysis has shown that even the Black-Scholes delta strategy at the implied volatility performs better and thus one should investigate a bit more about the meaning of a model based delta.

We remark that our FFT machinery allows us to compute the desired quantities, and therefore the variance of hedging error, for different values of strikes, the spacing depending on the choice of the algorithm parameters. As already said, our spacing around the ATM is about $2 \div 3$ percent points. In the second panel of Figures 1 and 2 we show the dependence on the strikes of the variance of the local optimal hedging strategy, for different values of the number of trading dates. We show the strikes from $K \simeq 86.31$ to $K \simeq 115.87$, passing trough the ATM value that is $K = 100$. We also show the corresponding call prices.

A natural quantity to which compare the standard deviation of hedging error is the call price. We note that the variance of hedging error reaches the maximum value for at the money strikes but, if compared with the call prices, the variance out of the money (OTM) has a greater relative weight. So for example for $N = 50$, in the case of the "local optimal" strategy, and for the case of NIG model, one has an ATM call price $C(K = 100) \simeq 4.34$ with the corresponding standard deviation of hedging error about equal to 2.06 while the price for an OTM strike is $C(K = 110.32) \simeq 0.99$ to which corresponds a standard deviation of the hedging error of $\simeq 1.69$. Therefore while the standard deviation can be of the same order of the call price ATM, in the OTM region, the standard deviation can be also greater than the price.

The same thing happens for the Merton model where instead the hedging error is lower. Thus for example the ATM standard deviation is $\simeq 1.39$ for a call price $C(K = 100) \simeq 4.99$ while the standard deviation for a call struck at $K = 110.32$ is about 1.01 to compare to the price $C(K = 110.32) \simeq 1.36$.

4.3 Robustness to the model parameters

In this section we perform an analysis of sensitivity of the hedging error to the realized model parameters in the following sense. We start with a vector ϱ_0 of model parameters and compute option prices and hedging strategies; such a vector may be thought as obtained through a calibration procedure. Then we let the underlying evolve under the same model, but with a different set of parameters ϱ. For example, in the case of the Black-Scholes model, this means that one performs the delta hedging at the implied volatility but the underlying moves with a different realized volatility (see for example [2] for such an experiment).

In our work we carry out this kind of analysis for NIG, VG and Merton models but we report only the results for Merton and NIG models as the VG case is very similar to the NIG one. Moreover we chose to analyze only the case of the "local optimal" strategy (2.6) (continuous line in the Figures 3, 4) performed at the calibrated parameters ϱ_0 and the implied Black-Scholes delta strategy (the dotted-line in the Figures).

Let us notice that the expectation of hedging error in such a case does not depend on the strategy but it is simply the difference between the expectation of the payoff computed with the calibrated distribution and the expectation computed with the realized distribution. This is true as the discounted price process of the underlying is a martingale.

Figure 3 shows the dependence of the standard deviation of the hedging error $\sqrt{(var[\varepsilon])}$ on the realized parameters for the NIG model. The calibrated parameters are $[\sigma_0, \theta_0, \nu_0] = [0.2, 0, 0.2]$. We chose to represent the results for three different moneyness $K_1 \simeq 90.65, K_2 = 100, K_3 \simeq 110.32$ to which correspond the NIG prices $C_1 \simeq 12.58, C_2 \simeq 6.36, C_3 \simeq 2.57$. The spot value of the underlying is $S_0 = 100$, $r = 0.04$ and the maturity of the option is $T = 0.5$. The number of hedging dates is $N = 12$, meaning that the portfolio is rebalanced every two weeks.

We suppose the underlying is driven by a NIG process with a realized set of parameter $[\sigma, \theta, \nu]$ which is different from the calibrated one. The three panels in Figure 3 represent the standard deviation of hedging error respectively to σ when $\theta = \theta_0$ and $\nu = \nu_0$ (Panel 1), to θ when $\sigma = \sigma_0$ and $\nu = \nu_0$ (Panel 2), and to ν when $\sigma = \sigma_0$ and $\theta = \theta_0$ (Panel 3).

The reason why we chose to analyze the case of $\theta_0 = 0$ is because the model parameters are directly related to the moments of the distribution of returns. Indeed, for such a value the skewness of the log-return distribution implied by the process is zero. In that case, the variance is given just by σ^2 while the percentage excess of kurtosis is exactly ν. In other words, moving σ, while holding fixed the other parameters, affects only the realized volatility. Ranging ν around ν_0 means to have a realized excess of kurtosis different from the one implied by the quoted option prices, but a realized volatility and a realized skewness (actually zero) equal to those implied from the market. Movements of the parameter θ are the most interesting as they introduce a realized skewness and a modification of both realized variance and kurtosis.

First of all notice that the two different strategies considered in our analysis (compare the dotted line and the continuous line) lead to results that are almost similar if compared to the differences deriving from a wrong choice of the parameters. Therefore, as already noted in [2], a wrong choice of the parameters has a stronger impact on the hedging performance than the choice of the particular strategy adopted.

From Figure 3 we see that the hedging error is strongly influenced by the difference from the realized and the implied volatility. In fact a mispecification of 0.01 for σ leads to a variation in $\sqrt{(var[\varepsilon])}$ of about 0.2, while the same variation of 0.01 in θ or ν gives a variation on $\sqrt{(var[\varepsilon])}$ of about $0.05 \div 0.07$. Notice that the standard deviation of hedging error is fairly linear in σ and also in ν, apart from small values of ν. Moreover, the first and the second panel of Figure 3 show that the performance of the hedging can exploit the fact that the realized variance and percentage excess of kurtosis are smaller than those used for the construction of the strategy. In contrast with this fact, from the second panel it emerges that a realized skewness different from zero, positive or negative, has a negative impact on the performance of hedging.

We performed the same kind of experiment for the Merton model. We show our results in Figure 4. The parameters of the model are $[\sigma_0, \lambda_0, \mu_0^J, \sigma_0^J] = [0.2, 1, 0, 0.1]$ to which correspond call prices $C_1 \simeq 13.08, C_2 \simeq 7.23, C_3 \simeq 3.25$ at the usual strikes $K_1 \simeq 90.65$, $K_2 = 100$ and $K_3 \simeq 110.32$. It is not possible in the Merton case to separate the effect of kurtosis and variance as in NIG. And in fact, even if the skewness is zero, as $\mu_0^J = 0$, the variance and excess of kurtosis move together and we are not able to separate the two effects simply moving the parameters independently.

In the case of Merton the variations due to a mispecification of all the parameters but λ, are of the same order, that is $0.1 \div 0.2$ for a variation of one point percent of the single parameter. The impact of the parameter λ is weaker of about two orders. The introduction of some realized skewness, moving μ^J away from μ_0^J, causes a behavior similar to that in the NIG model, namely impacting negatively on the performance of the hedging error, but with a stronger effect.

5 Conclusion

We studied the problem of hedging a contingent claim in incomplete models, when the hedging portfolio is re-balanced at a set of discrete dates. In particular, we are interested in measuring the final hedging error of a given strategy. We started with contingent claims with payoffs having an integral representation to compute the expectation and the variance of the hedging error using results from [2]. The method, which involves inverting Laplace/Fourier transforms, is implemented using the FFT technique, which is a popular algorithm adopted for pricing and calibration purposes. One of the contribution of this work is to show how to apply the FFT machinery also to the analysis of hedging error and in particular of its expectation and standard deviation.

With our apparatus we analyzed the performance of hedging error for different strategies: the model-delta, the Black-Scholes delta computed at the implied volatility and the local optimal strategy. The models considered for the study are the NIG, VG and Merton models but of course the methodology can be applied to all the class of Lévy models. We first studied the behavior of each strategy as the number of trading dates increases. Then we analyzed the sensitivity of hedging error to model parameters, which may be considered as a robustness test. The analysis shows that hedging adopting different strategies has in general a less impact on the hedging performance than that deriving from a mispecification of the model parameters. This kind of analysis is easier for the NIG or VG models for which the sensitivity on the parameters is essentially the sensitivity to the realized variance, skewness and percentage excess of kutosis. As expected, we see that the standard deviation of hedging error increases with the volatility of the underlying and the same happens when increasing the kurtosis. Hedging neglecting the skewness when the realized one is away from zero, positively or negatively, leads to worst performance.

There are different future directions that can be developed. First of all one can extend the methodology to a more general class of affine models and stochastic volatility models. One can also try to analyze other payoffs (see [16] and [15]). It should be also interesting to compare the hedging error in the presence of jumps, computed with our methodology, to the computation of the second moment of the asymptotic distribution as computed by Tankov et al. in [23].

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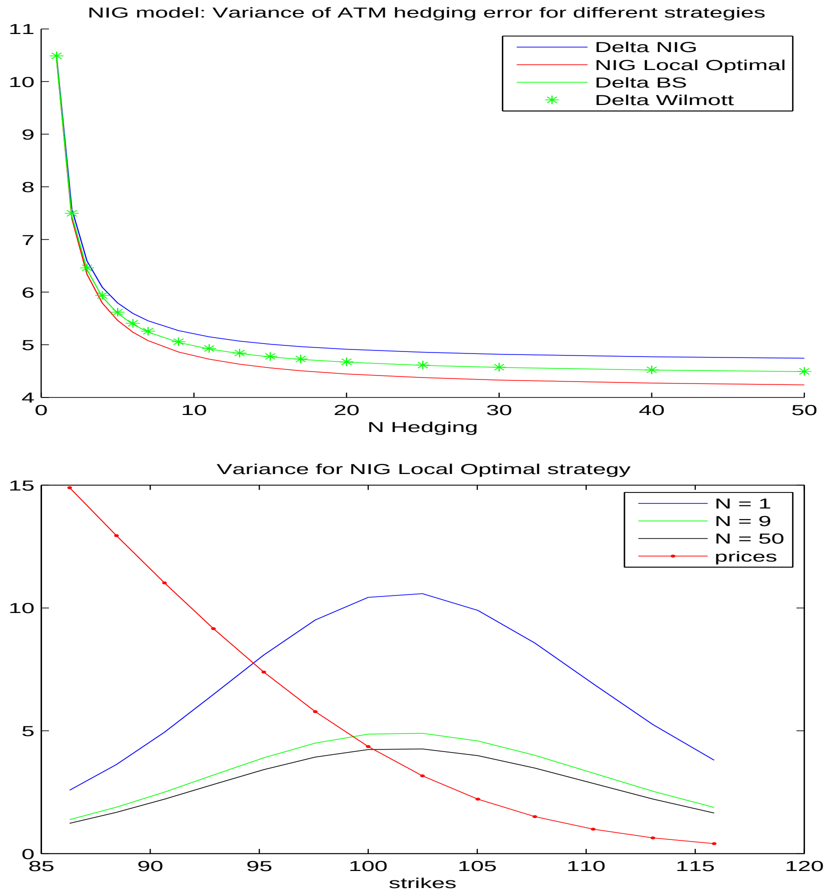

Figure 1 *Variance of the hedging error when the underlying is driven by a NIG process of parameter* $[\sigma, \theta, \nu] = [0.2, -0.1, 0.1]$. *The maturity of the option is* $T = 0.25$, *the spot value of the underlying is* $S_0 = 100$ *and the risk free interest rate is* $r = 0.04$. *The first panel shows the at the money (ATM) variance for different strategies as the number of trading dates increases. The "Delta BS" and the "Delta Wilmott" strategies are performed at ATM Black-Scholes implied volatility. The second panel shows the dependence of the local optimal strategy variance on different strikes for three values of the number of hedging dates. Moreover it is shown the price of the option at the same strikes.*

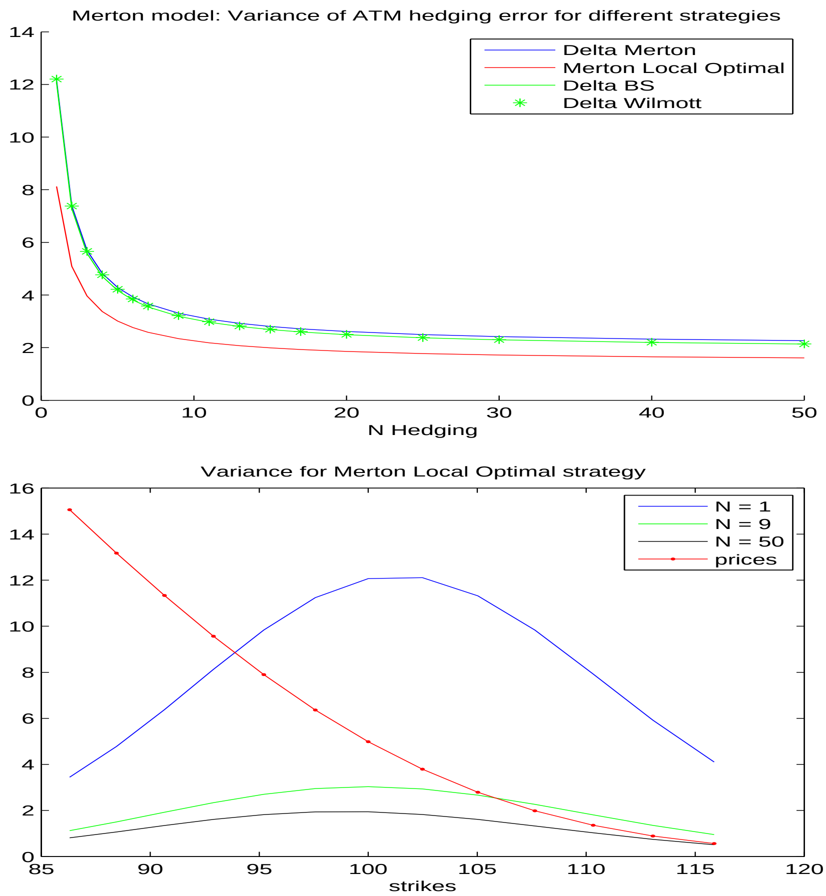

Figure 2 *Variance of the hedging error when the underlying is driven by a Merton process of parameter* $[\sigma, \lambda, \mu_{Jumps}, \sigma_{Jumps}] = [0.2, 1, -0.1, 0.05]$. *The maturity of the option is* $T = 0.25$, *the spot value of the underlying is* $S_0 = 100$ *and the risk free interest rate is* $r = 0.04$. *The first panel shows the at the money (ATM) variance for different strategies as the number of trading dates increases. The "Delta BS" and the "Delta Wilmott" strategies are performed at ATM Black-Scholes implied volatility. The second panel shows the dependence of the local optimal strategy variance on different strikes for three values of the number of hedging dates. Moreover it is shown the price of the option at the same strikes.*

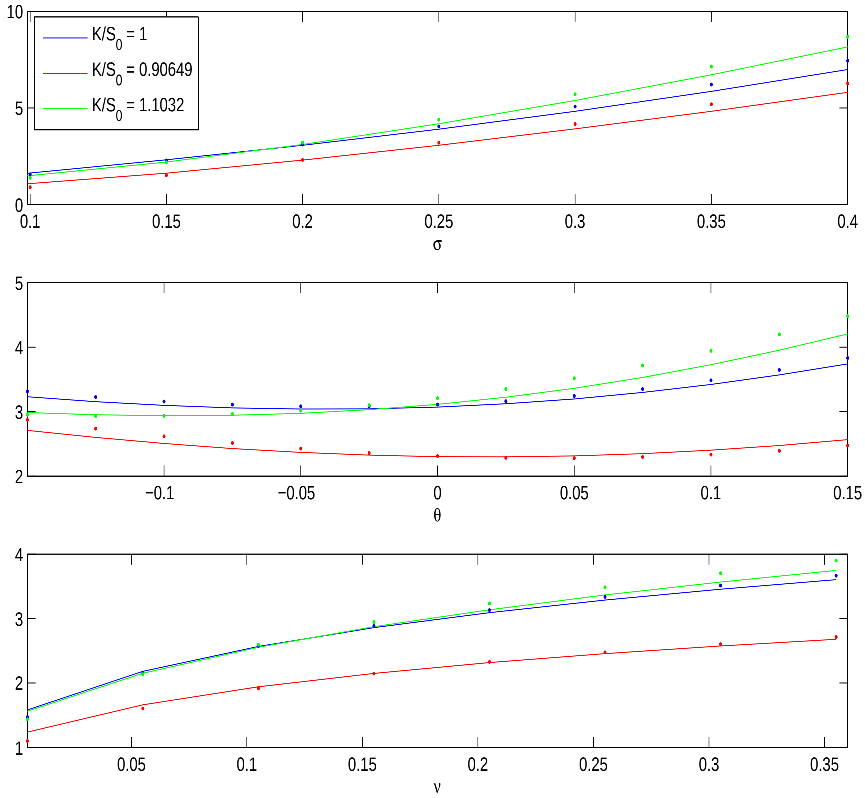

Figure 3 *Influence of the actual parameters to the performances of the NIG local optimal (continuous line) and Δ^{BS} (dotted line) strategies. The call prices correspond to the set of NIG parameters $[\sigma_0, \theta_0, \nu_0] = [0.2, 0, 0.2]$. The NIG local optimal strategy is computed at the same parameters, while the Δ^{BS} is computed at the BS implied volatilities depending on the strike. The figure shows three different moneyness $K_1 \simeq 90.65, K_2 = 100, K_3 \simeq 110.32$ to which correspond the prices $C_1 \simeq 12.58, C_2 \simeq 6.36, C_3 \simeq 2.57$. The other inputs are: $S_0 = 100$, $T = 0.5$, $r = 0.04$ and $N_{hedging} = 12$. The underlying is driven by a NIG process with a realized set of parameter $[\sigma, \theta, \nu]$ that is different from $[\sigma_0, \theta_0, \nu_0]$. The three panels represent the dependence of the standard deviation of hedging error to σ when $\theta = \theta_0$ and $\nu = \nu_0$ (Panel 1), θ when $\sigma = \sigma_0$ and $\nu = \nu_0$ (Panel 2), and ν when $\sigma = \sigma_0$ and $\theta = \theta_0$ (Panel 3).*

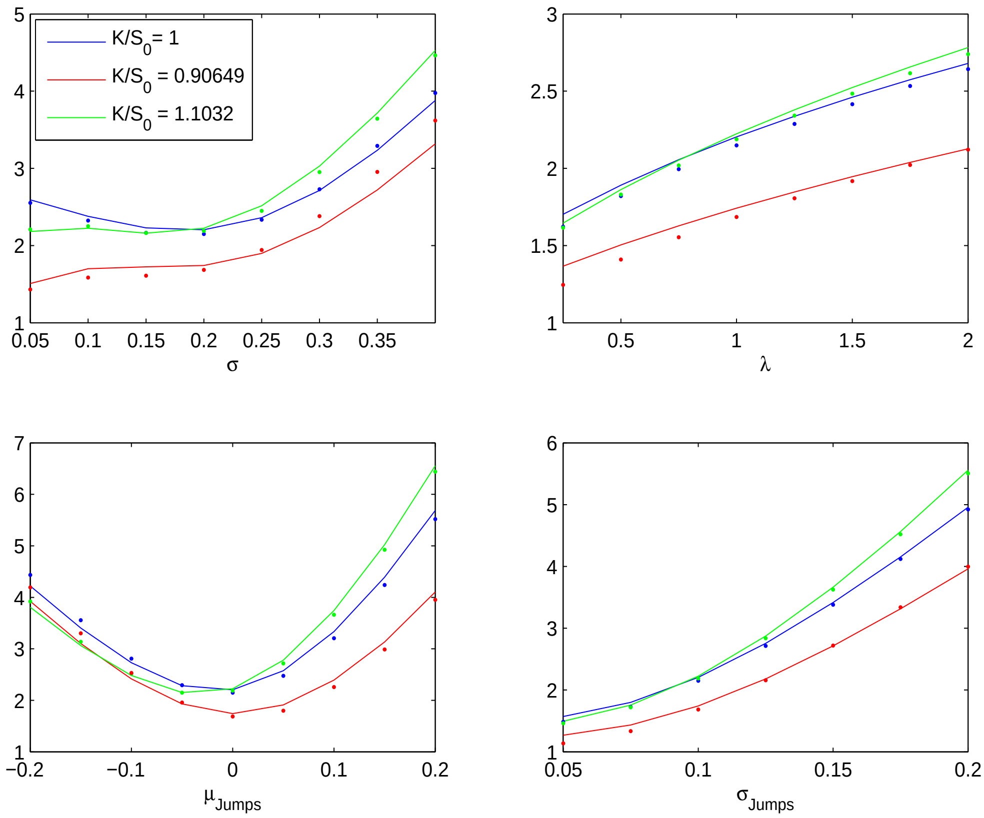

Figure 4 *Influence of the actual parameters to the performances of the Merton local optimal (continuous line) and* Δ^{BS} *(dotted line) strategies. The call prices correspond to the set of Merton parameters* $[\sigma_0, \lambda_0, \mu_0^{Jumps}, \sigma_0^{Jumps}] = [0.2, 1, 0, 0.1]$*. The Merton local optimal strategy is computed at those parameters while the* Δ^{BS} *is computed at the BS implied volatilities depending on the strike. The figure shows three different moneyness* $K_1 \simeq 90.65, K_2 = 100, K_3 \simeq 110.32$ *to which correspond the prices* $C_1 \simeq 13.08, C_2 \simeq 7.23, C_3 \simeq 3.25$*. The other inputs are:* $S_0 = 100$*,* $T = 0.5$*,* $r = 0.04$ *and* $N_{hedging} = 12$*. The underlying is driven by a Merton process with a realized set of parameter* $[\sigma, \lambda, \mu^{Jumps}, \sigma^{Jumps}]$ *that is different from* $[\sigma_0, \lambda_0, \mu_0^{Jumps}, \sigma_0^{Jumps}]$*. The four panels show the dependence of the standard deviation of hedging error due to movements of each single parameter while keeping the others fixed at their calibrated values.*

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