MODELING TIME AND MACROECONOMIC DYNAMICS

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Abstract. When analyzing dynamic macroeconomic models, a commonly held view is that assuming continuous or discrete time is a matter of convenience without any economic importance. The aim of this paper is to challenge this view by demonstrating that there is indeed a significant difference between a discrete and a continuous time version of the same model, in terms of structure, solution and economic interpretation. Specifically, we consider a general framework that incorporates many well known models, and show that the Euler equations of the two setups are different in the following way. While investment decisions in the continuous time setup depend on present rates of return, in the discrete time setup they depend on expected future rates of return. The reason for this is that in continuous time, the household's decisions are allowed to adjust continuously to possible changes in the economic environment, while in discrete time, households are committed to their decision until the beginning of the following period. Furthermore, we illustrate via some examples, that this fundamental difference between the two modeling assumptions has important implications for the stability properties of the steady state. In turn, this difference may imply contradicting macroeconomic policy recommendations under the two setups.

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1. Introduction

When analyzing dynamic macroeconomic models, the decision of whether to model time as continuous or discrete is often based upon the methodological needs of the researcher. Typically, a continuous time setting yields a set of equilibrium conditions that are easier to work with compared to those deriving from a discrete time setting. On the other hand, when the aim is to explore quantitative issues and the analysis needs to be done numerically, assuming discrete time is more appropriate since computers cannot do exact representations of continua. Although occasionally some economic interpretation is put forth to justify the use of one or the other, the general consensus is that there should not be any difference between neither the qualitative nor the quantitative results deriving from the two assumptions.

In this paper, we challenge this view by demonstrating that there is indeed a significant difference between a discrete and a continuous time version of the same model, in terms of structure, solution and economic interpretation. This is shown in the framework of a generic

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model that incorporates a large class of well-known dynamic macroeconomic models. We show that there are two sources of possible differences between the two setups. The first is the mere fact that the stability properties of systems described by differential equations (corresponding to the equilibrium conditions in continuous time) are different from those of systems described by difference equations (corresponding to the equilibrium conditions in discrete time). The second, and perhaps more important, is the fact that investment decisions in the continuous time setup depend on present rates of return, whereas in the discrete time setup they depend on expected future rates of return.

We begin by shortly describing the difference between systems of differential and difference equations. It is a well known mathematical result that the conditions required for stability (in the mathematical sense) of a system of difference equations are stricter than those required for stability of the corresponding system of differential equations. In terms of systems of macroeconomic variables, this result is important for specifying whether a steady state is locally indeterminate or not. In particular, even if there is no other difference between the equilibrium conditions of a model in continuous and discrete time, this mathematical result is enough to render a steady state indeterminate in continuous time, while it might be determinate under discrete time.

Next, we introduce a generic economy populated by a representative household that maximizes lifetime utility, over a vector of state variables and a vector of control variables, subject to its budget constraint. Time is modeled with discrete time intervals of length h. We then derive the equilibrium conditions that come from the household's maximization and take the limits of these for $h \to 0$ and $h \to 1$, in order to get the corresponding continuous and discrete time systems of equations. The second difference between the two settings stems from the difference in the two Euler equations that describe the intertemporal decision process of the households. Rewriting these in such a way that the time-preference discount rate is equal to the total rate of return to the state variable, we then explain how, in discrete time, households make investment decisions based on expected future rates of return while in continuous time they base their decisions on current rates of return. This result further implies that, even if the steady state of the economy is determinate under both modeling assumptions, there are possibly differences between the equilibrium paths leading to the steady state. Consequently, there might be differences in the volatility of the variables.

There are two (equivalent) ways of interpreting the difference between the discrete and continuous time models. Both derive from the fact that essentially, these are *two different models* with different fundamental modeling assumptions, rather than two versions of the same story. The first interpretation comes from realizing that in the discrete time model there is an inherent time-to-build delay since current stock variables (e.g. capital) become operative in the subsequent period, while in the continuous time model stock variables become operative instantaneously. The second interpretation comes from understanding the time line of events in the asset market (e.g. the market for capital goods). In the discrete time model, agents formulate their market demand by looking ahead and anticipating their optimal decision as of the end of the current period (basing their decisions on expected future returns on the asset). Alternatively, one could assume that prices and rates of return adjust so the agents optimize given the asset levels at the beginning of the period; the limit of such behavior, as the time period shrinks, corresponds to the standard continuous time model. In

Foley's (1975) terminology, with discrete time we have an "end-of-period" equilibrium, while with continuous time we have a "beginning-of-period" equilibrium, and these two have quite different properties.

Finally, to illustrate the importance of the differences between the two modeling assumptions, we present three examples of well known models that fit into our framework, namely the real business cycle model of Hansen (1985), the model of balanced-budget rules of Schmitt-Grohé and Uribe (1997) and the model with increasing returns of Benhabib and Farmer (1994) and Farmer and Guo (1994). The first example is used to give a more detailed and intuitive explanation for the differences between the two timing formulations. The last two of examples have been shown to exhibit indeterminacies; exploring this fact, we then demonstrate that for these models the difference between the two assumptions results in different ranges of indeterminacies, for reasonable parameterizations of the models.

Related literature that explicitly addresses the importance of how time is modeled in macroeconomics includes the following work. Foley (1975) considers a simple model with two assets (capital and money) and shows that the "beginning-of-period" and "end-of-period" specifications of a discrete time model are in general inconsistent with each other in the limit (i.e. as time becomes continuous). Carlstrom and Fuerst (2003) point out the possibility of the difference between discrete and continuous time modeling by studying the local determinacy of a monetary model for interest rate rules. Hintermaier (2003) shows that the existence of sunspot equilibria in discrete time dynamic stochastic general equilibrium models may depend on the length of the time period considered. Finally, Bambi and Licandro (2004) consider an extension of the continuous time model of Benhabib and Farmer (2004) with time-to-build delay and find that even small time-to-build delays rule out local indeterminacy.

2. STABILITY OF DYNAMIC SYSTEMS

We begin by exploring the differences in the stability of difference and differential equations. Consider the difference equation

$$w_{t+1} = Dw_t \tag{1}$$

where w is an $n \times 1$ vector of variables and D is a known $n \times n$ matrix. We can rewrite 1 as

$$\begin{aligned} \Delta w_{t+1} &\equiv w_{t+1} - w_t = (D - I_n)w_t \equiv Cw_t \Longrightarrow \\ \Delta w_{t+1} &= Cw_t \end{aligned}$$

Therefore, the corresponding differential equation will be

$$\dot{w}_t = Cw_t \tag{2}$$

Stability and indeterminacy of a system. In physics, a problem of great importance is that of determining the behavior of a system in the neighborhood of an equilibrium state. For example, suppose that a physical system is described by the vector w_t. If the system returns to the equilibrium state after a small disturbance, it is called *stable*, if not it is called *unstable*.

Similarly, in economics an important issue is to determine the behavior of a system in the neighborhood of a steady state. Typically, w_t corresponds to a vector that contains the predetermined (state) and the control (jump) variables. The steady state of the economic system is then called *indeterminate* if, given the predetermined variables, there exists more than one trajectory that leads to the steady state. The notion of indeterminacy of a steady state for an economic system is equivalent to the notion of stability of a physical system.[1] In this section we will use the term *stability* of a generic system w_t in this sense.

Let λ_i^C and λ_i^D be the eigenvalues of C and D correspondingly. Then the conditions for stability of the system defined by (1) is that $\left|\lambda_i^D\right| < 1$, for all i, and the conditions for stability of the system described by (2) is that $\operatorname{Re}\lambda_i^C < 0$, or equivalently that $\operatorname{Re}\lambda_i^D < 1$, for all i.[2] This means that the conditions for stability of a difference equation are stricter than the conditions for stability of the corresponding differential equation.

Turning into a typical dynamic macroeconomic model, suppose for example that the economy is described by a univariate state variable x_t and a univariate control variable y_t and that after linearizing the system of equilibrium conditions around a steady state, we can reduce the model in a system of two difference equations (if time is discrete)

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = D \begin{pmatrix} x_t \\ y_t \end{pmatrix} \tag{3}$$

or a system of differential equations (if time is continuous)

$$\begin{pmatrix} \dot{x}_t \\ \dot{y}_t \end{pmatrix} = C \begin{pmatrix} x_t \\ y_t \end{pmatrix} \tag{4}$$

In these two cases, the conditions for indeterminacy are $\left|\lambda_i^D\right| < 1$, for $i = 1, 2$ for the discrete time system (3) and $\operatorname{Re}\lambda_i^C < 0$, for $i = 1, 2$ for the continuous time system (4).

Clearly, even if it is the case that $C = D - I_2$, i.e. even if there is an exact correspondence between the continuous and the discrete time setting, the conditions for indeterminacy are not the same. Specifically, if the steady state is locally indeterminate for the discrete time setting, it will also be locally indeterminate in the continuous time setting, but the opposite is not true. In other words, *if $C = D - I_2$, indeterminacy would occur less often in the discrete time setting than in the continuous time setting.*

However, it is not always true that $C = D - I_2$ (in a later section we will show that for many well known models, this is indeed not true). When $C \neq D - I_2$, there is an inherent difference in the economic interpretation of the equilibrium conditions that reduce to the systems (3) and (4). In this case, the conditions required for indeterminacy in the two settings may, in principle, be completely unrelated.

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[1] This is a somewhat unfortunate correspondence of terminologies, because in economics we use the term *stability* to describe a system that is *saddle-path stable*, i.e. a system for which, given the predetermined variables, there is a unique trajectory that leads to the steady state.

[2] This is because the eigenvalues of the two matrices C and D are related in the following way. If λ_i^C are the eigenvalues of C then

$$\begin{aligned} 0 &= \det\left(C - \lambda_i^C I_n\right) = \det\left(D - I_n - \lambda_i^C I_n\right) \\ &= \det\left(D - \left(1 + \lambda_i^C\right) I_n\right) \Longrightarrow \\ \lambda_i^D &= 1 + \lambda_i^C \end{aligned}$$

3. A General Discrete Time Model

Foley (1975) asserts that "*No substantive prediction or explanation in a well-defined macroeconomic [discrete time] model should depend on the real time length of the period. [...] If the results of a [discrete time] model do not depend in any important way on the period, the model can be formulated as a continuous model. The method used to accomplish this is to retain the length of the period as an explicit variable in the mathematical formulation of a [discrete time] model and to make sure that it is possible to find meaningful limiting forms of the equations as the period goes to zero. [...] this procedure should be routinely applied as a test that any [discrete time] model is consistent and well formed where no particular calendar time is specified as a natural period*".

Following this precept, we first consider a general discrete time model, with an arbitrary period length h. The generic model is described by an $n \times 1$ vector of control (flow) variables, y_t and an $m \times 1$ vector of state (stock) variables x_t. The stock variables are measured at the beginning of the period, i.e. at the beginning of the time interval. The model might also contain other (flow) variables, summarized by an $l \times 1$ vector z_t, that might be endogenously determined but are not choice variables for the representative household. We consider the following maximization problem

$$\max_{\{y_t, x_{t+h}\}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho h} \right)^{\frac{t}{h}} u(y_t)h \tag{5}$$

$$s.t. \quad \sum_{i}^{m} (x_{i,t+h} - x_{i,t}) = hQ(y_t, x_t; z_t) \tag{6}$$

$$x_0 \text{ given}$$

This setup covers a wide range of standard dynamic macroeconomic models, such as variations of the real business cycle model (possibly with increasing returns to scale), as well as many standard monetary models.

In the above maximization problem, y_t is interpreted as the vector containing the rates of flow of the control variables that are assumed constant over period $\frac{t}{h}$, while x_t is interpreted as the vector of the levels of the state variables that are measured at the beginning of period $\frac{t}{h}$ (i.e. at the beginning of the interval $[t, (t+h)]$). In (5) we have multiplied $u(y_t)$ with h because the cumulative utility over a period is the product of the instantaneous rate of change times the length of the period. A similar argument applies to multiplying the right-hand-side of (6) with h. We also make the usual assumption for concavity of the utility function, although such an assumption does not have any direct implication for the issue we study here.

The Lagrangian of this problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho h} \right)^{\frac{t}{h}} \left[u(y_t)h - \lambda_t \left(\sum_{i}^{m} (x_{i,t+h} - x_{i,t}) - hQ(y_t, x_t; z_t) \right) \right]$$

and the first order conditions are

$$\begin{aligned} u_{y_j}(y_t) &= -\lambda_t Q_{y_j}(y_t, x_t; z_t), \quad \text{for all } j = 1, ..., n & (7) \\ \lambda_t &= \frac{1}{1+\rho h}\left(1 + hQ_{x_i}(y_{t+h}, x_{t+h}; z_{t+h})\right)\lambda_{t+h}, \quad \text{for all } i = 1, ..., m & (8) \\ \sum_i^m (x_{i,t+h} - x_{i,t}) &= hQ(y_t, x_t; z_t) & (9) \end{aligned}$$

Note that substituting (7) into (8) yields the usual Euler equations. Rearranging (8), we obtain[3]

$$\frac{\lambda_{t+h} - \lambda_t}{h} = \frac{\left(\rho - Q_{x_i}(y_{t+h}, x_{t+h}; z_{t+h})\right)\lambda_t}{1 + hQ_{x_i}(y_{t+h}, x_{t+h}; z_{t+h})} \tag{10}$$

Finally, the equilibrium conditions are summarized by the following set of equations

$$\begin{aligned} u_{y_j}(y_t) &= -\lambda_t Q_{y_j}(y_t, x_t; z_t) & (11) \\ \frac{\lambda_{t+h} - \lambda_t}{h} &= \frac{\left(\rho - Q_{x_i}(y_{t+h}, x_{t+h}; z_{t+h})\right)\lambda_t}{1 + hQ_{x_i}(y_{t+h}, x_{t+h}; z_{t+h})} & (12) \\ \sum_i^m \frac{x_{i,t+h} - x_{i,t}}{h} &= Q(y_t, x_t; z_t) & (13) \end{aligned}$$

The model closes with l additional equations that determine the evolution of z_t.

Note that to obtain the equilibrium conditions for the usual discrete time setup with a unit length time period, we set $h = 1$ in (11) - (13) to get

$$\begin{aligned} u_{y_j}(y_t) &= -\lambda_t Q_{y_j}(y_t, x_t; z_t) & (14) \\ \Delta\lambda_{t+1} &= \frac{\left(\rho - Q_{x_i}(y_{t+1}, x_{t+1}; z_{t+1})\right)\lambda_t}{1 + Q_{x_i}(y_{t+1}, x_{t+1}; z_{t+1})} & (15) \\ \sum_i^m \Delta x_{i,t+1} &= Q(y_t, x_t; z_t) & (16) \end{aligned}$$

where Δ denotes the difference operator.

4. THE STANDARD CONTINUOUS TIME MODEL

Our next step is work out the continuous time analogue of the discrete time model. To obtain the equilibrium conditions for the continuous time setup we let $h \to 0$ in (11) - (13) to get

$$\begin{aligned} u_{y_j}(y_t) &= -\lambda_t Q_{y_j}(y_t, x_t; z_t) & (17) \\ \dot{\lambda}_t &= \left(\rho - Q_{x_i}(y_t, x_t; z_t)\right)\lambda_t & (18) \\ \sum_i^m \dot{x}_t &= Q(y_t, x_t; z_t) & (19) \end{aligned}$$

[3]For the derivation, see appendix A.

The difference between the continuous and discrete time boils down to understanding the difference between (15) and (18). To work out the intuition for this, it is convenient to assume that there is only one (asset) stock variable x. We rewrite (12) as follows[4]

$$\rho = \frac{\lambda_{t+h} - \lambda_t}{h\lambda_t} + \frac{\lambda_{t+h}}{\lambda_t} Q_x \left(y_{t+h}, x_{t+h}; z_{t+h}\right)$$

Then, for $h \to 1$ we get

$$\rho = \frac{\Delta\lambda_{t+1}}{\lambda_t} + \frac{\lambda_{t+1}}{\lambda_t} Q_x \left(y_{t+1}, x_{t+1}; z_{t+1}\right) \tag{20}$$

and for $h \to 0$ we get

$$\rho = \frac{\dot{\lambda}_t}{\lambda_t} + Q_x \left(y_t, x_t; z_t\right) \tag{21}$$

The multiplier λ corresponds to the shadow price of the asset x. In equilibrium, this asset price adjusts such that the asset's total rate of return balances out the time-preference (discount) rate ρ. This is exactly the interpretation of equations (20) and (21). The right hand side of both relations represents the total rate of return to holding one unit of the asset x, decomposed into two components. The first term in both expressions represents the rate of change of asset gains. Turning into the second term, in the continuous time setting it represents the rate of change of the asset. In the discrete time setting, it represents the *future* rate of change of the asset, in terms of the current period's price λ_t.

The difference between the discrete and continuous time setup is thus the fact that the rate Q_x (that influences investment decisions) is *known* to the households at every instance when time is continuous, while when time is discrete, the households get a return on their investment based on next period's rate. In other words, in continuous time, if the rate Q_x changes, households may adjust their investment decisions instantaneously, while in discrete time they are "committed" to their decision until the beginning of the next period.

To understand better the intuition behind this difference, it is useful to closely look at the time line of events within one period (for the discrete time model) and then compare it with analogue continuous time setting. We do this in the next section, in the context of a familiar simple model which will facilitate the comparison.

5. A First Example: the Real Business Cycle Model

We start by briefly describing the model, which is along the lines of Hansen (1985). We set up the model in the general discrete time setting, i.e. with time periods of length h. The representative household maximizes lifetime discounted utility, subject to its budget constraint:

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \left[\log c_t - A n_t\right] h$$

$$s.t. \quad c_t h + k_{t+h} - k_t = h(r_t - \delta)k_t + h w_t n_t$$

$$k_0 \text{ given}$$

[4] The discussion follows the arguments of Obstfeld (1992).

where ρ is the preference discount rate. Capital depreciates at rate $0 < \delta < 1$. Consumption, instantaneous utility, labor income and capital income are flow variables. Therefore, at period $\frac{t}{h}$ defined by the interval $[t, (t+h)]$, the cumulative flow of any of these variable is the (fixed) rate of flow of the variable, i.e. c_t, $\log c_t - An_t$, $w_t n_t$, and $(r_t - \delta)k_t$ respectively, times the period's length h, where r_t and w_t denote the rates of return on capital and labor. Finally, the total capital over the same interval is $k_{t+h} - k_t$.[5] The firm has a production a Cobb-Douglas production function with constant returns to scale $F(k_t, n_t) = k_t^{s_k} n_t^{s_n}$, $s_k + s_n = 1$, and maximizes profits period by period. Since firm profits are a flow variable as well, the firm's problem is

$$\max h \left[F(k_t, n_t) - r_t k_t - w_t n_t\right]$$

which implies the first order conditions $r_t = F_k(k_t, n_t)$ and $w_t = F_n(k_t, n_t)$.

To write this example in the general form (5) - (6) we rewrite the households budget constraint as

$$k_{t+h} - k_t = \left((r_t - \delta)k_t + w_t n_t - c_t\right) h$$

So that the state variable is $x_t \equiv k_t$ and the control variables are $y_t \equiv (c_t, n_t)$. Also, $z_t \equiv (r_t, w_t, \tau_t)$. Furthermore,

$$\begin{aligned} Q(c_t, n_t, k_t; r_t, w_t, \tau_t) &\equiv (r_t - \delta)k_t + w_t n_t - c_t \\ u(c_t, n_t) &\equiv \log c_t - An_t \end{aligned}$$

Therefore, the equilibrium conditions are

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t \\ A &= \lambda_t w_t \\ \frac{\lambda_{t+h} - \lambda_t}{h} &= \frac{\rho - (r_{t+h} - \delta)}{1 + h(r_{t+h} - \delta)} \lambda_t \\ \frac{k_{t+h} - k_t}{h} &= (r_t - \delta)k_t + w_t n_t - c_t \\ r_t &= F_k(k_t, n_t) \\ w_t &= F_n(k_t, n_t) \end{aligned}$$

[5]Note that when $h = 1$ this is the typical disrete time model. The continuous time version can be obtained by taking limits as $h \to 0$.

$$\lim_{h \to 0} \left(\frac{1}{1 + \rho h}\right)^{\frac{t}{h}} = \exp(-\rho t)$$

so the objective becomes

$$U_t = \int_{t=0}^{\infty} \exp(-\rho t)(u(c_t) - v(n_t))dt$$

and the budget constraint

$$\begin{aligned} c_t + \frac{k_{t+h} - k_t}{h} &= (r_t - \delta)k_t + w_t n_t \Rightarrow \\ \frac{k_{t+h} - k_t}{h} &= (r_t - \delta)k_t + w_t n_t - c_t \Rightarrow \\ \dot{k}_t &= \lim_{h \to 0} \frac{k_{t+h} - k_t}{h} = (r_t - \delta)k_t + w_t n_t - c_t \end{aligned}$$

Concentrating on the Euler equation we have that for $h = 1$ and $h \to 0$

$$\begin{aligned}\rho &= \frac{\Delta\lambda_{t+1}}{\lambda_t} + \frac{\lambda_{t+1}}{\lambda_t}(r_{t+1} - \delta) \\ \rho &= \frac{\dot{\lambda}_t}{\lambda_t} + (r_t - \delta)\end{aligned}$$

To understand the key difference between discrete and continuous time let's consider the timeline of events within a period $[t, t+h]$. Since all the variables apart from capital are flows, the households work, receive income, save and consume continuously over the period, at fixed rates. On the other hand, while capital is being produced and accumulates during the current period, new additions to the capital stock only become operative (i.e. put into production) in the next period. In particular, capital k_t is being rented out once at the beginning of the period; whatever is saved throughout the period remains inoperative in the possession of consumers. At the end of the period, the rented (depreciated) capital returns to the possession of the households and is added to the newly accumulated capital. This new capital stock k_{t+h} remains in the possession of the households until the beginning of next period, when it is rented out again. For this reason, the households are interested in the return they will get for their capital once *all* of it becomes operative. Therefore, when optimizing in the current period, they choose how much to invest so that their subjective discount rate ρ is balanced out by the growth rate of capital gains, plus the *next period's* rental rate (in terms of this period's capital shadow price). This argument implies that in the discrete time model, there is a inherent time-to-build delay of one period, since it takes one period for capital to become operative and agents make their decisions based on this fact. Turning to the limiting case of $h \to 0$ (continuous time), capital is made operative at the very instance that it is produced (i.e. there is no time-to-build delay), so that at each instance, households decide how much of this capital to rent out according on the current rental rate they get.

To summarize, the two modeling assumptions (discrete or continuous time) essentially imply two different models with different dynamics and possibly different predictions. The next section provides two more examples where indeed the two models provide different conclusions regarding the local determinacy of the steady state.

6. More Examples: Local Indeterminacy

In this section, we present two examples of dynamic macroeconomic models, namely the model of balanced-budget fiscal policy of Schmitt-Grohé and Uribe (1997) and the model with increasing returns to scale as in Benhabib and Farmer (1994) and Farmer and Guo (1994). Both these models are well known for exhibiting local indeterminacies. By studying the dynamics of the continuous and discrete time versions we illustrate how there are certain (empirically plausible) parameter regions for which the predictions of the two models are contradicting.

6.1. A model with balanced-budget fiscal policy. We consider the model of Schmitt-Grohe and Uribe (1997), where time is measured in increments of length h. This model is an extension of the model of Hansen (1985) with a government which maintains a balanced budget and finances its expenditures by taxing labour income.

The economy is populated by a continuum of households, a firm and a government. The representative household maximizes lifetime discounted utility, subject to its budget constraint:

$$\begin{aligned} & & \max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} [\log c_t - A n_t]\, h \qquad (22)\\ s.t. \quad c_t h + k_{t+h} - k_t &=& h(r_t - \delta)k_t + h(1-\tau_t) w_t n_t \\ & & k_0 \text{ given} \end{aligned}$$

The notation and description of the model is identical to our previous example, with the added constraint for the government: in period t defined by the interval $[t,(t+h)]$ the *government* has a constant rate of flow of expenditures G (which is time invariant). The cumulative flow of government expenditures hG is financed by taxing labor income, i.e. $hG = h\tau_t w_t n_t$, where τ_t is the labor tax rate.

To write this example in the general form (5) - (6) we rewrite the households budget constraint, using the first order conditions from the firm's maximization, as

$$k_{t+1} - k_t = ((r_t - \delta)k_t + (1-\tau_t) w_t n_t - c_t)\, h$$

So that the state variable is $x_t \equiv k_t$ and the control variables are $y_t \equiv (c_t, n_t)$. Also, $z_t \equiv (r_t, w_t, \tau_t)$. Furthermore,

$$\begin{aligned} Q(c_t, n_t, k_t; r_t, w_t, \tau_t) &\equiv (r_t - \delta)k_t + (1-\tau_t) w_t n_t - c_t \\ u(c_t, n_t) &\equiv \log c_t - A n_t \end{aligned}$$

Therefore, the equilibrium conditions for the discrete time setting are

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t \\ A &= \lambda_t (1-\tau_t) w_t \\ \Delta\lambda_{t+1} &= \frac{\rho - (r_{t+1} - \delta)}{1 + (r_{t+1} - \delta)} \lambda_t \\ \Delta k_{t+1} &= (r_t - \delta)k_t + (1-\tau_t) w_t n_t - c_t \\ r_t &= F_k(k_t, n_t) \\ w_t &= F_n(k_t, n_t) \\ G &= \tau_t w_t n_t \end{aligned}$$

while in continuous time (as in Schmitt-Grohé and Uribe (1997)), the equilibrium conditions are

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t \\ A &= \lambda_t(1-\tau_t)w_t \\ \dot{\lambda}_t &= (\rho-(r_t-\delta))\lambda_t \\ \dot{k}_t &= (r_t-\delta)k_t+(1-\tau_t)w_t n_t - c_t \\ r_t &= F_k(k_t,n_t) \\ w_t &= F_n(k_t,n_t) \\ G &= \tau_t w_t n_t \end{aligned}$$

Let lower case letters denote the steady state values of variables and denote with $s_i = \delta k/F(k,n)$, $s_c = c/F(k,n)$. Log-linearizing the two systems around the steady state and eliminating all variables apart from the state variable and the Lagrange multiplier, we obtain for the continuous time setting

$$\begin{pmatrix} \dot{k}_t \\ \dot{\lambda}_t \end{pmatrix} = C \begin{pmatrix} k_t \\ \lambda_t \end{pmatrix} \tag{23}$$

where

$$C \equiv \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} (\rho+\delta)\frac{1-\tau}{s_k-\tau}-\delta & \frac{\delta}{s_i}\left[\frac{s_n(1-\tau)}{s_k-\tau}+s_c\right] \\ -(\rho+\delta)\frac{s_n\tau}{s_k-\tau} & -(\rho+\delta)\frac{s_n(1-\tau)}{s_k-\tau} \end{pmatrix} \tag{24}$$

and for the discrete time setting

$$\begin{pmatrix} \Delta k_{t+1} \\ \Delta \lambda_{t+1} \end{pmatrix} = B \begin{pmatrix} k_t \\ \lambda_t \end{pmatrix} \tag{25}$$

where[6]

$$B \equiv \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ \frac{c_{21}+c_{21}c_{11}}{1+\rho-c_{22}} & \frac{c_{22}+c_{21}c_{12}}{1+\rho-c_{22}} \end{pmatrix} \tag{26}$$

It is apparent from the previous expression that $B \neq C$, i.e. the discrete time system (25) is not the direct analogue of the continuous time system (23). In other words the stability dynamics of the two models will be different. The conditions for indeterminacy of the continuous time system are that $\mathrm{Re}\,\lambda_i^C < 0$, while the conditions for indeterminacy of the discrete time system are that $-2 < \lambda_i^B < 0$.

Fixing all the parameters of the model apart from the steady state level of labor tax rate τ, as in Schmitt-Grohé and Uribe (1997), i.e. setting $s_k = 0.3$, $\delta = 0.1$, $\rho = 0.04$, we find (numerically) that the discrete time model is indeterminate for the range $\tau \in (0.38, 0.75)$, while the continuous time model is indeterminate for $\tau \in (0.3, 0.75)$. In other words, there is no common prediction of the two models for the range $(0.3, 0.38)$.

[6] For the derivation see appendix B.

6.2. A model with increasing returns. In this section, we describe the economy of Benhabib and Farmer (1994) and Farmer and Guo (1994). It is very similar in structure to the standard real business cycle model. The difference lies in the aggregate production function that exhibits increasing returns to scale

$$F(k_t, n_t) = k_t^{\alpha} n_t^{\beta}$$

where $\alpha + \beta > 1$. This can be interpreted as a setup where competitive firms face constant returns technologies but the economy wide technology has increasing returns due to production externalities. A second interpretation assumes monopolistic competition with increasing returns to scale technologies in the intermediate goods sector. But these goods are combined to produce a final good in a perfectly competitive sector[7]. When calibrating the model's parameters, we follow Farmer and Guo (1994) in adopting the second interpretation.

The consumer problem and the resulting equilibrium conditions are invariant to the choice of production structure. The representative consumer faces a standard maximization problem as follows

$$\begin{aligned} & & \max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \left[\log c_t - A\frac{n_t^{1-\gamma}}{1-\gamma}\right] h \\ s.t. \quad c_t h + k_{t+h} - k_t &=& h(r_t - \delta)k_t + h w_t n_t + \pi_t h \\ & & k_0 \text{ given} \end{aligned}$$

where the only differences compared to the previous example are that utility is now non-linear in labour and that firm profits π_t are also present in the consumer's budget constraint. The correspondence to the general form (5) - (6) is straightforward. The state variable is $x_t \equiv k_t$, the control variables are $y_t \equiv (c_t, n_t)$ and $z_t \equiv (r_t, w_t, \pi_t)$ are additional endogenously determined variables. The rate of change in the state variable Q is given by

$$Q(c_t, n_t, k_t; \pi_t, r_t, w_t) \equiv (r_t - \delta)k_t + w_t n_t - c_t$$

and the utility function

$$u(c_t, n_t) \equiv \log c_t - A\frac{n_t^{1-\gamma}}{1-\gamma}$$

depends only on the control variables. Equilibrium conditions in discrete time are given by

$$\begin{aligned} \frac{1}{c_t} &=& \lambda_t \\ A n_t^{-\gamma} &=& \lambda_t w_t \\ \Delta\lambda_{t+1} &=& \frac{\rho - (r_{t+1} - \delta)}{1 + (r_{t+1} - \delta)}\lambda_t \\ \Delta k_{t+1} &=& (r_t - \delta)k_t + w_t n_t + \pi_t - c_t \end{aligned}$$

[7] For a detailed discussion of the model with increasing returns and the alternative production structures, see Benhabib and Farmer (1994).

and in continuous

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t \\ An_t^{-\gamma} &= \lambda_t w_t \\ \dot{\lambda}_t &= (\rho - (r_t - \delta))\lambda_t \\ \dot{k}_t &= (r_t - \delta)k_t + w_t n_t + \pi_t - c_t \end{aligned}$$

Additional restrictions are provided by the firm's maximization problem (these are the same for the two cases) and specify that factors are paid their shares in national income

$$\begin{aligned} r_t &= s_k \frac{F(k_t, n_t)}{k_t} \\ w_t &= s_n \frac{F(k_t, n_t)}{n_t} \end{aligned}$$

and profits are

$$\pi_t = F(k_t, n_t) - w_t n_t - r_t k_t$$

Let $s_i = \delta k / F(k, n)$ and $s_c = c/F(k, n)$. Log-linearizing the two systems around the steady state and eliminating all variables apart from the state variable and the Lagrange multiplier, we obtain for the continuous time setting

$$\begin{pmatrix} \dot{k}_t \\ \dot{\lambda}_t \end{pmatrix} = C \begin{pmatrix} k_t \\ \lambda_t \end{pmatrix}$$

where

$$C \equiv \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \frac{\rho+\delta}{s_k}(\alpha + \frac{\alpha\beta}{1-\beta-\gamma}) - \delta & \frac{\delta}{s_i}\left[\frac{\beta}{1-\beta-\gamma} + s_c\right] \\ -(\rho+\delta)(\alpha - 1 + \frac{\alpha\beta}{1-\beta-\gamma}) & -(\rho+\delta)\frac{\beta}{1-\beta-\gamma} \end{pmatrix}$$

and for the discrete time setting

$$\begin{pmatrix} \Delta k_{t+1} \\ \Delta \lambda_{t+1} \end{pmatrix} = B \begin{pmatrix} k_t \\ \lambda_t \end{pmatrix}$$

where

$$B \equiv \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ \frac{c_{21} + c_{21}c_{11}}{1+\rho-c_{22}} & \frac{c_{22}+c_{21}c_{12}}{1+\rho-c_{22}} \end{pmatrix}$$

Note that this relationship between B and C, the matrices describing equilibrium dynamics in discrete and in continuous time respectively, is common in both the examples considered.

We examine the presence of indeterminacies in the solution of the model under the two timing conventions. We fix all parameters to the values used by Farmer and Guo (1994) except for the parameter λ, which measures the degree of monopoly power in the markets for intermediate goods. Thus we set $\delta = 0.025$, $\rho = 0.01$, $\gamma = 0$. For factor shares we adopt the Baxter and King (1994) choices of $s_k = 0.3$ and $s_n = 0.63$. Finally, we let λ vary in the range $(0, 1)$ and set $\alpha = \frac{s_k}{\lambda}$ and $\beta = \frac{s_n}{\lambda}$.

Steady states are found to be determinate under both timing conventions for $\lambda < s_k$ or $\lambda > s_n$. When $s_k < \lambda < s_n$, the continuous time steady state is always indeterminate whereas the discrete time one can be either determinate or indeterminate. Obviously, whatever the choice of λ, the solutions are different, as can be seen by inspection of the matrices C and D.

7. Closing Comments

In this paper, we have explored the differences arising from modeling time as discrete or continuous in a wide class of dynamic macroeconomic models. We have shown that there two ways in which the analysis under the two setups might differ. The first is due to the differences arising from studying a set of variables described by a system of differential or difference equations. The second is due to the fact that in continuous time, investment decisions are made based on present rates of return and are allowed to adjust continuously, while in discrete time, investment decisions are made based on future rates of return, so that the households are committed to their decision until the next period when the new rate of return is realized. The differences between the two setups have important implications for the stability properties of steady states, as well as for the equilibrium paths leading to these steady states.

In further work on this issue, we are exploring whether it is possible to have a general framework which will nest a discrete and continuous time model that give the same qualitative and quantitative economic predictions. A possible direction for this is to construct a discrete time model with a generic period length h, which will also incorporate a time-to-build delay parameter, so that when taking appropriate limits, we can obtain a continuous time model which is equivalent to the discrete time model.

APPENDIX

A. DERIVATIONS FOR THE GENERAL CASE

A.1. Expression (10).

$$
\begin{aligned}
\lambda_t &= \frac{1}{1+\rho h}\left(1+hQ_{x_i}(y_{t+h},x_{t+h})\right)\lambda_{t+h} \Longrightarrow \\
\lambda_{t+h} &= \frac{1+\rho h}{1+hQ_{x_i}(y_{t+h},x_{t+h})}\lambda_t \Longrightarrow \\
\lambda_{t+h}-\lambda_t &= \frac{1+\rho h}{1+hQ_{x_i}(y_{t+h},x_{t+h})}\lambda_t - \lambda_t \Longrightarrow \\
&= \frac{(1+\rho h)\,\lambda_t - \lambda_t\left(1+hQ_{x_i}(y_{t+h},x_{t+h})\right)}{1+hQ_{x_i}(y_{t+h},x_{t+h})} \\
&= \frac{\lambda_t - \lambda_t - h\lambda_t Q_{x_i}(y_{t+h},x_{t+h}) + h\rho\lambda_t}{1+hQ_{x_i}(y_{t+h},x_{t+h})} \\
&= h\frac{-\lambda_t Q_{x_i}(y_{t+h},x_{t+h}) + \rho\lambda_t}{1+hQ_{x_i}(y_{t+h},x_{t+h})} \Longrightarrow \\
\frac{\lambda_{t+h}-\lambda_t}{h} &= \frac{-\lambda_t Q_{x_i}(y_{t+h},x_{t+h}) + \rho\lambda_t}{1+hQ_{x_i}(y_{t+h},x_{t+h})} \Longrightarrow \\
\frac{\lambda_{t+h}-\lambda_t}{h} &= \frac{(\rho - Q_{x_i}(y_{t+h},x_{t+h}))}{1+hQ_{x_i}(y_{t+h},x_{t+h})}\lambda_t
\end{aligned}
$$

B. DERIVATIONS FOR EXAMPLES

B.1. Example 2. The first reduced equation of the log-linearized equation for the discrete time setting is

$$\Delta k_{t+1} = c_{11}k_t + c_{12}\lambda_t$$

The second reduced log-linearized equation for the discrete time setting is

$$\lambda_{t+1} - \lambda_t = \frac{c_{21}}{1+\rho}k_{t+1} + \frac{c_{22}}{1+\rho}\lambda_{t+1}$$

which simplifies to

$$
\begin{aligned}
&-\frac{c_{21}}{1+\rho}\Delta k_{t+1} + \left(1-\frac{c_{22}}{1+\rho}\right)\Delta\lambda_{t+1} \\
= &-\frac{c_{21}}{1+\rho}k_t - \frac{c_{22}}{1+\rho}\lambda_t
\end{aligned}
$$

Thus.

$$\begin{aligned}
&\begin{pmatrix} -\frac{c_{21}}{1+\rho} & (1-\frac{c_{22}}{1+\rho}) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Delta k_{t+1} \\ \Delta \lambda_{t+1} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{c_{21}}{1+\rho} & -\frac{c_{22}}{1+\rho} \\ c_{11} & c_{12} \end{pmatrix}\begin{pmatrix} k_t \\ \lambda_t \end{pmatrix} \Longrightarrow \\
\begin{pmatrix} \Delta k_{t+1} \\ \Delta \lambda_{t+1} \end{pmatrix} &= \begin{pmatrix} -\frac{c_{21}}{1+\rho} & (1-\frac{c_{22}}{1+\rho}) \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} \frac{c_{21}}{1+\rho} & \frac{c_{22}}{1+\rho} \\ c_{11} & c_{12} \end{pmatrix}\begin{pmatrix} k_t \\ \lambda_t \end{pmatrix} \\
&= \frac{1}{(1-\frac{c_{22}}{1+\rho})}\begin{pmatrix} 0 & (1-\frac{c_{22}}{1+\rho}) \\ 1 & \frac{c_{21}}{1+\rho} \end{pmatrix}\begin{pmatrix} \frac{c_{21}}{1+\rho} & \frac{c_{22}}{1+\rho} \\ c_{11} & c_{12} \end{pmatrix}\begin{pmatrix} k_t \\ \lambda_t \end{pmatrix} \\
&= \begin{pmatrix} c_{11} & c_{12} \\ \frac{c_{21}(1+c_{11})}{1+\rho-c_{22}} & \frac{c_{21}c_{12}+c_{22}}{1+\rho-c_{22}} \end{pmatrix}\begin{pmatrix} k_t \\ \lambda_t \end{pmatrix} \\
&\equiv \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} k_t \\ \lambda_t \end{pmatrix}
\end{aligned}$$

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