Bond Immunization and Exchange Rate Risk: Some Further Considerations[*]

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Abstract

This research project seeks to address two critical problems in the theory of international bond pricing: 1) how can exchange rate risk be formally incorporated into standard bond valuation models?, and 2) how must strategies to "immunize" bonds against interest rate and inflation risk be modified to also incorporate exchange rate risk? Most of all, this study analyzes the mathematical properties of international bonds (e.g., Eurobonds). A special consideration is given to the two most important characteristics of debt securities – duration and convexity and through them to the various ways to immunize bonds and bond portfolios from real interest, inflation, and exchange rate risks. Fogler (1984) formally addressed the effects of changes in inflation and interest rates on bond prices. Unfortunately, exchange rate risk does not appear to have been formally incorporated into these previous models. Moreover, we correct a mathematical error in Fogler's analysis.

1. Introduction

This research project will address two critical questions in the theory of international bond pricing: 1) how can exchange rate risk be formally incorporated into standard bond valuation models?, and 2) can international bonds be "immunized" against interest rate, inflation, and exchange rate risks? Since most bonds provide fixed returns to investors in the form of coupon payments and principal, the primary risk to a holder of a domestic bond is that interest rates and inflation may increase. The consequence is a reduction in the bond's market price due to a decline in the purchasing power of future coupon payments and principal. Investors in international bonds, however, face the added risk that should domestic interest rates increase, the value of their currency will appreciate against the currency of the foreign bond. Thus, the value of the foreign-denominated coupon payments and principal will also decline in value.

Heretofore, researchers have been able to formally address the effects of changes in inflation and real interest rates on bond prices (Fogler 1984). The literature has also emphasized risk management techniques and "immunization" strategies to minimize both of these particular risks. Unfortunately, exchange rate risk does not appear to have been formally incorporated into these previous models.

2. Redington conditions for bond immunization

This study seeks to analyze the mathematical properties of the major international bonds issues such as foreign bonds and Eurobonds. A special consideration will be given to the two most important characteristics of debt securities – duration and convexity, and through them to the various ways to immunize bonds and bond portfolios from the previously mentioned risks.

Frank Mitchell Redington (1922) identified the two conditions for immunizing a bond portfolio (also called the "Redington conditions") which have been widely used and applied to managing bond portfolios in the insurance and banking industries. Many saving and loans banks and other financial institutions became financially stressed during the late-1980s because they failed to adhere to these simple conditions. For example, a bank leverages returns by issuing shorter-term liabilities (deposits) to fund longer-term assets (mortgages). While this strategy of maturity "mismatching" is fairly bounded, it is not an uncommon condition for many financial institutions (Hempel and Simonson 1999). Reddington formally defined two necessary conditions for bond immunization as follows:

● The first derivative of the assets with respect to the interest rate (r) should be equal to the first derivative of the liabilities with respect to r. That is changes in the assets are offset by changes in the liabilities:

$$W(r) = A(r) - L(r), \qquad \textbf{(1)}$$

where, $W(r)$ is the wealth or the net present value of the cash flows, $A(r)$ is the present value of the assets, and $L(r)$ is the present value of liabilities at the same point in time. Redington's initial assumption is that $A(r) = L(r)$ (De la Grandville 2000). This is also called an exact match of assets and liabilities (Fogler). Differentiating the net value of the cash flows in equation **(1)** with respect to the interest rate yields the first Redington condition:

$$\frac{dA}{dr} = \frac{dL}{dr} \text{ or } \frac{dA}{dr} - \frac{dL}{dr} = 0 \qquad \textbf{(2)}$$

● The second derivative of the assets with respect to the interest rate should be greater than the second derivative of the liabilities with respect to *r*, so that *W(r)* remains positive within any interval of change *dr*:

$$\frac{d^2A}{dr^2} > \frac{d^2L}{dr^2} \text{ or } \frac{d^2A}{dr^2} - \frac{d^2L}{dr^2} > 0 \qquad \textbf{(3)}$$

A bond's value is thus shielded or "immunized" from interest rate changes if both Reddington conditions are met.

3. Interest rate and inflation risk

According to the standard Fisher equation, the return on a risk-free investment includes the real rate of interest (*r'*) and the expected short-term rate of inflation (*i*):

$$(1+r) = (1+r')(1+i) \qquad \textbf{(4)}$$

Fogler (1984) examines the effect of both of these risks on the investor's wealth evaluated at any horizon point prior to maturity and denoted as H. A^H is the present value of the assets, L^H is the present value of the liabilities and W^H is the net present value of the cash flows, all of them evaluated at time H. Here A^H and L^H can be regarded as assets and liabilities that will be acquired/incurred at time H:

$$W^H = A^H - L^H =$$

$$= P(i,r') + Y(i,r') - L(i,r') \quad =$$

$$= \sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)} + \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t} - L(i,r') \qquad \textbf{(5)}$$

,where $P(i,r')$, $Y(i,r')$, and $L(i,r')$ is the present value of the bond, the reinvestment income and the value of the liabilities, evaluated at the time horizon. We can extend Fogler's results to continuous compounding, where we will use De la Grandville's derivations of bond prices and duration:

$$B(i) = \sum_{0}^{T} C \exp\{-rt\} \qquad \textbf{(6)}$$

where C is the bond's cash flow, and r is the continuously compounded interest rate. For the sake of simplicity, we assume a flat yield curve, where the interest rate is the same regardless of maturity.

If we use the above continuous compounding model our wealth equation **(5)** becomes:

$$W^H = A^H - L^H =$$

$$\sum_{H}^{M} C\exp\{r'(t-H) - i(t-H)\} + F\exp\{r'(M-H) - i(M-H)\} + \sum_{0}^{H} C\exp\{r'(H-t) + i(H-t)\} - L(i, r') \qquad \textbf{(7)}$$

Because duration and convexity measures as originally developed by actuaries, were designed to address immunization of risk free debt securities, only western European and Japanese government bonds with the highest debt ratings are be considered in this paper. Brady bonds and other developing countries obligations are not to be taken into consideration - since they have a considerable portion of default risk - although they present an interesting field for a research. On the other hand, German Pfandbrief Papier - which are highly liquid, very low-risk German mortgage bonds are relevant to this analysis. Fabozzi (2001) has observed that there have been no cases of default in Pfandbrief Papier since their first issue a century ago.

4. Bond immunization and exchange rate risk

Historically there have existed different exchange rate "regimes" with the "managed-float" regime currently favored by the German, Japanese, and American central bankers. Accordingly, currencies are generally allowed to trade against one another within some (broadly defined) range before central bankers intervene to try to reestablish exchange rate parity. In recent times, some governments intervene only in extreme circumstances while others follow a more active policy. Unstable economies and third world countries usually peg their currency against a major currency (e.g., the dollar) or a basket of currencies which are correlated with their economic circumstances. In most cases these countries are pressured to follow monetary policies that are similar to those of the pegged currency. This can create serious problems for the satellite country but in reality it is a better choice than hyperinflation and serious financial crises (Gandolfo 2001).

To that extent, we will now examine how exchange rate changes influence international bonds prices. We define the value of an international bond (e.g., a German government bond) at the horizon time, according to the following equation:

$$V = eB \tag{8}$$

where V is the market value of the bond in US dollars, e is the euro-dollar exchange rate and B is the market value of the bond denominated in euros (at the horizon moment):

$$B_H = \sum_{t=H+1}^{M} C[1+r]^{-(t-H)} + F[1+r]^{-(M-H)} \tag{9}$$

Using the standard Fisher equation **(9)** we can define a rate to discount bond cash flows such that:

$$V = e\left(\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)}\right) \tag{10}$$

By the above discussion we already know that our net worth is equal to the difference between the total present value of the assets and liabilities at the horizon, where the assets are a combination of the present value of the bond and the reinvestment coupon income at the time horizon. If we define e_1 and e_2 to be the exchange rates between the domestic legal tender and two different foreign currencies then:

$$W^H = A^H - L^H =$$

$$e_1[\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)} + \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}] - e_2 L(i,r') \quad \textbf{(11.A)}$$

Using continuous compounding we can transform **(11.A)** into the following equation:

$$e_1[\sum_H^M C\exp\{-r'(t-H) - i(t-H)\} + F\exp\{-r'(M-H) - i(M-H)\} + \sum_0^H C\exp\{r'(H-t) + i(H-t)\}] - e_2 L(i,r')$$

(11.B)

Equations **(11.A)** and **(11.B)** show how both assets and liabilities depend on the exchange rate. Incorporating the first Redington condition from equation **(2)** yields the following results:

$$\frac{\partial A^H}{\partial i} - \frac{\partial L^H}{\partial i} = 0 \quad \textbf{(12.A)}$$

$$\frac{\partial A^H}{\partial r'} - \frac{\partial L^H}{\partial r'} = 0 \quad \textbf{(12.B)}$$

$$\frac{\partial A^H}{\partial e} - \frac{\partial L^H}{\partial e} = 0 \quad \textbf{(12.C)}$$

Boundary condition: $A^H = L^H$.

The notation here is: *r'*-real rate of interest, *i*-inflation and *e*-exchange rate. Note that the boundary condition $A^H = L^H$ is equivalent to the Redington's initial assumption, as described above. If we then differentiate partially with respect to the real rate and inflation we get results similar to Fogler's, however now the exchange rate has been incorporated into the model. Assets and liabilities become functions of three variables: the real rate of interest, inflation and exchange rates. Here the additional effect of the exchange rate can be investigated and certain useful results can be derived as to whether immunization is possible. Notice that in the case where assets and liabilities are both functions of the same exchange rate (i.e., $e = e_1 = e_2$) partially differentiating **(11.A)** and **(11.B)** with respect to *e* produces a situation where the present value of the assets should exactly match the present value of the liabilities, which is a confirmation of our boundary condition. If, however, we are given different exchange rates, then we can partially differentiate only with respect to one of the given exchange rates. As we will show below, this situation poses a serious challenge to our model.

5. A correction of a Fogler's result

Since the next section of our analysis deals exclusively with calculus based applications, it should be mentioned here that one should be very careful when caring out (partial) differentiation. Fogler made the following mistake when calculating the partial derivative of the reinvestment income with respect to the inflation rate holding all else constant (p. 253): $\frac{\partial}{\partial i}Y(i) = \frac{\partial}{\partial i}\sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t} \underset{Fogler}{=} \sum_{t=1}^{H}(H-t)C[(1+r')(1+i)]^{H-t-1}$, while the actual result is $\frac{\partial}{\partial i}\sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t} = \sum_{t=1}^{H}(H-t)C[(1+r')]^{H-t}[(1+i)]^{H-t-1}$. We give the derivation of the actual result in the next paragraph.

Suppose we are given the reinvested income expression from equation **(5)**: $Y(i,r') = \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}$ and that we are asked to differentiate it partially with respect to the inflation rate (*i*). How do we proceed? One way to do this is to separate the relevant variable - (*i*) so that differentiation is simplified. Thus:

$Y(i,r') = \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t} = \sum_{t=1}^{H} C[(1+r')]^{H-t}[(1+i)]^{H-t}$. Now it is easier to see that the only pertinent part of the last equation is $[(1+i)]^{H-t}$ since we are treating all else constant. Hence:

$$\frac{\partial}{\partial i}Y(i) = \frac{\partial}{\partial i}\sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t} = \frac{\partial}{\partial i}\sum_{t=1}^{H} C[(1+r')]^{H-t}[(1+i)]^{H-t} =$$
$$= \sum_{t=1}^{H}(H-t)C[(1+r')]^{H-t}[(1+i)]^{H-t-1}.$$

6. Model results

6.1 First-Order Conditions: Duration

This section of the analysis will focus on the implications of introducing exchange rate risk into the model. In particular, we examine four cases where assets and liabilities are denominated in similar and differing currencies. We intentionally ignore hedging opportunities to underscore specific conditions where immunization is theoretically possible and where it is precluded.

***Case I*:** Assets and liabilities are denominated in the same foreign currency ($e = e_1 = e_2$).

Using the exchange rate wealth equation **(11.A)** and partially differentiating it with respect to the inflation rate, (using **(12.A)**):

$$\frac{\partial W^H}{\partial i} = \frac{\partial A^H}{\partial i} - \frac{\partial L^H}{\partial i} = 0$$

$$\frac{\partial}{\partial i}\{e_1[\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)} + \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}]\} - \frac{\partial}{\partial i}\{e_2 L(i,r')\} = 0$$

After simplifying the above expression we get:

$$(-t+H)\sum_{t=H+1}^{M} C[(1+r')]^{-(t-H)}[(1+i)]^{-(t-H+1)}+(-M+H)F[(1+r')]^{-(M-H)}(1+i)]^{-(M-H+1)}+(H-t)\sum_{t=1}^{H} C[(1+r')]^{H-t}[(1+i)]^{H-t-1}=\frac{\partial L(i,r')}{\partial i}$$

(13.1.1)

Similarly using **(11.A)** and **(12.B)** we obtain:

$$\frac{\partial W^H}{\partial r'}=\frac{\partial A^H}{\partial r'}-\frac{\partial L^H}{\partial r'}=0$$

$$\frac{\partial}{\partial r'}\{e_1[\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)}+F[(1+r')(1+i)]^{-(M-H)}+\sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}]\}-\frac{\partial}{\partial r'}\{e_2 L(i,r')\}=0$$

After simplification:

$$(-t+H)\sum_{t=H+1}^{M} C[(1+r')]^{-(t-H+1)}[(1+i)]^{-(t-H)}+(-M+H)F[(1+r')]^{-(M-H+1)}(1+i)]^{-(M-H)}+(H-t)\sum_{t=1}^{H} C[(1+r')]^{H-t-1}[(1+i)]^{H-t}=\frac{\partial L(i,r')}{\partial r'}$$

(13.1.2)

Differentiating partially with respect to the exchange rate yields us the following result (using **(11.A)** and **(12.C)**):

$$\frac{\partial W^H}{\partial e}=\frac{\partial A^H}{\partial e}-\frac{\partial L^H}{\partial e}=0$$

$$\frac{\partial}{\partial e}\{e[\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)}+F[(1+r')(1+i)]^{-(M-H)}+\sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}]\}-\frac{\partial}{\partial e}\{eL(i,r')\}=0,$$

which is equivalent to:

$$\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)}+F[(1+r')(1+i)]^{-(M-H)}+\sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}=L(i,r')$$

The above result can also be written as:

$A^H=L^H$ **(13.1.3)**

This condition shows how economic net worth is immunized against changes in any of the risk factors. In particular, exchange rate risk is precluded when both assets and liabilities are denominated in the sane currency.

Case II: Assets are denominated in a foreign currency and liabilities are denominated in the domestic currency.

In this situation, $e_2 = 1$ because our liabilities are denominated in the domestic currency and thus only assets are exposed to exchange rate risks.

Using **(11.A)** and **(12.A)** we get the following result:

$$e_1[(-t+H)\sum_{t=H+1}^{M} C[(1+r')]^{-(t-H)}[(1+i)]^{-(t-H+1)} + (-M+H)F[(1+r')]^{-(M-H)}(1+i)]^{-(M-H+1)} + (H-t)\sum_{t=1}^{H} C[(1+r')]^{H-t}[(1+i)]^{H-t-1}] = \frac{\partial L(i,r')}{\partial i}$$

(13.2.1)

Similarly using **(11.A)** and **(12.B)** we obtain:

$$e_1[(-t+H)\sum_{t=H+1}^{M} C[(1+r')]^{-(t-H+1)}[(1+i)]^{-(t-H)} + (-M+H)F[(1+r')]^{-(M-H+1)}(1+i)]^{-(M-H)} + (H-t)\sum_{t=1}^{H} C[(1+r')]^{H-t-1}[(1+i)]^{H-t}] = \frac{\partial L(i,r')}{\partial r'}$$

(13.2.2)

Differentiating partially with respect to the exchange rate e_1 yields us the following result (using **(11.A)** and **(12.C)**):

$$\frac{\partial W^H}{\partial e_1} = \frac{\partial A^H}{\partial e_1} - \frac{\partial L^H}{\partial e_1} = 0$$

$$\frac{\partial}{\partial e_1}\{e_1[\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)} + \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}]\} - \frac{\partial}{\partial e_1}\{L(i,r')\} = 0$$

$$\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)} + \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t} = 0$$

which is equivalent to $A^H = 0$ **(13.2.3)**

Here we end up with an impossible condition where $A^H=0$. We can conclude that immunization is impossible because, although the third equation of the system **(12)** is valid in the mathematical sense, it implies only a condition of extreme negative economic value.

Case III. The assets and liabilities are denominated in different currencies ($e_1 \neq e_2$).

Using **(11.A)** and **(12.A)** we get the following result:

$$e_1[(-t+H)\sum_{t=H+1}^{M} C[(1+r')]^{-(t-H)}[(1+i)]^{-(t-H+1)} + (-M+H)F[(1+r')]^{-(M-H)}(1+i)]^{-(M-H+1)} + (H-t)\sum_{t=1}^{H} C[(1+r')]^{H-t}[(1+i)]^{H-t-1}] = e_2 \frac{\partial L(i,r')}{\partial i}$$

(13.3.1)

Similarly using **(11.A)** and **(12.B)** we obtain:

$$e_1[(-t+H)\sum_{t=H+1}^{M} C[(1+r')]^{-(t-H+1)}[(1+i)]^{-(t-H)} + (-M+H)F[(1+r')]^{-(M-H+1)}(1+i)]^{-(M-H)} + (H-t)\sum_{t=1}^{H} C[(1+r')]^{H-t-1}[(1+i)]^{H-t}] = e_2 \frac{\partial L(i,r')}{\partial r'}$$

(13.3.2)

Differentiating partially first with respect to e_1 and then e_2 yields us the following two results (using **(11.A)** and **(12.C)**):

Result A: $\frac{\partial W^H}{\partial e_1} = \frac{\partial A^H}{\partial e_1} - \frac{\partial L^H}{\partial e_1}$

$$\frac{\partial}{\partial e_1}\{e_1[\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)} + \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}]\} - \frac{\partial}{\partial e_1}\{e_2 L(i,r')\} = 0$$

$$\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)} + \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t} = 0$$

that is equivalent to $A^H=0$. **(13.3.3.A)**

Result B: $\frac{\partial W^H}{\partial e_2} = \frac{\partial A^H}{\partial e_2} - \frac{\partial L^H}{\partial e_2}$

$$\frac{\partial}{\partial e_2}\{e_1[\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)} + \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}]\} - \frac{\partial}{\partial e_2}\{e_2 L(i,r')\} = 0$$

that is equivalent to $L(i,r') = 0$. **(13.3.3.B)**

Again, as in case II. there is a contradiction such that $A^H=0$ or $L^H=0$. Therefore, we can conclude that in this case immunization is not possible.

***Case IV*:** The assets are denominated in the domestic currency and the liabilities denominated in foreign currency. Thus we have e_1=1 and the liabilities vary with e_2.

Using **(11.A)** and **(12.A)** we get the following result:

$$(-t+H)\sum_{t=H+1}^{M} C[(1+r')]^{-(t-H)}[(1+i)]^{-(t-H+1)} + (-M+H)F[(1+r')]^{-(M-H)}(1+i)]^{-(M-H+1)} + (H-t)\sum_{t=1}^{H} C[(1+r')]^{H-t}[(1+i)]^{H-t-1} = e_2\frac{\partial L(i,r')}{\partial i}$$

(13.4.1)

Similarly using **(11.A)** and **(12.B)** we obtain:

$$(-t+H)\sum_{t=H+1}^{M} C[(1+r')]^{-(t-H+1)}[(1+i)]^{-(t-H)} + (-M+H)F[(1+r')]^{-(M-H+1)}(1+i)]^{-(M-H)} + (H-t)\sum_{t=1}^{H} C[(1+r')]^{H-t-1}[(1+i)]^{H-t} = e_2\frac{\partial L(i,r')}{\partial r'}$$

(13.4.2)

Differentiating partially with respect to the exchange rate e_2 yields us the following result (using **(11.A)** and **(12.C)**):

$$\frac{\partial W^H}{\partial e_2} = \frac{\partial A^H}{\partial e_2} - \frac{\partial L^H}{\partial e_2}$$

$$\frac{\partial}{\partial e_2}\{[\sum_{t=H+1}^{M} C[(1+r')(1+i)]^{-(t-H)} + F[(1+r')(1+i)]^{-(M-H)} + \sum_{t=1}^{H} C[(1+r')(1+i)]^{H-t}]\} - \frac{\partial}{\partial e_2}\{e_2 L(i,r')\} = 0$$

, which is equivalent to:

$$L(i,r') = 0 \qquad \textbf{(13.4.3)}$$

Again, as in case II. and case III. we reach a contradiction (L^H=0). We can also conclude that immunization is impossible here.

Since we have a system of equations, every equation in the system should be true if we want to conclude that the system is true. Thus, the above results lead us to the conclusion that the first order condition might hold true only in case I, where both the assets and the liabilities are denominated in the same foreign currency. In the other three cases more complex financial instruments than simple bonds should be used to satisfy the 1st Reddington Conditions of bond immunization. Such instruments can be obtained by creating a portfolio consisting of bonds and options, bonds and futures, or a combination of these. These cases can also be generalized to situations of continuous compounding instead which are presented in Appendix A.

6.2 Second Order Conditions: Convexity

The second order conditions will be such that the elasticity of the assets is greater than that of the liabilities:

$$\frac{\partial^2 A^H}{\partial r'^2} - \frac{\partial^2 L^H}{\partial r'^2} > 0 \qquad \textbf{(14.A)}$$

$$\frac{\partial^2 A^H}{\partial i^2} - \frac{\partial^2 L^H}{\partial i^2} > 0 \qquad \textbf{(14.B)}$$

$$\frac{\partial^2 A^H}{\partial e^2} - \frac{\partial^2 L^H}{\partial e^2} > 0 \qquad \textbf{(14.C)}$$

If we take the second derivatives of the exchange rate equations **(11.A)** and **(11.B),** we see that the above stated system of equations **(14.A)**, **(14.B), (14.C)** can never hold true, simply because in all of the four cases both $\frac{\partial^2 A^H}{\partial e^2}$ and $\frac{\partial^2 L^H}{\partial e^2}$ have values of zero. Thus, trying to satisfy **(14.C)** we reach the contradiction $0 > 0$. The implication is that immunization is not possible if we have an international portfolio where the assets or the liabilities, or both of them vary with exchange rates.

7. Conclusion

While exchange rate risk can be formally incorporated into the basic bond valuation model, there do not exist satisfactory theoretical conditions for simple bond portfolio immunization where assets or liabilities, or both are denominated in foreign currencies. We have showed that an investor dealing with international bonds or such portfolios cannot fully immunize his position against adverse changes in real interest, inflation and exchange rates. Only in case I, where both of the assets and liabilities are denominated in the same foreign currency, partial protection of the portfolio might be achieved without the help of more complex financial instruments. However, for the second Redington condition to hold true in case I, more complex assets should be used. A clear limitation of this analysis is that we have not introduced hedging opportunities nor have we explored empirical tests of these models which are all left for future investigations.

Appendix A

1. Both assets and liabilities varying with the same exchange rate.

2. Assets denominated in a foreign currency and liabilities denominated in the domestic medium of exchange.

3. Assets and liabilities denominated in different foreign currencies.

4. Assets denominated in the domestic currency and liabilities denominated in a foreign currency.

We consider three sub cases (A, B, C) for each of the above four cases. They correspond to **(11.B)** (the exchange rate wealth equation using continuous compounding) when differentiated using the system of equations – **(12.A)**, **(12.B)** and **(12.C)**.

1. A

$$(H-t)\sum_{H}^{M} C(t)\exp\{-r'(t-H)-i(t-H)\}+(H-M)F\exp\{-r'(M-H)-i(M-H)\}+(H-t)\sum_{0}^{H} C(t)\exp\{r'(H-t)+i(H-t)\}=\frac{\partial L(i,r')}{\partial i}$$

(15.1.1)

1. B

$$(H-t)\sum_{H}^{M} C(t)\exp\{-r'(t-H)-i(t-H)\}+(H-M)F\exp\{-r'(M-H)-i(M-H)\}+(H-t)\sum_{0}^{H} C(t)\exp\{r'(H-t)+i(H-t)\}=\frac{\partial L(i,r')}{\partial r'}$$

(15.1.2)

1. C

$$\sum_{H}^{M} C(t)\exp\{-r'(t-H)-i(t-H)\}+F\exp\{-r'(M-H)-i(M-H)\}+\sum_{0}^{H} C(t)\exp\{r'(H-t)+i(H-t)\}=L(i,r')$$

, which corresponds to:

$$A^{H}=L^{H} \qquad \textbf{(15.1.3)}$$

2. A

$$e_1[(H-t)\sum_{H}^{M}C(t)\exp\{-r'(t-H)-i(t-H)\}+(H-M)F\exp\{-r'(M-H)-i(M-H)\}+(H-t)\sum_{0}^{H}C(t)\exp\{r'(H-t)+i(H-t)\}]=\frac{\partial L(i,r')}{\partial i}$$

(15.2.1)

2. B

$$e_1[(H-t)\sum_{H}^{M}C(t)\exp\{-r'(t-H)-i(t-H)\}+(H-M)F\exp\{-r'(M-H)-i(M-H)\}+(H-t)\sum_{0}^{H}C(t)\exp\{r'(H-t)+i(H-t)\}]=\frac{\partial L(i,r')}{\partial r'}$$

(15.2.2)

2. C

$$\sum_{H}^{M}C(t)\exp\{-r'(t-H)-i(t-H)\}+F\exp\{-r'(M-H)-i(M-H)\}+\sum_{0}^{H}C(t)\exp\{r'(H-t)+i(H-t)\}=0$$

, which is equivalent to:

$A^H=0$ **(15.2.3)**

This, as we mentioned above, is not feasible.

3. A

$$e_1[(H-t)\sum_{H}^{M}C(t)\exp\{-r'(t-H)-i(t-H)\}+(H-M)F\exp\{-r'(M-H)-i(M-H)\}+(H-t)\sum_{0}^{H}C(t)\exp\{r'(H-t)+i(H-t)\}]=e_2\frac{\partial L(i,r')}{\partial i}$$

(15.3.1)

3. B

$$e_1[(H-t)\sum_{H}^{M}C(t)\exp\{-r'(t-H)-i(t-H)\}+(H-M)F\exp\{-r'(M-H)-i(M-H)\}+(H-t)\sum_{0}^{H}C(t)\exp\{r'(H-t)+i(H-t)\}]=e_2\frac{\partial L(i,r')}{\partial r'}$$

(15.3.2)

3. C

$$\sum_{H}^{M}C(t)\exp\{-r'(t-H)-i(t-H)\}+F\exp\{-r'(M-H)-i(M-H)\}+\sum_{0}^{H}C(t)\exp\{r'(H-t)+i(H-t)\}=0$$

(15.3.3.A)

Or

L (i, r’)=0 **(15.3.3.B)**

, which is equivalent to:

A^H=0 or L^H=0 and as we mentioned above this is contradiction.

4. A

$$(H-t)\sum_{H}^{M} C(t)\exp\{r'(t-H)-i(t-H)\}+(H-M)F\exp\{r'(M-H)-i(M-H)\}+(H-t)\sum_{0}^{H} C(t)\exp\{r'(H-t)+i(H-t)\}=e_2\frac{\partial L(i,r')}{\partial i}$$

(15.4.1)

4. B

$$(H-t)\sum_{H}^{M} C(t)\exp\{r'(t-H)-i(t-H)\}+(H-M)F\exp\{r'(M-H)-i(M-H)\}+(H-t)\sum_{0}^{H} C(t)\exp\{r'(H-t)+i(H-t)\}=e_2\frac{\partial L(i,r')}{\partial r'}$$

(15.4.2)

4. C

L (i, r’)=0 (not possible) **(15.4.3)**

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