Valuing American Style Options by Least Squares Methods

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Abstract

We investigate the finite sample performance of some recent Monte Carlo estimators under different market scenarios. We find that the accuracy and efficiency of these estimators are remarkable, even when more exotic financial instruments are considered. Finally, we extend the Glasserman and Yu (2004b) methodology to price Asian Bermudan options and basket options.

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1. Introduction

The enormous growth of structured products over the last years has transformed the option-pricing theory in one of the most dynamic areas in finance. It has now become essential for the business and finance industry to extend the traditional option pricing methodologies in order to price more exotic type of options with American style features.[1]

It is well known that pricing American options is fundamentally an optimal stopping problem. In fact while with an European options a payoff can only occur when the option expires, with American options a payoff can occur at any time during the option life, including the expiration date. This feature gives rise to a free boundary problem.

Different methodologies have been proposed for pricing American style options. The binomial method introduced by Cox, Ross, and Rubinstein (1979) is still the most widely used valuation model, because it is easy to implement and produces reasonably accurate results. However, it has a major drawback, namely the fact that a high degree of accuracy can only be achieved with a high number of time steps, which reduces the computational speed and results in considerable efficiency costs. Furthermore, it is very difficult, if not impossible to price derivatives such as the ones analysed in the second part of this paper.

Monte Carlo methods to price American style options seem to be now an active research area, the reason is mainly due to its suitability to price path dependent options and be employed to solve high dimensional problems (see for example Xiaoquin, 2001).

Recently, Longstaff and Schwartz (2001) suggest using Least squares approximation to approximate the option price on the continuation region and Monte Carlo methods to calculate the option value. They call this technique least square Monte Carlo approach (LSM). They also show that their methodology can be extended to price path dependent options and solve high dimensional problems. In their empirical analysis, the authors apply their method to price a wide class of derivatives instruments, and show that it yields the best combination of price accuracy and efficiency amongst the several methodologies they consider. However, apart from Proposition 1 and 2, very little is said about the statistical properties of the proposed estimator. Furthermore proofs of Proposition 1 and 2 do not consider the effect of anthitetic techniques.

Recently Clement et al (2002) address some of the above issues by undertaking a theoretical analysis of the LSM estimator, and show that the option price converges, in limit, to the true option price. However, the theoretical proof in Clement et al (2002) might have at least three limitations. Firstly, again, they do not consider the effect of anthitetic techniques in their proof. Secondly, their proof is based on a sequential limit rather than joint limit.[2] The latter might be rather odd on a practical ground. Finally, the LSM-estimator assumes constant volatility and this assumption is maintained when asymptotic convergence is proved.

Glasserman et al (2004a) consider the limitations in Clement et al (2002) and prove convergence of the LSM estimator as the number of paths and the number of basis functions increase together. They consider two cases for the underlying

[1] In fact, American style types of options are embedded in various structured product instruments.

[2] That is, they show convergence in two stages. First they fix the number of replications and let the number of basis functions go to infinite. Thereafter, they fix the number of basis functions and let the number of replications go to infinite.

stochastic process driving the stock prices, namely standard Brownian motion and Geometric Brownian motion and show that, in the geometric Brownian motion case, the number of paths must increase very fast with respect to the number of basis function.

Glasserman et al (2004b) show that under certain assumptions, the weighted Monte Carlo Estimator (WME) is equivalent to regression estimators and can produce less disperse estimates of the option price. Yet proofs of convergence of this class of estimators assume constant volatility and furthermore no finite-sample proof of the convergence of the proposed estimators is provided in that study.

In the last few years, the LSM and the WME estimators have raised great interest amongst practitioners working in the finance industry. The main reason for this, as mentioned above, is their suitability for pricing very exotic financial instruments. However, despite the notoriety of these methods, proofs of convergence of these estimators are still limited and based on different assumptions. The general objective of this paper is to analyse the finite sample approximation of the two estimators above. To achieve this objective, we allow for different market scenarios, different number of basis functions and different number of replications and measure the performance of these estimators by estimating the standard error of the regression. In this respect, this study extends previous empirical studies such as Xiooquin et al (2001), Morenos and Novas (2001) and Stentoft (2004).[3]

As shown in Glasserman and Yu (2004a) the choice of the basis function used in the regression is very important since (uniform) convergence of the option price to the true price can only be guaranteed if the polynomial basis spans the "true optimum". To address this issue, we consider different basis functions and suggest a possible "optimal polynomial basis".

Finally, our study is the first empirical study on the WME as in Glasserman et al (2004b) and it also extends that methodology to price options on a maximum of n assets and Bermudan-Asian options. We show that even when more difficult payoffs are considered, the WME estimator produces reasonably accurate prices.

[3] These studies consider finite sample approximations of the LS (2001) estimator, when both the number of basis functions and paths increase, but they do not consider different market scenarios. Since, as we mentioned above, these estimators raise great practical interest, we believe it might be of some interest to see how well they perform under different market conditions.

2. The LS (2001) Monte Carlo Method

In the following sections we briefly review the methodologies analysed in this paper and critically assess some of their relevant assumptions.

We consider a probability space $(\Omega, \mathrm{A}, \mathrm{P})$ and its discrete filtration $(F_i)_{i=0,\ldots,n}$, with n being an integer. Define with $X_0, X_1, \ldots X_n$ a R^d valued Markov chain representing the state variable recording all the relevant information on the price of a certain underlying asset. If an American option is exercised at time i, with $i = 0,1,\ldots,n$, its payoff is given by the following sequence of square integrable random variables $(\mathrm{P}^*_i)_{i=0,1,\ldots n}$, and we assume that the latter is an adapted process on F_i, such that for $i = 0,1,\ldots n$, $\mathrm{P}^*_i = \Theta(i, X_i)$, for some functions $\Theta(i,.)$. The focus here is on computing $\sup_{\tau\in\Gamma_{0,n}} E\mathrm{P}_\tau$, where Γ_i denote the randomised stopping times.

Define with $V_i(x)$, $x \in R^d$, the value of an option if exercised at time i under the state x. The value of this option within a dynamic programming framework can be written as:

$$V_i(x) = \sup_{\tau\in\Gamma} E[\Theta_\tau(X_\tau) \mid X_i = x] \qquad (1)$$

with

$$V_n(x) = \Theta_n(x) \qquad (2)$$

$$V_i(x) = \max\{\Theta_n(x), E[(V_{i+1}(X_{i+1}) \mid X_i = x]\} \quad (3)$$

Since the objective here is to determine V_0, this reduces to (i) approximating the conditional expectations in (3) in some ways, and (ii) obtaining a numerical (Monte Carlo) evaluation of the latter.

Since the payoff above is a square integrable variable, then $V_i(.)$ will be a function spanning the Hilbert space and we may consider approximating the conditional expectations in (3) by the orthogonal projection on the space generated by a finite number of basis functions ϕ_{ik}, $i = 1,\ldots,n$ and $k = 0,1,\ldots,K$, such that

$$V_n(x) = \phi_n(x) \tag{5}$$

$$V_i(x) = \max\{\phi_i(x), E[(V_{i+1}(X_{i+1}) \mid X_i = x]\} \tag{6}$$

Therefore the conditional expectations can now be approximated, for all i, by a simple regression approach:

$$V_{i+1}(X_{i+1}) \equiv \sum_{k=0}^{K} c_{i,k}\phi_{i,k}(X_i) + \varepsilon_{i+1} \tag{7}$$

In this way we have transformed the complex problem in (6) in a simpler one, requiring the estimation of the $K+1$ coefficients in (7). This can be easily achieved by using a least square approach as follows:

$$(c_{i0}^*, c_{i1}^*, ..., c_{iK}^*) = E[(V_{i+1}^*(X_{i+1})\phi_i(X_i)'][\phi_i(X_i)\phi_i(X)']^{-1} \tag{8}$$

One may also considering replacing (6) with its continuation value as:

$$\mathrm{K}_i(x) = E\{\max\{\varphi_{i+1}(X_{i+1}),[(\mathrm{K}_{i+1}(X_{i+1})\} \mid X_i = x]\} \tag{9}$$

with the final condition $\mathrm{K}_n(x) = 0$

And the least squares coefficients φ^* now solve

$$(\varphi_{i0}^*, \varphi_{i1}^*, ..., \varphi_{iK}^*) = E[(\mathrm{K}_{i+1}^*(X_{i+1})\varphi_i(X_i)'][\varphi_i(X_i)\varphi_i(X)']^{-1} \tag{10}$$

Once we have solved the conditional expectation problem by using a finite number of basis functions as in (7), the next step consists in evaluating it numerically. This can be done by simulating j paths of the Markov process X_i^j, with $j = 1,...,m$, and calculating, at each stopping time τ, recursively, the payoff $\mathrm{P}^j{}^*_{i\tau} = \phi(\tau, X_i^j)$.

Remark 1. Note that in (8) we assume that the coefficients are estimated using a sample, therefore to account for sample bias we have included residuals in (7). This allows equation (7) to be an exact approximation of the conditional expectations in (3)[4].

Assumption 1. For all $i = 0,...,n-1$, (i) $E(\varepsilon_{i+1} | X_i) = 0$, (ii) $E[(\phi_i(X_i)\phi_i(X_i)'] = 0$.

Remark 2. Assumption 1 (i) requires ε_{i+1} being strictly exogenous for all i. Proofs are given in Grosserman and Yu (2004a) and Longstaff and Schwartz (2001). Assumption 1 (ii) is slightly trickier and we shall come back to this issue in the next sections.

Under Assumption 1 (i) and (ii) as $m \to \infty$ $V_i^* | m |= V_i$, where V_i^* is the estimated option price, or also, once fixed m, that $\lim_{K\to\infty} E(\mathrm{P}_{i\tau} * | m | F_i) = E(P*_i | F_i)$, Clement et al (2002).

Remark 3.

Clement et al (2002) analytically show convergence of the LS estimator. They also prove that the rate of convergence of the LS estimator is tight. However, their proof is a sequential one and not a joint proof. Theoretically, the LS method can be seen as consisting of two stages. First, the evaluation of the conditional expectations, that can be regarded as an optimal stopping problem on τ. Second the estimation of the option value. Therefore, we can, first, check the convergence of the value function as the number of basis functions increases, for a given m. Finally, we can fix K, and check the convergence as m increases. We have already discussed about the limitations of this approach.

[4] Clement et al (2002) do not consider sample bias and assume that the coefficients can be exactly estimated by least square methods.

3. The Grasserman and Yu (2004b) Method

In equation (7) we approximated the conditional expectation by using a finite number of current basis functions (that is $\phi_i(X_i)$). However one would expect the option price at time $i+1$to be more closely correlated with the basis function $\phi_{i+1}(X_{i+1})$rather than $\phi_i(X_i)$ .Glasserman and Yu (2004b) develop a method based on weighted Monte Carlo simulation where the conditional expectation in (3) is approximated by $\phi_{i+1}(X_{i+1})$rather than $\phi_i(X_i)$. They show that their Monte Carlo scheme has a regression representation given by:

$$\hat{V}_{i+1}(X^j_{i+1}) = \sum_{k=0}^{K} \varpi_{ik}\phi_{i+1,k}(X^j_{i+1}) + \hat{\varepsilon}_{i+1} \qquad (11)$$

and the least squares estimator is, therefore, given by:

$$(\varpi^*_{i0}, \varpi^*_{i1}, ..., \varpi^*_{iK}) = [\sum_{j=1}^{m} (V^*_{i+1}(X^j_{i+1})\phi_{i+1}(X^j_{i+1})'][\sum_{j=1}^{m} \phi_{i+1}(X^j_{i+1})\phi_{i+1}(X^j_{i+1})']^{-1} \qquad (12)$$

Provided that Assumption 2 below holds:

Assumption 2. $E(\phi_{i+1}(X_{i+1}) \mid X_i) = \phi_i(X_i)$, for all i.

Remark 4. Grasserman and Yu (2004b) consider the following assumptions on $\hat{\varepsilon}_{i+1}$: (i), $E(\hat{\varepsilon}_{i+1}(\phi_{i+1}(X_{i+1}) - \phi_i(X_i)) = 0$, (ii) $E(\hat{\varepsilon}_{i+1} \mid X_i) = 0$. These assumptions together with Assumption 2 (i.e. martingale property of the basis function) guarantee that $V_i = \hat{V}_i$.

Glasserman and Yu (2004b) call this method regression later, since it involves using basis functions $\phi_{i+1}(X_{i+1})$. On the other hand, they call the LS (2001) method regression now since it uses basis functions $\phi_i(X_i)$.

4. Valuing American Put Options

In this section we apply the methodologies above to price American style put options. Although there are other applications of the Longstaff and Schwartz (2001) Monte Carlo method to price American style options (see for example Xiooquin et al, 2001 and Moreno and Novas, 2001), none of them has considered such a wide set of parameters as the present study does. We consider a wide range of parameters for strike, maturity and volatility. Allowing for a wide range of volatility parameters is particular important since both the methodologies assume constant volatility. Therefore it might be informative to investigate the performance of these methods when volatility changes. Finally, at the best of our knowledge this is the first study to empirically assess the method proposed in Glasserman and Yu (2004b).

As in Longstaff and Schwartz (2001), we implement the methodologies by using antithetic techniques (for example, 50,000 simulations plus 50,000 antithetic). As a benchmark, we consider the Binomial method with 10,000 time steps. For each set of parameters we report option prices obtained by using different basis functions and different polynomial order. We consider polynomial of second, third up to the fifth order. We have also considered different number of replications, that is 30,000, 50,000 and 150,000 paths. Results are available upon request. For each set of parameters, we calculate the bias with respect to a Binomial price. We also calculate the absolute best price across the four.

As we pointed out above (see equations (1)-(3) and (4)-(5)), the methods appear to suffer from two main biases. First there is a bias in the approximation of the conditional expectation and consequently in the estimation of the optimal stopping strategy. This will lead to underestimate the true option price. This bias should tend to zero as the number of basis function increases. There is also a second bias that results from using a sample to estimate the price of the option. This should tend to zero as the number of replication increases[5]. We present crude estimates of the first bias, by calculating the absolute error[6], and of the second bias by calculating the standard errors. Combining the increase in the number of basis with a correct increase in the

[5] There is also a bias introduced by replacing the American option problem with a series of Bermudan options. Glasserman et Yu (2004a) use martingales basis to deal with this issue.
[6] A better measure of bias based on mean square error (MSE) was proposed in Glasserman and Yu (2004a). The analysis of the MSE and martingales basis as suggested in Glasserman and Yu (2004a) is the aim of a companion paper.

number of replication, and assuming that the polynomial spans the "true optimum", then convergence of the estimated price to the true price should be guaranteed[7].

We only report prices when in the money options are considered since this should be the most interesting case to consider here. However we have also considered at the money and out of the money options. Results are available upon request[8].

Price estimates of the option using the LS (2001) method and associate bias are reported in Table 1, Table 3, and Table 5. We note that, in general, the bias approaches to zero as the number of basis increase, and convergence seems to be faster when simple exponential basis are used than in other cases[9]. This result contrasts with what reported in Longstaff and Schwartz (2001) since they find Laguerre basis over-performing the others. Generally three or four basis are sufficient to eliminate the bias when exponential basis are used.

Recently, Glasserman and Yu (2004b) suggest a Monte Carlo method based on martingales basis. We repeat exactly what we have done with the LS (2001) method with the Glasserman and Yu (2004b). To apply this method we need to satisfy the martingales assumption on the basis functions as requested by Assumption 2. This is rather demanding particularly when applying it to solve high dimensional problems. In the Geometric Brownian motion case, we specify the following basis function for ϕ_k, $\phi_k(t) = e^{kW(t)-k^2t/2}$, where W is a standard Brownian motion process. Results are reported in Table 7.

In general, the method, in terms of bias, produces rather accurate prices. The error is well inside a bid-ask price for similar traded options. However, the accuracy does not seem to be as good as with the LS (2001) method. Furthermore the bias does not seem to drop to zero as fast as with the LS (2001) method and it tends to be particularly relevant when volatility changes. Probably this is due to the martingale assumption made on the basis functions. In fact the latter might become too restrictive in this context (see Glasserman and Yu, 2004b).

[7] As we have already mentioned this is one of the objectives of this study.
[8] However, the first source of bias should not be very relevant in these particular cases.
[9] As pointed out in Glasserman and Yu (2004ab) the inclusion of too many bases functions may cause an over-fitting of the true price and consequently we may observe a non-monotone convergence. This problem seems not to be relevant in our empirical analysis probably due to the sufficiently low number of basis considered.

We now consider the second source of bias (i.e. sample bias) by calculating standard errors. These were obtained by replicating the option price one hundred times. Standard errors for the LS (2001) method are reported in Tables 2, Table 4, and Table 6. In general, standard errors are very low, and much lower than what reported in Longstaff and Schwartz (2001)[10]. Apart few cases with Hermite basis, standard errors do not vary noticeably across the different number of basis. This might suggest that the method, regardless the basis used, tends to produce estimates that are not very dispersed. In general, standard errors with exponential methods are at least as low as standard errors obtained with Laguerre basis. We confirm a significant reduction of the standard errors when the numbers of replications increase from 50,000 to 100,000[11]. Standard errors for the Glasserman and Yu (2004b) method follow a similar pattern. This result might imply that Theorem 1 in Glasserman and Yu (2004b) also holds when a multi-periods framework is considered.

6. Valuing American Bermuda Asian Options

In the following sections we consider the previous methodologies when pricing more complex options such as American Asian options and options written on a maximum of n assets. It is with this type of options that these methodologies become useful and interesting[12].

As in Longstaff and Schwartz (2001), we consider pricing an American Asian option having also an initial lockout period. Pricing these types of options is rather demanding since they contain two features. Firstly, the option features a lock out period. Secondly, it is path dependent since its value depends not just on the price of the underlying asset but also on the arithmetic average price. Therefore the continuation value function in this case will depend not just on the price of the underlying asset but also on the average price.

In order to use the options prices reported in Longstaff and Schwartz (2001) as benchmark, we consider an American call option that after an initial lock out period

[10] However as pointed out in Rasmussen (2002) it is likely that standard errors in that study were computed without using variance reduction techniques.
[11] Results available upon request.
[12] In fact standard applications on American put options may well be covered by standard Binomial methods. However, as noted in Caporale and Cerrato (2005), even in these standard applications there is a trade-off between price accuracy and number of time steps to be considered when using Binomial

of three months can be exercised at any time up to maturity T. We assume $T = 2$ years. The average is the (continuous) arithmetic average of the underlying stock price calculated over the lock out period. As in Longstaff and Schwartz (2001) the strike price is $100, the risk free rate of interest 0.06 and volatility 0.20. We use anthitetic technique plus different scenarios for the stock prices (S) and assume 200 steps for both stock price and average.

Results are reported in Table 9. Longstaff and Schwartz (2001) use the first eight Laguerre basis functions in their application[13] and 50,000 replications. We do exactly the same and report results in the 5th column of Table 9. Qualitatively, our results support those reported in Tables 3 of Longstaff and Schwartz (2001). However, we also change the number of replications and the basis functions. We note that by increasing the number of replications we obtain prices that are, generally, very close to the ones reported in the Table 3 of Longstaff and Schwartz (2001), with a slightly preference for Laguerre basis.

In Table 10, we extend the Glasserman and Yu (2004b) method to price American Asian options. We use Hermite basis (ϕ_{KH}) to satisfy Assumption 2 as follows, $f_k \phi_{KH}$, with $f_k = t^{k/2}$. The method seems to underestimate the true option price.[14]

7. Valuing American Basket Options

Finally, we consider solving high dimensional problems. We consider an American call option written on a maximum of five risky assets paying a proportional dividend. We assume that each asset return is independent from the other. Once again, we use the same parameter specifications as in Longstaff and Schwartz (2001) and Broadie and Glaserman (1997) such that we can use prices reported in these papers as benchmark.

Broadie and Glasserman (1997) use stochastic mesh to solve this type of problems and report confidence interval for the option prices. However, the

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methods. Caporale and Cerrato (2005) propose a dynamic programming approach based on binomial probabilities and they show that their approach is more efficient than an accelerated Binomial method.

[13] That is first two Laguerre basis on the stock price and average plus their cross products including an intercept.

computational time in that study appears to be a serious matter since it takes about 20 hours on a 266-Pentium 2 to achieve an accurate price.

Longstaff and Schwartz (2001) use their LS-Monte Carlo method and estimate a price with the same level of accuracy as in Brodie and Glasserman (1997), but in only two minutes. However, that study only considers Hermite polynomials.

We consider three different options with initial stock prices of 90,100, and 110 respectively[15]. The assets pay a 10% proportional dividend, the strike price of the option is 100, the risk free rate of interest is 10% and volatility is 20%. Confidence intervals reported in Brodie and Glasserman (1997) are [16.602, 16.710] when the initial asset value is 90; [26.101, 26.211] with initial asset value of 100, and finally [36.719, 36.842] when the initial value is 110.

The option prices in Longstaff and Schwartz (2001) are respectively, 16.657, 26.182, and 36.812 and they all fall within the Broadie and Glasserman `s confidence interval above. Using a 300MHz Pentium II processor the authors claim that they are able to achieve that accuracy in only 2 minutes.

We note that regardless of the number of replications or basis functions used, we achieve, in all the cases, a price that follows within the above interval. We also calculated the average time for the computation of the price by using a Pentium 4 1.6 Hz-M. The average time is about 8.2 seconds. The gain in terms of time seems to be greater when exponential basis are used rather than Hermite basis.

Finally, we extend the Glasserman and Yu (2004b) method to price these types of options. Once again we use Hermite basis as in the previous section to satisfy Assumption 2. We note that option prices estimates fall within the Broadie and Glasserman `s confidence interval when 50,000 paths are considered. In terms of computational speed, the method seems to be more demanding than the LSM method.

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[14] Note that we take the prices reported in Longstaff and Schwartz (2001) Table 3, column 6th, as true prices. However, prices reported in Table 9, column 7th, above are a good approximation of the ones reported in Longstaff and Schwartz (2001).

8.Conclusions

From an academic and even a practitioner`s point of view, pricing American options still remains an interesting research area, particularly when Monte Carlo techniques are used. This is due mainly to the flexibility of this method when used to solve high dimensional problems.

Recently, Longstaff and Schwartz (2001) and Glasserman and Yu (2004b) propose two methods based on simulations to price American options. Proofs of convergence of the LSM estimator are given in Clement et al (2002) and Egloff (2004), while proofs of convergence of the WME, as in Glasserman and Yu (2004b), are given in the same paper. However, those proofs are based on a set of assumptions that, in some cases might result rather restrictive.

The general objective of this paper is to undertake a large empirical analysis to investigate the finite sample approximations of these estimators. We consider different market scenarios, use different polynomial basis, and number of basis functions. Therefore, in this respect, the present study differs substantially from the previous empirical ones in the area since the latter are rather limited.

Other objectives are (i) estimating the bias induced by these methodologies, (ii) suggesting "optimal" basis functions. Finally, this is the first empirical study on the estimator proposed in Glasserman and Yu (2004b) and it extends that method to price exotic type of American options, and solve high dimensional problems.

Overall, we find that the option price estimate provided by these estimators is economically acceptable regardless the type of option considered. Large part of the sample bias can be eliminated with an acceptable number of replications (i.e. 100,000). However, in general, the LS (2001) estimator performs the best. With this estimator we found simple exponential basis functions to over-performing the others. Therefore, in practical applications, we recommend using this basis. In general, a number of basis equals to three, 100,000 replication and exponential basis appear to be sufficient for the method to eliminate the bias.

Two issues on the agenda for future research in this area. Firstly, finding martingales basis for the most common used basis functions, such as to satisfy Assumption 2 in this paper and consequently implementing the Glasserman and Yu (2004b) methodology. In fact, it is evident, from the empirical results in sections 7-8 that this is particularly important when the Glasserman and Yu (2004b) method is applied to high dimensional problems. This might also explain the weaker performance of that methodology with respect to the LSM methodology in that context. Secondly, considering the method proposed in Glasserman and Yu (2004a) and the measure of bias proposed in that paper to investigate finite sample approximations.

[15] Note that we assume the initial value of the asset to be the same for all the five stocks in the basket.

References

Abadir, K., and J., R., Magnus,2005, "Matrix Algebra", Volume 1, Cambridge University Press.

Brodie, M. and P., Glasserman, 1997, "A stochastic Mesh Method for Pricing High Dimensional American Options, Working Papers, Columbia University.

Caporale, M., G., and M., Cerrato, 2005, "Valuing American Put Options Using Chebyshev Polynomial Approximation" London Metropolitan University, Centre for International Capital Markets, Discussion Paper No.05-3

Clement, E., D., Lamberton and P., Protter, 2002, "An Analysis of a Least Squares Regression Method for American Option Pricing", Finance and Stochastic, 6, 449-471.

Cox, J., C., S.A.Ross and M.,Rubinstein, 1979, "Option Pricing:A Simplified Approach", Journal of Financial economics, 7, 229-263.

Egloff, D., 2003, "Monte Carlo Algorithms for Optimal Stopping and Statistical Learning", Working Paper, Zurich Kantonalbank, Zurich Switzerland.

Glasserman, P., and B., Yu, 2004a, "Number of Paths Versus Number of Basis Functions in American Option Pricing", The Annals of Applied Probability, Vol.14.

Glasserman, P., and B., Yu, 2004b, "Simulation for American Options:Regression Now or Regression Later?", in H., Niederreiter editor, Monte Carlo and Quasi Monte Carlo Methods, 2002.

Longstaff, F., A., and E., S., Schwartz, 2001,"Valuing American Options by Simulation: A Simple Least-Squares Approach", The Review of Financial Studies, Vol. 14, No.1.

Mitchell, O., H., 1881, "Note on Determinants of Powers", Amer.J.Math. 4, pp.341-344.

Moreno, M., and F., J., Novas, 2001,"On the Robustness of the Least Square Monte Carlo (LSM) for Pricing American Derivatives" Working Paper no.543, Department of Economics and Business, Universitat Pompeu Fabra.

Werther, T., 1993, "Generalized Vandermode Determinants over the Chebyshev Basis", International Computer Science Institute, working paper No.20.

Rasmussen, N., S., 2002, "Efficient Control Variates for Monte Carlo Valuation of American Options", Aarhus School of Business, Working Papers in Finance.

Stentoft, L., 2004, "Assessing the Least Squares Monte Carlo Approach to American Option Valuation", Review of Derivatives research, 7(3).

Xiaoqing, L., and L., K., Guan, 2001, "A Parsimonious Monte Carlo Method for Pricing American Style Options", Centre for Financial Engineering, Working Paper 10, National University of Singapore.

Table 1: 100,000 Paths with Exponential Basis. LS (2001) Method

Strike	Maturity	Volatility	Order				Binomial	Difference				Best
			2	3	4	5		2	3	4	5	
45	**0.0833**	**0.2**	4.996	4.9963	4.998	4.9962	5	0.00	0.00	0.00	0.00	0.00
45	**0.3333**	**0.2**	5.075	5.0863	5.077	5.0873	5.087	-0.01	0.00	-0.01	0.00	0.00
45	**0.5833**	**0.2**	5.267	5.2447	5.259	5.2594	5.265	0.00	-0.02	-0.01	-0.01	0.00
45	**0.0833**	**0.3**	5.052	5.0578	5.057	5.0435	5.06	-0.01	0.00	0.00	-0.02	0.00
45	**0.3333**	**0.3**	5.686	5.6985	5.713	5.7077	5.706	-0.02	-0.01	0.01	0.00	0.00
45	**0.5833**	**0.3**	6.219	6.2375	6.25	6.2307	6.244	-0.02	-0.01	0.01	-0.01	0.01
45	**0.0833**	**0.4**	5.265	5.2807	5.29	5.2942	5.286	-0.02	-0.01	0.00	0.01	0.00
45	**0.3333**	**0.4**	6.508	6.5139	6.508	6.5104	6.51	0.00	0.00	0.00	0.00	0.00
45	**0.5833**	**0.4**	7.364	7.3827	7.387	7.3748	7.383	-0.02	0.00	0.00	-0.01	0.00

Table 2: Standard Errors LS (2001) Method and Exponential Basis.

Strike	Maturity	Volatility	Order			
			2	3	4	5
45	**0.0833**	**0.2**	0.0017	0.0018	0.0022	0.0019
45	**0.3333**	**0.2**	0.0047	0.0040	0.0050	0.0053
45	**0.5833**	**0.2**	0.0065	0.0065	0.0060	0.0068
45	**0.0833**	**0.3**	0.0038	0.0019	0.0036	0.0039
45	**0.3333**	**0.3**	0.0065	0.0063	0.0061	0.0054
45	**0.5833**	**0.3**	0.0070	0.0076	0.0073	0.0065
45	**0.0833**	**0.4**	0.0052	0.0046	0.0046	0.0047
45	**0.3333**	**0.4**	0.0073	0.0063	0.0067	0.0062
45	**0.5833**	**0.4**	0.0074	0.0075	0.0083	0.0084

Table 3: 100,000 Paths with Laguerre Basis. LS (2001) Method

Strike	Maturity	Volatility	Order				Binomial	Difference				Best
			2	3	4	5		2	3	4	5	
45	**0.0833**	**0.2**	4.9964	4.997	4.9965	4.997	5	0.00	0.00	0.00	0.00	0.00
45	**0.3333**	**0.2**	5.0674	5.0748	5.0785	5.082	5.087	-0.02	-0.01	-0.01	0.00	0.00
45	**0.5833**	**0.2**	5.174	5.2112	5.2454	5.2518	5.265	-0.09	-0.05	-0.02	-0.01	0.01
45	**0.0833**	**0.3**	5.0134	5.0455	5.0469	5.0461	5.0597	-0.05	-0.01	-0.01	-0.01	0.01
45	**0.3333**	**0.3**	5.4138	5.4977	5.6559	5.6699	5.7059	-0.29	-0.21	-0.05	-0.04	0.04
45	**0.5833**	**0.3**	5.6949	5.7385	5.9944	6.1951	6.2438	-0.55	-0.51	-0.25	-0.05	0.05
45	**0.0833**	**0.4**	5.1981	5.2465	5.2443	5.2545	5.2863	-0.09	-0.04	-0.04	-0.03	0.03
45	**0.3333**	**0.4**	5.063	5.2776	5.69	6.1073	6.5096	-1.45	-1.23	-0.82	-0.40	0.40
45	**0.5833**	**0.4**	6.2169	6.1817	6.5022	6.9545	7.3829	-1.17	-1.20	-0.88	-0.43	0.43

Table 4: Standard Errors LS (2001) Method and Laguerre Basis

Strike	Maturity	Volatility	Order 2	3	4	5
45	**0.0833**	**0.2**	0.0017	0.0018	0.0019	0.0016
45	**0.3333**	**0.2**	0.0049	0.0049	0.0049	0.0048
45	**0.5833**	**0.2**	0.0060	0.0062	0.0060	0.0063
45	**0.0833**	**0.3**	0.0043	0.0035	0.0045	0.0046
45	**0.3333**	**0.3**	0.0070	0.0063	0.0060	0.0065
45	**0.5833**	**0.3**	0.0065	0.0068	0.0069	0.0067
45	**0.0833**	**0.4**	0.0050	0.0046	0.0047	0.0048
45	**0.3333**	**0.4**	0.0070	0.0072	0.0073	0.0072
45	**0.5833**	**0.4**	0.0092	0.0082	0.0093	0.0091

Table 5: 100,000 Paths with Hermite Basis. LS (2001) Method

Strike	Maturity	Volatility	2	3	4	5	Binomial	Difference 2	3	4	5	Best
45	**0.0833**	**0.2**	4.9963	4.9963	4.9963	4.9963	5	-0.0037	-0.0037	-0.0037	-0.0037	0.0037
45	**0.3333**	**0.2**	5.0823	5.0830	5.0841	5.0849	5.087	-0.0047	-0.0040	-0.0029	-0.0021	0.0021
45	**0.5833**	**0.2**	5.2538	5.2630	5.2635	5.2633	5.265	-0.0112	-0.0020	-0.0015	-0.0017	0.0015
45	**0.0833**	**0.3**	5.0514	5.0519	5.0534	5.0544	5.0597	-0.0083	-0.0078	-0.0063	-0.0053	0.0053
45	**0.3333**	**0.3**	5.6871	5.6941	5.6958	5.6982	5.7059	-0.0188	-0.0118	-0.0101	-0.0077	0.0077
45	**0.5833**	**0.3**	6.2185	6.2321	6.2363	6.2322	6.2438	-0.0253	-0.0117	-0.0075	-0.0116	0.0075
45	**0.0833**	**0.4**	5.2560	5.2842	5.2834	5.2831	5.2863	-0.0303	-0.0021	-0.0029	-0.0032	0.0021
45	**0.3333**	**0.4**	6.4865	6.5020	6.5056	6.4930	6.5096	-0.0231	-0.0076	-0.0040	-0.0166	0.004
45	**0.5833**	**0.4**	7.3657	7.3792	7.3870	7.3819	7.3829	-0.0172	-0.0037	0.0041	-0.0010	0.001

Table 6: Standard Errors LS (2001) Method and Hermite Basis

Strike	Maturity	Volatility	2	3	4	5
45	**0.0833**	**0.2**	0.0018	0.0019	0.0017	0.0050
45	**0.3333**	**0.2**	0.0052	0.0047	0.0050	0.0041
45	**0.5833**	**0.2**	0.0054	0.0058	0.0056	0.0055
45	**0.0833**	**0.3**	0.0036	0.0038	0.0037	0.0050
45	**0.3333**	**0.3**	0.0057	0.0054	0.0060	0.0060
45	**0.5833**	**0.3**	0.0071	0.0063	0.0072	0.0050
45	**0.0833**	**0.4**	0.0047	0.0041	0.0044	0.0028
45	**0.3333**	**0.4**	0.0066	0.0065	0.0069	0.0039
45	**0.5833**	**0.4**	0.0078	0.0079	0.0082	0.0056

Table 7: 100,000 Paths, GY (2004b) Method

Strike	Maturity	Volatility	Order 2	3	4	5	Binomial	Difference 2	3	4	5	Best
45	**0.083**	**0.2**	5	4.996	4.997	4.996	5	-0.0041	-0.0037	-0.0034	-0.0037	0.0034
45	**0.333**	**0.2**	5.08	5.092	5.08	5.086	5.087	-0.0051	0.0049	-0.0068	-0.001	0.001
45	**0.583**	**0.2**	5.26	5.248	5.252	5.259	5.265	-0.0062	-0.0172	-0.0126	-0.006	0.006
45	**0.083**	**0.3**	5.05	5.053	5.053	5.052	5.06	-0.0091	-0.0068	-0.007	-0.0076	0.0068
45	**0.333**	**0.3**	5.69	5.678	5.699	5.688	5.706	-0.018	-0.028	-0.0066	-0.0184	0.0066
45	**0.583**	**0.3**	6.22	6.229	6.212	6.228	6.244	-0.0269	-0.0146	-0.0315	-0.0155	0.0146
45	**0.083**	**0.4**	5.27	5.278	5.286	5.278	5.286	-0.0182	-0.0087	0.0001	-0.0083	0.0001
45	**0.333**	**0.4**	6.49	6.491	6.493	6.498	6.51	-0.0206	-0.0182	-0.0171	-0.0114	0.0114
45	**0.583**	**0.4**	7.37	7.367	7.352	7.347	7.383	-0.0161	-0.0157	-0.0309	-0.0357	0.0157

Table 8: Standard Errors GY(2004b) Method

Strike	Maturity	Volatility	Order 2	3	4	5
45	**0.083**	**0.2**	0.0019	0.0018	0.0022	0.0019
45	**0.333**	**0.2**	0.0050	0.0040	0.0050	0.0053
45	**0.583**	**0.2**	0.0059	0.0065	0.0060	0.0068
45	**0.083**	**0.3**	0.0046	0.0019	0.0036	0.0040
45	**0.333**	**0.3**	0.0070	0.0063	0.0061	0.0054
45	**0.583**	**0.3**	0.0059	0.0076	0.0073	0.0065
45	**0.083**	**0.4**	0.0054	0.0046	0.0046	0.0047
45	**0.333**	**0.4**	0.0073	0.0063	0.0067	0.0062
45	**0.583**	**0.4**	0.0084	0.0075	0.0083	0.0084

Table 9: American Bermudan Asian Options (LS 2001 Method)

	Expon.	Lagu.	Expon.	Lagu.	Expon.	Lagu.
S	**m = 30,000**		**M = 50,000**		**m = 75,000**	
80	0.9211	0.9218	0.937	0.945	0.9322	0.957
90	3.084	3.108	3.211	3.314	3.222	3.312
100	7.492	7.522	7.699	7.855	7.744	7.874
110	13.23	13.89	14.198	14.234	14.355	14.501
120	20.09	21.2	22.091	22.122	22.198	22.311

Note: S is the stock price, m the number of simulations, while Expon. and Lagu. are respectively exponential and Laguerre basis functions.

Table 10: American Bermudan Asian Options (GY, 2004b Method)

	Hermite		
S	**m = 30,000**	**M = 50,000**	**m = 75,000**
80	0.923	0.932	0.942
90	3.189	3.311	3.167
100	7.521	7.544	7.563
110	13.82	14.122	14.311
120	20.01	21.633	22.011

Note: S is the stock price, m the number of simulations.

Table 11: American Basket Option (LS 2001 Method)

	S	Expon.	Hermite	Expon.	Hermite	Expon.	Hermite
		m = 30,000		m = 50,000		m = 75,000	
	90	16.6895	16.677	16.6555	16.6171	16.6632	16.642
	100	26.1758	26.1744	26.1708	26.1033	26.0804	26.12
	110	36.7697	36.7642	36.7826	36.7482	36.8214	36.748
Average time		8.21	8.5	12.23	12.53	15.22	15.3

Note: S is the stock price, m the number of simulations, while Expon. and Hermite are respectively exponential and Hermite basis functions.

Table 12: American BasketOptions (GY, 2004b Method)

			Hermite	
	S	m = 30,000	m = 50,000	m = 75,000
	90	16.5935	16.623	16.4759
	100	26.0789	26.181	25.6802
	110	36.286	36.71	36.1032
Average time		9.05	13.3	16.9

Note: S is the stock price, m the number of simulations.