“Credit Cycles” in a OLG Economy with Money and Bequest (Preliminary draft)

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1. Introduction

In the 1997 paper, KM assume that the economy is populated by Infinitely Lived Agents (ILA). In this paper we develop an extended version of the original Kiyotaki and Moore’ s model (“Credit Cycles” Journal of Political Economy, vol. 105, no 2, April 1997) (hereafter KM) using an overlapping generation structure instead of the assumption of infinitely lived agents adopted by the authors. In each period the population consists of two classes of heterogeneous interacting agents, in particular: a financially constrained young agent (young farmer), a financially constrained old agent (old farmer), an unconstrained young agent (young gatherer), an unconstrained old agent (old gatherer). By assumption each young agent is endowed with one unit of labour. Heterogeneity is introduced in the model by assuming that each class of agents use different technologies to produce the same non durable good. If we study the effect of a technological shock it is possible to demonstrate that its effects are persistent over time in fact the mechanism that it induces is the reallocation the durable asset (“land”)among agents.

As in KM we develop a dynamic model in which the durable asset is not only an input for production processes but also collateralizable wealth to secure lenders from the risk of borrowers’default.

2. The environment

In each period there are four classes of agents. In order to simplify matters, we assume for the moment that there is only one (representative) agent per class. Therefore in t population consists of

- a financially constrained young agent (young farmer, YF)
- a financially constrained old agent (old farmer, OF)
- an unconstrained young agent (young gatherer, YG)
- an unconstrained old agent (old gatherer, OG).

A YF borrows from a YG. Being endowed with *inalienable human capital,* the former can get from the latter no more than the value of the collateralizable assets, i.e. the future value of the land he is currently owning: $b_t = \frac{q_{t+1}}{R} K_t^F$.

There are two types of goods, output ("fruit") and a non-reproducible asset ("land") whose total supply is fixed ($\bar{K}$). Output is produced by means of a technology which uses land and labour. By assumption each young agent is endowed with one unit of labour. Assuming that there is no disutility of labour, this endowment is supplied inelastically. By assumption farmers and gatherers have access to different technologies. The production function of the YF is: $y_t^F = \alpha K_{t-1}^F$ where y_t^F is output of the farmer in t, α is a positive technological parameter and K_{t-1}^F is land of the farmer in t-1. The production function of the YG is: $y_t^G = G\left(K_{t-1}^G\right) = G\left(\bar{K} - K_{t-1}^G\right)$ and $G(.)$ is increasing, concave and satisfies the Inada conditions.Both farmers and gatherers work when young and consume when old.

The paper is organized as follows. OLG-KM economy with money and bequest. Money is a reserve of value and a way of leaving a bequest. Heterogeneity...

3. The farmer/borrower

For simplicity we assume that the agent does not consume when young. Hence preferences are defined over consumption and bequest of the agent when old. Adopting a Cobb-Douglas specification of the utility function, preferences of the farmer are represented by

$$U^F = \gamma \ln c_{t,t+1}^F + (1-\gamma) \ln a_{t+1}^F \tag{1}$$

where $0 < \gamma < 1$; $c^F_{t,t+1}$ is consumption of the agent of generation t in t+1, a^F_{t+1} is bequest left by agent of generation t in t+1 to his child. [1]

The farmer maximizes utility subject to three constraints: the flow-of-funds (FF) constraint of the YF, the FF constraint of the OF and the financing constraint (see appendix A for the derivation).

The FF constraint of the YF in t (in real terms) is:

$$q_t \left(K^F_t - K^F_{t-1} \right) + m^F_{t,t} \leq b_t + a^F_t \tag{2}$$

where $q_t := \dfrac{Q_t}{P_t}$ is the real price of land,[2] $m^F_{t,t} := \dfrac{M^F_{t,t}}{P_t}$ are real money balances of the YF; b_t is credit and a^F_t is bequest, i.e. "wealth" inherited by the YF. According to 2, the "resources" of the YF, of internal or external origin (a^F_t and b_t respectively), can be employed to "invest", $q_t \left(K^F_t - K^F_{t-1} \right)$ – i.e. to change the landholding – and hold money balances.

The YF may be financially coinstrained. The financing constraint can be expressed as

$$b_t \geq \frac{q_{t+1}}{R} K^F_t \tag{3}$$

where $q_{t+1} := \dfrac{Q_{t+1}}{P_{t+1}}$ is the real price of land in the future, known in advance, and R is the real (gross) interest rate.

In t, the YF uses labour and land K^F_t to produce output which will become available in $t+1$: $y^F_{t+1} = \alpha K^F_t$.When old, the farmer's resources consist of output (produced when young) and money balances.These resources can be employed to repay the loan (if the YF were a borrower), consume and leave a bequest. Therefore the FF constraint of the OF in $t+1$ in real terms is:

$$c^F_{t,t+1} + a^F_{t+1} + b_t R \leq \alpha K^F_t + m^F_{t,t+1} \tag{4}$$

where $b_t R$ is the repayment of the loan; $m^F_{t,t+1} = \dfrac{M^F_{t,t+1}}{P_{t+1}}$are real money balances of the OF of generation t in t+1.

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[1] In the case of bequest, the notation is unambiguous. The bequest left by the agent of generation t in t+1 (i.e. when old) to his child can be denoted by $a^F_{t,t+1}$. The bequest received by agent of generation t+1 in t+1 (i.e. when young) is $a^F_{t+1,t+1}$. Of course the two notions amount to the same magnitude, i.e. $a^F_{t,t+1} = a^F_{t+1,t+1} = a^F_{t+1}$.

[2] Following KM, we purposedly adopt a notation reminiscent of Tobin's q.

The farmer maximizes 1 subject to 2 4 and 3. The Lagrangian is:

$$\begin{aligned}\mathcal{L} &= \gamma \ln c_{t,t+1}^F + (1-\gamma)\ln a_{t+1}^F + \lambda_t^F \left[b_t + a_t^F - q_t\left(K_t^F - K_{t-1}^F\right) - m_{t,t}^F\right] + \\ &+ \lambda_{t+1}^F \left[\alpha K_t^F + m_{t,t+1}^F - c_{t,t+1}^F - a_{t+1}^F - b_t R\right] + \phi_t \left[b_t - \frac{q_{t+1}}{R} K_t^F\right]\end{aligned}$$

from which one gets the $FOCS$:

$$\begin{aligned}(iF)\ \frac{\partial \mathcal{L}}{\partial c_{t,t+1}^F} &= 0 \Rightarrow \frac{\gamma}{c_{t,t+1}^F} = \lambda_{t+1}^F \\ (iiF)\ \frac{\partial \mathcal{L}}{\partial a_{t+1}^F} &= 0 \Rightarrow \frac{1-\gamma}{a_{t+1}^F} = \lambda_{t+1}^F \\ (iiiF)\ \frac{\partial \mathcal{L}}{\partial K_t^F} &= 0 \Rightarrow -\lambda_t^F q_t + \lambda_{t+1}^F \alpha = \phi_t \frac{q_{t+1}}{R} \\ (ivF)\ \frac{\partial \mathcal{L}}{\partial b_t} &= 0 \Rightarrow \phi_t = \lambda_{t+1}^F R - \lambda_t^F\end{aligned}$$

From *(iF)* and *(iiF)* follows that $\lambda_{t+1}^F \neq 0$. Therefore the FF constraint of the OF is binding.

Substituting *(ivF)* into *(iiiF)* and rearranging:

$$\frac{\mu_t}{\alpha - q_{t+1}} = \frac{\lambda_{t+1}^F}{\lambda_t^F} \tag{5}$$

where $\mu_t := q_t - \dfrac{q_{t+1}}{R}$ is the downpayment, i.e. the amount of internal finance the borrower has to accumulate in order to get a loan equal to $\dfrac{q_{t+1}}{R}$ (per unit of land he wants to purchase). The downpayment is always positive.

We assume that

(A1) $q_{t+1} \neq \alpha$. In this case, from 5 follows that $\lambda_t^F \neq 0$. Therefore the FF constraint of the YF is also binding.

Substituting 5 into the (ivF) one gets:

$$\phi_t = \left(R\frac{\mu_t}{\alpha - q_{t+1}} - 1\right)\lambda_t^F$$

We assume that

(A2) $q_{t+1} \neq \alpha/R$ or $R\mu_t \neq \alpha - q_{t+1}$. Therefore $\phi_t \neq 0$. Hence the financing constraint is binding:

$$b_t = \frac{q_{t+1}}{R} K_t^F \tag{6}$$

The farmer is the borrower. Therefore the gatherer is the lender.

Thanks to assumptions (A1) and (A2) all the constraints are binding. Substituting 3 into 2 and 4 respectively, in the case of binding constraints, one gets:

$$\mu_t K_t^F + m_{t,t}^F = a_t^F + q_t K_{t-1}^F \tag{7}$$

$$c_{t,t+1}^F + a_{t+1}^F = (\alpha - q_{t+1}) K_t^F + m_{t,t+1}^F \tag{8}$$

Equation 7 provides a different interpretation of the FF constraint of the young: the YF employs bequests and *collateralizable wealth* $q_t K_{t-1}^F$ to put aside internal finance and hold money balances.

From *(iF)* and *(iiF)* and the FF constraint 8 we derive the optimal consumption and the optimal bequest of the OF:

$$c_{t,t+1}^F = \gamma \left[(\alpha - q_{t+1}) K_t^F + m_{t,t+1}^F \right] \tag{9}$$

$$a_{t+1}^F = (1-\gamma) \left[(\alpha - q_{t+1}) K_t^F + m_{t,t+1}^F \right] \tag{10}$$

Thanks to the Cobb-Douglas specification of prefences, consumption and bequest are a fraction γ and $1-\gamma$ respectively of the *resources* available in t+1 to the OF, e_{t+1}^F,where

$$e_{t+1}^F = (\alpha - q_{t+1}) K_t^F + m_{t,t+1}^F$$

From the YF constraint in t 2one gets:

$$K_t^F = \frac{a_t^F + q_t K_{t-1}^F - m_{t,t}^F}{\mu_t} \tag{11}$$

Notice now that from 10 follows that the optimal bequest of the OF of generation t-1 in t is

$$a_t^F = (1-\gamma) \left[(\alpha - q_t) K_{t-1}^F + m_{t-1,t}^F \right]$$

where $m_{t-1,t}^F = \dfrac{M_{t-1,t}^F}{P_t}$are real money balances of the OF of generation t-1 in t. Substituting this expression into 11 and rearranging one gets:

$$K_t^F = \frac{\left[(1-\gamma)\alpha + \gamma q_t \right] K_{t-1}^F + (1-\gamma) m_{t-1,t}^F - m_{t,t}^F}{\mu_t} \tag{12}$$

which is the law of motion of the land of the farmer.

Money plays two different and contrasting roles with respect to landholding.On the one hand, *given the bequest,* the higher money of the young $m_{t,t}^{F}$, the lower landholding: In fact resources of the young (bequest and credit) can be devoted either to money or landholding; On the other hand, the higher money of the old $m_{t-1,t}^{F}$, the higher resources available to him and the higher bequest and landholding.

In the special and convenient case in which $m_{t-1,t}^{F} = m_{t,t}^{F} = m_{t}^{F}$,i.e. when the old farmer is exchanging money only with the young farmer (more on this in section...), $(1-\gamma)\, m_{t-1,t}^{F} - m_{t,t}^{F} = (1-\gamma)\, m_{t}^{F} - m_{t}^{F} = -\gamma m_{t}^{F}$:The second effect is offset by the first so that in the end the accumulation of money affects negatively land holding. Hence, recalling that $\mu_t = q_t - \dfrac{q_{t+1}}{R}$ 12 boils down to

$$K_t^F = \frac{\left[(1-\gamma)\,\alpha + \gamma q_t\right] K_{t-1}^F - \gamma m_t^F}{q_t - \dfrac{q_{t+1}}{R}} \tag{13}$$

4. The gatherer/lender

Following the same modelling route of the previous section, we assume that preferences of the gatherer are represented as follows

$$U^G = \gamma \ln c_{t,t+1}^G + (1-\gamma) \ln a_{t+1}^G \tag{14}$$

where $c_{t,t+1}^G$ and a_{t+1}^G are consumption and bequest of the OG.Being unconstrained from the financial point of view, the gatherer maximizes utility subject to the FF constraints of the YG and of the OG (see appendix A for the derivation).

The FF constraint of the YG in t is

$$m_{t,t}^G + b_t + q_t \left(K_t^G - K_{t-1}^G\right) \leq a_t^G \tag{15}$$

According to 15, the *resources* of the YG, of internal origin (a_t^G), can be employed to "invest", $q_t \left(K_t^G - K_{t-1}^G\right)$,i.e. to change the landholding, extend credit and hold money balances.

In t, the YG uses labour and land K_t^G to produce output which will become available in $t+1$: $y_{t+1}^G = G\left(K_t^G\right)$ .When old, the gatherer's resources consist of output (produced when young), the reimbursement of debt and money balances.These resources can be employed to consume and leave a bequest. Therefore the FF constraint of the OG in $t+1$ in real terms is:

$$c_{t,t+1}^{G} + a_{t+1}^{G} \leq G\left(K_t^G\right) + Rb_t + m_{t,t+1}^{G} \tag{16}$$

The gatherer maximizes 14 subject to 15 16. The Lagrangian is:

$$\begin{aligned}\mathcal{L} \;=\;& \gamma \ln c_{t,t+1}^{G} + (1-\gamma)\ln a_{t+1}^{G} + \lambda_t^G\left[a_t^G - q_t\left(K_t^G - K_{t-1}^G\right) - m_{t,t}^G - b_t\right] \\ & +\lambda_{t+1}^G\left[G\left(K_t^G\right) + Rb_t + m_{t,t+1}^G - c_{t,t+1}^G - a_{t+1}^G\right]\end{aligned}$$

$$\begin{aligned}(iG)\ \frac{\partial \mathcal{L}}{\partial c_{t,t+1}^G} &= 0 \Rightarrow \frac{\gamma}{c_{t,t+1}^G} = \lambda_{t+1}^G \\ (iiG)\ \frac{\partial \mathcal{L}}{\partial a_{t+1}^G} &= 0 \Rightarrow \frac{1-\gamma}{a_{t+1}^G} = \lambda_{t+1}^G \\ (iiiG)\ \frac{\partial \mathcal{L}}{\partial K_t^G} &= 0 \Rightarrow \lambda_{t+1}^G G'\left(K_t^G\right) = \lambda_t^G q_t \\ (ivG)\ \frac{\partial \mathcal{L}}{\partial b_t} &= 0 \Rightarrow \lambda_t^G = \lambda_{t+1}^G R\end{aligned}$$

From the FOCS it is clear that all the constraints are binding. Moreover, from (iiiG) and (ivG) follows $q_t = \dfrac{G'\left(K_t^G\right)}{R}$.Since the total amount of "land" is fixed by assumption, $K_t^F = \bar{K} - K_t^G$, $G'\left(K_t^G\right) = G'\left(\bar{K} - K_t^F\right)$. In nthe following, in order to save on notation, we will write $G'\left(\bar{K} - K_t^F\right) = g\left(K_t^F\right), g' = -G'' > 0$.Therefore we can write

$$q_t = \frac{g\left(K_t^F\right)}{R} \tag{17}$$

Since the financing constraint is binding, the amount of credit extended by the YG in t is equal to the present value of land in t+1: $b_t = \dfrac{q_{t+1}}{R} K_t^F$. Taking into account 6, from 16, *(iG)* and *(iiG)* we derive the optimal consumption and the optimal bequest of the OG:

$$c_{t,t+1}^G = \gamma\left[G\left(K_t^G\right) + q_{t+1}K_t^F + m_{t,t+1}^G\right] \tag{18}$$

$$a_{t+1}^G = (1-\gamma)\left[G\left(K_t^G\right) + q_{t+1}K_t^F + m_{t,t+1}^G\right] \tag{19}$$

Thanks to the Cobb-Douglas specification of prefences, consumption and bequest are a fraction γ and $1-\gamma$ respectively of the *resources* available in t+1 to the OG, e_{t+1}^G,where

$$e_{t+1}^G = G\left(K_t^G\right) + q_{t+1}K_t^F + m_{t,t+1}^G$$

5. Playing with constraints

Since the total amount of "land" is fixed by assumption, $K_t^F = \bar{K} - K_t^G$. Hence an increase of landholding for the farmer can occur only if there is a corresponding decrease of landholding for the gatherer: $K_t^F - K_{t-1}^F = -\left(K_t^G - K_{t-1}^G\right)$. Taking this fact into account, summing side by side the (binding) FF constraints of the young agents 2 and 15, i.e.

$$q_t\left(K_t^F - K_{t-1}^F\right) + m_{t,t}^F = b_t + a_t^F$$

$$m_{t,t}^G + b_t + q_t\left(K_t^G - K_{t-1}^G\right) = a_t^G$$

one gets

$$m_{t,t}^F + m_{t,t}^G = a_t^F + a_t^G \tag{20}$$

In words: the total amount of bequest obtained by the young agents is equal to the total amount of *money of the young agents*. In the special case in which bequest is left exclusively in terms of money, i.e. $m_{t,t}^F = a_t^F$ and $m_{t,t}^G = a_t^G$, investment of the farmer is financed exclusively by means of credit, i.e. $q_t\left(K_t^F - K_{t-1}^F\right) = b_t$

Updating 20 we obtain

$$m_{t+1,t+1}^F + m_{t+1,t+1}^G = a_{t+1}^F + a_{t+1}^G \tag{21}$$

Summing side by side the (binding) FF constraints of the old agents 4 and 16, i.e.

$$c_{t,t+1}^F + a_{t+1}^F + b_t R = y_{t+1}^F + m_{t,t+1}^F$$

$$c_{t,t+1}^G + a_{t+1}^G = y_{t+1}^G + Rb_t + m_{t,t+1}^G$$

yields

$$c_{t,t+1}^F + c_{t,t+1}^G + a_{t+1}^G + a_{t+1}^F = y_{t+1}^G + y_{t+1}^F + m_{t,t+1}^G + m_{t,t+1}^F \tag{22}$$

In words: aggregate output and real *money balances of the old agents* is equal to the sum of aggregate consumption and aggregate bequest.

We assume equilibrium on the goods market, i.e.

$$c_{t,t+1}^F + c_{t,t+1}^G = y_{t+1}^G + y_{t+1}^F \tag{23}$$

Taking 23 into account, 22 boils down to

$$m_{t,t+1}^F + m_{t,t+1}^G = a_{t+1}^F + a_{t+1}^G \tag{24}$$

i.e. the total amount of bequest left by the old agents is equal to the total amount of *money of the old agents*. From 21 and 24 we get

$$m^F_{t+1,t+1} + m^G_{t+1,t+1} = m^G_{t,t+1} + m^F_{t,t+1} \tag{25}$$

i.e the total amount of money of the young agents in t+1 must be equal to the total amount of money of the old agents of generation t in t+1.

6. Money trickles down

In our economy money "trickles down" from one period to the next and from one agent to the other. In fact a network of money transfers is taking place from the pool of monetary resources of one agent to the pool of another agent. In principle we distinguish three types of transfers:

- "within generations" or *horizontal transfers*, i.e. transfers between agents of the same generation but of different types (farmers and gatherers). Horizontal transfers are the monetary counterpart of transactions between agents of different types concerning goods (fruit) or land. Therefore they are motivated by agents' decisions to consume and invest, i.e. modify landholdings;

- "between generations" or *vertical transfers*, i.e. transfers between agents of different generations but of the same type (old and young agents).Vertical transfers coincides with bequests, which are motivated by intergenerational altruism.

- Government transfers, i.e. monetized subsidies to the old.

In order to describe the way in which money spreads in the economy, let's take a look at table 1. In each row we report the inflows and outflows which show up in the FF constraints of the agents in period t+1. The amount in the inflow cell is equal to the amount in the outflow cell. For instance, the first row represents the FF constraint of the YF in t+1 : $a^F_{t+1} = q_{t+1}\left(K^F_{t+1} - K^F_t\right) + m^F_{t+1,t+1} - b_{t+1}$.[3] In other words, we have rewritten in a suitable form equation 2. The third row is the sum of rows 1 and 2 (i.e. of the young agents), the sixth row is the sum of rows

[3] Matter of factly b_{t+1} represent an inflow for the YF in t+1. It shows up as a negative component in the outflow cell for convenience.

4 and 5 (i.e. of the old agents). Therefore, the table contains adapted equations 2 15 21 4 16 22.

	inf lows	outflows
YF	a_{t+1}^F	$q_{t+1}\left(K_{t+1}^F - K_t^F\right) + m_{t+1,t+1}^F - b_{t+1}$
YG	a_{t+1}^G	$-q_{t+1}\left(K_{t+1}^F - K_t^F\right) + m_{t+1,t+1}^G + b_{t+1}$
$\sum$	$a_{t+1}^G + a_{t+1}^F$	$m_{t+1,t+1}^G + m_{t+1,t+1}^F$
OF	$y_{t+1}^F + m_{t,t+1}^F$	$c_{t,t+1}^F + a_{t+1}^F + b_t R$
OG	$y_{t+1}^G + m_{t,t+1}^G$	$c_{t,t+1}^G + a_{t+1}^G - Rb_t$
$\sum$	$y_{t+1}^F + m_{t,t+1}^F + y_{t+1}^G + m_{t,t+1}^G$	$c_{t,t+1}^F + a_{t+1}^F + c_{t,t+1}^G + a_{t+1}^G$

Let's assume that $y_{t+1}^F - c_{t,t+1}^F = s_{t,t+1}^F > 0$, i.e. the OF consumes less than the output he has produced. In a sense he is "saving" the amount $s_{t,t+1}^F$. Market clearing on the goods market implies $s_{t,t+1}^G = -\left(c_{t,t+1}^G - y_{t+1}^G\right) = -s_{t,t+1}^F < 0$ i.e. the OG consumes more than the output he has produced. He is "dissaving" the amount $-\left(c_{t,t+1}^G - y_{t+1}^G\right)$. In other words, the OG has excess consumption $c_{t,t+1}^G - y_{t+1}^G$.

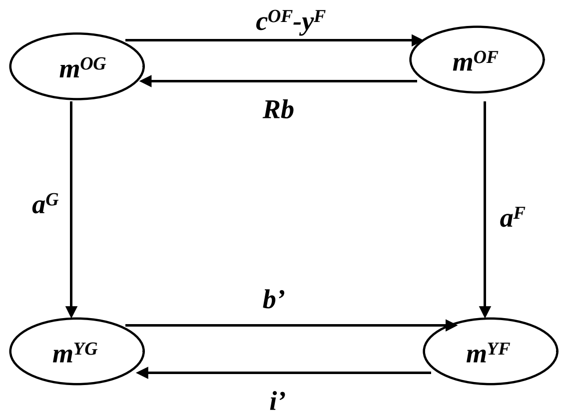
The OF sells $s_{t,t+1}^F$ units of output to the OG in order to let him consume in excess of his output. The OG pays this output by means of money. Therefore, after the transaction, the OF has money balances equal to $m_{t,t+1}^F + \left(c_{t,t+1}^G - y_{t+1}^G\right)$. This money is used to reimburse debt $b_t R$ and leave the bequest a_{t+1}^F. Accounts are consistent: In fact $c_{t,t+1}^G - y_{t+1}^G = y_{t+1}^F - c_{t,t+1}^F$ so that $m_{t,t+1}^F + y_{t+1}^F - c_{t,t+1}^F = a_{t+1}^F + b_t R$ which is the FF of the OF.

The YF receives a_{t+1}^F from his parents and b_{t+1} from the YG and employs these resources to invest, i.e. $q_{t+1}\left(K_{t+1}^F - K_t^F\right)$. The difference between bequest and credit on the one hand and investment on the other consists of money balances $m_{t+1,t+1}^F = a_{t+1}^F + b_{t+1} - q_{t+1}\left(K_{t+1}^F - K_t^F\right)$ that the farmer holds idle when young (since he does not consume) in order employ them when old to access consumption and leave a bequest. Notice that, since $a_{t+1}^F = (1-\gamma)\left[(\alpha - q_{t+1}) K_t^F + m_{t,t+1}^F\right]$,

$$m_{t+1,t+1}^F = [(1-\gamma)\alpha + \gamma q_{t+1}] K_t^F + (1-\gamma) m_{t,t+1}^F - \mu_{t+1} K_{t+1}^F$$

This equation links the money of the young farmer to the money of the old farmer in period t+1. It is 12 rewritten and updated. In figure 1 we represent the horizontal and vertical transfers taking places among private agents.

where m^{OG} stands for money of the OG and therefore coincides with $m_{t,t+1}^G$.Analogously $m^{OF} = m_{t,t+1}^F; m^{YG} = m_{t+1,t+1}^G; m^{YF} = m_{t+1,t+1}^F; c^{OG} - y^G = c_{t,t+1}^G - y_{t+1}^G; b' =$



$b_{t+1}; i' = q_{t+1}\left(K^F_{t+1} - K^F_t\right)$. Both excess consumption of the OG $c^{OG} - y^G$ and investment of the YF i' are positive by assumption.

Thanks to the Cobb-Douglas specification of the utility function,from the FOCs (iF)(iiF) and (iG)(iiG) one gets

$$c^i_{t,t+1} = \frac{\gamma}{1-\gamma} a^i_{t+1} \qquad i = F, G \tag{26}$$

Substituting 26 and the market clearing condition 23 into 22 we obtain

$$m^G_{t,t+1} + m^F_{t,t+1} = \frac{1-\gamma}{\gamma}\left(y^G_{t+1} + y^F_{t+1}\right) \tag{27}$$

Total real money balances are proportional to aggregate output. Equation 27 is a sort of quantity theory of money in this context.

In the following we will write:

$$m^F_{t,t+1} = \frac{1-\gamma}{\gamma\left(1+\sigma_{t+1}\right)}\left(y^G_{t+1} + y^F_{t+1}\right) \tag{28}$$

where $\sigma_{t+1} := \dfrac{m^G_{t,t+1}}{m^F_{t,t+1}}$.

In principle there is no reason to assume that the money the young agent has must be equal to the money of the old agent of the same class. This equality holds in the aggregate (see equation 25) but not for each class of agent. We assume however exactly this: $m^F_{t+1,t+1} = m^F_{t,t+1} = m^F_{t+1}$ and $m^G_{t+1,t+1} = m^G_{t,t+1} = m^G_{t+1}$ in order to simplify the analysis. Therefore we can write

$$m^F_t = \frac{1-\gamma}{\gamma(1+\sigma_t)}\left[\alpha K^F_{t-1} + G\left(\bar{K} - K^F_{t-1}\right)\right] \tag{29}$$

Moreover we assume the following $M^i_{t+1} = M^i_t\left(1+\eta^i_{t+1}\right)$, $i = F, G$. Therefore

$$m^i_{t+1} = m^i_t \frac{1+\eta^i_{t+1}}{1+\pi_{t+1}} \qquad i = F, G$$

The ratio of money of the gatherer to money of the farmer will be denoted by

$$\sigma_t = \frac{M^G_t}{M^F_t} = \frac{m^G_t}{m^F_t}$$

Hence

$$\sigma_{t+1} = \frac{M^G_{t+1}}{M^F_{t+1}} = \frac{m^G_{t+1}}{m^F_{t+1}} = \frac{m^G_t}{m^F_t}\frac{1+\eta^G_{t+1}}{1+\eta^F_{t+1}} = \sigma_t\frac{1+\eta^G_{t+1}}{1+\eta^F_{t+1}}$$

Equation ... is the law of motion of the composition of money. This ratio is constant iff $\eta^G_{t+1} = \eta^F_{t+1}$.

7. A simple welfare criterion

The utility function 1 is a logarithmic transformation of $U^F = \left(c^F_{t,t+1}\right)^\gamma \left(a^F_{t+1}\right)^{1-\gamma}$and therefore represents the same preferences. In order to compute indierct utility we plug opimal consumption and bequests into the function, obtaining $U^F = \gamma^\gamma(1-\gamma)^{1-\gamma} e^F_{t+1}$ i.e. indirect utility is increasing linearly with resources e^F_{t+1} of the old farmer where $e^F_{t+1} = (\alpha - q_{t+1}) K^F_t + m^F_{t,t+1}$.

Following the same reasoning, we can draw the conclusion that $U^G = \gamma^\gamma(1-\gamma)^{1-\gamma} e^G_{t+1}$ i.e. indirect utility is increasing linearly with resources e^G_{t+1} of the old gatherer where $e^G_{t+1} = G\left(K^G_t\right) + q_{t+1}K^F_t + m^G_{t,t+1}$.

A measure of society's welfare can be roughly be

$$U^S = U^F + U^G = \gamma^\gamma(1-\gamma)^{1-\gamma}\left(e^F_{t+1} + e^G_{t+1}\right)$$

i.e. society's well being is increasing with the sum of resources of the farmer and the gatherer.

Notice now that $e^F_{t+1} + e^G_{t+1} = \alpha K^F_t + G\left(K^G_t\right) + m^F_{t,t+1} + m^G_{t,t+1}$. The term $q_{t+1}K^F_t = Rb_t$, i.e. debt service, is a positive component of the gatherer's resources and a negative component of the farmer's resources. It cancels out in the aggregate.

Notice, moreover, that $m^G_{t,t+1} + m^F_{t,t+1} = \frac{1-\gamma}{\gamma}\left(y^G_{t+1} + y^F_{t+1}\right)$. Substituting this expression into the expression above and rearranging we get: $e^F_{t+1} + e^G_{t+1} = \frac{1}{\gamma}\left[\alpha K^F_t + G\left(K^G_t\right)\right]$. Hence

$$U^S = U^F + U^G = \left(\frac{1-\gamma}{\gamma}\right)^{1-\gamma}\left[\alpha K^F_t + G\left(K^G_t\right)\right]$$

i.e. society's well being is increasing with aggregate output.

Maximization of society's welfare therefore occurs when the marginal productivity of the farmer equals that of the gatherer's, i.e.

$$\alpha = G'\left(K^G_t\right) \tag{30}$$

i.e. when $K^G_t = G'^{-1}(\alpha) = \bar{K} - K^F_t$. Hence $K^F_f = \bar{K} - G'^{-1}(\alpha)$. In this case

$$y_f = y^G_f + y^F_f = G'\left(\bar{K} - K^F_f\right) + \alpha K^F_f = y\left(K^F_f\right)$$

is the maximum aggregate output society can obtain.

The same conclusion in KM.

Notice that from equation... follows that the only level of q_t such that the first best would be obtained is

$$q_f = \frac{\alpha}{R}$$

which we have ruled out (see assumption A2 above) because it would imply that the financing constraint is not binding. In other words, the first best could be attained only if the financing constraint were not binding. From the FF constraints it turns out that in the steady state $b = a^F - m^F$and $a^F = (1-\gamma)\left(y^F_f - Rb + m^F\right)$. Finally, from the quantity theory $m^F = \frac{1-\gamma}{\gamma(1+\sigma)}y_f$. Substituting we get:

$$b_f = \frac{1-\gamma}{1+\sigma}\left(\sigma y^F_f - y^G_f\right)$$

8. Dynamics

The dynamics of the macroeconomy are described by equation 13, i.e the law of motion of the farmer's land, equation 17, which links the asset price to the farmer's land, and equation 29, i.e. the quantity theory of money. The state variables are K_t^F, q_t and m_t^F.We list the equations below for the reader's convenience.

$$K_t^F = \frac{\left[(1-\gamma)\alpha + \gamma q_t\right] K_{t-1}^F - \gamma m_t^F}{q_t - \frac{q_{t+1}}{R}}$$

$$q_t = \frac{g\left(K_t^F\right)}{R}$$

$$(1+\sigma) m_t^F = \frac{1-\gamma}{\gamma}\left[\alpha K_{t-1}^F + G\left(\bar{K} - K_{t-1}^F\right)\right]$$

Plugging the third equation into the first one, the system boils down to

$$K_t^F = \frac{\left[\frac{(1-\gamma)\alpha\sigma}{1+\sigma} + \gamma q_t\right] K_{t-1}^F - \frac{1-\gamma}{1+\sigma} G\left(\bar{K} - K_{t-1}^F\right)}{q_t - \frac{q_{t+1}}{R}}$$

$$q = \frac{g\left(K^F\right)}{R}$$

Substituting the second equation into the first one and noting that $q_{t+1} = \frac{g\left(K_{t+1}^F\right)}{R}$ the system boils down to

$$\left[\frac{g\left(K_t^F\right)}{R} - \frac{g\left(K_{t+1}^F\right)}{R^2}\right] K_t^F - \left[\frac{(1-\gamma)\alpha\sigma}{1+\sigma} + \frac{\gamma g\left(K_t^F\right)}{R}\right] K_{t-1}^F + \frac{1-\gamma}{1+\sigma} G\left(\bar{K} - K_{t-1}^F\right) = 0 \tag{31}$$

which is a non linear second order difference equation in the state variable K_t^F in implicit form.

8.1. A convenient special case: linear technology

In order to try and solve the model we have to specify the functional form of the gatherer's production function. Since the beginning (see section 2) we have assumed that $G(.)$ is well behaved, i.e increasing and concave. As a preliminary step in the analysis, however, it is worth exploring the properties of the simplest case, i.e. of a linear technology:

$$G\left(\bar{K}-K_{t-1}^{F}\right)=\beta\left(\bar{K}-K_{t-1}^{F}\right); \qquad \beta>0$$

In this case, of course $g\left(K_t^F\right)=G'\left(\bar{K}-K_t^F\right)=\beta$. Hence, from equation 17 one gets

$$q_s=\frac{\beta}{R}$$

and

$$\mu_s=q_s\left(1-\frac{1}{R}\right)=\frac{\beta}{R}\left(1-\frac{1}{R}\right)$$

In this particular scenario, therefore, the price of land and the downpayment are given and constant: In a sense we are switching off the interaction between changes in landholding and the dynamics of the asset price, a crucial feature of the present and of the original KM model. This property, however, simplifies the analysis to the greatest extent.

In the case of a linear technology, from 31 we get

$$K_t^F=\kappa_0K_{t-1}^F-\kappa_1 \tag{32}$$

where

$$\kappa_0=(1-\gamma)\frac{\alpha\sigma+\beta}{\mu_s(1+\sigma)}+\gamma\frac{q_s}{\mu_s}$$
$$\kappa_1=\frac{1-\gamma}{\mu_s(1+\sigma)}\beta\bar{K}$$

and $\dfrac{q_s}{\mu_s}=\left(1-\dfrac{1}{R}\right)^{-1}$.

The law of motion 32 is a first order linear difference equation. Let's assume

A3 $(1-\gamma)\dfrac{\alpha\sigma+\beta}{1+\sigma}+\gamma q_s>\mu_s$

Since $K_{t-1}^F\geq 0$, $K_t^F\geq 0$,the phase diagram of the difference equation is

$$K_t^F=0 \ \text{ for } K_{t-1}^F\leq\frac{\kappa_1}{\kappa_0}$$
$$K_t^F=\kappa_0K_{t-1}^F-\kappa_1 \ \text{ for } \ \frac{\kappa_1}{\kappa_0}<K_{t-1}^F<\frac{\bar{K}+\kappa_1}{\kappa_0}$$
$$K_t^F=\bar{K} \ \text{ for } \ \frac{\bar{K}+\kappa_1}{\kappa_0}\leq K_{t-1}^F<\bar{K}$$

where

$$\frac{\kappa_1}{\kappa_0} = \frac{(1-\gamma)\,\beta\bar{K}}{(1-\gamma)\,(\alpha\sigma+\beta)+\gamma q_s\,(1+\sigma)}$$
$$\frac{\bar{K}+\kappa_1}{\kappa_0} = \frac{[\mu_s\,(1+\sigma)+(1-\gamma)\,\beta]\,\bar{K}}{(1-\gamma)\,(\alpha\sigma+\beta)+\gamma q_s\,(1+\sigma)}$$

The phase diagram is piecewise linear. The first segment is a portion of the x-axis. The second segment is an upward sloping straight line with negative intercept and slope greater than one. The third segment is a portion of the straight line of equation $K_t^F = \bar{K}$ (see figure...).

The steady state is

$$K_s^F = \bar{K}\xi\,(\sigma)$$

where

$$\xi\,(\sigma) = \frac{(1-\gamma)\,\beta}{(1-\gamma)\,(\alpha\sigma+\beta)+\left[\frac{1}{R}-(1-\gamma)\right]q_s\,(1+\sigma)}$$

In words, the steady state value of farmer's land is a portion ξ of total land which is decreasing with σ. Since the slope of the phase diagram at the intersection point is greater than one, the steady state is unstable.

The steady state of m^F is

$$m_s^F = \frac{1-\gamma}{\gamma\,(1+\sigma)}\left[(\alpha-\beta)\,K_s^F+\beta\bar{K}\right] = m\,(\sigma)$$

The real money balances are a function of σ. A sufficient condition for the steady state real money balances to be decreasing with σ is $\alpha > \beta$.

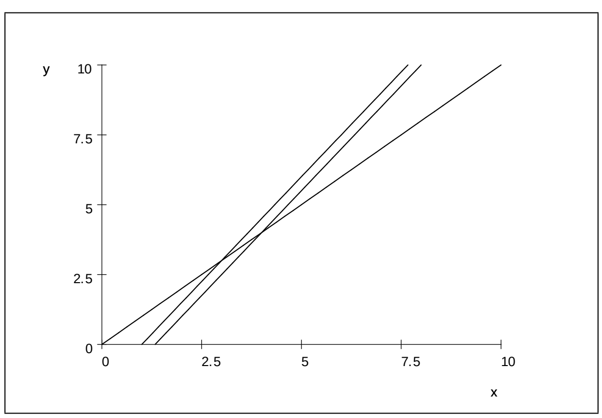
In order to assess the effects of a nominal shock, let's assume that the ratio of the gatherer's to the farmer's money is given, say σ_0. This means that the rates of change of the farmer's and of the gatherer's money are the same: $\eta_0^G = \eta_0^F = \eta_0$. Hence the inflation rate is $\pi_0 = \eta_0$.Let's assume, moreover, that, by chance, the initial condition is the steady state $K_0^F = \bar{K}\xi\,(\sigma_0)$ so that $m_0^F = m\,(\sigma_0)$.

Let's assume now that the central bank increases the rate of change of money of the farmer and of the gatherer in the same proportion $\eta_1^G = \eta_1^F = \eta_1 > \eta_0$. The ratio σ_0 does not change. Also K_0^F and m_0^F do not change. The effect of such a move is to increase the rate of inflation to $\pi_1 = \eta_1$. The real interest rate remains unchanged $R = \dfrac{1+i_0}{1+\pi_0} = \dfrac{1+i_1}{1+\pi_1}$ but the nominal interest rate goes up in the same proportion as the inflation rate.

Suppose now that the central bank adopts a *differentiated policy move*. For instance the rate of growth of money of the gatherer becomes $\eta_1^G > \eta_0$ while the rate of growth of money of the farmer remains unchanged $\eta_0^F = \eta_0$. The ratio goes up to σ_1 and stays there even if the rate of growth of money of the gatherer goes down to η_0 thereafter. Hence $K_1^F = \bar{K}\xi(\sigma_1) < K_0^F$. Recall, however, that the equilibrium is unstable. Hence the economy is spiralling up until $K_\infty^F = \bar{K}$ is reached (see figure.. again). In other words, the farmer purchases all the land. Hence $m_\infty^F = \dfrac{1-\gamma}{\gamma(1+\sigma)}\alpha\bar{K}$. We can draw therefore the following conclusion

Remark 1. *If a policy move does not change the ratio of moneys of the farmer and of the gatherer σ, i.e. if the central bank changes the rates of growth of the two monetary aggregates by the same amount, monetary policy is superneutral, i.e. the allocation of land to the farmer and the gatherer does not change, real variables are unaffected and the only effect of the policy move is an increase in the rate of inflation, which is pinned down to the (uniform) rate of change of money(s), and of the nominal interest rate. If, on the other hand, the move is differentiated, i.e. the central bank changes the rates of growth of the two monetary aggregates by different amounts so that the rates of growth are heterogeneous, monetary policy is not superneutral, i.e. the allocation of land changes and real variables are permanently affected, even if the rates of growth of the two aggregates go back to the original value afterwards.*

In the case of a linear technology and an unstable steady state, in the end the economy converges to one of the polar cases. If the initial condition is in between, we can easily conclude that such a move is welfare reducing, i.e. the economy distances itself from the first best.



8.2. Steady states

Let's go back to the original dynamic system. An interesting way of computing the steady state is the following. Rewriting the system ignoring time indices and recalling that, in the steady state $\mu = q\delta$ with $\delta = 1 - \frac{1}{R}$ we get

$$K^F = \frac{\left[(1-\gamma)\alpha + \gamma q\right] K^F - \gamma m^F}{q\delta}$$
$$q = \frac{g\left(K^F\right)}{R}$$
$$(1+\sigma) m^F = \frac{1-\gamma}{\gamma}\left[\alpha K^F + G\left(\bar{K} - K^F\right)\right]$$

Plugging the third equation into the first one, the system boils down to

$$q = \theta\left[h\left(K^F\right) - \sigma\alpha\right]$$
$$q = \frac{g\left(K^F\right)}{R}$$

where

$$\theta = \frac{1-\gamma}{(\gamma-\delta)(1+\sigma)}$$
$$h\left(K^F\right) = \frac{G\left(\bar{K} - K^F\right)}{K^F}$$

i.e. a system of two equations in $K^F$and q. Notice that $h\left(K^F\right)$ is clearly decreasing with K^F.

The first equation yields two different curves on the $\left(K^F, q\right)$ plane depending upon the relative value of γ and δ.

In the case $\gamma > \delta$, the curve is downward sloping. In the opposite case, $\gamma < \delta$ the curve is upward sloping. In both cases, the curve and crosses the x-axis when $K^F = h^{-1}(\sigma\alpha)$.

The two curves can be interpreted as isoclines.

Defining $K_t^F \cong K^F + \dot{K}^F$ where $\dot{K}^F = \frac{dK^F}{dt}$ is the time derivative of K^F, equation (13) can be reformulated in continuous time as follows:

$$\dot{K}^F = \left[\frac{(1-\gamma)\alpha\sigma}{(1+\sigma) q\delta} + \frac{\gamma}{\delta} - 1\right] K^F - \frac{1-\gamma}{(1+\sigma) q\delta} G\left(\bar{K} - K^F\right) \qquad (33)$$

Defining $q_t \cong q + \dot{q}$ where $\dot{q} = \dfrac{dq}{dt}$ is the time derivative of q, equation (17) can be reformulated in continuous time as follows:

$$\dot{q} = q - g(K^F) \tag{34}$$

We end up therefore with the dynamical system of two non linear differential equations in the state variables K and q.

Let's focus on (33) first. It is a first order non-linear differential equation in K.

Imposing the steady state condition $\dot{K}^F = 0$ one gets the isocline

$$q = \theta \left[h\left(K^F\right) - \sigma\alpha\right]$$

which is represented as an upward sloping of downward sloping (depending on gamma and delta) curve on the $\left(q, K^F\right)$ plane

Consider now (34). Imposing the steady state condition $\dot{q} = 0$ one gets the isocline

$$q = \frac{g\left(K^F\right)}{R}$$

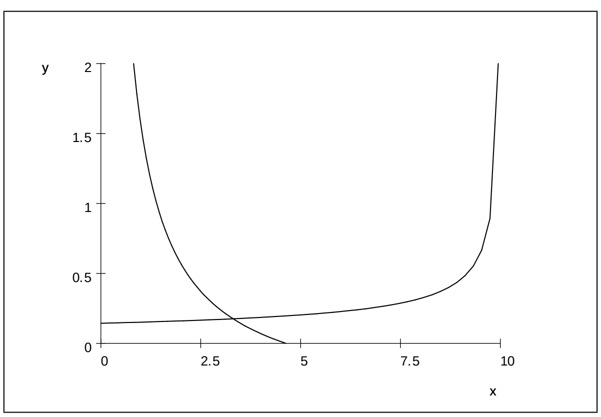
The isocline is upward sloping on the $\left(q, K^F\right)$ plane

The steady state of the system can be found at the intersection of the two isoclines.

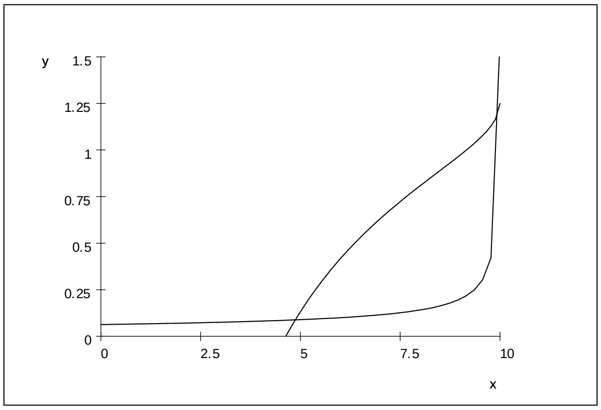
It is unique if the curve is downward sloping, there are two steady states in the opposite case.

For instance, in the case of a Cobb-Douglas production function $G\left(\bar{K} - K^F\right) = \sqrt{\bar{K} - K^F}$ we can have two cases.

Case 1. gamma=0.5, R=1.1 (delta=0.1), sigma 1, alfa=0.5, Kbar=10



Case 2. gamma=0.5, R=2.5 (delta=0.6), sigma 1, alfa=0.5, Kbar=10



A. Constraints at current and constant prices

In the following we denote magnitudes at current (constant) prices with capital (small) letters.

We assume that each young farmer is endowed at birth with *bequest* A_t^F. The YF employs the bequest he got and credit B_t (since it turns out that he is a borrower) to invest in land – $Q_t\left(K_t^F - K_{t-1}^F\right)$ – *and hold money balances* $M_{t,t}^F$. Since the young does not derive utility from consumption, *money is a reserve of value for the YF: He carries money over from youth to old age in order to use it as a means of payment in the latter stage of his life, i.e. to access consumption when old.*

The flow-of-funds (*fof*) constraint of the YF in t is:

$$Q_t\left(K_t^F - K_{t-1}^F\right) + M_{t,t}^F \leq B_t + A_t^F \tag{35}$$

Dividing by P_t and rearranging we get:

$$q_t\left(K_t^F - K_{t-1}^F\right) + m_{t,t}^F \leq b_t + a_t^F \tag{36}$$

The YF may be financially coinstrained. The financing constraint can be expressed as follows:

$$B_t \geq \frac{Q_{t+1}}{1+i_t} K_t^F$$

where i_t is the nominal interest rate. In words: The YF gets a loan in t greater or equal to the present value of *collateralizable wealth, i.e. of the market value* *in t+1 of land owned in t.* If the constraint is binding, $B_t = \dfrac{Q_{t+1}}{1+i_t} K_t^F$. In real terms: $b_t = \dfrac{Q_{t+1}}{P_t (1+i)} K_t^F$. Multiplying and dividing the expression above by P_{t+1}

$$b_t = \frac{q_{t+1}}{R} K_t^F$$

where $R := (1+i) / (1+\pi_{t+1})$ is the real (gross) interest rate and $1+\pi_{t+1} := P_{t+1}/P_t$ is the (gross) rate of inflation.

In t, the YF uses labour and land K_t^F to produce output which will become available in $t+1$, $y_{t+1}^F = \alpha K_t^F$.When old, the farmer has an "inflow" equal to the revenues from sale of output (produced when young) and money balances. Part of the money balances are carried over from youth, part are conferred to the old by the Government as a (monetized) transfer payment. [4]The "outflow" consists of the repayment of the loan (if the YF were a borrower), consumption and bequest. Therefore the fof constraint of the OF in $t+1$ in nominal terms is:

$$P_{t+1} c_{t,t+1}^F + A_{t+1}^F + B_t (1+i_t) \leq P_{t+1} y_{t+1}^F + M_{t,t+1}^F$$

Dividing by P_{t+1} and recalling that $y_{t+1}^F = \alpha K_t^F$

$$c_{t,t+1}^F + a_{t+1}^F + b_t R \leq \alpha K_t^F + m_{t,t+1}^F$$

Thanks to assumptions A1 and A2, all the constraints are binding, i.e.

$$\begin{gathered} q_t \left(K_t^F - K_{t-1}^F \right) + m_{t,t}^F = b_t + a_t^F \\ b_t = \frac{q_{t+1}}{R} K_t^F \\ c_{t,t+1}^F + a_{t+1}^F + b_t R = \alpha K_t^F + m_{t,t+1}^F \end{gathered}$$

---

[4] Let's assume

$$M_{t+1}^F = M_t^F + T_{t+1}$$

where T_{t+1} is a monetized transfer payment to the old agent. The transfer is proportional to the individual money holdings, i.e.

$$T_{t+1} = \eta^F M_t^F$$

Hence

$$m_{t+1}^F = \frac{M_{t+1}^F}{P_{t+1}} = \frac{\left(1+\eta^F\right) M_t^F}{P_{t+1}} = \frac{1+\eta^F}{1+\pi_{t+1}} m_t^F$$

Substituting the second constraint into the first and the third one gets

$$\mu_t K_t^F + m_{t,t}^F = a_t^F + q_t K_{t-1}^F$$
$$c_{t,t+1}^F + a_{t+1}^F = (\alpha - q_{t+1}) K_t^F + m_{t,t+1}^F$$

where $\mu_t := q_t - \frac{q_{t+1}}{(1+i_t)}(1+\pi_{t+1}) = q_t - \frac{q_{t+1}}{R}$ is the downpayment, i.e. the amount of internal finance the borrower has to accumulate in order to get a loan equal to $\frac{q_{t+1}}{R}$ (per unit of land he wants to purchase). The fof constraint of the YF $\mu_t K_t^F + m_{t,t}^F = a_t^F + q_t K_{t-1}^F$provides a different interpretation of the fof constraint: the YF employs bequests and *collateralizable wealth* $q_t K_{t-1}^F$ to put aside internal finance and hold money balances.

The flow-of-funds constraint of the YG in t is

$$Q_t \left(K_t^G - K_{t-1}^G\right) + B_t + M_{t,t}^G \leq A_t^G$$

In real terms

$$q_t \left(K_t^G - K_{t-1}^G\right) + b_t + m_{t,t}^G \leq a_t^G$$

Since $b_t = \frac{q_{t+1}}{R} K_t^F$ and $K_t^G - K_{t-1}^G = -\left(K_t^F - K_{t-1}^F\right)$

$$-\mu_t K_t^F + m_t^G = a_t^G - q_t K_{t-1}^F \tag{37}$$

When old, the gatherer employs "income", the repayment of the loan, money carried out from youth and transfers to consume and leave a bequest. Therefore the flow-of-funds constraint of the OG in $t+1$ is:

$$P_{t+1} c_{t,t+1}^G + A_{t+1}^G = P_{t+1} y_{t+1}^G + B_t (1+i_t) + M_{t,t+1}^G$$

Dividing by P_{t+1} and recalling that $y_{t+1}^G = G\left(K_t^G\right)$ we get

$$c_{t,t+1}^G + a_{t+1}^G = G\left(K_t^G\right) + R b_t + m_{t,t+1}^G$$

B. A different trickling process

Money "trickles down" from one period to the next and from one agent to the other. In order to describe a different way by which money spreads in the economy, let's take a look at table 2. In each row we report the inflows and outflows which show up in the FF constraints of the agents in period t+1. The amount in the inflow cell is equal to the amount in the outflow cell. For instance, the first row represents the FF constraint of the YF in t+1. In this scenario, however, output is available to the young, not to the old of the previous generation

$$
\begin{array}{lll}
 & \text{inf lows} & \text{outflows} \\
YF & a^F_{t+1} + y^F_{t+1} & q_{t+1}\left(K^F_{t+1} - K^F_t\right) + m^F_{t+1,t+1} - b_{t+1} \\
YG & a^G_{t+1} + y^G_{t+1} & -q_{t+1}\left(K^F_{t+1} - K^F_t\right) + m^G_{t+1,t+1} + b_{t+1} \\
\sum & a^G_{t+1} + a^F_{t+1} + y^F_{t+1} + y^G_{t+1} & m^G_{t+1,t+1} + m^F_{t+1,t+1} \\
OF & m^F_{t,t+1} & c^F_{t,t+1} + a^F_{t+1} + b_t R \\
OG & m^G_{t,t+1} & c^G_{t,t+1} + a^G_{t+1} - Rb_t \\
\sum & m^F_{t,t+1} + m^G_{t,t+1} & c^F_{t,t+1} + a^F_{t+1} + c^G_{t,t+1} + a^G_{t+1}
\end{array}
$$

Summing side by side the (binding) FF constraints of the old agents yields

$$c^F_{t,t+1} + c^G_{t,t+1} + a^G_{t+1} + a^F_{t+1} = m^G_{t,t+1} + m^F_{t,t+1} \tag{38}$$

In words: real *money balances of the old agents* is equal to the sum of aggregate consumption and aggregate bequest.

We assume equilibrium on the goods market, i.e.

$$c^F_{t,t+1} + c^G_{t,t+1} = y^G_{t+1} + y^F_{t+1} \tag{39}$$

As to the young agents

$$a^G_{t+1} + a^F_{t+1} + y^F_{t+1} + y^G_{t+1} = m^G_{t+1,t+1} + m^F_{t+1,t+1}$$

i.e. the total amount of bequest left by the old agents + total output is equal to the total amount of *money of the young agents*. Hence

$$m^F_{t+1,t+1} + m^G_{t+1,t+1} = m^G_{t,t+1} + m^F_{t,t+1} \tag{40}$$

i.e the total amount of money of the young agents in t+1 must be equal to the total amount of money of the old agents of generation t in t+1 as in the case of section 5.

Thanks to the Cobb-Douglas specification of the utility function,from the FOCs (iF)(iiF) and (iG)(iiG) one gets

$$c^i_{t,t+1} = \frac{\gamma}{1-\gamma} a^i_{t+1} \qquad i = F, G \tag{41}$$

Moreover

$$
\begin{aligned}
a_{t+1}^{F} &= (1-\gamma)\left(m_{t,t+1}^{F} - Rb_{t}\right) \\
a_{t+1}^{G} &= (1-\gamma)\left(m_{t,t+1}^{G} + Rb_{t}\right)
\end{aligned}
$$

Substituting ... and the market clearing condition into... we obtain

$$
m_{t,t+1}^{G} + m_{t,t+1}^{F} = \frac{1}{\gamma}\left(y_{t+1}^{G} + y_{t+1}^{F}\right) \tag{42}
$$

Total real money balances are proportional to aggregate output. Equation ... is a sort of quantity theory of money in this context, even if formally different from the one obtained in section 6