

Macroeconomic Volatility in a Model of the UK Gilt-edged Bond Market 1979-2004.

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Abstract

This paper develops an econometric model of the UK economy and uses this to explain the behavior of the gilt-edged bond market. These macro-finance models typically assume a homoscedastic error process, but I allow volatility to be conditioned by the underlying level of inflation. Empirically, this conditional heteroscedasticity appears to be very significant in UK macroeconomic data. This type of specification is now standard in the mainstream finance literature, and I show that it has regular asymptotic properties even in the presence of unit roots, unlike the standard homoscedastic macro-finance model. The empirical version of this model provides a much better explanation of the UK data than the standard macro-finance model. This research opens the way to a much richer term structure specification, incorporating the best features of both macro-finance and mainstream finance models. Moreover, it also yields insights into the behavior of the macroeconomy, particularly with respect to the behaviour of inflationary expectations and heteroscedastic nature of the data. This model rationalizes the much lower level of output; interest rate and inflation volatility seen in recent years, attributing it to the fall in the underlying rate of inflation.

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1 Introduction

The volatility of the world's leading economies has been much lower over the last two decades than over the previous two. Nowhere has this fall been more dramatic than in the UK. During the 1970s the UK had the most volatile economy in the G7 but over the last decade it has had one of the least volatile, as the OECD recognized in its June 2005 Review of the UK economy. There is no shortage of explanations for this phenomenon. In the UK, the move to inflation targeting in 1992 and the new monetary policy arrangements of 1997 are just the front runners King (2003). France and Italy have joined the EMU. However, the fact that this is an international phenomenon and is also marked in economies such as the US that have not formally changed their monetary policy arrangements, should lead us to look for common factors.

The work reported in this paper is based on a hypothesis that also seems to explain the fall in volatility in the US (Spencer (2004)). Robert Engle's Engel (1982) paper on conditional heteroscedasticity in UK inflation data noted that besides autoregressive conditional heteroscedasticity (ARCH), the volatility of inflation might also depend upon the prevailing level of inflation. This observation is due to Milton Friedman, who argued in his Nobel lecture Friedman (1977) that the variability of inflation, output and other macroeconomic variables seemed to be related to the level of inflation itself. There is an interesting theoretical (see for example Ball (1992)) and extensive empirical literature (see for example Brunner and Hess (1993), Holland (1995) and Caporale and McKiernan (1997)) on this phenomenon. There is also an emerging literature on the effect of declining macroeconomic volatility on the equity risk premium (Lettau, Ludvigson, and Wachter (2004), Brandt and Wang (2003)). However, as far as I am aware no one has tried to test this hypothesis on bond market data. Indeed, although there are now a growing number of papers that follow (Ang and Piazzesi (2003)) and model the effect of macroeconomic variables on US interest rates and the yield curve, these all assume that volatility is constant as in Vasicek (1977)¹. This paper extends this homoscedastic macro-finance model to allow the volatility of the macroeconomic variables to be conditioned by the underlying rate of inflation, modelled as a latent variable.

This specification provides a neat solution to the problems posed by unit roots in interest rate data. As Kozicki and Tinsley (2001) and Dewachter and Lyrio (2006) observe, macroeconomic data are characterized by a non-stationary common trend which seems to be explained by market perceptions of the underlying rate of inflation. This situation is very familiar to macroeconomic modelers but poses difficult problems for term structure researchers. That is because asymptotic (long maturity) forward rates and yields are not properly defined if the short term (spot) interest rate is driven by a random walk (a homoscedastic unit root process). Campbell, Lo, and MacKinlay (1996) show that when the spot rate follows a random walk, forward rates and yields fall to larger and larger negative values as maturity lengthens, without a well-defined limit. This reflects the fact that the variance of a random walk increases with the forecast horizon without limit. Because the bond price function is convex in the state variables this volatility is valuable to the investor and is offset by the forward rate.

With the notable exception of Dewachter and Lyrio (2006), macro-finance modelers have tried to avoid the unit root problem by assuming that the underlying inflation

¹Notable examples include Rudebusch and Wu (2003) and Rudebusch and Aruoba (2003). The model of Dewachter and Lyrio (2006) is similar, but developed in continuous rather than discrete time.

variable follows a near-unit root (AR(1)) rather than a unit root (I(0)) process (Ang and Piazzesi (2003), Rudebusch and Wu (2003)). However, if this variable is stationary, it mean-reverts to a constant rather than the variable end-point suggested by unit root macroeconomic models. As Dewachter and Lyrio (2006) note, this means that it cannot be interpreted as a long run inflation expectation. In this kind of model, inflationary expectations are ultimately anchored to a constant which cannot be influenced by monetary policy.

In contrast, Dewachter and Lyrio (2006) proceed by distinguishing the historical (or state) probability measure \mathcal{P} that drives the macroeconomic data from the risk-neutral measure \mathcal{Q} that determines asset prices. To do this, they employ the ‘essentially affine’ model of Duffee (2002), which assumes that the difference between the two measures is due to the effect that the variables of the model have on ‘the price of risk’: the expected excess returns that the market offers for exposure to risk. They assume that the underlying inflation variable follows a random walk under the historical measure but is mean reverting under \mathcal{Q} . This risk adjustment drives a wedge between the two asymptotes, so that the spot rate asymptote is a random walk under \mathcal{P} , but is constant under \mathcal{Q} . This conveniently makes the forward rate (and yield) asymptote constant. However, for this to happen it is necessary to assume that a shift in the inflationary trend moves the associated price of risk and the risk premium in the *opposite* direction. Although it is not possible to rule this behavior out a priori, this should arguably be tested empirically rather than assumed *a priori*. Moreover, even if the data exhibit a near-unit root rather than a unit root under \mathcal{Q} , so that the asymptotic forward rate is constant, this adopts an extremely large negative value.

These awkward characteristics reflect the basic mathematical problem with the homoscedastic framework: it allows yields to become negative. For that reason the mainstream finance literature typically uses heteroscedastic (stochastic volatility) interest rate models such as that of Cox, Ingersoll, and Ross (1985). The variance of the spot rate is proportional to the spot rate in this type of model, ruling out negative spot and forward rates. This paper adapts this model for use with discrete time macroeconomic data. This specification is supported by the empirical finding that the common stochastic trend drives the volatility as well as the central tendency in these data. Technically, this trend is a martingale (a unit root process that admits heteroscedasticity) but not a random walk (a martingale with constant variance). This model naturally generates a sensible forward rate asymptote without placing constraints on the roots of system or the price of risk.

To investigate these issues empirically, this paper develops a macro-finance specification which conditions both the central tendency and the variance structure of the model on a non-stationary nominal latent variable. A second latent (but stationary) variable is introduced to handle unobservable real rate of return influences. This model is the macro-finance analogue of the preferred model of Dai and Singleton (2000), (2004), which as they say: ‘builds upon a branch of the finance literature that posits a short-rate process with a single stochastic central tendency and volatility’. The latent variables are estimated using the Extended Kalman Filter, which is also standard in the finance literature. This specification can accommodate unit and near unit roots under both measures while generating admissible variance & asymptotic term structures. It encompasses the standard macro-finance model, which is decisively rejected by the data.

The paper is set out as follows. The next section describes the macroeconomic model and its stochastic structure, supported by appendix 1. Section 3, supported by appendices 2 and 3, derives the bond pricing model. It discusses the problems posed

by the unit root in the standard macro-finance specification and shows how these are avoided in the Dai and Singleton (2000) $\mathbb{E}\mathbb{A}_1(N)$ version. The two respective empirical models are compared in Section 5. Section 6 offers a brief conclusion.

2 The model framework

This framework consists of a heteroscedastic macroeconomic Vector Autoregression (VAR) augmented by two latent variables, which is specified under physical or historical probability measure and a yield model which is specified under the risk neutral measure.

2.1 The macroeconometric model

The macro-model is based on the specification developed by Svensson (1999); Rudebusch (2002); Smets (1999) and others. It represents the behavior of the macroeconomy in terms of the output gap (g_t); the annual CPI inflation rate (π_t) and the 3 month Treasury Bill rate ($r_{1,t}$). These are part of an n -vector z_t of macroeconomic variables driven by the difference equation system:

$$z_t = K + \Phi_0 y_t + \sum_{l=1}^L \Phi_l z_{t-l} + G \eta_t \quad (1)$$

where G is a lower triangular matrix, η_t is an n -vector of orthogonal errors and y_t is a k -vector of latent factors. These follow the first order process:

$$y_t = \theta + \Xi y_{t-1} + \varepsilon_t \quad (2)$$

where ε_t is an k -vector of orthogonal errors and $\Xi = \text{Diag}\{\xi_1, \dots, \xi_k\}^2$. It is assumed that z_t is observed without measurement error and that y_t is known to the monetary authorities and markets but has to be inferred by the econometrician. I do this using the Extended Kalman Filter (Harvey (1989), Duffee and Stanton (2004)) as described in appendix 4 .

The specific model developed in this paper defines $z_t = \{g_t, \pi_t, r_{1,t}\}$ and $y_t = \{y_{1,t}, y_{2,t}\}$ (with $n = 3$ and $k = 2$). (Preliminary regression analysis suggested $l = 3$, giving $N = 11$.) I assume that the inflation asymptote π^* is equal to $y_{1,t}$ plus a shift constant φ_1 . Similarly the real interest asymptote is $y_{2,t}$ plus another shift constant φ_2 . The equilibrium conditions $\pi_t^* = y_{1,t} + \varphi_1$; $g_t^* = 0$ and $r_t^* = y_{1,t} + y_{2,t} + \varphi_1 + \varphi_2$ are enforced by imposing a set of restrictions on (1):

$$\begin{aligned} \Phi_0 &= (I - \sum_{l=1}^L \Phi_l) R; \quad K = \Phi_0 \varphi; \\ \text{where : } \varphi' &= \{\varphi_1, \varphi_2\}; \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned} \quad (3)$$

to give the steady state solution $z_t^* = (I - \sum_{l=1}^L \Phi_l)^{-1} \Phi_0 (y_t + \varphi) = R(y_t + \varphi)$.

This system can be consolidated by defining $x_t = \{y_t', z_t'\}'$; $v_t = \{\varepsilon_t', \eta_t'\}'$; and combining (1) and (2) to get the l -the order difference system:

$$x_t = \begin{bmatrix} \theta \\ K + \Phi_0 \theta \end{bmatrix} + \sum_{l=1}^L \Gamma_l x_{t-l} + w_t \quad (4)$$

²In this paper, $\text{Diag}\{y\}$ represents a matrix with the vector y in the diagonal and zeros elsewhere. 0_a denotes an $(a \times 1)$ zero vector; $0_{a,b}$ the zero matrix of dimension $a \times b$; and I_a an a^2 identity matrix.

where:

$$w_t = \Gamma v_t; \quad \Gamma = \begin{bmatrix} I_k & 0_{k,n} \\ \Phi_0 & G \end{bmatrix};$$

$$\Gamma_1 = \begin{bmatrix} \Xi & 0_{k,n} \\ \Phi_0 \Xi & \Phi_1 \end{bmatrix}; \Gamma_l = \begin{bmatrix} 0_{k^2} & 0_{k,n} \\ 0_{n,k} & \Phi_l \end{bmatrix}; l = 2, \dots, L.$$

The yield model employs the state space form, obtained by arranging this as first order difference system describing the dynamics of the state vector (see appendix 1):

$$X_t = \Theta + \Phi X_{t-1} + W_t \quad (5)$$

where $X_t = \{y'_t, z'_t, \dots, z'_{t-l}\}'$ is the state vector, $W_t = C.\{\varepsilon'_t, \eta'_t, 0_{1,N-k-n}\}'$ and Θ, Φ & C are defined in appendix 1. X_t has dimension $N = k + nl$.

The macroeconomic data are shown in chart 1. I use the Retail Price Index excluding mortgage interest payments (RPIX) to measure inflation (π_t). This was the policy objective (with a target rate of 2.5 %) between November 1992 and April 2004, when it was replaced by the Consumer Price Index. As in previous macro-finance studies, inflation is measured on an annual basis. The three month Treasury Bill rate is used to represent the spot rate (r_t). Both of these series were taken from Datastream. Quarterly estimates of the GDP output gap (g_t) were kindly provided by Oxford Economic Forecasts³. These data dictated the use of a quarterly time frame. The macro data are shown in chart 1⁴.

The gilt-edged yield data were taken from the Banks of England's website and are derived using the methodology discussed in Anderson and Sleath (2001). To represent this curve I use 1,2,3,5,7,10 and 15 year maturities, the longest one for which a continuous series is available. These yield data are available on a monthly basis, but the macroeconomic data dictated a quarterly time frame (1961Q4-2004Q1, a total of 170 periods). The quarterly yield data are shown in chart 2. The 15 year yield is shown at the back of the chart, while the shorter maturity yields are shown at the front. Table 2 shows the means, standard deviations and first order autocorrelation coefficients of these data, as well as KPSS and ADF test results. The inflation and interest rates all exhibit non-stationarity. Further tests show that inflation and interest rates are cointegrated. Consequently, this paper follows Dewachter and Lyrio (2006) in analyzing a macroeconometric model characterized by a common stochastic trend.

2.2 The stochastic structure

The standard macro-finance model assumes that the volatility structure is homoscedastic and Gaussian: $W_t \sim N(0_N, \Omega)$. However, mainstream finance models usually assume that volatility is stochastic. In the affine model developed by Duffie and Kan (1996) and Dai and Singleton (2000) & (2002), conditional heteroscedasticity in the errors is driven by square root processes in the state variables⁵. In the 'admissible' version of this specification developed by Dai and Singleton (2000), regularity or

³This measure is based on the production function approach, building up potential GDP from estimates of the capital stock, labour force and productivity, and then subtracting GDP to obtain the estimate of the output gap. This was used in preference to the OECD measure based on the trend filtering approach, since this indicates that output was above trend in 2004, in contrast to the impression given by the behaviour of inflation and other macroeconomic variables.

⁴These are annual rates in percentages. In the empirical model these were appropriately converted to quarterly decimal fractions by dividing by 400.

⁵Preliminary tests showed no significant evidence of Autoregressive Conditional Heteroscedasticity (ARCH) in this quarterly data set.

admissibility conditions are imposed to ensure that the variance structure remains non-negative definite. Variations in the risk premia depend entirely upon variations in volatility in these models. In the ‘Essentially Affine’ model of Duffee (2002) state variables can affect risk premia through the price of risk as well as through volatility. In the notation of Dai and Singleton (2002) an admissible essentially affine model with N state variables and m independent square root factors conditioning volatility is classed as $\mathbb{EA}_m(N)$. Thus the standard macro-finance model (which is ‘essentially affine’ and homoscedastic) is denoted $\mathbb{EA}_0(N)$.

This paper compares $\mathbb{EA}_0(N)$ with the $\mathbb{EA}_1(N)$ specification: a model with a single stochastic volatility term. Both of these generate affine yield models. That is because the probability distributions underpinning these models are all ‘exponential-affine’ in the sense of Duffee, Filipovic, and Schachtermayer (2003). They define a process as exponential-affine under any measure \mathcal{M} if the conditional Moment Generating Function (MGF) for Z_{t+1} ($L^{\mathcal{M}}[u, Z_t] = E_t^{\mathcal{M}}[\exp[u' Z_{t+1}] | Z_t]$, which is the Laplace Transform of the density of Z_{t+1} where u is a vector of Laplace coefficients) is an exponential-affine function of Z_t . This makes bond prices exponential-affine (i.e. loglinear) and yields affine in the state variables as shown in appendix 2. For example $\mathbb{EA}_0(N)$ assumes that $\varepsilon_{1,t}$ is normally distributed with mean zero and standard deviation δ_{01} and we use the formula for the MGF of a normal variable:

$$E_t[\exp[\nu \cdot y_{1,t+1} | y_{1,t}]] = \exp[\nu(\theta_1 + \xi_1 y_{1,t}) + \frac{1}{2} \nu^2 \delta_{01}] \quad (6)$$

which is familiar from the expression for the expected value of a lognormally distributed variable. E_t denotes the expectation under the measure \mathcal{P} describing the state price density.

In $\mathbb{EA}_1(N)$, volatility is driven by the first latent variable $y_{1,t}$ which is independent of the other state variables. This is the single factor specification of Cox, Ingersoll, and Ross (1985). Using this to drive the volatility of the remaining $(k + n - 1)$ volatility terms makes the model admissible in the sense of Dai and Singleton (2000) and (2002). In discrete time, $y_{1,t+1}$ has a non-central χ^2 conditional distribution conditional upon $y_{1,t}$:

$$y_{1,t+1} \sim \chi^2[2cy_{1,t+1}; c\theta_1; 2\xi_1 cy_{1,t}] \quad (7)$$

where $2c$ is the scale factor, $2\xi_1 cy_{1,t}$ is the non-centrality parameter and $c\theta_1$ shows the degrees of freedom. This process is of the exponential-affine class because its conditional Moment Generating Function (MGF) is an exponential function of $y_{1,t}$:

$$E_t[\exp[\nu \cdot y_{1,t+1} | y_{1,t}]] = \exp\left[\frac{\nu \xi_1 y_{1,t}}{1 - \nu/c} - c\theta_1 \ln\left[1 - \frac{\nu}{c}\right]\right] \quad (8)$$

provided that: $\nu < c$ (Johnson and Kotz (1970)). Differentiating the MGF (8) w.r.t. ν once, twice and then setting ν to zero gives the conditional mean and variance:

$$E_t[y_{1,t+1} | y_{1,t}] = \theta_1 + \xi_1 y_{1,t}; \quad V_t[y_{1,t+1} | y_{1,t}] = \delta_{01} + \delta_{11} y_{1,t}; \quad (9)$$

where : $\delta_{01} = \theta_1/c, \delta_{11} = 2\xi_1/c$.

This distribution approaches the Gaussian either as $c\theta_1$ approaches infinity (with $\xi_1 cy_{1,t}$ held constant) or as $\xi_1 cy_{1,t}$ approaches infinity (with $c\theta_1$ constant). This paper investigates the limit of a unit root. In this case the degree of freedom parameter is zero, the situation studied by Seigel (1979) and (in the case of the non-central

Gamma distribution) Gouriéroux and Jasiak (2002). Key results are reported in Chapter 29 of Johnson and Kotz (1970). In the limit:

$$\xi_1 = 1; \quad \theta_1 = \delta_{01} = 0; \quad c = 2/\delta_{11}; \quad (10)$$

the MGF simplifies to:

$$E_t[\exp[\nu \cdot y_{1,t+1} | y_{1,t}]] = \exp\left[\frac{\nu y_{1,t}}{1 - \nu/c}\right]; \quad (11)$$

and (9) & (11) simplify to:

$$E_t[y_{1,t+1} | y_{1,t}] = y_{1,t}; \quad V_t[y_{1,t+1} | y_{1,t}] = \frac{1}{2} \delta_{11} y_{1,t}. \quad (12)$$

This process is a martingale: the expectation of any future value is equal to the current value. However, unlike the random walk model, the error variance is also proportional to this value. Models (6), (9) and (12) can be represented as a discrete first order process:

$$y_{1,t+1} = \theta_1 + \xi_1 y_{1,t} + w_{1,t+1} \quad (13)$$

In $\mathbb{E}\mathbb{A}_1(N)$, this stochastic trend also conditions the volatility of the other variables. It is ordered as $x_{1,t} = y_{1,t}$, the first variable in x_t . The other contemporaneous variables are put into an $n - 1$ vector $(x_{2,t}) : x_t = \{y_{1,t}, x'_{2,t}\}'$. Similarly: $v_t = \{w_{1,t}, v'_{2,t}\}'$ and $w_t = \{w_{1,t}, w'_{2,t}\}'$. I partition (4) conformably and write Γ as:

$$\Gamma = \begin{bmatrix} I_k & 0_{k,n} \\ \Phi_0 & G \end{bmatrix} = \begin{bmatrix} 1 & 0_{1,(k+n-1)} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$$

Γ_{22} is an $(k + n - 1)^2$ lower triangular matrix with unit diagonals and Γ_{21} is a $(k + n - 1)$ column vector. The errors in $x_{2,t+1}$ are decomposed into components that are related to $w_{1,t+1}$ & $y_{1,t}$ and an orthogonal component $v_{2,t+1}$:

$$w_{2,t+1} = \Gamma_{21} w_{1,t+1} + \Gamma_{22} (\sqrt{y_{1,t}} u_{2,t+1} + v_{2,t+1}) \quad (14)$$

where:

$$\begin{aligned} E_t[u_{2,t+1}] &= E_t[v_{2,t+1}] = 0_{(k+n-1)}; \quad E[v_{2,t+1} u'_{2,t+1}] = 0_{(k+n-1)^2}; \\ E[v_{2,t+1} w_{1,t+1}] &= E[u_{2,t+1} w_{1,t+1}] = 0_{(k+n-1)}; \\ E[v_{2,t+1} v'_{2,t+1}] &= \Delta_{02}; \quad E[u_{2,t+1} u'_{2,t+1}] = \Delta_{12}; \\ \Delta_{s2} &= \text{Diag}[\delta_{s2}, \dots, \delta_{sn}]; \quad s = 0, 1. \end{aligned}$$

Similarly, writing $X_t = \{y_{1,t}, X'_{2,t}\}'$ and partitioning W_t, Θ, Φ, C conformably (appendix 1), (5) becomes:

$$\begin{bmatrix} y_{1,t+1} \\ X_{2,t+1} \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \Theta_2 \end{bmatrix} + \begin{bmatrix} \xi_1 & 0'_{N-1} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} y_{1,t} \\ X_{2,t} \end{bmatrix} + \begin{bmatrix} w_{1,t+1} \\ W_{2,t+1} \end{bmatrix} \quad (15)$$

In this paper subscripts 1 and 2 denote partitions of N (or n) dimensional vectors and matrices into 1 and $N - 1$ (or $n - 1$). The stochastic structure for (15) is described in appendix 1. I also normalize the second latent variable by assuming: $\theta_2 = 0$ (φ_2 and θ_2 play a similar role and cannot be separately identified). This means that $y_{2,t}$ reverts to a zero mean and the spot rate asymptote is thus $r_t^* = y_{1,t} + \varphi_1 + \varphi_2$.

This model is admissible in the sense of Dai and Singleton (2000). That is because $y_{1,t}$, the variable driving volatility has a non-central χ^2 distribution and is non-negative, keeping the variance structure non-negative. It is estimated by quasi-maximum likelihood and the Extended Kalman filter (which gives the optimal linear

estimate of the latent variables in this situation). At the estimation stage it is assumed that $y_{1,t}$ is approximately normal (appendix 4)⁶. Using the mean and variance given by (9) or (12):

$$y_{1,t+1} \sim N(\theta_1 + \xi_1 y_{1,t}, \delta_{01} + \delta_{11} y_{1,t}) \quad (16)$$

This Gaussian specification is also used (with $\delta_{11} = 0$) in place of (8) to generate the bond prices in the $\mathbb{E}\mathbb{A}_0(N)$ model (6).

2.3 The risk neutral measure

The aim of this paper is to use these exponential-affine MGFs to model the macro-economy and the yield curve jointly. The macro model is naturally defined under the state probability measure \mathcal{P} , but assets are priced under the risk neutral measure \mathcal{Q} . This is obtained by adjusting the state probabilities multiplicatively by a state-dependent subjective-utility weight N_{t+1} (with the logarithm n_{t+1}):

$$E_t^{\mathcal{Q}}[X_{t+1} | X_t] = E_t[N_{t+1} X_{t+1} | X_t] \quad (17)$$

where $E_t^{\mathcal{Q}}$ denotes the expectation under \mathcal{Q} . For the homoscedastic model: $-n_{t+1} = \omega_t + \lambda_t' v_{t+1}$, where λ_t is an $(k+n) \times 1$ vector of coefficients related to the prices of risk associated with shocks to x_{t+1} . In the basic Affine model these are constant and in the ‘Essentially Affine’ specification of Duffee (2002) they are linear in x_t . Define the $N \times 1$ deficient vector $\Lambda_t = [\lambda_t', 0'_{N-(k+n)}]'$ and partition this conformably with (15):

$$\begin{bmatrix} \lambda_{1,t} \\ \Lambda_{2,t} \end{bmatrix} = \begin{bmatrix} \lambda_{10} \\ \Lambda_{20} \end{bmatrix} + \begin{bmatrix} \lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} y_{1,t} \\ X_{2,t} \end{bmatrix} \quad (18)$$

The error structure of the $\mathbb{E}\mathbb{A}_1(N)$ model is richer than that of $\mathbb{E}\mathbb{A}_0(N)$ and the effects of $V_{2,t+1}$ and $U_{2,t+1}$ on n_{t+1} should both be allowed for when specifying n_t . This can be achieved by multiplying the former by $\Lambda_{2,t}' C_{22}$ and the latter by $Z_{21}' C_{22}$, where Z_{21} is an $N-1$ vector resembling Λ_{21} . However Z_{21} and Λ_{21} are not separately identified - they capture the effect of $y_{1,t}$ on the risk premia working through volatility and the price of risk respectively. To identify the model and facilitate comparisons with $\mathbb{E}\mathbb{A}_0(N)$ I set $Z_{21} = 0_{N-1}$ and use the model:

$$-n_{t+1} = \omega_t + \lambda_{1,t} y_{1,t} + \Lambda_{2,t}' C_{22} V_{2,t+1} \quad (19)$$

Admissibility under \mathcal{Q} in $\mathbb{E}\mathbb{A}_1(N)$ requires $\Lambda_{12} = 0_{N-1}$. I follow Dewachter and Lyrio (2006) in using this restriction in the $\mathbb{E}\mathbb{A}_0(N)$ model as well, to save degrees of freedom and help nest it within $\mathbb{E}\mathbb{A}_1(N)$. λ_{11} is redundant in $\mathbb{E}\mathbb{A}_1(N)$ since the effect of $y_{1,t}$ on the risk premia comes through its effect on volatility not the price of risk. Setting this to zero:

$$\lambda_{1,t} = \lambda_{10}. \quad (20)$$

⁶The approximation is a reasonable one in this case, since the empirical values of the non-centrality parameter are large, except at the low point in 2000Q1. This distribution approaches the normal for large values of the non-centrality parameter. Johnson and Kotz (1970) note that this effect is clearly apparent in their figure 29.2, which shows the density for selected values of this parameter over the range [2,15] (given zero degrees of freedom). For model M1, reported below the non-centrality parameter $4y_{1,y}/\delta_{11}$ varies from 10.8 in 2001Q1 to 383.1 in 1981Q3.

3 Bond prices in the $EA_0(N)$ and $EA_1(N)$ models

The stochastic processes of the previous section are exponential-affine and admissible under measure \mathcal{P} and appendix 3 shows that (19) and (20) preserve these properties under measure \mathcal{Q} . The dynamics of X_t under \mathcal{Q} is generated by a process resembling (15):

$$\begin{bmatrix} y_{1,t+1} \\ X_{2,t+1} \end{bmatrix} = \begin{bmatrix} \theta_1^{\mathcal{Q}} \\ \Theta_2^{\mathcal{Q}} \end{bmatrix} + \begin{bmatrix} \xi_1^{\mathcal{Q}} & 0'_{N-1} \\ \Phi_{21}^{\mathcal{Q}} & \Phi_{22}^{\mathcal{Q}} \end{bmatrix} \begin{bmatrix} y_{1,t} \\ X_{2,t} \end{bmatrix} + \begin{bmatrix} w_{1,t+1}^{\mathcal{Q}} \\ W_{2,t+1}^{\mathcal{Q}} \end{bmatrix} \quad (21)$$

where the time t expectations of the error terms are zero under \mathcal{Q} . Appendix 3 shows that $\Phi_{22}^{\mathcal{Q}} = \Phi_{22} - \Upsilon_2$ is the same for both models. The other parameters differ and are denoted with a superscript \mathcal{Q}_m for each model $EA_m(N) : m = 0, 1$. They are shown in the second and third columns of the table:

Table 1: Dynamic coefficients for different models and measures

measure: \mathcal{P}	\mathcal{Q}	\mathcal{Q}
model:	$EA_0(N)$	$EA_1(N)$
θ_1	$\theta_1^{\mathcal{Q}_0} = \theta_1 - \delta_{01} \lambda_{10}$	$\theta_1^{\mathcal{Q}_1} = \frac{\theta_1}{1 + \lambda_{10}/c}$
ξ_1	$\xi_1^{\mathcal{Q}_0} = \xi_1 - \delta_{01} \lambda_{11}$	$\xi_1^{\mathcal{Q}_1} = \frac{\xi_1}{(1 + \lambda_{10}/c)^2}$
Θ_2	$\Theta_2^{\mathcal{Q}_0} = \Theta_2 - \Upsilon_0 - \delta_{01} \lambda_{10} C_{21}$	$\Theta_2^{\mathcal{Q}_1} = \Theta_2 - \Upsilon_0 - \frac{\lambda_{10}}{(c + \lambda_{10})} \theta_1 C_{21}$
Φ_{21}	$\Phi_{21}^{\mathcal{Q}_0} = \Phi_{21} - \Upsilon_1 - \delta_{01} \lambda_{11} C_{21}$	$\Phi_{21}^{\mathcal{Q}_1} = \frac{\Phi_{21}}{(1 + \lambda_{10}/c)^2} - \Upsilon_1$
Φ_{22}	$\Phi_{22}^{\mathcal{Q}} = \Phi_{22} - \Upsilon_2$	$\Phi_{22}^{\mathcal{Q}} = \Phi_{22} - \Upsilon_2$

where $\Upsilon_i = \Sigma_0 \Lambda_{2i}$, $i = 0, 1, 2$ are matrices of parameter products showing how the stochastic structure and the price of risk affect the change of measure. The solution for $EA_1(N)$ is regular provided that:

$$c + \lambda_{10} > 0. \quad (22)$$

Appendix 2 shows that an exponential-affine MGF yields an exponential-affine bond price (affine yield) model of the form:

$$P_{\tau,t} = \exp[-\gamma_{\tau} - \Psi'_{\tau} X_t]; \quad \tau = 1, \dots, M. \quad (23)$$

Taking logs and maturity differencing gives the forward rate:

$$f_{\tau,t} = \gamma_{\tau+1} - \gamma_{\tau} + [\Psi_{\tau+1} - \Psi_{\tau}]' X_t; \quad \tau = 1, \dots, M. \quad (24)$$

where: $p_{\tau,t} = \ln P_{\tau,t}$. The price coefficients are derived in appendix 3. At this stage, it is convenient to partition Ψ_{τ} conformably with (15) as $\Psi_{\tau} = \{\psi_{1,\tau}, \Psi'_{2,\tau}\}'$. This system is time-recursive. It is also recursive in the sense that $\Psi_{2,\tau}$ does not depend upon $\psi_{1,\tau-1}$ (or $\gamma_{\tau-1}$). This sub-structure is also common to both models:

$$\begin{aligned} \Psi_{2,\tau} &= (\Phi_{22}^{\mathcal{Q}})' \Psi_{2,\tau-1} + J_{2,r} \\ &= (I - (\Phi_{22}^{\mathcal{Q}})')^{-1} (I - ((\Phi_{22}^{\mathcal{Q}})')^{\tau}) J_{2,r} \end{aligned} \quad (25)$$

I assume that the roots of this system are stable under \mathcal{Q} , so this has the asymptote:

$$\Psi_2^* = \lim_{\tau \rightarrow \infty} \Psi_{2,\tau} = (I - (\Phi_{22}^{\mathcal{Q}})')^{-1} J_{2,r} \quad (26)$$

where $\Phi_{22}^{\mathcal{Q}}$ is defined in table 1.

3.1 The $EA_0(N)$ specification

In $\mathbb{EA}_0(N)$, the first slope coefficient follows the recursion:

$$\psi_{1,\tau} = \xi_1^{\mathcal{Q}_0} \psi_{1,\tau-1} + \Phi_{12}^{\mathcal{Q}_0} \Psi_{2,\tau-1}. \quad (27)$$

while the intercepts follow:

$$\begin{aligned} \Delta\gamma_\tau = \gamma_\tau - \gamma_{\tau-1} &= \Psi'_{2,\tau-1} \Theta_2^{\mathcal{Q}_0} + \psi_{1,\tau-1} \theta_1^{\mathcal{Q}_0} - \frac{1}{2} \Psi'_{2,\tau-1} \Sigma_0 \Psi_{2,\tau-1} - \frac{1}{2} \delta_{01} [\psi_{1,\tau-1} + \Psi'_{2,\tau-1} C_{21}]^2 \\ \tau &= 2, \dots, M. \end{aligned} \quad (28)$$

with the parameters defined in table 1. The two quadratic terms show the Jensen effects implied by the constant volatility specification.

Suppose that $\xi_1^{\mathcal{Q}_0} = 1$ but that the other roots of the system are stable. Then (27) yields a well-defined asymptote for $\Psi_{2,\tau}$, but (27) has a unit root and in the limit $\psi_{1,\tau}$ trends in line with τ : $\lim_{\tau \rightarrow \infty} (\psi_{1,\tau+1} - \psi_{1,\tau}) = \Phi_{12}^{\mathcal{Q}_0} (I - (\Phi_{22}^{\mathcal{Q}_0})')^{-1} J_{2,r}$. Substituting these coefficients into (28) and (27) shows that the asymptotic behavior of the forward rate is determined by the final term in (28), which is the Jensen effect associated with $\psi_{1,\tau} y_{1,t}$. This behaves as $(-\frac{1}{2} \delta_{01}^2 \tau^2)$ in the limit.

In the specification of Dewachter and Lyrio (2006), y_1 has a unit root under \mathcal{P} ($\theta_1 = 0, \xi_1 = 1$) but is mean-reverting under \mathcal{Q} : ($|\xi_1^{\mathcal{Q}_0}| = |\xi_1 - \delta_{01} \lambda_{11}| < 1$). In this case, y_1 obeys a near-unit root process and ψ_1^* and f^* are constant. However, because $\xi_1^{\mathcal{Q}_0}$ is close to unity empirically⁷, these asymptotes adopt very large numerical values (respectively positive and negative). Moreover, this specification requires the restriction $\lambda_{11} > 0$. This restriction seems hard to justify a priori and ideally it should be tested in a framework (like $EA_1(N)$) that does not require it as a regularity condition. For example it means that the last term in the expression for the risk premium :

$$\begin{aligned} \rho_{\tau,t} &= -\Psi'_{2,\tau-1} (\Upsilon_0 + \delta_{01} \lambda_{10} C_{21} + (\Upsilon_1 + \delta_{01} \lambda_{11} C_{21}) y_{1,t} + \Upsilon_2 X_{2,t}) \\ &\quad - \psi_{1,\tau-1} \delta_{01} (\lambda_{10} + \lambda_{11} y_{1,t}) \end{aligned} \quad (29)$$

is negative. This shows the the excess return that investors expect on a portfolio that is only exposed to movements in $y_{1,t}$ ⁸. If $\xi_1^{\mathcal{Q}_0}$ is close to unity, this portfolio is closely approximated by a long term bond, since ψ_1^* is extremely large relative to Ψ_2^* for long maturities. In other words, long term bond yields are approximately linear in $y_{1,t}$. In the US at least, their excess returns tend to be positively related to the yield gap, which can be approximated by $y_{1,t} - r_{1,t}$ in this framework. This suggests that the long term risk premium is positively related to $y_{1,t}$ (and negatively related to $r_{1,t}$).

3.2 The $EA_1(N)$ specification

In contrast, $\mathbb{EA}_1(N)$ generates a well-defined yield curve asymptote without any restrictions on $\Phi^{\mathcal{Q}}$. That is because the volatility of $y_{1,t}$ is linear in $y_{1,t}$ and so the equation determining its slope coefficient includes non-linear Jensen terms:

$$\psi_{1,\tau} = \frac{\xi_1 [\psi_{1,\tau-1} + \lambda_{10} + \Psi'_{2,\tau-1} C_{21}]}{1 + [\psi_{1,\tau-1} + \lambda_{10} + \Psi'_{2,\tau-1} C_{21}]/c} - \frac{\xi_1 \lambda_{10}}{1 + \lambda_{10}/c} - \Psi'_{2,\tau-1} \Upsilon_1 - \frac{1}{2} \Psi'_{2,\tau-1} \Sigma_1 \Psi_{2,\tau-1}. \quad (30)$$

⁷For example, the values of δ_{01} and λ_{11} for M0 reported in table 4b imply $\xi_1^{\mathcal{Q}_0} = 0.9940$.

⁸This term is the product of $\psi_{1,\tau-1}$ (which shows the exposure of the τ -maturity log price to unanticipated shocks to $y_{1,t}$) and the associated factor risk premium (the excess return that investors expect for bearing this exposure): $\rho_t^1 = -\delta_{01} (\lambda_{10} + \lambda_{11} y_{1,t})$.

To ensure a regular solution I assume that (22) holds and similarly:

$$[\psi_{1,\tau-1} + \lambda_{10} + \Psi'_{2,\tau-1}C_{21}] + c > 0. \quad (31)$$

As Campbell, Lo, and MacKinlay (1996) note in a similar heteroscedastic yield curve model⁹, the slope parameter ψ_1^* is determined by a quadratic rather than a linear equation and is well-defined even if $|\xi_1^{\mathcal{Q}_1}| \geq 1^{10}$. For the intercept:

$$\Delta\gamma_\tau = (\Theta_2 - C_{21}\theta_1 - \Upsilon_0)' \Psi_{2,\tau-1} - \frac{1}{2} \Psi'_{2,\tau-1} \Sigma_0 \Psi_{2,\tau-1} + c\theta_1 \ln \left[\frac{c + \psi_{1,\tau-1} + \lambda_{10} + \Psi'_{2,\tau-1}C_{21}}{c + \lambda_{10}} \right]. \quad (32)$$

Unit roots are not a problem in $\mathbb{E}A_1(N)$, indeed, they greatly simplify the model structure. Because the volatility of $y_{1,t}$ is proportional to $y_{1,t}$, the associated Jensen effects are found in (30), but not in the equations defining the intercept (32) and the forward rate (24). Substituting $\theta_1 = 0$ shows that γ_τ and hence f^* is independent of ψ_1^* :

$$f^* = (\Theta_2 - \Upsilon_0)' \Psi_2^* - \frac{1}{2} \Psi_2^{*\prime} \Sigma_0 \Psi_2^* \quad (33)$$

where $\Psi_2^* = (I - (\Phi_{22}^{\mathcal{Q}})')^{-1} J_{2,r}$. The forward and expected spot rate asymptotes f^* and r_t^* behave very differently in the unit root $EA_1(N)$ framework: the former is constant, but the latter is driven by a martingale, with the implication that its asymptote changes in line with the inflation trend $y_{1,t}$ ¹¹.

The risk premium for the $\mathbb{E}A_1(N)$ model is:

$$\begin{aligned} \rho_{\tau,t} = & c\theta_1 \ln \left(\frac{c(c + \psi_{1,\tau-1} + \lambda_{10} + \Psi'_{2,\tau-1}C_{21})}{(c + \lambda_{10})(c + \psi_{1,\tau-1} + \Psi'_{2,\tau-1}C_{21})} \right) \\ & - \left\{ \frac{\xi_1 c \lambda_{10} (\Psi'_{2,\tau-1}C_{21} + \psi_{1,\tau-1}) (2c + \lambda_{10} + \Psi'_{2,\tau-1}C_{21} + \psi_{1,\tau-1})}{(c + \lambda_{10} + \Psi'_{2,\tau-1}C_{21} + \psi_{1,\tau-1})(c + \lambda_{10})(c + \psi_{1,\tau-1} + \Psi'_{2,\tau-1}C_{21})} \right\} y_{1,t} \\ & - \Psi'_{2,\tau-1} (\Upsilon_0 + \Upsilon_2 X_{2,t} + \Upsilon_1 y_{1,t}) \end{aligned} \quad (34)$$

The non-linear term on the first line vanishes if there is a unit root ($\theta_1 = 0$). The non-linear term on the second line compensates for exposure to shifts in $y_{1,t}$ while the linear term on the last line is the compensation for the bond's exposure to shifts in $X_{2,t}$. The latter is negligible for a portfolio or security like an ultra-long bond, with a yield that mimics $y_{1,t}$. Both of the non-linear effects disappear if $\lambda_{10} = 0$. The second is positively related to the inflationary trend if $\lambda_{10} \leq 0$; and negatively

⁹Thier model uses a normal approximation to the Cox, Ingersoll, and Ross (1985) process describing the spot rate, due originally to Pearson and Sun (1994).

¹⁰Substituting (26) into (30) gives ψ_1^* as the solution to: $\psi_1^* = \frac{\xi_1(\psi_1^* + \lambda_{10} + \Psi_2^{*\prime}C_{21})}{1 + (\psi_1^* + \lambda_{10} + \Psi_2^{*\prime}C_{21})/c} - \frac{\xi_1 \lambda_{10}}{1 + \lambda_{10}/c} - \Psi_2^{*\prime} \Upsilon_1 - \frac{1}{2} \Psi_2^{*\prime} \Sigma_1 \Psi_2^*$. This may be arranged as: $0 = \vartheta^2 + \vartheta(c(1 - \xi_1) - \zeta) - c\zeta$ where: $\vartheta = \psi_1^* + \lambda_{10} + \Psi_2^{*\prime}C_{21}$; $\zeta = \frac{\lambda_{10}(1 - \xi_1) + \lambda_{10}/c}{1 + \lambda_{10}/c} + \Psi_2^{*\prime}(C_{21} - \Upsilon_1) - \frac{1}{2} \Psi_2^{*\prime} \Sigma_1 \Psi_2^*$. The intercept term $c\zeta$ shows the product of the roots and is a very large negative number empirically. Consequently, one root is a large negative and the other a large positive number. Phase analysis reveals that the recursion (30) selects the positive root.

¹¹Jensen effects associated with the volatility of y_1 (working through its price coefficient ψ_1^*) drive a wedge between r_t^* and f^* . Condition (30) ensures that ψ_1^* increases to point at which the effect of changes in $y_{1,t}$ on the first moments of the system and r_t^* are exactly offset by the Jensen effects working through the second moments.

related if $\lambda_{10} \geq 0$ since:

$$\frac{d^2 \rho_t}{dy_{1,t} d\lambda_{10}} = -\xi_1 c^2 \left(\frac{C_{21} \Psi'_{2,\tau-1} + \psi_{1,\tau-1}}{(c + \lambda_{10})^2} \right) \left(\frac{2c + 2\lambda_{10} + C_{21} \Psi'_{2,\tau-1} + \psi_{1,\tau-1}}{(c + \lambda_{10} + C_{21} \Psi'_{2,\tau-1} + \psi_{1,\tau-1})^2} \right) \leq 0$$

given (22) and (31).

4 Specification tests

Preliminary tests indicated the presence of a unit root in the macroeconomic and yield data as well as suggesting a third order lag structure for (1). Consequently this research focussed on the $\mathbb{E}\mathbb{A}_0(11)$ and $\mathbb{E}\mathbb{A}_1(11)$ specifications, and followed Dewachter and Lyrio (2006) by imposing $\xi_1 = 1$ ¹². It was found that the risk adjustments associated with the real interest rate (the first rows of Υ_1 and Υ_2) were very poorly determined in both specifications and could be eliminated without significantly reducing the likelihood. The resulting empirical version of the $\mathbb{E}\mathbb{A}_0(11)$ specification is reported as M0 and that for $\mathbb{E}\mathbb{A}_1(11)$ as M1 in table 3. The parameter values for these two models are reported in tables 3a and 3b. M0 uses 66 parameters ($\Delta_0(5), \Upsilon_0(4), \Upsilon_1(3), \Upsilon_2(12), P(7), G(3), \Phi(28), \varphi(2), \lambda_{10}$ and λ_{11}) and has a loglikelihood of 6865.2, as shown in table 3. M1 uses 69 parameters ($\Delta_0(4), \Delta_1(5), \Upsilon_0(4), \Upsilon_1(3), \Upsilon_2(12), P(7), G(3), \Phi(28), \varphi(2)$ and λ_{10}) but has a much higher loglikelihood: 6942.8. These two likelihood values are not directly comparable because the models are not nested ($\Delta_1(5) = 0_5$ in M0 and $\lambda_{11} = \delta_{01} = 0$ in M1). However the difference between them certainly raises questions about the validity of M0.

To find out precisely why M0 does so poorly, I constructed the encompassing model reported as M2 in table 3. Note that apart from the error model (5) and (25) are common to both specifications. However, (28) encompasses (32) while (30) and (27) are non-nested. M2 is formed by taking $\mathbb{E}\mathbb{A}_1(11)$ and first replacing (32) by (28). This introduces the parameter δ_{01} into M2. Adding the constant term $-\delta_{01}\lambda_{11}$ into the left hand side of (30) then introduces λ_{11} and allows this hybrid equation to encompass (27). Thus M2 has 71 parameters and encompasses M0 and M1. Yet its loglikelihood is hardly any higher than that of M1 and a standard $\chi^2(2)$ likelihood ratio test of M1 against M2 gives a very high acceptance value. The reason for this is both simple and instructive. Non-zero values of δ_{01} in (28) have the effect of introducing quadratic terms in $\psi_{1,\tau}$ into the intercept and hence the forward rate structure. This effect is strongly rejected by the data. The optimal value of the dummy parameter δ_{01} is almost identically zero and positive values depress the likelihood sharply.¹³

However, δ_{01} dictates the variance of the inflationary trend in M0 and must assume a strictly positive value in that model. For this model to fit the data, the effect of δ_{01} and ψ_1 on the forward rate structure must be offset by variations in other parameters. M0 also performs poorly relative to M1 and M2 because it does not allow for the conditional heteroscedasticity of the macroeconomic variables. Consequently, the $\chi^2(5)$ likelihood ratio test statistic ($\chi^2(5) = 155.9, p = 0$) indicates an overwhelming rejection of M0¹⁴, the standard macro-finance model.

¹²Subsequent diagnostic tests show that the local maximum is $\xi_1 = 1.0105$ (for M1) which lies just outside the range consistent with macroeconomic stability. A grid search over the range 0.9 to 1.0 strongly suggests that the corner solutions $\xi_1 = 1.0$ reported here for M0 and M1 are both global maxima.

¹³Since the second additional parameter λ_{11} only enters M2 in the product $\delta_{01}\lambda_{11}$, this has negligible effect on the fit.

¹⁴M0 has 5 more degrees of freedom than M2 because it employs the restrictions ($\Delta_1 = 0_5$).

4.1 The empirical macro-model

In view of the poor performance of the $\mathbb{EA}_0(11)$ specification, I now focus upon the detailed results for the $\mathbb{EA}_1(11)$ specification: M1. At the core of this model there is a heteroscedastic macro VAR with a steady state solution dictated by the restrictions (3). These ensure that the equilibrium inflation rate is in line with the nominal factor $y_{1,t}$ (plus φ_1) while the spot rate is in line with this implicit inflation rate and the real factor $y_{2,t}$ (plus φ_2). Model estimates of these factors are shown in Chart 3, along with their 95% confidence intervals. Most of the work is done by the nominal trend. Since this has a unit root it can be interpreted as the market's long run inflation expectation. This moves down in response to the deep recessions of the early 1980s and early 1970s. However, the most dramatic fall appears to have occurred in 1997-98 following the independence of the Bank of England. In contrast, the adoption of monetary targets in November 1992 appears to have had little effect.

Since $y_{1,t}$ has a non-central χ^2 distribution, the downside variance is smaller than the upside, but this asymmetry is only apparent at the end of the estimation period when the underlying inflation rate is low. This variable is drives the conditional heteroscedasticity in the macro variables. Their one-quarter-ahead forecasts values and 95% confidence intervals are shown in chart 4. This heteroscedasticity is particularly pronounced in the case of the rate of inflation since the parameter δ_{04} is effectively zero making the variance of inflation proportional to the inflation trend. This is particularly low over the last five years of the estimation period, consistent with the *ex post* stability of inflation over this period (chart 4b).

How firmly does this stochastic trend anchor inflation and interest rates? This question depends upon whether output, inflation & interest rates the real factor $y_{2,t}$ are cointegrated with the non-stationary nominal factor $y_{1,t}$. This was checked by running ADF tests on the residuals of these four equations, which decisively reject non-stationarity. These variables adjust surprisingly quickly and smoothly to their equilibrium values. This is clear from the impulse responses, which show the dynamic effects of innovations in the macroeconomic variables on the system. Because these innovations are correlated empirically, we work with orthogonalized innovations using the triangular factorization defined in (4). The orthogonalized impulse responses show the effect on the macroeconomic system of increasing each of these shocks by one percentage point for just one period using the Wald representation of the system as described for example in Hamilton (1994).

This arrangement is affected by the ordering of the macroeconomic variables in the vector x_t , making it sensible to order the variables in terms of their likely degree of exogeneity or sensitivity to contemporaneous shocks. The unobservables y_t are supposed to reflect exogenous expectational influences and are ordered first in the sequence. This means that η_t (which are orthogonal to y_t) can be interpreted as surprises in output, inflation and interest rates - shocks that are not anticipated by the markets. Following convention the output gap is ordered before inflation. Interest rates are placed after these variables on the view that monetary policy reacts relatively quickly to disturbances in output and prices.

Chart 5 reports the results of this exercise. The continuous line shows the effect of each surprise on the spot rate, the dashed line the output effect and the dotted line the inflation effect. Elapsed time is measured in quarters. Panel (i) shows that a shock to the stochastic trend increases output, inflation and interest rates immediately. By construction, there is a one-for-one effect on inflation and interest rates in the long run and no effect on output. The latent variables both act as leading indicators for output, inflation and interest rates. In the case of $y_{2,t}$ the output effect reverses after

the third year in response to an increase in real rates, setting up a damped cyclical variation. The system is back close to its initial level after 10 years. The other panels show similarly fast responses. Panel (iii) shows the effect of an output shock. The initial inflation and interest rate effects are similar, but output falls back after three years. Panel (iv) shows the effect of an inflation surprise. The initial effect on the spot rate is muted, so real interest rates fall. However output falls, reflecting real balance effects. The fall in output then has the effect of reversing the rise in inflation and interest rates, setting up damped cycles in these variables. The effect of a rise in the spot rate is shown in the final panel. The initial effect is to depress output, and then inflation responds with a short lag.

Taken together, this model gives a plausible description of the macroeconomic dynamics, in contrast to many VAR-type results (Grilli and Roubini (1996)). Its use of Kalman filters to pick up the effect of unobservable expectational influences seems to solve the notorious price puzzle - the tendency for increases in policy interest rates to anticipate inflationary developments and apparently cause inflation. The nominal filter dictates the long run equilibrium of the macroeconomy and its volatility. These effects are persistent, but the responses of the macroeconomic variables to deviations between inflation and its long run trend ($\pi_t - y_{1,t}$) as well as surprises in output and interest rates are remarkably rapid. They are largely exponential in nature, suggesting that monetary policy has been effective in securing its objectives quickly, without significant policy reversals or cycles.

4.2 The empirical yield model

The behavior of the yield curve is dictated by the factor loadings. These are depicted in Chart 6, as a function of maturity (expressed in quarters). The first panel shows the loadings on $y_{1,t}$ (broken line) and $y_{2,t}$ (continuous line). The second panel shows the loadings on π (dotted line), g (broken line) and the spot rate (continuous line). The spot rate provides the link between the macroeconomic model and the term structure. Since it is the 3 month yield, this variable has a unit coefficient at a maturity of one quarter and other factors have a zero loading. The spot rate loadings decline over the next few years, reflecting the adjustment of the spot rate towards $y_{1,t}$ and $y_{2,t}$. This variable determines the slope of the short-term yield curve. Five year maturity yields are strongly influenced by the behavior of the real rate factor $y_{2,t}$. The loading on this factor then fades gradually over the longer maturities, allowing this to act as a ‘curvature’ factor. In contrast, the loading upon $y_{1,t}$ moves up to unity and then increases gradually with maturity over the 2 to 15 year maturity range, so that it acts as a ‘level’ factor. The loadings for output and inflation have a humped shape, but are relatively small.

Importantly, the effect of the inflation trend $y_{1,t}$ on the risk premia is not restricted in any way in the \mathbb{EA}_1 framework. As noted earlier, λ_{10} largely determines the effect of the inflationary trend on the risk premium at the long end. This parameter is significantly negative in M1, meaning that the inflationary trend $y_{1,t}$ has a positive effect on the risk premia. Reflecting the above-mentioned results of Duffee (2002) and others for the US, this is matched by a negative spot rate effect, so that the time variation in the 15 year premia is largely captured by: $y_{1,t} - r_{1,t}$. Other influences are small. However, as explained, λ_{10} is forced to adopt a small positive value in M0 in order to keep the model dynamics stable under \mathcal{Q} , so that the effect of this trend on the long term risk premium is negative in that model.

5 Conclusion

This paper models the relationship between the gilt edged market and the UK economy in a way that rules out arbitrage and allows for the likely non-stationarity of the data. To do this I adapt the \mathbb{EA}_1 model of the mainstream finance literature, handling the asymptotic problems posed by the stochastic trend using a latent variable with stochastic first and second moments. This variable dictates the asymptotic behavior of inflation and short term interest rates but does not influence the asymptotic forward rate, which is constant in this framework.

The \mathbb{EA}_1 specification provides a sensible description of the term structure without restraining the parameters dictating the dynamic responses, under either historical or risk-neutral measures. It gives a better representation of the behavior of the economy and the bond market than does the standard macro-finance model, which is decisively rejected by the data. The rejection of \mathbb{EA}_0 reflects its failure to allow for the conditionally heteroscedasticity nature of the macroeconomic data as well as its awkward yield characteristics. Compared to the mainstream finance model of the bond market, the macro-finance version of the \mathbb{EA}_1 model can use a relatively large number of factors (11) because the parameters of the model are informed by macroeconomic as well as yield data (with a total of 1020 data points). However, the adjustment speed means that the behavior of the yield curve is largely dictated by three factors: the inflation asymptote, the real factor and the spot rate. The model is consistent with the traditional three-factor finance specification in this respect, but links these factors into the behavior of the macroeconomy.

Besides providing a sensible model of the term structure, the empirical results reported here provide insights into the working of the UK monetary system and the macroeconomy. In particular, the significance of the latent variables strongly suggests that to understand the behavior of the economy over this period it is important to allow for extraneous influences on inflationary expectations and real rates of return¹⁵. The dramatic fall in the underlying rate of inflation when the new monetary arrangements were introduced in May 1997 provides a good example of this kind of exogenous influence. The degree and persistence of the conditional heteroscedasticity revealed in UK macroeconomic data is also important from a policy perspective, suggesting that low inflation can help stabilize the economy. This research opens the way to a much richer term structure specification, incorporating the best features of both macro-finance and mainstream finance models.

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¹⁵This reflects the basic empirical finding of US macro-finance studies, that macro variables alone cannot provide an adequate explanation of long term yields. Extraneous variables can only influence an arbitrage free (non-defaultable) bond market if they influence the short term interest rate, indicating that they should be included in the macromodel.

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6 Appendix 1: The state-space representation of the model

Stacking (4) puts the system into state space form (5), where :

$$\Theta' = \{\theta, K + \Phi_0\theta, 0_{N-k-n,1}\}; \quad (35)$$

$$\Phi = \begin{bmatrix} \Xi & 0_{k,n} & \dots & 0_{k,n} & 0_{k,n} \\ \Phi_0\Xi & \Phi_1 & \dots & \Phi_{l-1} & \Phi_l \\ 0_{n,k} & I_n & \dots & 0_{n^2} & 0_{n^2} \\ 0_{n,k} & 0_{n^2} & \dots & I_n & 0_{n^2} \end{bmatrix} = \begin{bmatrix} \xi_1 & 0'_{N-1} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}.$$

where the last matrix partitions Φ conformably with (15). Similarly:

$$C = \begin{bmatrix} I_k & 0_{k,n} & 0_{k,(N-k-n)} \\ \Phi_0 & G & 0_{n,(N-k-n)} \\ 0_{(N-k-n),k} & 0_{(N-k-n),n} & 0_{(N-k-n)^2} \end{bmatrix} = \begin{bmatrix} 1 & 0'_{N-1} \\ C_{21} & C_{22} \end{bmatrix}.$$

where the last matrix partitions C conformably with (15). Comparing this with the partition (35), note that:

$$\Phi_{21} = C_{21}\xi_1. \quad (36)$$

The error structure of (15) follows from (14) as:

$$\begin{aligned} W_{2,t+1} &= C_{21}w_{1,t+1} + \sqrt{y_{1,t}}U_{2,t-1} + V_{2,t+1} \\ V_{2,t+1} &\sim N(0_{N-1}, \Delta_{02}); \quad U_{2,t+1} \sim N(0_{N-1}, \Delta_{12}) \end{aligned} \quad (37)$$

where: $D_{s2} = \text{Diag}[\delta_{s2}, \dots, \delta_{s,(k+n)}, 0_{1,N-k-n}]$; $s = 0, 1$. This implies the conditional MGFs:

$$\begin{aligned} E_t[\exp[\nu_2' C_{22} V_{2,t+1}]] &= \exp\left[\frac{1}{2}[\nu_2' \Sigma_0 \nu_2]\right]; \\ E_t[\exp[\nu_2' C_{22} U_{2,t+1}]] &= \exp\left[\frac{1}{2}[\nu_2' \Sigma_1 \nu_2]\right]; \end{aligned} \quad (38)$$

where: $\Sigma_s = C_{22} D_{s2} C_{22}'$; $s = 0, 1$.

7 Appendix 2: Exponential-affine bond price models

This appendix shows how exponential-affine bond price (affine yield) models and their risk premia can be derived when the MGF of the state vector is exponential-affine under both \mathcal{P} and \mathcal{Q} . The MGF for measure \mathcal{P} is $E_t[\exp[\nu'X_{t+1}] | X_t]$ where ν is a vector of Laplace constants. Using (17) allows us to shift to \mathcal{Q} :

$$\begin{aligned} L[\nu, X_t; \Lambda] &= E_t^{\mathcal{Q}}[\exp[\nu'X_{t+1}] | X_t] \\ &= E_t[\exp[n_{t+1} + \nu'X_{t+1}] | X_t] \end{aligned} \quad (39)$$

where Λ contains the relevant parameters of Λ_t . Recall that the defining characteristic of these processes is that the MGF for X_{t+1} is an exponential-affine function of X_t :

$$L[\nu, X_t; \Lambda] = \exp[a(\nu; \Lambda) + b(\nu; \Lambda)'X_t] \quad (40)$$

Setting Λ to zero gives the MGF under \mathcal{P} :

$$\begin{aligned} L[\nu, X_t; 0] &= E_t[\exp[\nu'X_{t+1}] | X_t] \\ &= \exp[a(\nu; 0) + b(\nu; 0)'X_t] \end{aligned} \quad (41)$$

The MGF of an exponential-affine process can be used to generate the parameters and risk premia of the associated yield model, as well as its moments. First, (39) can be used as a moment generating function to find the expected value of X_t under \mathcal{Q} . This is generated by an affine process resembling (15). Second, the MGF can be used to generate the arbitrage-free exponential-affine bond price models of the form (23). To obtain the price coefficients, substitute the risk neutral expectation (17) into the pricing kernel (or discounted risk neutral expectation, (Campbell, Lo, and MacKinlay (1996), Cochrane (2000)) :

$$P_{\tau,t} = e^{-r_{1,t}} E_t^{\mathcal{Q}}[P_{\tau-1,t+1} | X_t]; \quad \tau = 1, \dots, M. \quad (42)$$

Using (23) to replace $P_{\tau-1,t+1}$ in (42) represents $P_{\tau,t}$ by (39) or (40) with $\nu = -\Psi_{\tau-1}$:

$$\begin{aligned} P_{\tau,t} &= e^{-r_{1,t}} E_t^{\mathcal{Q}}[\exp[-\gamma_{\tau-1} - \Psi'_{\tau-1} X_{t+1}] | X_t]; \\ &= e^{-\gamma_{\tau-1} - r_{1,t}} L[-\Psi_{\tau-1}, X_t; \Lambda] \end{aligned} \quad (43)$$

$$= \exp[-\gamma_{\tau-1} - J'_r X_t + a(-\Psi_{\tau-1}; \Lambda) + b(-\Psi_{\tau-1}; \Lambda)'X_t]. \quad (44)$$

where $J_s = \{j_{1,s}, J'_{2,s}\}$ is a selection vector such that:

$$x_{s,t} = J'_s X_t = J'_{2,s} X_{2,t}; \quad s = 2, \dots, N.$$

Equating the intercept and slope coefficients in the exponent with those in (23) gives a recursion relationship for the parameters:

$$\begin{aligned} \Psi_{\tau} &= J_r - b(-\Psi_{\tau-1}; \Lambda) \\ \gamma_{\tau} &= \gamma_{\tau-1} - a(-\Psi_{\tau-1}; \Lambda) \\ \tau &= 2, \dots, M. \end{aligned}$$

Since $-p_{\tau,t} = r_{1,t}$ for $\tau = 1$, the coefficients of this system have the starting values:

$$\gamma_1 = 0; \quad \Psi_1 = J_r. \quad (45)$$

The coefficients Ψ_τ and γ_τ of (23) are functions of (Θ^Q, Φ^Q) and determine the τ -period discount yield:

$$\begin{aligned} r_{\tau,t} &= -p_{\tau,t}(\Theta^Q, \Phi^Q)/\tau & (46) \\ &= \alpha_\tau(\Theta^Q, \Phi^Q) + \beta'_\tau(\Theta^Q, \Phi^Q)X_t; \text{ where :} \\ \alpha_\tau &= \gamma_\tau(\Theta^Q, \Phi^Q)/\tau; \quad \beta_\tau = \Psi_\tau(\Theta^Q, \Phi^Q)/\tau. \end{aligned}$$

The slope coefficients of the yield system β_τ are known as ‘factor loadings’ and depend critically upon the eigenvalues of the adapted macroeconomic system (21). Stacking the m yield equations (46) and adding an error vector e_t gives a multivariate regression model for the m -vector of yields r_t :

$$r_t = \alpha(\Theta^Q, \Phi^Q) + B(\Theta^Q, \Phi^Q)'X_t + e_t \quad (47)$$

$$= \alpha_0(\Theta^Q, \Phi^Q) + B'_0(\Theta^Q, \Phi^Q)y_t + \sum_{l=1}^L B'_l(\Theta^Q, \Phi^Q)z_{t+1-l} + e_t \quad (48)$$

$$e_t \sim N(0, \bar{P});$$

$$\bar{P} = \text{Diag}[\rho_1, \rho_2, \dots, \rho_m].$$

where e_t is an error vector. The standard assumption in macro-finance models is that this represents measurement error which is orthogonal to the errors W_t in the macroeconomic system (5).

Finally, (40) can be used to generate the risk premia. These depend upon the gross expected rate of return, which is obtained by taking the expected payoff on a τ -period bond after one period $E_t[P_{\tau-1,t+1}]$ and dividing by its current price $P_{\tau,t}$. Taking the natural logarithm expresses this as a percentage return and subtracting the spot rate $r_{1,t}$ then gives the expected excess return or risk premium: $\rho_{\tau,t} = \log E_t[P_{\tau-1,t+1}] - \log[P_{\tau,t}] - r_{1,t}$. Setting $E_t[P_{\tau-1,t+1}] = e^{-\gamma_{\tau-1}} \exp[a(-\Psi_{\tau-1}; 0) + b(-\Psi_{\tau-1}; 0)'X_t]$ (using (39), (41) and (23)) and substituting (44) into the second term shows that this depends entirely upon Λ , which determines the difference between the two measures :

$$\begin{aligned} \rho_{\tau,t} &= \log L[-\Psi_{\tau-1}, X_t; 0] - \log L[-\Psi_{\tau-1}, X_t; \Lambda] & (49) \\ &= a(-\Psi_{\tau-1}; 0) - a(-\Psi_{\tau-1}; \Lambda) + (b(-\Psi_{\tau-1}; 0) - b(-\Psi_{\tau-1}; \Lambda))'X_t \\ &\tau = 1, \dots, M. \end{aligned}$$

The next appendix shows how these results can be applied to models $\mathbb{E}\mathbb{A}_0(N)$ and $\mathbb{E}\mathbb{A}_1(N)$.

8 Appendix 3 : The $\mathbb{E}\mathbb{A}_0(N)$ and $\mathbb{E}\mathbb{A}_1(N)$ specifications

This appendix shows how the MGF (39) can be used to derive the adapted dynamic systems, bond prices and risk premia for models $\mathbb{E}\mathbb{A}_0(N)$ and $\mathbb{E}\mathbb{A}_1(N)$.

First, write the probability density of X_{t+1} as the product of the marginal density of $y_{1,t+1}$ and the conditional density of $X_{2,t+1} | y_{1,t+1}$, obtained by substituting (15) into (15):

$$\begin{aligned} X_{2,t+1} &= \Theta_2 + \Phi_{21}y_{1,t} + C_{21}w_{1,t+1} + \Phi_{22}X_{2,t} + C_{22}(\sqrt{y_{1,t}}U_{2,t+1} + V_{2,t+1}) \\ &= F_2 + C_{21}y_{1,t+1} + \Phi_{22}X_{2,t} + C_{22}(\sqrt{y_{1,t}}U_{2,t+1} + V_{2,t+1}) \end{aligned} \quad (50)$$

(using 36) where $F_2 = \Theta_2 - C_{21}\theta_1$. Putting (19) and (50) into (39) and noting that

the $y_{1,t+1}$; $V_{2,t+1}$ and $U_{2,t+1}$ are independent:

$$\begin{aligned}
L[\nu, X_t; \Lambda] &= E_t[\exp[-(\omega_t + \lambda_{1,t}y_{1,t+1} + \Lambda'_{2,t}C_{22}V_{2,t+1}) \\
&\quad + \nu_1y_{1,t+1} + \nu'_2(F_2 + C_{21}y_{1,t+1} + \Phi_{22}X_{2,t} + C_{22}(V_{2,t+1} + \sqrt{y_{1,t}}U_{2,t+1}))] \\
&= \exp[-\omega_t + \nu'_2(F_2 + \Phi_{22}X_{2,t})] \\
&\quad \times E_t[\exp[(\nu_2 - \Lambda_{2,t})'C_{22}V_{2,t+1}]] \times E_t[\exp[\nu'_2C_{22}U_{2,t+1}\sqrt{y_{1,t}}]] \\
&\quad \times E_t[\exp[(\nu_1 - \lambda_{1,t} + \nu'_2C_{21})y_{1,t+1} | y_{1,t}]] \tag{51}
\end{aligned}$$

In $\mathbb{EA}_0(N)$ these errors are all Gaussian and are evaluated using (??), (8) and (38):

$$\begin{aligned}
L[\nu, X_t; \Lambda] &= \exp[-\omega_t + \nu'_2(F_2 + \Phi_{22}X_{2,t}) + (\nu_1 - \lambda_{1,t} + \nu'_2C_{21})(\theta_1 + \xi_1y_{1,t})] \\
&\quad \times \exp\left[\frac{1}{2}[(\nu_2 - \Lambda_{2,t})'\Sigma_0(\nu_2 - \Lambda_{2,t}) + \frac{1}{2}\delta_{01}[\nu_1 - \lambda_{1,t} + \nu'_2C_{21}]^2]\right] \tag{52}
\end{aligned}$$

The probability density integral $L[0_N, X_t; \Lambda]$ is normalized to unity using: $\omega_t = \frac{1}{2}(\Lambda'_{2,t}\Sigma_0\Lambda_{2,t} + \frac{1}{2}\delta_{01}\lambda_{1,t}^2) - \lambda_{1,t}(\theta_1 + \xi_1y_{1,t})$. This restriction simplifies the MGF to:

$$\begin{aligned}
L[\nu, X_t; \Lambda] &= \exp[\nu'_2(F_2 + \Phi_{22}X_{2,t}) - \nu'_2\Sigma_0\Lambda_{2,t} + (\nu_1 + \nu'_2C_{21})(\theta_1 + \xi_1y_{1,t}) \\
&\quad + \frac{1}{2}\nu'_2\Sigma_0\nu_2 + \frac{1}{2}\delta_{01}[\nu_1 + \nu'_2C_{21}]^2 - \delta_{01}\lambda_{1,t}[\nu_1 + \nu'_2C_{21}]] \tag{53}
\end{aligned}$$

Differentiating w.r.t. $\{\nu_1, \nu_2\}$ and setting these parameters to zero gives the dynamic system under \mathcal{Q}_1 reported in (21) and table 1. This uses the risk adjustments (18). The price coefficient systems (25), (27) and (28) follow by substituting $\nu = -\Psi_{\tau-1}$ and (18) into (53), putting this in (43) and equating the coefficients of X_t in the exponent with those in (23). The risk premia follow by substituting $\nu = -\Psi_{\tau-1}$ into (49) and obtaining $L[\nu, X_t; \Lambda]$ by setting the risk coefficients to zero.

For the $\mathbb{EA}_1(N)$ model, the density of $y_{1,t+1}$ is given by (8). Using this to evaluate the expectation in the last line in (51):

$$\begin{aligned}
L[\nu, X_t; \Lambda] &= \exp[-\omega_t + \nu'_2(F_2 + \Phi_{22}X_{2,t})] \tag{54} \\
&\quad \times \exp\left\{\frac{1}{2}[(\nu_2 - \Lambda_{2,t})'\Sigma_0(\nu_2 - \Lambda_{2,t}) + y_{1,t}\nu'_2\Sigma_1\nu_2]\right\} \\
&\quad \times \exp\left\{\left[\frac{\xi_1[\nu_1 - \lambda_{1,t} + \nu'_2C_{21}]y_{1,t}}{1 - [\nu_1 - \lambda_{1,t} + \nu'_2C_{21}]/c}\right] - c\theta_1 \ln[1 - [\nu_1 - \lambda_{1,t} + \nu'_2C_{21}]/c]\right\}.
\end{aligned}$$

This probability density is normalized using:

$$\omega_t = \frac{1}{2}\{\Lambda'_{2,t}\Sigma_0\Lambda_{2,t} - \frac{\lambda_{1,t}\xi_1y_{1,t}}{1 + \lambda_{1,t}/c} - c\theta_1 \ln[1 + \lambda_{1,t}/c]\}. \tag{55}$$

Substituting this back:

$$\begin{aligned}
L[\nu, X_t; \Lambda] &= \exp\left\{\nu'_2F_2 + \frac{1}{2}\nu'_2\Sigma_0\nu_2 - \nu'_2\Sigma_0\Lambda_{2,t} - c\theta_1 \ln[1 - [\nu_1 - \lambda_{1,t} + \nu'_2C_{21}]/c]\right. \\
&\quad \left.+ c\theta_1 \ln[1 + \lambda_{1,t}/c] + \nu'_2\Phi_{22}X_{2,t}\right. \tag{56} \\
&\quad \left.+ y_{1,t} \left[\frac{\xi_1[(\nu_1 - \lambda_{1,t} + \nu'_2C_{21})y_{1,t}]}{1 - [\nu_1 - \lambda_{1,t} + \nu'_2C_{21}]/c} + \frac{\lambda_{1,t}\xi_1}{1 + \lambda_{1,t}/c} + \frac{1}{2}\nu'_2\Sigma_1\nu_2\right]\right\}.
\end{aligned}$$

Substituting (18) then yields:

$$\begin{aligned}
L[\nu, X_t; \Lambda] = & \exp\{\nu'_2 F_2 + \frac{1}{2}\nu'_2 \Sigma_0 \nu_2 - \nu'_2 \Upsilon_0 - c\theta_1 \ln[1 - (\nu_1 - \lambda_{10} + \nu'_2 C_{21})/c] \\
& + c\theta_1 \ln[1 + \lambda_{10}/c] + \nu'_2 (\Phi_{22} - \Upsilon_2) X_{2,t} \\
& + y_{1,t} \left[\frac{\xi_1 (\nu_1 - \lambda_{10} + \nu'_2 C_{21})}{1 - (\nu_1 - \lambda_{10} + \nu'_2 C_{21})/c} + \frac{\xi_1 \lambda_{10}}{1 + \lambda_{10}/c} - \nu'_2 \Upsilon_2 + \frac{1}{2}\nu'_2 \Sigma_1 \nu_2 \right]\}.
\end{aligned} \tag{57}$$

Differentiating w.r.t. $\{\nu_1, \nu_2\}$ and setting these parameters to zero gives the dynamic system under \mathcal{Q}_1 reported in (21) and table 1. The price coefficient systems (25), (30) and (32) follow by substituting $\nu = -\Psi_{\tau-1}$, into (39) substituting this into (43) and equating the coefficients of X_t in the exponent with those in (23).

9 Appendix 4 : The Kalman filter and the likelihood function

In this model the unobservable variables are modelled using the Extended Kalman Filter with (2) (Harvey (1989), Duffee and Stanton (2004)). This method assumes that the revisions are approximately normally distributed:

$$\begin{aligned}
\varepsilon_{t+1} & \sim N(0, Q_t) \\
\text{where : } Q_t & = Q_0 + Q_1 y_{1,t} \\
Q_i & = \text{Diag}\{\delta_{i0}^2, \delta_{i1}^2\}; \quad i = 1, 2.
\end{aligned}$$

I represent expectations conditional upon the information available to the econometrician with a 'hat' (so that $\hat{y}_t = \hat{E}_t y_t; \hat{y}_{s,t} = \hat{E}_t y_s; s \geq t$) and define the covariance matrices:

$$\hat{P}_t = \hat{E}_t (y_t - \hat{y}_t)(y_t - \hat{y}_t)'; \tag{58}$$

$$\begin{aligned}
\hat{P}_{t+1,t} & = \hat{E}_t (y_{t+1} - \hat{y}_{t+1,t})(y_{t+1} - \hat{y}_{t+1,t})' \\
& = \Xi \hat{P}_t \Xi' + Q_t;
\end{aligned} \tag{59}$$

where:

$$y_{t+1,t} = \hat{E}_t y_{t+1} = \theta + \Xi \hat{y}_t. \tag{60}$$

Similarly, using (1):

$$z_{t+1} = K + \hat{z}_{t+1,t} + G\eta_{t+1} + \Phi_0(y_{t+1} - \hat{y}_{t+1,t}) \tag{61}$$

where:

$$\hat{z}_{t+1,t} = \Phi_0 \hat{y}_{t+1,t} + \sum_{l=1}^L \Phi_l z_{t+1-l}; \tag{62}$$

and using (47):

$$r_{t+1} = \hat{r}_{t+1,t} + B'_0(y_t - \hat{y}_{t+1,t}) + B'_1(z_{t+1} - \hat{z}_{t+1,t}) + e_{t+1} \tag{63}$$

where:

$$\hat{r}_{t+1,t} = \alpha + B'_{0\hat{y}t+1,t} + B'_1 \hat{z}_{t+1,t} + \sum_{l=2}^L B'_l z_{t+2-l}. \tag{64}$$

The t -conditional covariance matrix for this $t+1$ system is:

$$\begin{bmatrix} \sum_{rr} \\ \sum_{rz} \\ \sum_{ry} \end{bmatrix} \begin{bmatrix} \sum_{rz} \\ \sum_{zz} \\ \sum_{zy} \end{bmatrix} \begin{bmatrix} \sum_{ry} \\ \sum_{zy} \\ \sum_{yy} \end{bmatrix} = \hat{E}_t \begin{bmatrix} r_{t+1} - \hat{r}_{t+1,t} \\ z_{t+1} - \hat{z}_{t+1,t} \\ y_{t+1} - y_{t+1,t} \end{bmatrix} \cdot \begin{bmatrix} r_{t+1} - r_{t+1,t} & z_{t+1} - z_{t+1,t} & y_{t+1} - y_{t+1,t} \end{bmatrix} \tag{65}$$

where:

$$\begin{aligned}
\sum_{rr} &= \bar{P} + B_1 S_t B_1' + (B_1 \Phi_0 + B_0) \hat{P}_{t+1,t} (B_1 \Phi_0 + B_0)' \\
\sum_{rz} &= B_1 S_t + (B_1 \Phi_0 + B_0) \hat{P}_{t+1,t} \Phi_0' \\
\sum_{ry} &= (B_1 \Phi_0 + B_0) \hat{P}_{t+1,t} \\
\sum_{zz} &= \Phi_0 \hat{P}_{t+1,t} \Phi_0' + S_t \\
\sum_{zy} &= \Phi_0 \hat{P}_{t+1,t} \\
\sum_{yy} &= \hat{P}_{t+1,t} \\
S_t &= G[S_0 + S_1 y_{t,t}] G'
\end{aligned}$$

where $S_i = \text{Diag}\{\delta_{i1}^2, \dots, \delta_{in}^2\}$; $i = 0, 1$. This allows the expectations to be updated as:

$$\hat{y}_{t+1} = \hat{y}_{t+1,t} + [\sum_{yr} \sum_{yz}] \left[\begin{array}{cc} \sum_{rr} & \sum_{rz} \\ \sum_{zr} & \sum_{zz} \end{array} \right]^{-1} \begin{bmatrix} r_{t+1} - \hat{r}_{t+1,t} \\ z_{t+1} - \hat{z}_{t+1,t} \end{bmatrix} \quad (66)$$

$$\hat{P}_t = \hat{P}_{t+1,t} - [\sum_{yr} \sum_{yz}] \left[\begin{array}{cc} \sum_{rr} & \sum_{rz} \\ \sum_{zr} & \sum_{zz} \end{array} \right]^{-1} \left[\begin{array}{c} \sum_{yr} \\ \sum_{yz} \end{array} \right] \quad (67)$$

The (log) likelihood for period $t + 1$ is thus:

$$L_{t+1} = k - \frac{1}{2} \ln \left(\text{Det} \left[\begin{array}{cc} \sum_{rr} & \sum_{rz} \\ \sum_{zr} & \sum_{zz} \end{array} \right] \right) \quad (68)$$

$$- \frac{1}{2} [r_{t+1} - r_{t+1,t} \quad z_{t+1} - z_{t+1,t}] \left[\begin{array}{cc} \sum_{rr} & \sum_{rz} \\ \sum_{zr} & \sum_{zz} \end{array} \right]^{-1} \begin{bmatrix} r_{t+1} - \hat{r}_{t+1,t} \\ z_{t+1} - \hat{z}_{t+1,t} \end{bmatrix}. \quad (69)$$

The loglikelihood for the full sample follows by iterating (59), (60), (66) and (67) forward given suitable starting values; substituting (62) and (64) then averaging (68) over $t = 1, \dots, T$.

Table 2: Data Summary Statistics 1979Q4-2004Q2

	π	g	r_1	r_4	r_8	r_{12}	r_{20}	r_{28}	r_{40}	r_{68}
<i>Mean</i>	5.16216	-2.28634	8.85314	8.51549	8.58569	8.64373	8.7401	8.8052	8.80275	8.43876
<i>Std.</i>	3.86526	2.36017	3.72524	3.15488	2.98795	2.91517	2.88369	2.90559	2.91513	2.71249
<i>Skew.</i>	1.90093	-0.25013	0.378187	0.16422	0.122675	0.11968	0.09619	0.07360	0.07454	0.07454
<i>Kurt.</i>	3.3376	1.5923	0.9843	0.8858	0.9994	1.1071	1.1480	1.1128	1.3689	1.3689
<i>Auto.</i>	0.9921	0.4632	0.9815	0.9892	0.9923	0.9944	0.9953	0.9963	0.9969	0.9971
<i>KPSS</i>	0.7611	0.2677	0.941	1.0010	1.1030	1.1260	1.1394	1.1413	1.1384	1.1174
<i>ADF</i>	-4.1295	-3.1735	-2.3045	-2.1995	-1.9975	-1.7735	-1.3685	-1.2255	-1.0665	-0.9975

Output gap (g) is from Oxford Economic Forecasts; Inflation (π) and 3 month Treasury bill rate (r) are from Datastream. Yield data are UK Gilt edged discount bond equivalent data compiled by Bank of England. *Mean* denotes sample arithmetic mean expressed as percentage p.a.; *Std.* standard deviation and *Skew.* & *Kurt.* are standard measures of skewness (third moment) and excess kurtosis (fourth moment). *KPSS* is the Kwiatowski et al (1992) statistic testing the null hypothesis of level stationarity and *ADF* is the Adjusted Dickey-Fuller statistic testing the null hypothesis of non-stationarity. The 5% significance levels are 0.463 and 2.877 respectively.

Table 3: Model Evaluation

Model	Parameters		Loglikelihood		
	(M) Specification*	k(M)	k(2)-k(M)	L(M)	2x(L(2)-L(M))
0	EA ₀ (N)	65	6	6865.20	155.80
	χ_{95}				12.59
	p				0.00
1	EA ₁ (N)	69	2	6942.80	0.60
	χ_{95}				5.99
	p				0.74
2	Hybrid	71		6943.10	

(*) Model specification EA_m(N), where:

m = number of variables conditioning volatility

N = number of state variables.

Table 4a: The dynamic structure of Model M1

(asymptotic t-values in parentheses.)

Parameter	M0	M1	Parameter	M0	M1
Φ_1			Φ_2		
$\phi_{1,11}$	0.9107 (6.03)	0.9080 (6.33)	$\phi_{2,11}$	0.2614 (1.93)	0.2273 (1.81)
$\phi_{1,12}$	0.0975 (4.06)	0.1051 (4.33)	$\phi_{2,12}$	-0.1635 (2.21)	-0.1629 (2.21)
$\phi_{1,13}$	-0.0414 (1.19)	-0.0357 (1.02)	$\phi_{2,13}$	-0.0201 (2.01)	-0.0166 (1.47)
$\phi_{1,21}$	0.5489 (6.00)	0.5175 (5.66)	$\phi_{2,21}$	-0.3842 (3.33)	-0.3924 (3.53)
$\phi_{1,22}$	0.5175 (9.66)	0.4861 (8.56)	$\phi_{2,22}$	0.4844 (4.93)	0.4724 (5.19)
$\phi_{1,23}$	0.0411 (0.10)	0.0410 (0.09)	$\phi_{2,23}$	-0.0147 (0.23)	-0.0147 (0.21)
$\phi_{1,31}$	-.0302 (0.41)	-0.0290 (0.37)	$\phi_{2,31}$	0.0973 (1.83)	0.0928 (1.84)
$\phi_{1,32}$	0.0028 (0.01)	0.0030 (0.01)	$\phi_{2,32}$	0.0641 ()	0.0568 (1.11)
$\phi_{1,33}$	0.8429 (6.14)	0.8231 (6.12)	$\phi_{2,33}$	-0.0026 (0.02)	-0.0084 (0.60)
Parameter	M0	M1			
φ_1	0.4111 (4.51)	0.34734 (4.33)			
φ_2	2.5444 (13.59)	2.2297 (14.83)			
ξ_2	0.8632 (25.92)	0.8692 (23.81)			

Table 4b: The variance and risk structure of Model M1
(asymptotic t-values in parentheses.)

Parameter	M0	M1	Parameter	M0	M1
Δ_0			Δ_1		
δ_{01}	6.309×10^{-4} (2.35)	(-)	δ_{11}	(-)	0.0133 (2.63)
δ_{02}	3.0081×10^{-5} (1.27)	7.8417×10^{-6} (0.95)	δ_{12}	(-)	0.0298 (2.17)
δ_{03}	3.5608×10^{-6} (2.66)	1.5213×10^{-7} (1.09)	δ_{13}	(-)	0.0161 (4.36)
δ_{04}	6.9169×10^{-6} (4.02)	2.5801×10^{-10} (-)	δ_{13}	(-)	0.0211 (2.31)
δ_{05}	3.4596×10^{-6} (2.16)	3.0976×10^{-8} (1.00)	δ_{15}	(-)	0.0161 (2.00)

Υ_0			Υ_1		
$v_{0,1}$	-5.375×10^{-4} (1.57)	2.767×10^{-5} (0.18)	$v_{1,1}$	(-)	(-)
$v_{0,2}$	3.651×10^{-4} (2.11)	-1.9328×10^{-3} (6.21)	$v_{1,2}$	-0.0253 (1.04)	-0.0392 (0.95)
$v_{0,3}$	1.059×10^{-3} (3.42)	6.4071×10^{-4} (5.44)	$v_{1,3}$	-0.0374 (1.64)	-0.2202 (2.99)
$v_{0,4}$	2.5221×10^{-3} (4.28)	2.4000×10^{-3} (4.27)	$v_{1,4}$	0.0777 (1.97)	0.0627 (1.68)

Parameter	M0	M1	Parameter	M0	M1
Υ_{22}					
$v_{22,12}$	0.1559 (2.48)	0.2033 (2.89)	$v_{22,22}$	-0.0332 (10.85)	-0.0184 (10.42)
$v_{22,13}$	0.6927 (1.45)	0.0556 (1.58)	$v_{22,23}$	-0.1135 (1.64)	-0.0672 (1.99)
$v_{22,14}$	0.1228 (2.72)	0.04953 (2.77)	$v_{22,24}$	-0.0188 (1.56)	-0.0046 (4.67)
$v_{22,32}$	0.0139 (3.03)	0.0214 (1.91)	$v_{22,42}$	0.1324 (3.55)	0.1136 (3.17)
$v_{22,33}$	0.0139 (0.17)	0.0102 (0.32)	$v_{22,43}$	0.0283 (0.29)	0.0730 (0.78)
$v_{22,34}$	-0.0453 (0.66)	-0.0512 (0.65)	$v_{22,44}$	-0.0841 (0.99)	-0.0931 (2.04)

	M0	M1		M0	M1
P			G		
ρ_1	5.6928×10^{-4} (4.33)	5.7071×10^{-4} (4.08)	g_{21}	-0.3741 (7.23)	-0.3375 (7.02)
ρ_2	2.6701×10^{-4} (1.86)	2.5516×10^{-4} (1.38)	g_{31}	0.2295 (3.76)	0.2713 (4.00)
ρ_3	3.0499×10^{-4} (10.11)	3.4917×10^{-4} (10.50)	g_{32}	0.4444 (6.93)	0.4185 (5.91)
ρ_4	1.9221×10^{-4} (10.78)	2.1341×10^{-4} (10.15)	Λ		
ρ_5	2.8221×10^{-4} (4.55)	2.8401×10^{-4} (3.99)	λ_{10}	1.2771×10^{-4} (2.05)	-31.7885 (20.10)
ρ_6	2.8651×10^{-4} (6.64)	1.2156×10^{-4} (6.00)	λ_{11}	9.4951 (1.99)	(-)
ρ_7	8.1120×10^{-4} (10.59)	5.9992×10^{-4} (10.45)			

Chart 1: Macroeconomic variables

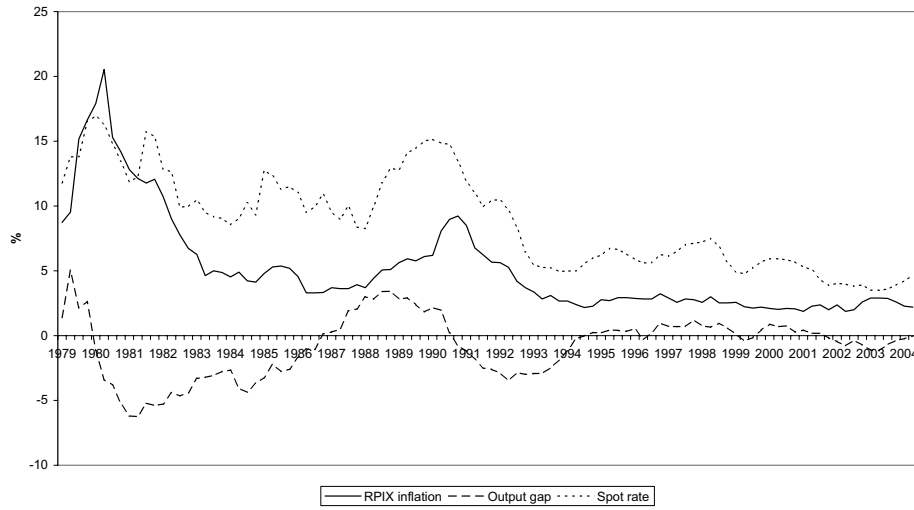


Chart 2: Gilt edged discount yields

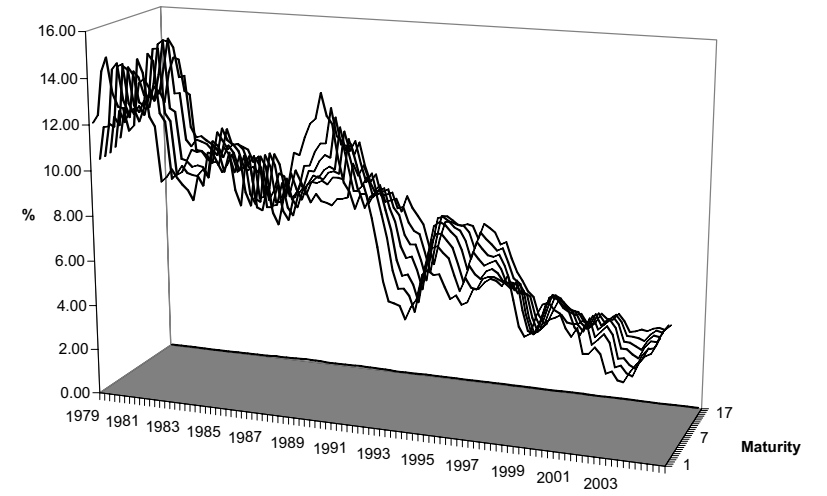


Chart 3(a) Inflation factor
(with 95% confidence band)



Chart 3(b) Real rate factor
(with 95% confidence band)

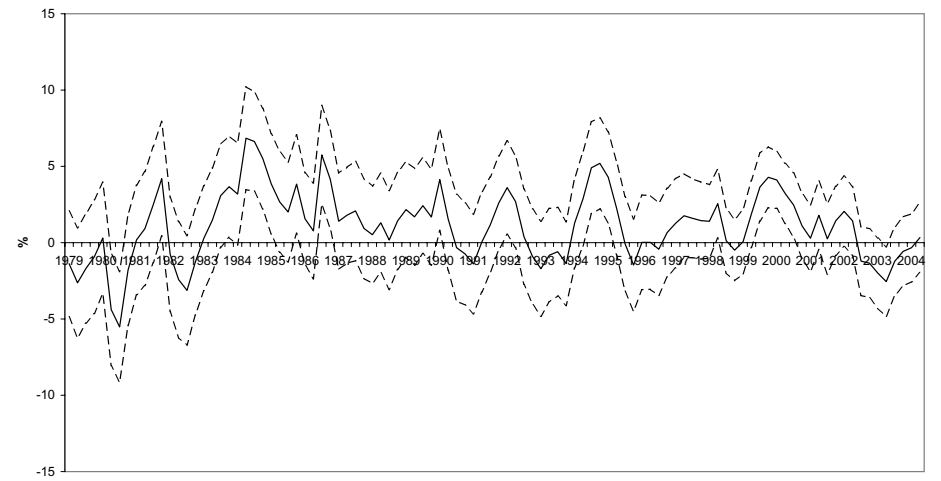


Chart 4a: Output gap variability
(One step ahead estimate plus 95% confidence interval)

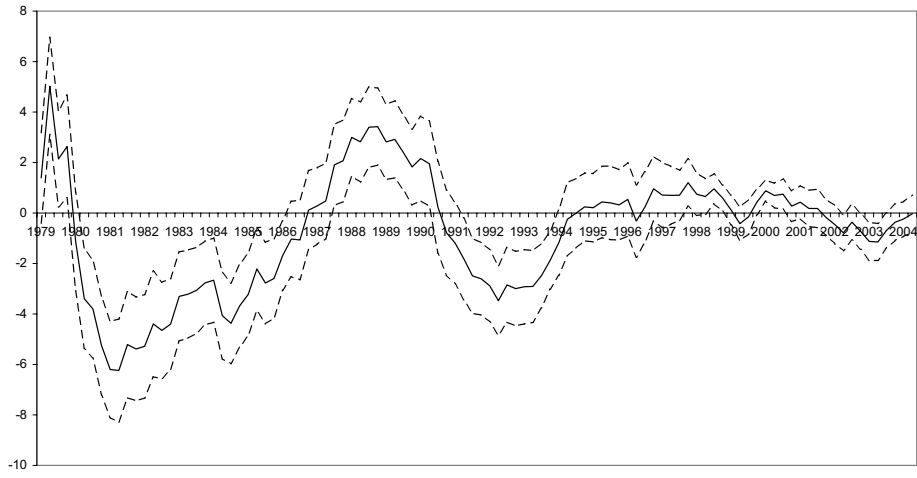


Chart 4b: Inflation variability
(One step ahead estimate plus 95% confidence interval)

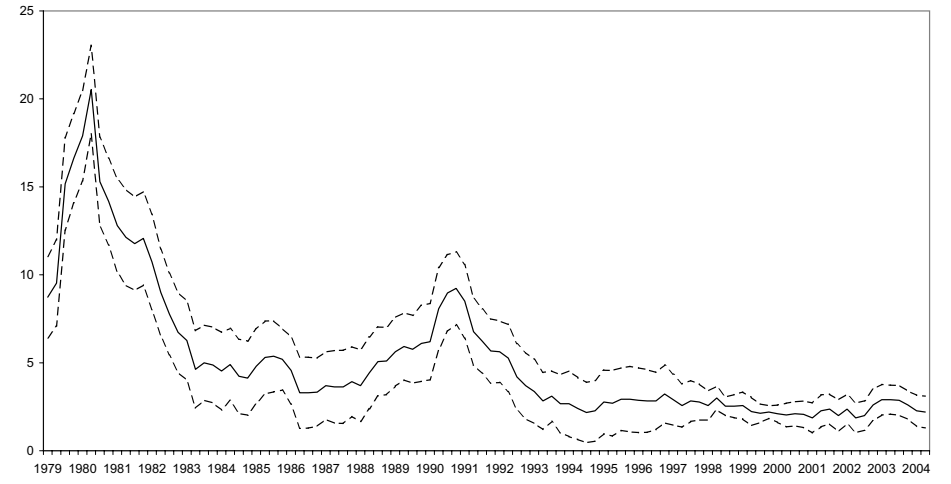


Chart 4c: Spot rate variability
(One step ahead estimate plus 95% confidence interval)

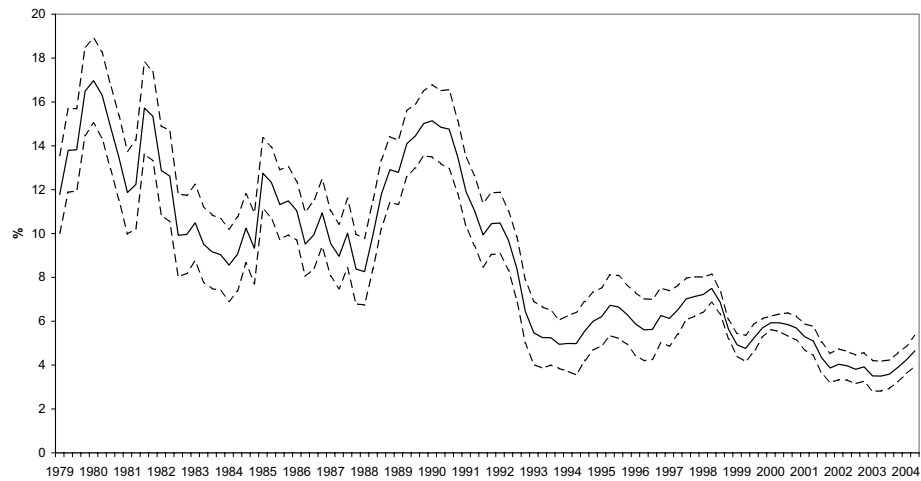
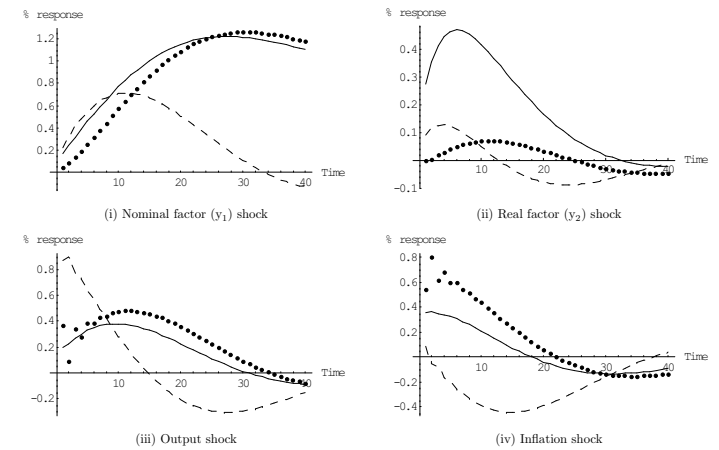
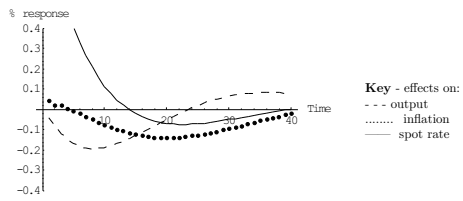


Chart 5: Model M1 macroeconomic impulse responses

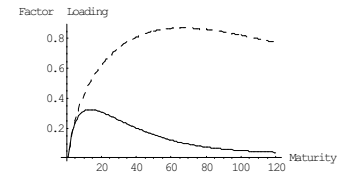




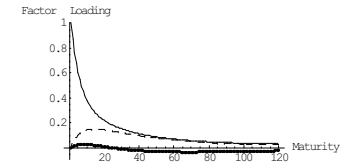
(iv) Spot rate shock

Each panel shows the effect of a shock to one of the five orthogonal innovations (ε, η) shown in (1) and (2). These shocks increase each of the five driving variables in turn by one percentage point compared to its historical value for just one period. Since y_1 is a martingale, the first shock (ε_1) has a permanent effect on inflation and interest rates, while other shocks are transient. The continuous line shows the effect on the spot rate, the dashed line the effect on output and the dotted line the effect on inflation. Elapsed time is measured in quarters.

Chart 4: Model M1 Factor loadings



y_1 (—) and y_2 (---)



output (---) inflation (....) and spot rate (—)

The factor loadings show the cumulative effect of changes in the three macro variables on yields at different maturities (quarters).