CENTER FOR APPLIED ECONOMICS AND POLICY RESEARCH

CAEPR Working Paper
#020-2009

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Seok Gil Park
Indiana University

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Abstract

Recently, central banks expanded their balance sheets by unconventional actions, including credit easing operations. Although such quasi-fiscal operations are significant in size and assumed to be crucial for the economy's recovery, little theory is available to explain the possible macroeconomic consequences of these operations. The main contribution of this paper is to show that quasi-fiscal shocks may affect inflation in plausible cases by utilizing a simple DSGE model that embraces the budgetary independence of the central banks. In the active quasi-fiscal policy regime, the shocks in the central bank's earnings alter the private agent's portfolio between consumption and the nominal money balance, thus affecting inflation. Conventional macroeconomic models have implicitly assumed policy regimes in which the aforementioned mechanism does not restrict equilibria; however, this paper shows that such assumptions generally are not guaranteed to hold. The extensions of the basic model show that quasi-fiscal shocks may produce undesirable effects, such as inflation following deflationary monetary policy during the implementation of exit strategy.

Keywords: Central bank balance sheet; Quasi-fiscal policy; Policy interactions; Central bank's budgetary independence;
JEL classification: E31; E58; E63

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*Correspondence: Department of Economics, Indiana University, Bloomington, IN 47405, USA. *E-mail address:* segpark@indiana.edu. I am grateful to my advisor, Eric Leeper, for his encouragement and support. I would like to thank Joon Park, Lars Svensson, Todd Walker, Tack Yun and the seminar participants at the Federal Reserve Board of Governors, Indiana University, the 2009 Midwest Macroeconomics Meeting and Sveriges Riksbank for their helpful comments and suggestions. All remaining errors are mine.

1 Introduction

Recently, central banks expanded the size of their balance sheets in an attempt to mitigate the financial turmoil that began in 2008. In order to accomplish this goal, various unconventional operations were implemented, such as a liquidity provision to households and businesses, bailouts to financial institutions and foreign exchange swaps with foreign central banks. All of the above-mentioned operations inflated the central banks' balance sheets, and the magnitude of these operations was significant.[1] These operations can be referred to as 'quasi-fiscal activities' because they do not conform to traditional monetary policy, which is used to stabilize inflation by controlling the policy interest rate.[2] Instead of being innate to central banks, most of these activities can be implemented by fiscal authorities. In this paper, a quasi-fiscal policy is defined as any policy action that affects the central banks' balance sheets, with the exception of the traditional monetary policy mentioned above. For example, since credit easing operations alter the composition of the central banks' asset accounts, they are considered quasi-fiscal policies. If losses are incurred from the central banks' assets and the fiscal authorities decide not to compensate the losses, then the fiscal authorities' decision is also a quasi-fiscal policy because it decreases the central banks' capital account.

Although current quasi-fiscal operations are significant in size and assumed to be crucial for economic recovery, little economic theory is available to explain the possible macroeconomic consequences of these operations. Specifically, the effect of quasi-fiscal policies on inflation needs to be explained because deflation has been one of the main concerns of policymakers during the implementation of such policies. This paper proposes a simple dynamic stochastic general equilibrium (DSGE) model, which incorporates quasi-fiscal policies in order to address the following questions: 1) Do quasi-fiscal policies and the central banks' balance sheets affect inflation? and 2) Do the central banks' balance sheets have implications for policy interactions between the central banks and fiscal authorities?

Sargent and Wallace (1981) and Leeper (1991) connected monetary and fiscal policy by showing that one policy may impose restrictions on the other policy, and that the two policies should interact in a coherent way in order to deliver a unique equilibrium. In this conventional approach to policy interaction, the budget constraints of the central banks and fiscal authorities are consolidated into a single equation. In other words, conventional models

[1] For example, the Federal Reserve System's asset account was \$894 billion at the end of 2007, but increased to \$2,266 billion at the end of 2008.

[2] Mackenzie and Stella (1996) defined a quasi-fiscal activity as "an operation or measure carried out by a central bank ... with an effect that can, in principle, be duplicated by budgetary measures ... and that has or may have an impact on the financial operations of the central bank" (p. 17).

implicitly assume that the fiscal authorities acknowledge the central banks' liabilities and assets as their own liabilities and assets, and that the central banks' losses are automatically compensated by the fiscal authorities. Due to these assumptions, in conventional models, the budget constraint of the central banks does not impose restrictions on the equilibrium.

However, questions can be raised about this conventional assumption. Stella and Lönnberg (2008) surveyed 135 central banks and discovered that laws did not always guarantee the fiscal authorities' responsibility for the central banks' liabilities, and that the fiscal authorities are not always prompt in recapitalizing the central banks. Specifically, Section 4 shows that the Federal Reserve System may suffer negative net profits due to risky assets and that such losses may not be fully covered by the treasury. In this regard, this paper relaxes the conventional assumption by elaborating on the institutional details that state that the central banks' flow budget constraint is separate from the fiscal authorities' flow budget constraint. In addition, this paper's public sector model includes the fiscal authorities' transfer rule for recapitalizing the central banks, and the transfer rule is the key quasi-fiscal policy in the benchmark model.

Furthermore, this paper departs from conventional models by assuming that the real values of the central banks' and fiscal authorities' liabilities have finite upper bounds, which are the expected present values of their future earnings. Due to this assumption, the peculiar equilibria, where the fiscal authorities and central banks can run a Ponzi scheme on each other and the real value of the central banks' capital grows (or shrinks) infinitely, are excluded from this paper. By utilizing this assumption and the institutional details of the flow budget constraints, we show that the intertemporal equilibrium condition from the central banks' budget constraint (the central banks' net liability valuation formula) may restrict equilibrium inflation in a certain policy regime. In other words, the model in this paper includes two intertemporal equilibrium conditions (from the public sector's flow budget constraints) and three policy instruments (monetary, fiscal and quasi-fiscal), whereas one intertemporal equilibrium condition and two policy instruments (monetary and fiscal) exist in the conventional model.

The main result of this paper shows that a new policy regime exists and that it delivers a unique stationary equilibrium path. In this new policy regime, while quasi-fiscal policy is active, monetary and fiscal policies are 'passive'.[3] The active quasi-fiscal policy means that the fiscal authorities do not stabilize the central banks' real capital and do not increase the fund transfer to the central banks when losses are incurred. In this case, the central

[3] Active and passive policies are defined by Leeper (1991). That is, an active policy is not constrained by the private agent's optimization (thus equilibrium conditions), states of economy and other policies.

banks' net liability valuation formula restricts the equilibrium inflation. Since the other two policies are passively adjusted in order to satisfy the equilibrium conditions, inflation is uniquely determined by the central banks' net liability valuation formula in this regime.

The economic mechanism underlying the active quasi-fiscal policy regime is the private agent's (household's) portfolio adjustment between the central bank's liability and consumption. Suppose that an unanticipated shock occurs to the central bank's earnings. Then, households expect that the real value of the central bank's net liability will deviate from the expected present value of the central bank's future earnings if the general price level does not change. However, the portfolio adjustment between the central bank's liability and the other nominal asset (the fiscal authority's liability) cannot be utilized to restore the equilibrium since the values of the two assets are synchronized by a common price, the general price level. Therefore, households will try to adjust their portfolio between the central bank's liability and consumption. This mechanism is the driving force behind inflation (or deflation) induced by quasi-fiscal shocks in the active quasi-fiscal policy regime.

The benchmark model is extended by including explicit examples of quasi-fiscal policy, such as a credit easing operation. In the first extension, the central bank holds long-term government bonds during the implementation of the exit strategy. That is, in response to economic recovery, the central bank may try to withdraw liquidity from the economy by increasing the short-term interest rate. In the active quasi-fiscal policy regime, such deflationary policy shock induces inflation via the devaluation of the central bank's long-term bond holdings when the central bank's capital is negative.

The second extension analyzes issues of the credit easing operation by including the central bank's risky loan to the private agent in the model. The results show that the loss incurred by the risky asset induces deflation when the central bank's capital is positive. In sum, the quasi-fiscal shocks may induce unintended effects on inflation. Therefore, if policymakers are concerned about the perverse effects of quasi-fiscal shocks, then the fiscal authority and central bank should coordinate to stabilize the central bank's capital in a systematic way.

A few studies have been completed that shed light on the effects of concerns on the central banks' balance sheets. Jeanne and Svensson (2007) showed that if the central banks suffer losses when their capital falls under a fixed level, then the central banks' commitment to escape from the liquidity trap is more credible. On the other hand, Sims (2003) showed that the central banks' balance sheet concerns might undermine the central banks' abilities to prevent inflation. Berriel and Bhattarai (2009) showed that the optimal monetary policy is significantly different when the central bank's budget constraint is separate from the fiscal authorities' budget constraint. Specifically, as the central banks place higher effective weight on inflation in the loss function, the variation in inflation decreases.

One of the differences between this paper and previous literature on the central banks' balance sheets is that a new type of equilibrium exists even if the policy interest rate does not depend upon the status of the central banks' balance sheets. For example, in Jeanne and Svensson (2007) and Berriel and Bhattarai (2009), the central banks' loss function includes a deviation of the central banks' real capital. In these cases, the central banks' monetary policy behavior may be restricted by the balance sheet concerns. However, the monetary policy behavior in this paper follows a simple Taylor rule. In other words, the central bank, in this paper, will not generate seigniorage in response to its balance sheet concerns.

The remainder of this paper is organized as follows. In Section 2, we build the model for the rational-expectations general equilibrium in exact nonlinear forms. In Section 3, the equilibrium conditions are linearized around the deterministic steady state in order to derive analytic solutions. In addition, we explain the equilibrium in the active quasi-fiscal policy regime in this section. In Section 4, the plausibility of the active quasi-fiscal policy regime is discussed by utilizing the actual balance sheet status of the Federal Reserve System. The benchmark model is extended to include the exit strategy in Section 5 and issues in credit easing operations are explored in Section 6. Section 7 concludes the paper.

2 The model

2.1 Private agent's optimizing behavior

This model is a closed economy dynamic stochastic general equilibrium (DSGE) model with money in the utility function. Output is given by the exogenous endowment process $\{y_t\}_{t=0}^{\infty}$ for simplicity. The representative agent solves the following problem:

$$\max_{\{c_t, M_t, B_t^A\}_{t=0}^{\infty}} \quad E_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + v\left(\frac{M_t}{P_t}\right) \right\} \right], \; 0 < \beta < 1 \tag{1}$$

$$s.t. \; c_t + \frac{M_t}{P_t} + \frac{B_t^A}{P_t} \leq y_t - \tau_t + \frac{M_{t-1}}{P_t} + \frac{(1+i_{t-1})B_{t-1}^A}{P_t}, \; c_t, M_t \geq 0, \; \forall t \geq 0,$$

$$M_{-1} + (1+i_{-1})B_{-1}^A > 0$$

where $u(\cdot)$ is the utility function for consumption, $v(\cdot)$ is the utility function for the real money balance, c_t is the single consumption goods, τ_t is the lump-sum tax, M_t is the nominal money balance, the agent holds B_t^A units of the government's risk-free one-period nominal bonds, P_t is the price level, $E_t[\cdot]$ is the expectation based on the information available in period t, i_t is the risk-free net nominal interest rate and the initial level of financial wealth $M_{-1} + (1+i_{-1})B_{-1}^A$ is given exogenously. The first order condition yields the following Euler equations:

$$\frac{v'(m_t)}{u'(c_t)} + E_t\left[X_{t,t+1}\frac{P_t}{P_{t+1}}\right] = 1, \tag{2}$$

$$\frac{1}{R_t} = E_t\left[X_{t,t+1}\frac{P_t}{P_{t+1}}\right], \tag{3}$$

$$X_{t,t+1} \equiv \beta\frac{u'(c_{t+1})}{u'(c_t)}, R_t \equiv 1 + i_t,$$

where $u'(\cdot)$ and $v'(\cdot)$ refers to the first order derivative of $u(\cdot)$ and $v(\cdot)$, m_t is the real money balance (M_t/P_t), $X_{t,t+1}$ is the real pricing kernel (or the discount factor) between period t and $t+1$ and R_t is the risk-free gross nominal interest rate. Equation (2) governs the money demand of the agent and Equation (3) leads to the Fisher equation.

In order to ensure the existence of the agent's unique optimal choice, the transversality condition should be assumed. Specifically, the agent cannot borrow a greater amount than the finite present value of her future income. This optimal condition is expressed in the following condition:

$$w_t^A \equiv \frac{M_{t-1} + R_{t-1}B_{t-1}^A}{P_t} \geq -\sum_{T=t}^{\infty} E_t\left[X_{t,T}(y_T - \tau_T)\right] > -\infty, \forall t, \tag{4}$$

where w_t^A is the agent's real financial wealth at period t and $X_{t,T} \equiv \prod_{s=t}^{T-1} X_{s,s+1}$. In the optimal path, the flow and intertemporal budget constraint are satisfied by equality and it can be shown that (1) and (4) lead to the transversality condition for the agent as follows:

$$\lim_{T\to\infty} E_t[X_{t,T}w_T^A] = 0, \forall t. \tag{5}$$

If the limit term in Equation (5) is strictly positive, then the agent accumulates an additional financial asset and it rolls over permanently. Obviously, this scenario is not optimal for the agent because the agent sacrifices her consumption without any compensation. On the other hand, if the limit term in Equation (5) is negative, then the agent is allowed to borrow an additional unit of debt and roll it over permanently. This scenario is ruled out by the No-Ponzi scheme assumption, which can be seen in Condition (4).

The market clearing conditions are imposed for the general equilibrium. Specifically, the goods market, the nominal money and the government bonds market should be cleared as follows:

$$c_t = y_t, \tag{6}$$

$$M_t^s = M_t, \tag{7}$$

$$B_t^s = B_t^A + B_t^C, \forall t, \tag{8}$$

where M_t^s is the nominal money supply, B_t^s is the government nominal bonds supply and B_t^C is the central bank's demand for the government nominal bonds.[4] Then, after imposing market clearing conditions, the Euler equations turn into the following equilibrium conditions under the standard assumptions on preferences:

$$\frac{M_t^s}{P_t} = f^*\left(\frac{i_t}{1+i_t}u'(y_t)\right) \equiv f(y_t, i_t), \tag{9}$$

$$\frac{1}{R_t} = E_t\left[\beta\frac{u'(y_{t+1})}{u'(y_t)}\frac{P_t}{P_{t+1}}\right], \forall t, \tag{10}$$

where $f^*(\cdot)$ refers to the inverse function of $v'(\cdot)$. Equation (9) states that the supply of the real money balance should be equal to the agent's real money demand and Equation (10) is the Fisher equation. In order to completely define the rational-expectations general equilibrium, the policy behavior of the fiscal authority and the central bank should be specified. Section 2.2 and 2.3 will explain the public sector model and the policy behavior.

2.2 Budgetary independence of central banks

This section introduces the budgetary independence of the central bank, which is a key difference between this model and conventional models. While the policy independence of the central bank refers to the central bank's autonomy in deciding policy variables, such as the policy interest rate, budgetary independence implies that the central bank's flow budget constraint and intertemporal equilibrium condition are separate from those of the fiscal authority. That is, the concept of budgetary independence has two aspects. First, the central bank's balance sheet is isolated from the balance sheet of the fiscal authority. Due to this aspect, the flow budget constraint of the consolidated government in conventional models is divided into two equations: one for the central bank and one for the fiscal authority.

[4] Determination of B_t^C will be explained in the next section.

Second, a set of assumptions separates the unified intertemporal equilibrium condition (IEC) from the consolidated government budget constraint into two IECs. One IEC is from the central bank's flow budget constraint and equilibrium conditions and the other is from the fiscal authority's budget constraint and equilibrium conditions.

The first aspect of budgetary independence elaborates on the institutional details of the central bank's balance sheet; therefore, the first aspect reflects a fact. The second aspect consists of assumptions on the agent's expectations in regard to the public institutions' liabilities. This set of assumptions, which I will refer to as 'the budgetary independence condition,' is a crucial requirement for the results of this paper. That is, the budgetary independence condition imposes an additional restriction on the equilibrium and is a main deviation from conventional models. This section also provides arguments that the budgetary independence condition is a valid requirement for the public sector model.

2.2.1 Separation of flow budget constraint

The first aspect of budgetary independence is explained using the balance sheets of the public institutions. In Table 1, the simplified balance sheets of the public institutions and the agent are illustrated as a benchmark case. Note that Table 1 omits the central bank's various accounts such as risky loans to the financial sector, which are accrued by quasi-fiscal activities. Specifically, the benchmark central bank only has monetary liability and one-period risk-free government bonds as assets. The purpose of Table 1 is to show how the balance sheets of public institutions are separated in the benchmark case. Based upon these simple benchmark balance sheets, future extensions can accommodate accounts from quasi-fiscal actions, such as liquidity provisions to the financial sector.

Table 1. Balance sheets of each entity[5]

Fiscal authority (A)		Central bank (B)		Public sector (A+B)		Private agent	
Asset	Liability	Asset	Liability	Asset	Liability	Asset	Liability
0	B_t		M_t	0	M_t	M_t	0
	$(= B_t^C + B_t^A)$	B_t^C			B_t^A	B_t^A	
	Capital		Capital		Capital		Capital
	$-B_t$		$B_t^C - M_t$		$-(M_t + B_t^A)$		$M_t + B_t^A$

[5] This balance sheet is effective at the end of the period t. If the balance sheet is measured at the beginning of period t, the capital of the fiscal authority is $-R_{t-1}B_{t-1}$, the central bank's capital is $R_{t-1}B_{t-1}^C - M_{t-1}$ and the agent's capital is $M_{t-1} + R_{t-1}B_{t-1}^A$.

According to Table 1, the fiscal authority issues bonds and sells them to the agent (B_t^A) or to the central bank (B_t^C). B_t is the total fiscal authority bonds outstanding, B_t^C is the central bank's holdings of the fiscal authority bonds and B_t^A is the agent's holdings of the fiscal authority bonds. If the fiscal authority's capital account is negative, then the fiscal authority has had an accumulated fiscal deficit, which was financed by debt issuance. The central bank issues a nominal money balance (M_t) and accumulates the fiscal authority's bond holdings (B_t^C) in return. Note that the central bank uses the fiscal authority bond holdings (B_t^C) as an instrument for open market operations. Specifically, the central bank sells (buys) fiscal authority bonds when the central bank decreases (increases) the money supply (M_t). The third column shows the consolidated public sector balance sheet, which is a balance sheet of the consolidated government in conventional models. This conventional balance sheet can be derived by adding the fiscal authority's balance sheet to the central bank's balance sheet.[6] In the last column, the private agent's balance sheet is a mirror image of the consolidated public sector balance sheet.

Table 2. Changes in each balance sheet between the period $t-1$ and t.

Fiscal authority (A)		Central bank (B)	
Asset	Liability	Asset	Liability
0	$B_t - B_{t-1}$		$M_t - M_{t-1}$
		$B_t^C - B_{t-1}^C$	
	Capital		Capital
	$P_t(s_t - \kappa_t)$		$P_t\kappa_t$
	$-i_{t-1}B_{t-1}$		$+i_{t-1}B_{t-1}^C$
Public sector (A+B)		**Private agent**	
Asset	Liability	Asset	Liability
0	$M_t - M_{t-1}$	$M_t - M_{t-1}$	0
	$B_t^A - B_{t-1}^A$	$B_t^A - B_{t-1}^A$	
	Capital		Capital
	$P_t s_t$		$P_t(y_t - \tau_t - c_t)$
	$-i_{t-1}B_{t-1}^A$		$+i_{t-1}B_{t-1}^A$

In order to derive flow budget constraints, the changes in the balance sheets are illustrated in Table 2. Table 2 shows the income and expenditure of each entity during the period t (flow variables), whereas Table 1 is a snapshot of each entity's financial status at the end of period t (stock variables). The fiscal surplus s_t is the lump-sum tax τ_t less the fiscal

[6] B_t^C is cancelled out when the two balance sheets are consolidated.

authority expenditure in units of real goods,[7] and κ_t refers to the transfer from the fiscal authority to the central bank in the real goods unit.[8] According to Table 2, the fiscal authority receives nominal income from the lump-sum tax, then spends for the government expenditure, transfer to the central bank and interest payments. The nominal income of the central bank is the transfer from the fiscal authority and interest earnings from the fiscal authority bonds holding. Finally, the agent receives nominal income $(P_t y_t + i_{t-1} B^A_{t-1})$, then spends for consumption and pays the lump-sum tax.

Since flow budget constraints show possible changes in the balance sheets, the constraints are derived by equating the changes in the asset account to the sum of the changes in the liability and capital accounts. From the (A+B) column in Table 2, a consolidated government flow budget constraint is derived as in Equation (11).

$$\underbrace{0}_{\Delta Asset} = \underbrace{\frac{B^A_t - B^A_{t-1}}{P_t} + \frac{M_t - M_{t-1}}{P_t}}_{\Delta Liability} + \underbrace{s_t - \frac{i_{t-1} B^A_{t-1}}{P_t}}_{\Delta Capital}. \tag{11}$$

The central bank's flow budget constraint is derived from the Figure 2 column (B) as in Equation (12), while the flow budget constraint of the fiscal authority as found in Equation (13) is derived from column (A).

$$\underbrace{\frac{B^C_t - B^C_{t-1}}{P_t}}_{\Delta Asset} = \underbrace{\frac{M_t - M_{t-1}}{P_t}}_{\Delta Liability} + \underbrace{\frac{i_{t-1} B^C_{t-1}}{P_t}}_{\Delta Capital} + \kappa_t, \tag{12}$$

$$\underbrace{0}_{\Delta Asset} = \underbrace{\frac{B_t - B_{t-1}}{P_t}}_{\Delta Liability} + \underbrace{s_t - \kappa_t - \frac{i_{t-1} B_{t-1}}{P_t}}_{\Delta Capital}, \tag{13}$$

Note that Equation (11) is a sum of Equations (12) and (13). Since κ_t is the transfer term between the two public institutions, it is cancelled out when the budget constraints are consolidated into Equation (11). In addition, since the total fiscal authority bonds B_t are divided into the central bank's holding B^C_t and the agent's holding B^A_t $(B_t = B^C_t + B^A_t)$, then only the agent's holding B^A_t remains in the consolidated budget constraint.

[7] The government spending is assumed to be zero for all time periods. No result relies on this assumption.

[8] Central banks generally submit their net profits to fiscal authorities. In this case, κ_t is a negative number.

2.2.2 Separation of intertemporal equilibrium condition

For the second aspect of the central bank's budgetary independence, an extra set of conditions is assumed for the agent's expectations in regard to the two public institutions' relationship. This set of conditions, which is shown in Conditions (15) and (16), is required in order to separate the unified IEC from the consolidated government budget constraint into two IECs. In the conventional models, such as are found in Woodford (2001), the unified IEC (government debt valuation formula) is as follows:

$$\frac{M_{t-1} + R_{t-1}B_{t-1}^{A}}{P_t} = \sum_{T=t}^{\infty} E_t\left[X_{t,T}\left(\frac{i_T}{1+i_T}f(y_T, i_T) + s_T\right)\right]. \tag{14}$$

The unified IEC (14) can be derived by solving forward the consolidated government budget constraint (11), while the equilibrium conditions including the agent's transversality condition (5) are substituted into the constraint.

The Proposition 1 states that the unified IEC is separated into two IECs when the agent expects that the central bank and the fiscal authority have bounded real values for their liabilities.

Proposition 1 *Suppose the Conditions (15) and (16) are true,* $\left|w_t^A\right| < \infty$[9] *and* P_t*,* $X_{t,T} > 0$ *for all* t *and* $T > t$*,*

$$d_t^C \equiv \frac{M_{t-1} - R_{t-1}B_{t-1}^{C}}{P_t} \leq \sum_{T=t}^{\infty} E_t\left[X_{t,T}\left(\frac{i_T}{1+i_T}f(y_T, i_T) + \kappa_T\right)\right] < \infty, \tag{15}$$

$$d_t^F \equiv \frac{R_{t-1}B_{t-1}^{A} + R_{t-1}B_{t-1}^{C}}{P_t} \leq \sum_{T=t}^{\infty} E_t\left[X_{t,T}(s_T - \kappa_T)\right] < \infty, \tag{16}$$

then the unified IEC (14) is separated into two IECs.

Proof. Note that $d_t^{C(F)}$ refers to the real net liability of the central bank (the fiscal authority) at period t and $w_t^A = d_t^C + d_t^F$. Intuitively, the Conditions (15) and (16) mean that the real values of the central bank and the fiscal authority liabilities cannot be greater

[9] Arguably, the real value of government liability (w_t^A) is bounded. One can imagine equilibria where the lump-sum tax to output ratio is unbounded. In these equilibria, it may be possible that the government liability to output ratio as well as the lump-sum tax to output ratio are explosive, while the tax to debt ratio remains finite. Canzoneri et al.'s (2001) results showed the Ricardian equivalence under these equilibria. However, if a distortionary tax is introduced, then the scenario is not feasible. This paper excludes these bizarre equilibria by assumption.

than the expected present value of the future earnings. Appendix A shows that the following two equations are equivalent to Conditions (15) and (16).

$$\lim_{T\to\infty} E_t\left[X_{t,T} d_T^C\right] = 0, \tag{17}$$

$$\lim_{T\to\infty} E_t\left[X_{t,T} d_T^F\right] = 0, \forall t, T > t. \tag{18}$$

Then, the flow budget constraints (12) and (13) are turned into the following IECs by imposing equilibrium conditions ((9) and (10)) and the budgetary independence condition ((17) and (18)).

$$\frac{M_{t-1} - R_{t-1}B_{t-1}^C}{P_t} = \sum_{T=t}^{\infty} E_t\left[X_{t,T}\left(\frac{i_T}{1+i_T} f(y_T, i_T) + \kappa_T\right)\right], \tag{19}$$

$$\frac{R_{t-1}B_{t-1}^A + R_{t-1}B_{t-1}^C}{P_t} = \sum_{T=t}^{\infty} E_t\left[X_{t,T}(s_T - \kappa_T)\right]. \tag{20}$$

Note that Equation (14) is a sum of Equations (19) and (20). ■

I refer to the IEC (19) as 'the central bank's net liability valuation formula' because Equation (19) indicates that the real value of the central bank's net liability is equal to the expected present value of the seigniorage ($\sum_{T=t}^{\infty} E_t[X_{t,T}\frac{i_T}{1+i_T}\frac{M_T}{P_T}]$) and the transfer earnings from (or payment to) the fiscal authority ($\sum_{T=t}^{\infty} E_t[X_{t,T}\kappa_T]$).[10] The IEC (20) states that the real value of the fiscal authority's total liability is supported by the current and future fiscal surplus less the transfer to the central bank. It should also be noted that Equations (19) and (20) are not constraints, but are equilibrium conditions from the Euler equations, the agent's optimal conditions, the agent's expectations on the policy institutions and flow budget constraints.

Remark 1 *The crux of assumptions (15) and (16) is that the expected present value of the transfers between the two institutions is finite.*

In this model, the present values of the seigniorage and fiscal surplus converge to finite values. The convergence of the present value of the seigniorage revenue can be shown by the

[10] In order to ensure $P_t > 0$, it is further assumed that the expected present values of the seigniorage, the fiscal surplus and the transfer are consistent with the positive price level. For example, in Equation (19), the positive central bank capital ($M_{t-1} - R_{t-1}B_{t-1}^C < 0$) implies that the expected present value of future transfer from the central bank to the fiscal authority is greater than the seigniorage revenue.

agent's intertemporal budget constraint, which is Inequality (21).

$$\sum_{T=t}^{\infty} E_t \left[X_{t,T} \left(c_T + \frac{i_T}{1+i_T} \frac{M_T}{P_T} \right) \right] \leq w_t^A + \sum_{T=t}^{\infty} E_t \left[X_{t,T} \left(y_T - \tau_T \right) \right]. \tag{21}$$

The right side of Inequality (21) is positive and bounded, which is implied by the agent's No-Ponzi scheme assumption (4) and the assumption $\left| w_t^A \right| < \infty$. Since both consumption and the real balance holding are positive, then the present value of the seigniorage is finite in any equilibrium. The present value of the fiscal surplus ($\sum_{T=t}^{\infty} E_t[X_{t,T} s_T]$) should be also finite in the IEC (14). Since the terms other than the present value of transfer are all finite in (15) and (16), the validity of assumptions (15) and (16) hinges on the convergence of the expected present value of the transfers.

A sufficient condition for the convergence of the transfer in the deterministic case is presented in Condition (22). Using the ratio test for the convergence of the infinite series, we get

$$\limsup \left| X_{T,T+1} \frac{\kappa_{T+1}}{\kappa_T} \right| < 1, \quad |\kappa_T| < \infty. \tag{22}$$

Condition (22) means that the transfer may grow faster than the real interest rate for only a finite number of time periods. If the transfer grows faster than the real rate for an infinite number of time periods, then the real values of the public institutions' liabilities will grow (or shrink) indefinitely by the flow budget constraints. In this case, the exploding transfer should ultimately be financed by the exploding B^C since the convergence of the seigniorage and surplus implies that the asymptotic growth rates of the seigniorage and surplus will fall below the real interest rate. The conventional models, without the budgetary independence of the central banks, have not excluded this peculiar scenario. In this perspective, the additional set of assumptions (15) and (16) is a lax requirement for the public sector model because the exploding liabilities scenario is hardly imaginable. Indeed, it is observed that restrictions (15) and (16) have been in effect, and that the restrictions become more plausible with the recent trend of the central banks' policy independence.[11] These two rationales are elaborated as follows.

First, the value of the transfers between the governments and central banks (or banknote issuing commercial banks) have been limited in the history of central banking. In the early stage of the central banks' evolution, they were bestowed a charter to issue monetary liability

[11] Sims (2003) claimed that among two types of central banks, type E banks are more probable to have the budgetary independence than type F banks. However, since the given arguments are well applicable to type F banks, this paper claims that the budgetary independence is universal to the central banks.

in return for financial favors to the governments. Although the early central banks provided financial services to the governments, the governments usually did not support the early central banks when they were in distress. This feature was revealed by the public's distrust of banknotes in the face of financial crises (Goodhart, 1988, p. 19-20). For example, the failure of the John Law's Banque Royale in France (1720) and the suspension of gold convertibility in England (1797-1819) showed that governments have refused (or been unable) to support the values of the central banks' liabilities. If the governments' transfers to the central banks have infinite present values, then the government should have injected gold (or silver) into the central banks' vaults in order to ensure the convertibility of the banknotes. However, the exact opposite scenario has occurred in history. That is, the governments often suspended convertibility during times of fiscal need. Also, during some occasions, such as during the first world war in Germany, the central banks transferred funds to the governments on a large scale, but such operations came to an end with the occurrence of hyperinflation and regime changes.

Second, as more emphasis is placed on the central banks' policy independence, the budgetary independence assumptions (15) and (16) (or equivalently, (17) and (18)) have become more plausible. That is, Sims (2003) claimed that a trend has occurred to prohibit the compulsory accumulation of the fiscal authority debts by the independent central bank, and that the transactions between the fiscal authority and the independent central bank should be mere by-products of monetary policy. The intuition on conditions (17) and (18) conforms to the above statements. Specifically, (17) and (18) imply that neither the central bank nor the fiscal authority can run a Ponzi scheme on the other entity. The Ponzi scheme scenario is obviously unacceptable to the central banks which have policy independence where acquisition of government debts only occur for monetary policy purposes. In other words, the policy independent central bank is expected to reject extreme paths of transfer which violate the condition (22) and the budgetary independence requirement. In sum, the budgetary independence is ubiquitous for the central banks and a sensible assumption because the concept is well applicable to both the early central banks as well as to the modern central banks which have policy independence.

2.3 Policy behavior

Although Equations (19) and (20) are equilibrium conditions, various policy mixtures can deliver the equilibrium price level P_t. In conventional literature, generally, monetary policy determines the path of the policy interest rate, while fiscal policy decides the path of the fiscal surplus $\{s_t\}$. A description of monetary and fiscal policy behavior is sufficient for determining a unique equilibrium price level in conventional literature because the two terms sufficiently characterize the policy behaviors in (14). However, in this paper, another policy rule must also be established in order to determine the equilibrium as there is an additional unspecified policy variable $\{\kappa_t\}$. This section proposes a rule for the fiscal authority's transfer to the central bank $\{\kappa_t\}$. The rule is referred to as quasi-fiscal policy in the benchmark model.

2.3.1 Clarification on quasi-fiscal policy

Before the policy rules can be specified, we must clarify the concept of quasi-fiscal policy, both in general and in particular for this model. The term quasi-fiscal policy covers a wide range of policy behaviors. As it is already defined in the introduction, quasi-fiscal policy refers to any action, except monetary policy, that affects the central bank's balance sheet. Additional clarifications can be made with respect to the following questions: 1) Which public institution decides quasi-fiscal policy?, 2) What are the policy instruments? and 3) Why does this model focus on the transfer between the two institutions as a representative of quasi-fiscal policy?

In regard to the authority in charge of quasi-fiscal policy, both the fiscal authority and the central bank may, in general, decide quasi-fiscal policy. For example, if the central bank decides to provide liquidity to the financial sector, the policy may be the central bank's decision. However, as it has already been noted, the transfer from the fiscal authority to the central bank is also quasi-fiscal policy. If the fiscal authority decides not to transfer funds to the central bank, then the fiscal authority determines the quasi-fiscal policy in this case.

Policy instruments can also vary based on specific cases. In order to provide liquidity to the financial sector, the central bank can purchase risky assets from the financial sector. In this case, quasi-fiscal policy is implemented by adjusting the central bank's asset account. The sterilized foreign exchange rate intervention is another example in which the central bank's asset account (foreign reserve) is the policy instrument. When a quasi-fiscal activity is aimed at recapitalizing the central bank, the transfer rule from the fiscal authority to the central bank can be used as an instrument.

The benchmark model of this paper excludes all other quasi-fiscal policies and focuses only on the transfer from the fiscal authority to the central bank. This modeling choice is related to the notion of policies' activeness. Leeper (1991) stated that active policy is not constrained by the state of the economy. Specifically, in Leeper (1991), fiscal policy is active when the fiscal authority pays little attention to the level of the real value of government debt when it determines the fiscal policy instrument, the lump-sum tax. In other words, the level of government debt is one of the states of the economy in conventional models. In this paper, the state of the economy includes the level of government debt as well as the central bank's capital, which is obvious from the IEC (19) and (20). Then, the modeling choice is to determine which particular quasi-fiscal policy is most appropriate with which to target the central bank's capital. Targeting the central bank's capital means that the fiscal authority uses the quasi-fiscal policy instrument in order to maintain a certain level of the central bank's capital. Therefore, if the fiscal authority does not target the central bank's capital, then the quasi-fiscal policy is active.

One of the most direct ways to target the central bank's capital is to transfer funds from the fiscal authority to the central bank because the central banks do not impose direct taxes to the agent in general. Therefore, an appropriate way to model the central bank's capital targeting behavior is the transfer rule. This rule can be used because the other quasi-fiscal policy actions, such as the liquidity provision, have other cardinal policy motivations. For instance, a key motivation of the liquidity provision is to stabilize the financial sector, and it is not generally used for adjusting the central bank's capital level. Although the benchmark model does not include various quasi-fiscal policies, the framework will be extended to include explicit examples of quasi-fiscal policies in Section 5 and 6.

2.3.2 Policy rules

Linear policy rules are introduced in order to specify the policy behaviors. First, monetary policy follows the simple Taylor rule. That is, the central bank sets the risk-free gross interest rate (policy interest rate) in response to contemporary inflation as follows:

$$R_t = \alpha_0 + \alpha \pi_t + \theta_t, \tag{23}$$

$$\theta_t = \rho_\theta \theta_{t-1} + \varepsilon_{\theta t}, \ (0 \leq \rho_\theta < 1)$$

where α_0 and α are policy parameters, θ_t refers to the monetary policy shock which follows the AR(1) process and $\varepsilon_{\theta t}$ is white noise. Note that the central bank does not adjust the policy interest rate in response to its capital level. In other words, the model central bank does not try to target its capital by printing money.

Second, the fiscal authority determines the transfer to the central bank (κ_t) by following the next rule.

$$\kappa_t = \kappa_0 + \gamma_Q \left(\overline{cap} - b^C_{t-1} + m_{t-1} \right) + \xi_t, \tag{24}$$

$$\xi_t = \rho_\xi \xi_{t-1} + \varepsilon_{\xi t}, \ (0 \leq \rho_\xi < 1)$$

where κ_0 and γ_Q are policy parameters, ξ_t is the AR(1) random policy shock, $b^C_{t-1} - m_{t-1}(\equiv B^C_{t-1}/P_{t-1} - M_{t-1}/P_{t-1})$ is the real capital of the central bank, $\overline{cap}$ is the central bank's real capital target and $\varepsilon_{\xi t}$ is white noise. The quasi-fiscal policy rule (24) states that the transfer between the central bank and the fiscal authority may or may not be responsive to the level of the central bank's real capital. When the central bank's real capital falls below the target level ($\overline{cap}$), the fiscal authority's transfer to the central bank increases if the parameter γ_Q is strictly positive. If the parameter γ_Q is zero, then the fiscal authority does not adjust its transfer to the central bank in response to the level of the central bank's capital. The quasi-fiscal policy shock ξ_t is the random part of the transfer.

Finally, the fiscal authority determines fiscal surplus s_t by the rule (25). Given this rule, the fiscal surplus may or may not be constrained by the level of the fiscal authority's debt.

$$s_t = \gamma_0 + \gamma_F b^A_{t-1} + \psi_t, \tag{25}$$

$$\psi_t = \rho_\psi \psi_{t-1} + \varepsilon_{\psi t},\ (0 \leq \rho_\psi < 1)$$

where γ_0 and γ_F are the policy parameters, b^A_t is the real value of the fiscal authority's debt which is held by the agent (B^A_t/P_t), ψ_t is an AR(1) fiscal policy shock and $\varepsilon_{\psi t}$ is white noise.

2.4 Equilibrium

The rational-expectations general equilibrium is defined as the sequence of price $\{P_t\}$, the sequences of policy variables $\{R_t, s_t, \kappa_t\}$ and the sequences of allocations $\{c_t, M_t, M^s_t, B^A_t, B^C_t, B^s_t\}$. The sequences satisfy equilibrium conditions ((9) and (10)), the market clearing conditions ((6), (7) and (8)), the public sector budget constraints ((12) and (13)) and the policy rules ((23), (24) and (25)), given the exogenous processes $\{y_t, \theta_t, \xi_t, \psi_t\}$. In addition, the optimal condition (5) and the budgetary independence condition ((17) and (18)) are satisfied in the equilibrium.

This paper focuses on the equilibrium in which the central bank's net liability valuation formula (19) uniquely determines the price level. Similar to Cochrane (2001)'s characterization of the fiscal theory of the price level, the price level is determined by the ratio of the central bank's net liability to the present value of the central bank's future earnings in such equilibrium. The complete characterization of the policy regime in which such equilibrium realizes will be explored in Section 3 within the context of linearized system.

3 Linearized system

In this section, the equilibrium conditions are linearized around the deterministic steady state while the equilibrium conditions are expressed in exact nonlinear forms in Section 2. Specifically, this section's analysis is restricted to the equilibria with the bounded inflation rate and real values of bonds (b_t^A and b_t^C).[12] In addition, this section assumes the log utility function and a constant output for simplicity.

$$u(c_t) = \log c_t,\ v(\frac{M_t}{P_t}) = \chi \log \frac{M_t}{P_t},\ y_t = y, \forall t, \tag{26}$$

where χ is a parameter for the marginal rate of substitution between consumption and the real money balance. In this simple case, the equilibrium conditions from the private agent's behavior ((9) and (10)) become the next two equations.

$$\frac{M_t}{P_t} = \chi y \frac{R_t}{R_t - 1}, \tag{27}$$

$$\frac{1}{R_t} = \beta E_t \left[\frac{1}{\pi_{t+1}} \right], \tag{28}$$

where the inflation rate $\pi_{t+1} \equiv P_{t+1}/P_t$. Another notable simplification from the log-utility and constant output assumption is that the seigniorage and real pricing kernel are constant for each period.

$$\frac{i_t}{1+i_t} f(y_t, i_t) = \frac{i_t}{1+i_t} \chi y \frac{1+i_t}{i_t} = \chi y, \tag{29}$$

$$X_{t,t+1} = \beta \frac{u'(y_{t+1})}{u'(y_t)} = \beta,\ \forall t. \tag{30}$$

3.1 Policy interactions

The linear dynamic system is derived from the equilibrium conditions of the private agent's optimization ((27) and (28)), the policy rules ((23), (24) and (25)) and the public sector's flow budget constraints ((12) and (13)), after they have been linearized. In order to apply Sims' (2001) algorithm, the system is summarized in a matrix form as follows.

$$\Gamma_0 x_{t+1} = \Gamma_1 x_t + \Phi_0 z_{t+1} + \Phi_1 z_t + \Pi_{t+1}, \tag{31}$$

[12] If we restrict the analysis to the equilibria with the bounded inflation, then the real values of bonds (b_t^A and b_t^C) should also be bounded due to the assumptions of Proposition 1.

where the vector of the endogenous variables is $x_t \equiv (\hat{\pi}_t, \hat{b}_t^C, \hat{b}_t^A)'$, the vector of the exogenous shocks is $z_t \equiv (\hat{\theta}_t, \hat{\xi}_t, \hat{\psi}_t)'$, the forecast error vector is $\Pi_{t+1} \equiv (\eta_{t+1}, 0, 0)'$, $\eta_{t+1} \equiv \pi_{t+1} - E_t[\pi_{t+1}]$, Γ and Φ are 3×3 parameter matrices, and the $\hat{}$ notation is used for the linear deviation from the deterministic steady state.

Proposition 2 *Three regions of policy parameter space (regimes) deliver a unique stationary solution of the system (31).*

Proof. In order to uniquely determine the system's (31) solution, only one of eigenvalues of the transition matrix $\Gamma_0^{-1}\Gamma_1$ should be greater than one in the absolute value because only one endogenous forecast error (η_{t+1}) exists in the system.[13] The three eigenvalues are as follows:

$$\alpha\beta, \beta^{-1} - \gamma_Q, \beta^{-1} - \gamma_F.$$

That is, the existence and uniqueness of the solution depends upon the policy parameters (α, γ_Q and γ_F) and the deep behavioral parameter (β). In addition, each policy parameter has one associated eigenvalue within the system. If an active policy is defined as the policy for which the associated eigenvalue is unstable, then it is obvious that only one of the three policies should be active in order for the system to have a unique solution. The three policy regimes that determine a unique stationary equilibrium path are as follows.

$$\begin{aligned} \text{AMP regime:} \quad & |\alpha\beta| > 1, \left|\beta^{-1} - \gamma_Q\right| < 1, \left|\beta^{-1} - \gamma_F\right| < 1, \\ \text{AQFP regime:} \quad & |\alpha\beta| < 1, \left|\beta^{-1} - \gamma_Q\right| > 1, \left|\beta^{-1} - \gamma_F\right| < 1, \\ \text{AFP regime:} \quad & |\alpha\beta| < 1, \left|\beta^{-1} - \gamma_Q\right| < 1, \left|\beta^{-1} - \gamma_F\right| > 1. \end{aligned}$$

■

The active monetary policy (AMP) and active fiscal policy (AFP) regimes are well-known policy regimes. In the AMP regime, the monetary policy actively sets the policy interest rate in order to stabilize inflation, while passive fiscal policy supports the real value of the fiscal authority's liability. In the AFP regime, the expected path of the fiscal surplus determines inflation, while the monetary policy passively accommodates the determined inflation path by adjusting the nominal money balance. A novel finding of this paper is the equilibrium in the active quasi-fiscal policy (AQFP) regime and the requirements for the passive quasi-fiscal policy when other policies are active. In the AQFP regime, the active quasi-fiscal policy determines inflation and other policies are passive in the same way as in the other

[13] The detailed derivation of the solution is described in Appendix B.

regimes. When the quasi-fiscal policy is passive, then it tries to support the real value of the public sector liability.

A brief intuition of this policy interaction can be found in the following equilibrium conditions.

$$E_t\left[\hat{\pi}_{t+1}\right] = \alpha\beta\hat{\pi}_t + \beta\hat{\theta}_t, \tag{32}$$

$$\frac{-\left(m - Rb^C\right)}{\pi^2}\hat{\pi}_t = \sum_{T=t}^{\infty} E_t\left[\beta^{T-t}\left(\chi y + \hat{\kappa}_T\right)\right], \tag{33}$$

$$\frac{-\left(Rb^C + Rb^A\right)}{\pi^2}\hat{\pi}_t = \sum_{T=t}^{\infty} E_t\left[\beta^{T-t}\left(\hat{s}_T - \hat{\kappa}_T\right)\right], \tag{34}$$

where (32) is the linearized Fisher equation after substituting the monetary policy rule and m, R, b^C, b^A and π are the steady state values. Equations (33) and (34) are simplifications of the IECs (19) and (20) using Equations (29) and (30).[14]

First, if monetary policy is active ($|\alpha\beta| > 1$), then the expectational difference equation (32) can be utilized to determine π_t. In this case, for the existence and uniqueness of the solution, the transfer $\{\kappa_T\}$ should support the real value of the central bank's liability (left side of IEC (33)) by being endogenously determined through the IEC (33). Meanwhile, the fiscal surplus $\{s_T\}$ should be passively adjusted in order to support the real value of the fiscal authority's liability in the IEC (34).

Second, active quasi-fiscal policy ($\left|\beta^{-1} - \gamma_Q\right| > 1$) implies that the transfer $\{\kappa_T\}$ is not responsive to the state of the central bank's capital. Then, $\{\kappa_T\}$ can be regarded as an exogenous process in the IEC (33). Therefore, the IEC (33) imposes a restriction on inflation since all variables other than inflation are predetermined or exogenous. The passive monetary policy in Equation (32) ($|\alpha\beta| < 1$) means that the expectational difference equation (32) does not restrict inflation π_t. Again, the passive fiscal policy supports the real value of the fiscal authority's liability so that the IEC (34) does not bind inflation.

Third, when the fiscal policy is active, the monetary and quasi-fiscal policies should be passive. Specifically, Equation (32) should not impose an additional restriction on inflation and the transfer $\{\kappa_T\}$ should be passive in a way that it is endogenously determined by the IECs (33) and (34), simultaneously. Since the endogenous process $\{\kappa_T\}$ appears in both the IECs, the two IECs jointly determine inflation. In other words, the unified IEC (14) is the single equation to be imposed on the inflation process. Note that this equilibrium is the same as in the fiscal theory of the price level.

[14] The IECs (33) and (34) are derived by linearizing the IECs (19) and (20) around the steady state by assuming that the economy was in the steady state at period $t-1$.

Another notable feature of the policy interaction is that passive monetary policy does not need to raise the seigniorage revenue in order to cover the decreased transfer from the fiscal authority or the decreased fiscal authority surplus. The real seigniorage revenue is even a fixed number (χy) for all of the periods in this model. Even if the seigniorage is a function of the nominal interest rate under general circumstances, it is still the case that the nominal interest rate is not responsive to inflation when monetary policy is passive. Therefore, in this case, the present value of the seigniorage in the IEC (33) can be treated as a fixed number. In sum, it is a misleading statement that passive monetary policy raises the seigniorage in order to balance the intertemporal budgets. Instead, the passive central bank adjusts the nominal money balance in order to accommodate the real money market clearing condition (9) in accordance with the equilibrium price level, which is determined elsewhere.

3.2 Equilibrium in the active quasi-fiscal policy regime

As it is discussed in Proposition 2, if the quasi-fiscal policy is active while the monetary and fiscal policies are passive, then a unique stationary equilibrium inflation path exists. The nature of this equilibrium is different from that of AMP and AFP regimes as the quasi-fiscal policy shock affects surprise inflation and the direction of the effect depends upon the steady state value of the central bank's capital.[15] For a simple illustration, the following special case is considered:

$$\alpha, \gamma_Q, \rho_\theta, \rho_\xi, \rho_\psi = 0, \left|\beta^{-1} - \gamma_F\right| < 1. \tag{35}$$

The central bank pegs the policy interest rate ($\alpha = 0$), the fiscal authority's transfer to the central bank is not responsive to the level of the central bank's capital ($\gamma_Q = 0$), the policy shocks are independent and identically distributed (i.i.d.) and the fiscal authority sufficiently adjusts the lump-sum tax in response to the level of its own debt held by the agent. In this case, surprise inflation is a function of the innovation in the quasi-fiscal policy (shock in the transfer, $\varepsilon_{\xi t}$).

$$\eta_t = \frac{\pi^2}{Rb^C - m}\varepsilon_{\xi t}, \tag{36}$$

where R is the steady state value of the gross interest rate, b^C is the steady state real value of the central bank's holding of the fiscal authority bonds, m is the steady state value of the real money balance and $Rb^C - m$ is the steady state value of the central bank's real capital.

Inflation is the sum of the expected inflation and surprise inflation. Since the surprise

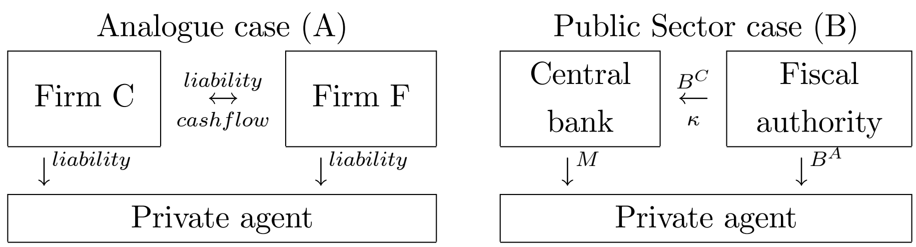
[15] Equilibria in AMP and AFP regimes are fundamentally the same as in conventional models, such as in Leeper (1991). Thus, those equilibria are not discussed in this paper.

inflation was solved in Equation (36), then inflation is determined as follows in the AQFP regime.

$$\hat{\pi}_t = \beta\varepsilon_{\theta t-1} + \frac{\pi^2}{Rb^C - m}\varepsilon_{\xi t}. \tag{37}$$

Note that $\beta\varepsilon_{\theta t-1}$ is the expected inflation $E_{t-1}[\hat{\pi}_t]$ in this case.[16] The solution in Equation (37) implies that the contractionary monetary policy shock (positive $\varepsilon_{\theta t-1}$) increases inflation in the next period via the expected inflation. The result is consistent with the simple fiscal theory of the price level (Leeper, 1991) and the 'Unpleasant Monetarist Arithmetic' (Sargent and Wallace, 1981).

Figure 1. Analogy diagram



The economic mechanism through which quasi-fiscal policy shocks affect inflation is explained using an analogy between this model and a private firm's asset pricing case. In Figure 1's Analogue case (A), the private agent holds liabilities from Firm C and Firm F. A cash flow contract also exists between the firms and, in return of the contract, one firm holds the liability of the other firm. Suppose that the cash flow from Firm F to Firm C unexpectedly decreases. Then, the agent will try to adjust her portfolio by selling Firm C's liability and buying Firm F's liability since the shock is favorable to Firm F's value. The portfolio adjustment will continue until the real value of each firm's liability becomes equal to the present value of each firm's earnings. In the end, the relative price of Firm C's liability with respect to the price of Firm F's liability will fall.

The structure and procedure are similar in Figure 1's Public Sector case (B). The private agent holds the liabilities from the central bank (M) and the fiscal authority (B^A), while an expected path of the cash transfer exists between the two public institutions (κ) and the central bank holds the fiscal authority's liability (B^C). Suppose that a similar shock exists as in the analogy case. That is, a negative quasi-fiscal policy shock decreases the transfer from the fiscal authority to the central bank unexpectedly. Then, a portfolio adjustment

[16] This point can be shown by setting $\alpha = 0$ in Equation (32).

will occur on the part of the private agent. However, fundamental differences exist between the Public Sector and Analogue cases that make the Public Sector case special. That is, the public sector model determines inflation, not asset prices.

A crucial difference between the cases is that the relative price of the central bank's liability with respect to the fiscal authority's risk-free liability is fixed to one in the Public Sector case, while the prices of Firm C's and Firm F's liabilities are determined independently. That is, it is obvious from the representative agent's budget constraint (1) that one unit of B_t^A is guaranteed to be exchanged for one unit of M_t. Since the values of the nominal money and government debts are synchronized,[17] it is useless to adjust the portfolio by exchanging M_t for B_t^A when a quasi-fiscal shock exists. Instead of exchanging M_t with B_t^A, the agent has the option to choose consumption. When the agent feels that the the central bank's liability has changed, then she may try to adjust her consumption. This portfolio adjustment on the part of the agent is the driving force behind inflation in the active quasi-fiscal policy regime.

Another key characteristic of the portfolio adjustment is that the central bank's capital account ($R_{t-1}B_{t-1}^C - M_{t-1}$) matters for the agent's portfolio determination. This feature also stems from the fact that the values of M, B^A and B^C are synchronized in the public sector model. On the other hand, the agent would not be concerned with how much of Firm F's liability is held by Firm C when she assesses the impact of the shock on Firm C's value. The negative shock, which decreases the cash flow to Firm C, is always unfavorable to Firm C's value. Although Firm C's asset value (the value of Firm F's liability within Firm C's asset account) reflects the impact of the shock, the impact is exogenous to the agent's decision on Firm C's liability because the asset value is determined independently by the asset pricing equation for Firm F's liability.

In the public sector model, however, the impact of a similar shock on the central bank's value via the value of B^C cannot be exogenous to the agent's decision on the central bank's liability. As the price of B^C is synchronized with the price of M, the portfolio adjustment for M will also affect the value of B^C through the common price $1/P_t$. Therefore, the agent takes the magnitude of both M and B^C into account when she reacts to the shock. Specifically, the negative quasi-fiscal shock is unfavorable to the value of M, but favorable to the value of B^C. Thus, in an overall assessment, the negative quasi-fiscal shock is favorable to the central bank's liability when the central bank has positive capital ($R_{t-1}B_{t-1}^C - M_{t-1} > 0$).

Due to the special features of the public sector model, a surprise deflation (inflation) exists in response to the negative quasi-fiscal policy shock when the central bank's capital is positive (negative), $RB^C - M > 0$ ($RB^C - M < 0$). This statement can be verified by the

[17] The common price of M_t, B_t^A and B_t^C in units of consumption goods is $1/P_t$.

analytic solution to the system, which is found in Equation (36). When the transfer from the fiscal authority to the central bank surprisingly decreases, the central bank's liability (M) becomes more attractive to the agent, as discussed previously. Then, the agent tries to increase the nominal money holding by decreasing consumption. Since output is fixed in this endowment economy, the fall in demand causes deflation. Meanwhile, the future lump-sum tax is expected to be passively adjusted in order to support the real value of the fiscal authority's liability, so that the valuation formula for the fiscal authority's liability (20) does not restrict the equilibrium inflation.

A mechanical explanation as to what procedure the quasi-fiscal shock induces in the AQFP regime is given as follows. The first step is the public sector's response to the quasi-fiscal shock. Suppose that the policy interest rate is pegged, no policy shocks exist other than the i.i.d. quasi-fiscal shock during the current period, the lump-sum tax and the transfer are initially expected to remain at the steady state level, and the path of the price level is given parametrically to public institutions. For algebraic simplicity, the steady state inflation is assumed to be zero ($P_t = P, \forall t$). Then, by the money market equilibrium condition (27), the nominal money balance is fixed at the steady state level for all time periods. Also, the laws of motion for B^C and B^A, which restrict the public sector's responses, are:

$$\frac{B_t^C}{P} = \frac{RB_{t-1}^C}{P} + \kappa_t, \tag{38}$$

$$\frac{B_t^A}{P} = \frac{RB_{t-1}^A}{P} - s_t, \tag{39}$$

where R refers to the policy interest rate, which is pegged at the steady state level. Equations (38) and (39) are derived by simplifying the flow budget constraints of the public institutions.

Suppose further that the transfer κ_t surprisingly decreases by one consumption good unit and that the price level is normalized to unity. Then, B_t^C also must be decreased by one consumption good unit. That is, the central bank sells one unit of the fiscal authority's debts at the bonds market and submits the received cash to the fiscal authority. In response to this shock, since the future transfers are expected to remain at the steady state level, it is expected that the central bank's holding of the fiscal authority's debt will fall below of the steady state level as follows.

$$E_t\left[B_T^C\right] = B^C - R^{T-t}, \tag{40}$$

where B^C is the steady state level of the central bank's holding of the fiscal authority's debt. Then, in this case, the budgetary independence condition ((17) and (18)) is violated as follows:

$$\lim_{T\to\infty} E_t\left[X_{t,T}d_T^C\right] = \lim_{T\to\infty} R^{t-T}\left(M - RB^C\right) + 1 = +1, \tag{41}$$

$$\lim_{T\to\infty} E_t\left[X_{t,T}d_T^F\right] = \lim_{T\to\infty} R^{t-T}\left(RB^A + RB^C\right) - 1 = -1. \tag{42}$$

The second step of the procedure is the agent's reaction to the violation of the budgetary independence condition. The intuition of Equations (41) and (42) is that the fiscal authority takes the central bank's real capital in one unit of real goods and does not eventually repay it. If the sales department takes the producing department's resource within the same firm, then the news would not affect the price of the firm's stock. However, if Firm F takes Firm C's real capital, then the stock prices of the two firms will be adjusted through the agent's portfolio adjustments. In the public sector model, the central bank and fiscal authority are two distinct institutions, as they have distinct liabilities and the present value of the transfer between the institutions is finite. Thus, the violation of conditions ((17) and (18)) induces the agent's portfolio adjustment and the general price level P_t can no longer be fixed. That is, through the portfolio adjustment, the price level is changed up to the point that the conditions ((17) and (18)) are restored. Then, the future lump-sum tax will be changed according to the fiscal policy rule.

Two more important features of the equilibrium exist. First, the model's economic mechanism for inflation does not rely on the central bank's money creation in response to the transfer to the government. In the first step of the procedure, the central bank fixed the nominal money balance, but inflation (deflation) is still induced by the quasi-fiscal shock. A similar statement can be made for the fiscal theory of the price level. In the active fiscal policy regime, the tax shock does not require the creation of money in order to have an impact on the price level. Otherwise, the tax shock induces the agent's portfolio adjustment, and, thus, the price changes. Therefore, this model's application is never restricted to extreme situations, such as a large scale transfer between two public institutions.

Second, the equilibrium in the AQFP regime is observationally equivalent to the equilibria in the AMP and AFP regimes. Suppose that a quasi-fiscal policy shock causes inflation in the AQFP regime. Then, the nominal money balance must be increased passively in order to clear the money market equilibrium condition (27) and the future fiscal surplus must be passively decreased in accordance with the price change by the fiscal policy rule (25). The required tax change in the present value can be shown by adding (19) to (20). This observational relationship between inflation, the nominal money balance and the fiscal surplus is the same as in the other regimes, although the causal relationships are opposite. For instance, in the AMP regime, an expansion of the nominal money balance (a negative monetary policy shock) causes inflation and the fiscal surplus is passively decreased. In the AFP regime, the decrease in the fiscal surplus causes inflation and the nominal money balance is passively increased.

4 Discussion: Plausibility of the AQFP regime

The active quasi-fiscal policy regime becomes more plausible in major economies as the major central banks, such as the Federal Reserve System, are more involved in quasi-fiscal activities. The crucial rationale for the activeness of quasi-fiscal policy in the U.S. is the fact that the treasury does not guarantee full automatic compensation for losses incurred from the quasi-fiscal activities of the Federal Reserve System. Suppose that the Federal Reserve System suffers losses from risky assets and the treasury does not automatically transfer funds to the system's capital account. Then, the transfer rule (24) implies that γ_Q is sufficiently small and the quasi-fiscal policy is arguably active.

The argument stems from the institutional feature that the funds transfer κ_t is asymmetric between the cases where the Federal Reserve System has positive and negative profits. As is usual for the central banks, the Federal Reserve System remits the positive net profit to the treasury on a regular basis, while maintaining a certain level of capital. This institutional feature may mislead the readers to argue that quasi-fiscal policy should always be passive. However, in a contradictory case where the Federal Reserve System has a negative net profit, the remittance from the treasury to the Federal Reserve System is not guaranteed.[18] Instead, the system will suspend its transfer to the treasury until the system has a positive profit.[19] This case matches with the policy parameter $\gamma_Q = 0$ in which the quasi-fiscal policy is active. The Joint Statement by the Department of the Treasury and the Federal Reserve in March 2009 broadly stated that fiscal policy should be responsible for the credit program and the system should avoid risks from policy operations. However, it is still true that the statement is silent about the treasury's practical responsibility in regard to the system's losses.

[18] Actually, the asymmetry in the disposal of the central banks' losses is ubiquitous around the world (Stella and Lönnberg, 2008).

[19] This strategy is stated on the Board of Governors of the Federal Reserve System's website under the title 'Credit and Liquidity Programs and the Balance Sheet'.

Table 3. Consolidated balance sheet of the Federal Reserve System (in billion dollars)

Asset[20]	9/30/09	12/27/06	Liability	9/30/09	12/27/06
Securities	1,593	779	Federal Reserve notes	874	783
Short term lending	264	0	Reserve Balances	848	13
Targeted lending	84	0	Treasury deposits	273	4
Emergency lending	101	0	Others	98	44
Others	102	96	Total	2,093	844
Total	2,144	875	Capital	51	31

Data source: Federal Reserve Statistical Release H.4.1 and Bernanke (2009c)

The following numerical example illustrates a specific case in which the quasi-fiscal policy becomes active. Table 3 shows the actual balance sheets of the Federal Reserve System before and after the financial turmoil in 2008. Notably, the Federal Reserve System funded various lending programs by increasing the reserve balances. While the magnitude of the security holdings increased, its maturity composition also changed significantly. That is, only 48% of the total securities ($372 billion) had a long-term remaining maturity (longer than 1 year) in 2006, whereas 93% of the total securities ($1,477 billion) were long-term bonds in 2009.

The assets acquired from the lending programs bear risks. Stella (2009) projected possible losses of the Federal Reserve System by assuming various default rates. Utilizing Stella's (2009) calculations and assumptions, the Federal Reserve System may lose $59 billion from the emergency lending programs and $5 billion from the targeted lending programs. Even though the interest rate risks for the security holdings and interest payments for the excess reserves were not considered, the results indicate that the potential loss of the Federal Reserve System could be roughly $64 billion. If the Federal Reserve System's income falls below $64 billion during the period in which the system suffers losses, then the quasi-fiscal policy becomes active. Since the system's average annual net income from 2003 to 2008 was $30 billion,[21] there is a non-trivial possibility that quasi-fiscal policy could become active in the U.S.

[20] Short term lending programs provide liquidity to financial institutions, which include discount window lendings, Term Auction Facility and etc. Targeted lending programs try to ease credit conditions in the commercial paper market as well as the credit market for households and small businesses. Emergency lending programs support specific financial institutions in distress, such as AIG and Bear Stearns.

[21] The Average of the 'Net income prior to distribution' item at the Federal Reserve Banks Combined Financial Statements, Annual Report of Board of Governors of the Federal Reserve System (various years).

5 Application 1: Exit strategy

Exit strategy refers to plans to reverse unconventional quasi-fiscal operations, such as credit easing operations, as economic recovery and inflation become imminent. Specifically, the Federal Reserve System will be required to "reduce excess reserve balances" of depository institutions or "neutralize their potential effects on broader measures of money and credit and thus, on aggregate demand and inflation" (Bernanke, 2009c). Roughly explained, the exit strategy is the plan to decrease the size of the central bank's balance sheet by selling the assets that have been acquired via unconventional operations.

This section analyzes the exit strategy's potential effects on inflation by extending the benchmark model to include the long-term bonds. Although the primary policy intent of the exit strategy is to avoid high inflation due to large-scale excess liquidity in the economy, this paper explores a different aspect of the strategy. Suppose that the central bank increases the short-term policy interest rate during the implementation of the exit strategy. Since the central bank holds the long-term nominal bonds, an increase in the short-term rate may devaluate the central bank's long-term bond holdings. Then, through a portfolio adjustment of the private agent as seen in Section 3, the valuation loss of the central bank may affect inflation in the AQFP regime. The interesting feature of this mechanism is that an increase in the short-term rate (deflationary policy) may lead to inflation when the central bank's capital is negative.

In this extension, the long-term government nominal bonds are introduced. Specifically, $B_{1,t}$ is the face value of one-period nominal bonds issued at t period and $B_{2,t}$ refers to the face value of two-period nominal bonds issued at t period. The short-term bonds $B_{1,t}$ mature at $t+1$ period and are redeemed for the nominal money value $R_{1,t}B_{1,t}$ at $t+1$ period. The long-term bonds $B_{2,t}$ mature at $t+2$ period and are redeemed for the nominal money value $R_{2,t}B_{2,t}$ at maturity. $R_{1,t}$ and $R_{2,t}$ are the gross nominal interest rates for the short- and long-term bonds, respectively.

At t period, the representative agent sells the premature long-term bonds ($B_{2,t-1}$) and purchases the new long-term bonds ($B_{2,t}$). The premature long-term bonds at period t ($B_{2,t-1}$) are claims for the nominal money value $R_{2,t-1}B_{2,t-1}$ at $t+1$ period. Instead of trading the premature long-term bonds, the private agent has the option to invest in the short-term bonds at t period. Since there should be no arbitrage opportunities, the two risk-free investment choices (premature long-term bonds and short-term bonds) should have the same return. Therefore, the market price of the premature long-term bonds at t period is $R_{2,t-1}R_{1,t}^{-1}$. For example, if a private agent invests $R_{2,t-1}R_{1,t}^{-1}$ dollars when purchasing one contract of the premature long-term bonds at t period, then she will receive $R_{2,t-1}$ dollars at $t+1$ period, since that is what the contract guarantees. In other words, the gross return of investment on the premature long-term bonds is $R_{1,t}$, which is equivalent to the return of the short-term bonds between periods t and $t+1$.

By including the long-term bonds, the agent's optimization problem is revised as follows:

$$\begin{aligned} \max_{\{c_t, M_t, B_{1,t}^A, B_{2,t}^A\}_{t=0}^{\infty}} \quad & E_0\left[\sum_{t=0}^{\infty} \beta^t \log c_t + \chi \log \frac{M_t}{P_t}\right],\ 0<\beta<1 \\ & s.t.\ c_t + \frac{M_t}{P_t} + \frac{B_{1,t}^A}{P_t} + \frac{B_{2,t}^A}{P_t} \leq y_t - \tau_t + \frac{M_{t-1}}{P_t} + \frac{R_{1,t-1}B_{1,t-1}^A}{P_t} + \frac{R_{2,t-1}R_{1,t}^{-1}B_{2,t-1}^A}{P_t}, \\ & c_t, M_t \geq 0,\ \forall t \geq 0,\ M_{-1} + R_{1,-1}B_{1,-1}^A + R_{2,-1}R_{1,0}^{-1}B_{2,-1}^A > 0 \end{aligned} \tag{43}$$

where $B_{\cdot,t}^A$ refers to the agent's holdings of the short- and long-term bonds. If the constant output assumption is maintained as seen in Section 3, then the Euler equations are Equations (27), (28)[22] and the following additional equation:

$$\begin{aligned} \frac{1}{R_{2,t}} &= E_t\left[\beta^2 \frac{P_t}{P_{t+2}}\right] \\ &= \frac{1}{R_{1,t}} E_t\left[\frac{1}{R_{1,t+1}}\right] + COV_t\left[\frac{\beta}{\pi_{t+1}}, \frac{\beta}{\pi_{t+2}}\right]. \end{aligned} \tag{44}$$

The additional Euler equation (44) implies the expectations theory of the term structure of interest rates. In regard to the market clearing condition, the bonds market clearing condition is revised as follows, while the money and goods market clearing conditions remain the same as in the previous sections.

$$B_{1,t}^s = B_{1,t}^A + B_{1,t}^C, \tag{45}$$

$$B_{2,t}^s = B_{2,t}^A + B_{2,t}^C, \tag{46}$$

where $B_{\cdot,t}^s$ refers to the short- and long-term bond supplies and $B_{\cdot,t}^C$ is the central bank's demand for the bonds.[23]

The public sector model should also be extended. First, the central bank holds the long-term government bonds, while the fiscal authority issues the long-term bonds. Parallel to the agent's case, the central bank sells the premature long-term bonds in the market and

[22] Since the short-term gross interest rate was denoted by R_t in the previous sections, R_t is equivalent to $R_{1,t}$ in this section.

[23] In addition, the definition of the agent's real financial wealth should be revised to include the long-term bonds.

purchases the newly issued long-term bonds. Then, the flow budget constraints of the central bank and the fiscal authority are as follows:

$$\frac{B_{1,t}^C - R_{1,t-1}B_{1,t-1}^C}{P_t} + \frac{B_{2,t}^C - R_{2,t-1}R_{1,t}^{-1}B_{2,t-1}^C}{P_t} = \frac{M_t - M_{t-1}}{P_t} + \kappa_t, \tag{47}$$

$$0 = \frac{B_{1,t} - R_{1,t-1}B_{1,t-1}}{P_t} + \frac{B_{2,t} - R_{2,t-1}R_{1,t}^{-1}B_{2,t-1}}{P_t} + s_t - \kappa_t. \tag{48}$$

The quasi-fiscal policy rule (24) and the fiscal policy rule (25) should be revised with long-term bonds. The revised rules are:

$$\kappa_t = \kappa_0 + \gamma_Q \left(\overline{cap} - b_{1,t-1}^C - b_{2,t-1}^C + m_{t-1}\right) + \xi_t, \tag{49}$$

$$s_t = \gamma_0 + \gamma_F \left(b_{1,t-1}^A + b_{2,t-1}^A\right) + \psi_t. \tag{50}$$

Second, the fiscal authority and central bank decide the maturity structure of their liabilities and assets. Cochrane (2001) showed that the maturity structure of the government nominal debt affects the equilibrium price level sequence. In this model extension, the following maturity structure is suggested:

$$B_{2,t} = \delta B_{1,t}, \ (\delta \geq 0), \tag{51}$$

$$B_{2,t}^C = (\delta + \mu_t)\, B_{1,t}^C, \tag{52}$$

$$\mu_t = \rho_\mu \mu_{t-1} + \varepsilon_{\mu t}, \ (0 \leq \rho_\mu < 1),$$

where δ is a policy parameter, $\varepsilon_{\mu t}$ is white noise and μ_t stands for the AR(1) exogenous policy shock that affects the central bank's asset maturity structure. In this specification of the maturity structure, the fiscal authority maintains a fixed ratio (δ) between the face values of the short- and long-term bonds. The central bank maintains the same ratio (δ)[24] for its assets in the steady state, while the bank's asset maturity structure deviates from the fixed ratio (δ) when the policy shock μ_t exists.

The aforementioned maturity policy is such that the central bank (the fiscal authority) adjusts the short- and long-term assets (debts) that are outstanding at each time period. Conversely, in Cochrane's (2001) cases with the outstanding long-term bonds, it was assumed that trade and redemption of premature long-term bonds were not allowed. Since this model allows such transactions, the total face value of the outstanding assets is exposed to the

[24] The basic results of the model remain the same when the fiscal authority's debt maturity ratio is different from the asset maturity ratio of the central bank.

valuation risk when the central bank adjusts its asset balances. That is, if the current short-term interest rate ($R_{1,t}$) increases, then the value of the premature long-term bonds decreases through the decreased market price ($R_{2,t-1}R_{1,t}^{-1}$).

This set-up is appropriate for analyzing issues of the exit strategy. When economic recovery and inflation are imminent, the Federal Reserve System may try to withdraw liquidity from the economy by selling its assets. During that process, it is plausible that the Federal Reserve System will be required to sell premature long-term bonds because a significant portion of the system's assets is contained within long-term bonds. Since the exit strategy increases the short-term interest rate, the valuation loss from trading the premature long-term bonds is relevant in the exit strategy.

By following the same steps as seen in Section 3, a linearized dynamic system is constructed for the vector of the endogenous variables $(\hat{\pi}_t, \hat{b}_{1,t}^C, \hat{b}_{1,t}^A)'$. The detailed specification of the system and solution are described in Appendix C. If the policy parameter is a special case of the AQFP regime as seen in Condition (35), then the surprise inflation is

$$\eta_t = \frac{\pi^2}{(1+\delta)R_1 b_1^C - m}\left(\varepsilon_{\xi t} - \frac{\delta b_1^C}{\pi}\varepsilon_{\theta t}\right), \tag{53}$$

where b_1^C is the steady state value (real) of the central bank's holdings for the short-term bonds. Note that the solution (53) is equivalent to the solution in Section 3 (36) when no long-term bonds are outstanding ($\delta = 0$).

The results show that when the central bank's steady state capital level is negative ($(1+\delta)R_1 b_1^C - m < 0$), then the positive monetary policy shock ($\varepsilon_{\theta t} > 0$) induces inflation ($\eta_t > 0$) in the AQFP regime. The economic mechanism is similar to that seen in Section 3. That is, the positive monetary policy shock devaluates the central bank's asset accounts. Then, in the AQFP regime, the agent will try to hold less money (when the central bank's capital is negative) since the policy shock has a negative effect on the value of the central bank's net liability. Therefore, the positive monetary policy shock induces surprise inflation. However, this outcome may not be desirable to the central bank because a positive monetary policy shock is generally intended for mitigating inflation.

Jeanne and Svensson (2007) pointed out that the negative capital of the central bank may undermine the bank's independence from the fiscal authority. However, it is also true that the central bank can still operate with negative capital. The results of this section show that negative capital may not be desirable to the central banks, but for a different reason. That is, a tightening monetary policy shock may ignite further inflation when the capital is negative. This issue is relevant to the central banks with negative capital because the fiscal authorities in such countries generally do not provide automatic fiscal back-ups in order to target the central bank's capital, thus the quasi-fiscal policy is arguably active.

6 Application 2: Credit easing

This section extends the benchmark model by including a credit easing operation. Credit easing is a policy strategy that attempts "to improve the functioning of credit markets and to increase the supply of credit to households and businesses" (Bernanke, 2009b). The Federal Reserve System implemented credit easing operations by purchasing risky assets from private agents. Therefore, this section introduces a risky asset into the central bank's asset account. As a result, it will be shown that the loss incurred by the central bank's operations affect inflation in the AQFP regime.

In this extension of the model, the central bank makes a one-period loan (L_t) to the agent, which is equivalent to the supply of the credit to the households and businesses in a credit easing operation. That is, the agent receives money in the same amount as the loan L_t at period t, then repays the loan in the next period with an interest payment. The gross interest rate for L_t at period $t+1$ is denoted as R^L_{t+1}, which is unknown at period t. In other words, the return for the loan is a random variable and the loan is risky.

The additional first order condition for L_t from the agent's optimization problem is as follows:

$$E_t \left[X_{t,t+1} \frac{P_t}{P_{t+1}} R^L_{t+1} \right] = 1. \tag{54}$$

By assuming further that the nominal pricing kernel $X_{t,t+1}\frac{P_t}{P_{t+1}}$ and the gross return of the loan R^L_{t+1} are uncorrelated, it is implied that

$$E_t \left[R^L_{t+1} \right] = R_t. \tag{55}$$

As a convenient specification of R^L_{t+1} that satisfies (55), I propose the following specification

$$R^L_{t+1} = R_t + \varepsilon_{Lt+1}, \tag{56}$$

where ε_{Lt} is white noise and refers to the random part of the return. The random part of the return ε_{Lt} is an exogenous process that the central bank cannot control. In this case, ε_{Lt} represents a random shock that occurs to the central bank's earnings due to random fluctuations in asset prices that are uncorrelated to its policy.

The central bank may finance the loan either by printing money or by selling fiscal authority bonds. If the loan is financed by issuing money, then the operation constitutes the monetary policy shock (θ_t) since the policy interest rate should be decreased while the nominal money balance increases. In order to focus on the equilibrium in the AQFP regime and the effect of the quasi-fiscal policy shock, let's suppose that the central bank fixes the level of the nominal money balance and sells fiscal authority bonds to finance the loan.[25] Then, in this model, the credit easing operation alters the composition of the central bank's assets by exchanging the risk-free fiscal authority debt with the risky loan to the agent. In regard to the agent's portfolio, the agent's asset (B_t^A) and liability (L_t) increase simultaneously due to the operation. In sum, the agent borrows from the central bank and invests in the fiscal authority debt while the fiscal authority debt held by the central bank decreases.

In actual policy implementations, the liquidity provided by the central bank is to be used to ease the credit condition among private agents instead of being invested in the fiscal authority debt. Since this paper is based on a representative agent model without financial frictions, the credit crunch issue is out of this paper's scope. Nevertheless, since the focus of this extension is on showing how the possible loss from the operation may affect inflation in a simple model, then the simplified model is still valid for analysis. In addition, the modeling set-up is such that the credit easing operation is implemented by changing the central bank's asset composition. Bernanke (2009a) stated that "Federal Reserve's credit easing approach focuses on the mix of loans and securities that it holds and on how this composition of assets affects credit conditions," whereas, in a quantitative easing operation "the composition of loans and securities ... is incidental." Therefore, the set-up reflects the actual policy intention of the Federal Reserve System.

In order to focus on the possible effects of losses incurred by the operation, policy decisions to adjust the level of the loan are also excluded from the model. That is, the central bank is assumed to maintain the constant level of the loan's real value. ($\frac{L_t}{P_t} = l, \forall t$) In other words, the central bank does not have the policy freedom to adjust the level of the loan due to an exogenous policy consideration (easing the credit condition), which cannot be explicitly modeled in this extension. The real value of the loan can be modeled to follow an exogenous stochastic process such as the AR(1) process, but the main result will remain the same. Then, with a minor extension to Equation (12), the central bank's budget constraint

[25] In the Federal Reserve System's case, the loan to the private agent was financed both ways. That is, the excess reserve held by banks increased significantly, which supported the asset purchases of the Federal Reserve System.

becomes the new accounting identity as follows.

$$\underbrace{\frac{B_t^C - B_{t-1}^C}{P_t} + \frac{L_t - L_{t-1}}{P_t}}_{\Delta Asset} = \underbrace{\frac{M_t - M_{t-1}}{P_t}}_{\Delta Liability} + \underbrace{\frac{i_{t-1}B_{t-1}^C + \left(i_{t-1} + \varepsilon_{Lt}\right) L_{t-1}}{P_t} + \kappa_t}_{\Delta Capital}. \tag{57}$$

Suppose further that the policy regime is the AQFP regime as in (35). Then, the surprise inflation is a linear function of the transfer shock ($\varepsilon_{\xi t}$), as well as the random return of the loan (ε_{Lt}). Note that, in this case, the term quasi-fiscal shock includes a policy shock ($\varepsilon_{\xi t}$) and an exogenous shock (ε_{Lt})

$$\eta_t = \frac{\pi^2}{Rb^C + Rl - m}\left(\varepsilon_{\xi t} + \frac{l}{\pi}\varepsilon_{Lt}\right). \tag{58}$$

A notable conclusion to this extension is that the loss incurred by the credit easing operation ($\varepsilon_{Lt} < 0$) has a deflationary effect when the central bank's steady state real capital ($Rb^C + Rl - m$) is positive. The economic mechanism underlying this result is the agent's portfolio adjustment as explained in Section 3. That is, the surprise earning shock (negative) of the central bank makes the nominal money more valuable to the agent in an overall assessment. Thus, the agent attempts to hold more money. Note that the result is analogous to the fiscal theory of the price level in the sense that a tax cut (or increase in a lump-sum transfer) induces inflation[26] when the fiscal policy is active.

The policy implication of the result is obvious. If policymakers are concerned with deflation (or inflation) and the quasi-fiscal policy is active, then any loss incurred from the credit easing operation should be immediately compensated by the fiscal authority as in (59)

$$\varepsilon_{\xi t} = -\frac{l}{\pi}\varepsilon_{Lt}. \tag{59}$$

The proposed policy (59) is actually guaranteed in the United Kingdom for the 'Asset Purchase Facility' of the Bank of England.[27] On the other hand, if the quasi-fiscal policy is passive, then the transfer from the fiscal authority to the central bank sufficiently reacts to the changes in the central bank's capital. In other words, in order to isolate the possible effects of the quasi-fiscal shocks on inflation, the fiscal authority's automatic and immediate funds transfer to the central bank is required regardless of whether the quasi-fiscal policy is active or passive. Therefore, the fiscal authority's role in offsetting the shocks in the central

[26] The steady state real capital of the consolidated government is generally negative ($-Rb^A - m < 0$).

[27] Regardless of whether the Bank of England has an active quasi-fiscal policy, it is "indemnified by the Treasury from any losses arising out of or in connection with the Facility" (Bank of England, 2009, p.1).

bank's capital is a prerequisite to price stabilization.

Traditionally, it has been argued that the quasi-fiscal operations of the central bank may undermine monetary policy's credibility and independence. Focusing on this point, Goodfriend (1994, 2009) claimed that an 'accord' between the two institutions is needed in order to ensure the central banks' independence. Although our reasoning is different, our analytic results support Goodfriend's (2009) suggestion. That is, this paper shows that the quasi-fiscal shocks affect inflation when the fiscal authority does not offset the shocks by targeting the central bank's capital. Therefore, policy coordination between the two institutions is indispensable for price stability.

7 Conclusion

Conventional macroeconomic models treat the central bank's balance sheet as superficial information that does not affect the equilibrium. However, this paper shows that quasi-fiscal policies, which alter the central bank's balance sheet, affect the equilibrium inflation in a certain policy regime. In the active quasi-fiscal policy regime, the transfer rule between the fiscal authority and central bank does not automatically stabilize the central bank's real capital while monetary and fiscal policies are passive. In this policy regime, the quasi-fiscal shocks, including the losses incurred from the credit easing operations, affect inflation via the private agent's portfolio adjustment. Previous studies, which have not considered this possibility, implicitly assume that the quasi-fiscal policy is always passive. However, in general circumstances, the policy's passivity is not guaranteed. In other words, it can be an illusionary belief that the fiscal authority will always automatically support the central bank's losses considering the actual laws on the central banks.

The crucial assumption needed for these results is the budgetary independence of the central banks. The budgetary independence of the central banks implies that the real value of central banks' net liability has a finite upper bound, which is the present value of seigniorage and transfer earnings. This paper also shows that the condition is satisfied under general circumstances. When the budgetary independence condition is satisfied, an additional equilibrium condition exists, which is referred to as the central bank's net liability valuation formula, and the formula may or may not restrict inflation process depending on the activeness of quasi-fiscal policy.

The economic mechanism underlying the equilibrium in the active quasi-fiscal policy regime is the private agent's response to the quasi-fiscal shocks. When a shock occurs to the central bank's earnings in this policy regime, the agent tries to adjust her portfolio between the central bank's liability and consumption, thus the shock affects inflation. This mechanism is comparable to the asset pricing relationships, which equate the value of an asset with the present value of the asset's current and future earnings. That is, the central bank's net liability valuation formula is an asset pricing equation where the general price level is the price of the bank's liability with respect to consumption.

By extending the benchmark model, this paper shows that the issues of an exit strategy and a credit easing operation may affect inflation in undesirable ways. If the policy regime is active quasi-fiscal policy during the implementation of the exit strategy, then the monetary policy shock may affect inflation through the central bank's net liability valuation formula. Notably, the contractionary monetary policy shock induces inflation when the central bank's capital is negative. If the central bank suffers losses from the credit easing operation in the active quasi-fiscal policy regime, then the loss induces deflation. Therefore, the fiscal authority and central bank should coordinate, by stabilizing the central bank's capital, in order to isolate the perverse effects of quasi-fiscal shocks on inflation.

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Appendix A. Part of the proof for Proposition 1

Conditions (15) and (16) imply that Conditions (17) and (18) hold. By summing the flow budget constraints (12) and (13) over the period from t to $T-1$ after taking the discounted expected value (at period t) of the constraints, the following equations are true for all $T \geq t+1$.

$$E_t\left[X_{t,T} d_T^C\right] = d_t^C - \sum_{s=t}^{T-1} E_t\left[X_{t,s}\left(\frac{i_s}{1+i_s} f(y_s, i_s) + \kappa_s\right)\right], \tag{A1}$$

$$E_t\left[X_{t,T} d_T^F\right] = d_t^F - \sum_{s=t}^{T-1} E_t\left[X_{t,s}\left(s_s - \kappa_s\right)\right]. \tag{A2}$$

From Conditions (15) and (16),

$$d_t^C - \sum_{s=t}^{\infty} E_t\left[X_{t,s}\left(\frac{i_s}{1+i_s} f(y_s, i_s) + \kappa_s\right)\right] \leq 0, \tag{A3}$$

$$d_t^F - \sum_{s=t}^{\infty} E_t\left[X_{t,s}\left(s_s - \kappa_s\right)\right] \leq 0. \tag{A4}$$

As $T \to \infty$ in (A1) and (A2),

$$\lim_{T\to\infty} E_t\left[X_{t,T} d_T^C\right] = d_t^C - \sum_{s=t}^{\infty} E_t\left[X_{t,s}\left(\frac{i_s}{1+i_s} f(y_s, i_s) + \kappa_s\right)\right], \tag{A5}$$

$$\lim_{T\to\infty} E_t\left[X_{t,T} d_T^F\right] = d_t^F - \sum_{s=t}^{\infty} E_t\left[X_{t,s}\left(s_s - \kappa_s\right)\right]. \tag{A6}$$

Equations (A3)-(A6) imply

$$-\infty < \lim_{T\to\infty} E_t\left[X_{t,T} d_T^C\right] \leq 0, \tag{A7}$$

$$-\infty < \lim_{T\to\infty} E_t\left[X_{t,T} d_T^F\right] \leq 0. \tag{A8}$$

since $\left|w_t^A\right| < \infty$ and the Conditions (15) and (16) imply $d_t^C, d_t^F > -\infty$. Due to the agent's transversality condition (5),

$$\lim_{T\to\infty} E_t\left[X_{t,T} d_T^C\right] + \lim_{T\to\infty} E_t\left[X_{t,T} d_T^F\right] = 0, \tag{A9}$$

since $w_T^A = d_T^C + d_T^F$. Therefore,

$$\lim_{T\to\infty} E_t\left[X_{t,T} d_T^C\right] = 0, \tag{A10}$$

$$\lim_{T\to\infty} E_t\left[X_{t,T} d_T^F\right] = 0. \tag{A11}$$

Conversely, Conditions (A10) and (A11) imply that assumptions (15) and (16) hold. Suppose that Conditions (A10) and (A11) are true. From the budget constraints, Equations (A1) and (A2) are true. As $T \to \infty$ in Equations (A1) and (A2), Conditions (A3) and (A4) hold by equality. Then, since d_t^C and d_t^F are finite, Conditions (15) and (16) are necessarily true. Therefore, Conditions (15) and (16) are equivalent to Conditions (A10) and (A11).

Appendix B. Solution of the dynamic system in Section 3

In order to arrange the model with a linear dynamic system, this paper linearizes the Fisher equation (28) and the budget constraints ((12) and (13)). These three equations form a dynamic system in matrix equation (A12). The first row of (A12) is from the Fisher equation, the second row is from the central bank budget constraint and the third row is from the fiscal authority budget constraint. The money market equilibrium condition (27) and the policy rules ((23), (24) and (25)) are substituted into the three equations. In the coefficient matrices (Γ_0, Γ_1, Φ_0 and Φ_1), the variables without time index refer to the steady state values.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \varphi_1 & 1 & 0 \\ \varphi_4 & 1 & 1 \end{bmatrix}}_{\Gamma_0} \underbrace{\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{b}^C_{t+1} \\ \hat{b}^A_{t+1} \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} \alpha\beta & 0 & 0 \\ \varphi_2 & \beta^{-1} - \gamma_Q & 0 \\ \varphi_5 & \beta^{-1} - \gamma_Q & \beta^{-1} - \gamma_F \end{bmatrix}}_{\Gamma_1} \underbrace{\begin{bmatrix} \hat{\pi}_t \\ \hat{b}^C_t \\ \hat{b}^A_t \end{bmatrix}}_{x_t}$$
$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ \varphi_3 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}}_{\Phi_0} \underbrace{\begin{bmatrix} \hat{\theta}_{t+1} \\ \hat{\xi}_{t+1} \\ \hat{\psi}_{t+1} \end{bmatrix}}_{z_{t+1}} + \underbrace{\begin{bmatrix} \beta & 0 & 0 \\ \alpha^{-1}\varphi_2 & 0 & 0 \\ \alpha^{-1}\varphi_5 & 0 & 0 \end{bmatrix}}_{\Phi_1} \underbrace{\begin{bmatrix} \hat{\theta}_t \\ \hat{\xi}_t \\ \hat{\psi}_t \end{bmatrix}}_{z_t} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \hat{\eta}_{t+1}}_{\Pi_{t+1}}, \tag{A12}$$

$$\text{where } \varphi_1 \equiv \frac{Rb^C - m}{\pi^2} + \frac{\alpha\chi y}{(R-1)^2}, \varphi_2 \equiv \frac{\alpha}{\pi}\left[\frac{\chi y}{(R-1)^2} + b^C\right] - \frac{\alpha\gamma_Q\chi y}{(R-1)^2}, \varphi_3 \equiv \frac{-\chi y}{(R-1)^2},$$
$$\varphi_4 \equiv \frac{Rb^A + Rb^C}{\pi^2}, \varphi_5 \equiv \frac{\alpha}{\pi}(b^A + b^C) - \frac{\alpha\gamma_Q\chi y}{(R-1)^2}.$$

If the vectors x_t and z_t are stacked, then the system (A12) becomes Equation (A13).

$$\underbrace{\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{b}^C_{t+1} \\ \hat{b}^A_{t+1} \\ \hat{\theta}_{t+1} \\ \hat{\xi}_{t+1} \\ \hat{\psi}_{t+1} \end{bmatrix}}_{Y_{t+1}} = \underbrace{\begin{bmatrix} \alpha\beta & 0 & 0 & \beta & 0 & 0 \\ \varphi_6 & \beta^{-1} - \gamma_Q & 0 & \varphi_7 & \rho_\xi & 0 \\ \varphi_8 & 0 & \beta^{-1} - \gamma_F & \varphi_9 & 0 & -\rho_\psi \\ 0 & 0 & 0 & \rho_\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_\xi & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_\psi \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \hat{\pi}_t \\ \hat{b}^C_t \\ \hat{b}^A_t \\ \hat{\theta}_t \\ \hat{\xi}_t \\ \hat{\psi}_t \end{bmatrix}}_{Y_t}$$

$$+ \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\varphi_1 & 0 & 0 & \varphi_3 & 1 & 0 \\ \varphi_1 - \varphi_4 & 0 & 0 & -\varphi_3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \eta_{t+1} \\ 0 \\ 0 \\ \varepsilon_{\theta t+1} \\ \varepsilon_{\xi t+1} \\ \varepsilon_{\psi t+1} \end{bmatrix}}_{\zeta_t}, \tag{A13}$$

$$\begin{aligned} \text{where } \varphi_6 &\equiv \varphi_2 - \alpha\beta\varphi_1, \varphi_7 \equiv \rho_\theta\varphi_3 - \beta\varphi_1 + \alpha^{-1}\varphi_2, \\ \varphi_8 &\equiv \alpha\beta(\varphi_1 - \varphi_4) - \varphi_2 + \varphi_5, \varphi_9 \equiv -\rho_\theta\varphi_3 + \alpha^{-1}\varphi_8. \end{aligned}$$

The eigenvalues of transition matrix A can be read off from the diagonal elements. By the eigenvector decomposition, matrix A can be decomposed into $P\Lambda P^{-1}$ where Λ is a diagonal matrix with eigenvalues, and the columns of the P matrix are the right eigenvectors of A. P^{-1} can be expressed as matrix (A14).

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & \frac{\beta}{\alpha\beta - \rho_\theta} & 0 & 0 \\ p_1 & 1 & 0 & p_2 & p_3 & 0 \\ p_4 & 0 & 1 & p_5 & 0 & p_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{A14}$$

where $p_1 \equiv \frac{-\varphi_6}{\alpha\beta - \beta^{-1} + \gamma_Q}, p_2 \equiv \frac{1}{\beta^{-1} - \gamma_Q - \rho_\theta}\left(\varphi_7 - \frac{\beta\varphi_6}{\alpha\beta - \beta^{-1} + \gamma_Q}\right), p_3 \equiv \frac{\rho_\xi}{\beta^{-1} - \gamma_Q - \rho_\xi},$

$$p_4 \equiv \frac{-\varphi_8}{\alpha\beta - \beta^{-1} + \gamma_F}, p_5 \equiv \frac{1}{\beta^{-1} - \gamma_F - \rho_\theta}\left(\varphi_9 - \frac{\beta\varphi_8}{\alpha\beta - \beta^{-1} + \gamma_F}\right), p_6 \equiv \frac{-\rho_\psi}{\beta^{-1} - \gamma_F - \rho_\psi}.$$

The solution of the system is derived by eliminating the explosive eigenvalues. If the j'th eigenvalue is explosive, then the solution is such that $P^{j\cdot}Y_t = 0$ where $P^{j\cdot}$ refers to the j'th row of P^{-1}. The linear mappings from the exogenous shock to the endogenous forecast error are given by $P^{j\cdot}C\zeta_t = 0$.

Appendix C. Solution of the dynamic system in Section 5

The linear dynamic system in Section 5 is derived by linearizing the Fisher equation (28) and the budget constraints ((47) and (48)). The money market equilibrium condition (27), the term structure equilibrium condition (44) and the policy rules ((23), (49), (50), (51) and (52)) are substituted into the system of three equations.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \varphi_1 & 1 & 0 \\ \varphi_8 & 1 & 1 \end{bmatrix}}_{\Gamma_0} \underbrace{\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{b}^C_{1,t+1} \\ \hat{b}^A_{1,t+1} \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} \alpha\beta & 0 & 0 \\ \varphi_2 & \beta^{-1} - \gamma_Q & 0 \\ \varphi_9 & \beta^{-1} - \gamma_Q & \beta^{-1} - \gamma_F \end{bmatrix}}_{\Gamma_1} \underbrace{\begin{bmatrix} \hat{\pi}_t \\ \hat{b}^C_{1,t} \\ \hat{b}^A_{1,t} \end{bmatrix}}_{x_t} \tag{A15}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ \varphi_3 & \varphi_4 & \frac{1}{1+\delta} & 0 \\ \varphi_{10} & 0 & \frac{1}{1+\delta} & \frac{-1}{1+\delta} \end{bmatrix}}_{\Phi_0} \underbrace{\begin{bmatrix} \hat{\theta}_{t+1} \\ \hat{\mu}_{t+1} \\ \hat{\xi}_{t+1} \\ \hat{\psi}_{t+1} \end{bmatrix}}_{z_{t+1}} + \underbrace{\begin{bmatrix} \beta & 0 & 0 & 0 \\ \varphi_5 & \varphi_6 & 0 & 0 \\ \varphi_{11} & \varphi_{12} & 0 & 0 \end{bmatrix}}_{\Phi_1} \underbrace{\begin{bmatrix} \hat{\theta}_t \\ \hat{\mu}_t \\ \hat{\xi}_t \\ \hat{\psi}_t \end{bmatrix}}_{z_t} + \underbrace{\begin{bmatrix} 1 \\ \varphi_7 \\ \varphi_{13} \end{bmatrix}}_{\Pi_{t+1}} \hat{\eta}_{t+1},$$

$$
\begin{aligned}
\text{where } \varphi_1 &\equiv \frac{1}{1+\delta}\left[\frac{R_1 b_1^C - m}{\pi^2} + \frac{\alpha\chi y}{(R_1-1)^2}\right], \varphi_2 \equiv \frac{1}{1+\delta}\left[\frac{\alpha}{\pi}\left(\frac{\chi y}{(R_1-1)^2} + b_1^C\right) - \frac{\alpha\gamma_Q\chi y}{(R_1-1)^2}\right], \\
\varphi_3 &\equiv \frac{-1}{1+\delta}\left[\frac{\chi y}{(R_1-1)^2} + \frac{\delta b_1^C}{\pi}\right], \varphi_4 \equiv \frac{-b_1^C}{1+\delta}, \varphi_5 \equiv \alpha^{-1}\varphi_2 + \frac{\delta\rho_\theta b_1^C}{(1+\delta)\pi}, \\
\varphi_6 &\equiv \frac{(\beta^{-1}-\gamma_Q)b_1^C}{1+\delta}, \varphi_7 \equiv -\frac{(1+\alpha\beta)\delta b_1^C}{\beta(1+\delta)\pi}, \varphi_8 \equiv \frac{R_1 b_1^A + R_1 b_1^C}{(1+\delta)\pi^2}, \\
\varphi_9 &\equiv \frac{1}{1+\delta}\left[\frac{\alpha(b_1^A + b_1^C)}{\pi} - \frac{\alpha\gamma_Q\chi y}{(R_1-1)^2}\right], \varphi_{10} \equiv \frac{-\delta(b_1^A + b_1^C)}{(1+\delta)\pi}, \varphi_{11} \equiv \alpha^{-1}\varphi_9 + \frac{\delta\rho_\theta(b_1^A + b_1^C)}{(1+\delta)\pi}, \\
\varphi_{12} &\equiv \frac{(\gamma_F - \gamma_Q)b_1^C}{1+\delta}, \varphi_{13} \equiv -\frac{(1+\alpha\beta)\delta(b_1^A + b_1^C)}{\beta(1+\delta)\pi}.
\end{aligned}
$$

By stacking the vectors x_t and z_t, the system (A15) becomes Equation (A16).

$$
\underbrace{\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{b}^C_{1,t+1} \\ \hat{b}^A_{1,t+1} \\ \hat{\theta}_{t+1} \\ \hat{\mu}_{t+1} \\ \hat{\xi}_{t+1} \\ \hat{\psi}_{t+1} \end{bmatrix}}_{Y_{t+1}} = \underbrace{\begin{bmatrix} \alpha\beta & 0 & 0 & \beta & 0 & 0 & 0 \\ \varphi_{14} & \beta^{-1}-\gamma_Q & 0 & \varphi_{15} & \varphi_{16} & \frac{\rho_\xi}{1+\delta} & 0 \\ \varphi_{17} & 0 & \beta^{-1}-\gamma_F & \varphi_{18} & \varphi_{19} & 0 & \frac{-\rho_\psi}{1+\delta} \\ 0 & 0 & 0 & \rho_\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_\xi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_\psi \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \hat{\pi}_t \\ \hat{b}^C_{1,t} \\ \hat{b}^A_{1,t} \\ \hat{\theta}_t \\ \hat{\mu}_t \\ \hat{\xi}_t \\ \hat{\psi}_t \end{bmatrix}}_{Y_t}
$$

$$
+ \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varphi_7 - \varphi_1 & 0 & 0 & \varphi_3 & \varphi_4 & \frac{1}{1+\delta} & 0 \\ \varphi_{20} & 0 & 0 & \varphi_{10} - \varphi_3 & -\varphi_4 & 0 & \frac{-1}{1+\delta} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \eta_{t+1} \\ 0 \\ 0 \\ \varepsilon_{\theta t+1} \\ \varepsilon_{\mu t+1} \\ \varepsilon_{\xi t+1} \\ \varepsilon_{\psi t+1} \end{bmatrix}}_{\zeta_t}, \quad \text{(A16)}
$$

$$
\begin{aligned}
\text{where } \varphi_{14} &\equiv \varphi_2 - \alpha\beta\varphi_1, \varphi_{15} \equiv \rho_\theta\varphi_3 - \beta\varphi_1 + \varphi_5, \varphi_{16} \equiv \rho_\mu\varphi_4 + \varphi_6, \\
\varphi_{17} &\equiv \alpha\beta(\varphi_1 - \varphi_8) - \varphi_2 + \varphi_9, \varphi_{18} \equiv \rho_\theta(\varphi_{10} - \varphi_3) + \beta(\varphi_1 - \varphi_8) - \varphi_5 + \varphi_{11}, \\
\varphi_{19} &\equiv -\rho_\mu\varphi_4 - \varphi_6 + \varphi_{12}, \varphi_{20} \equiv \varphi_1 - \varphi_7 - \varphi_8 + \varphi_{13}.
\end{aligned}
$$

Then, the transition matrix A is decomposed into $P\Lambda P^{-1}$ where the columns of P are right eigenvectors of A. The inverse matrix of P is as follows:

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & \frac{\beta}{\alpha\beta-\rho_\theta} & 0 & 0 & 0 \\ p_1 & 1 & 0 & p_2 & p_3 & p_4 & 0 \\ p_5 & 0 & 1 & p_6 & p_7 & 0 & p_8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{aligned} \text{where } p_1 &\equiv \frac{-\varphi_{14}}{\alpha\beta-\beta^{-1}+\gamma_Q}, p_2 \equiv \frac{1}{\beta^{-1}-\gamma_Q-\rho_\theta}\left(\varphi_{15}-\frac{\beta\varphi_{14}}{\alpha\beta-\beta^{-1}+\gamma_Q}\right), p_3 \equiv \frac{\varphi_{16}}{\beta^{-1}-\gamma_Q-\rho_\mu}, \\ p_4 &\equiv \frac{\rho_\xi(1+\delta)^{-1}}{(\beta^{-1}-\gamma_Q-\rho_\xi)}, p_5 \equiv \frac{-\varphi_{17}}{\alpha\beta-\beta^{-1}+\gamma_F}, p_6 \equiv \frac{1}{\beta^{-1}-\gamma_F-\rho_\theta}\left(\varphi_{18}-\frac{\beta\varphi_{17}}{\alpha\beta-\beta^{-1}+\gamma_F}\right), \\ p_7 &\equiv \frac{\varphi_{19}}{\beta^{-1}-\gamma_F-\rho_\mu}, p_8 \equiv \frac{-\rho_\psi(1+\delta)^{-1}}{\beta^{-1}-\gamma_F-\rho_\psi}. \end{aligned}$$

When j'th eigenvalue is explosive, the linear mappings from the exogenous shocks to the endogenous forecast error are given by $P^{j\cdot}C\zeta_t = 0$ where $P^{j\cdot}$ refers to the j'th row of P^{-1}.