A NOTE ON THE PSEUDO-SPECTRA AND THE PSEUDO-COVARIANCE GENERATING FUNCTIONS OF ARMA PROCESSES*

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Abstract

Although the spectral analysis of stationary stochastic processes has solid mathematical foundations, this is not the case for non-stationary stochastic processes. In this paper, the algebraic foundations of the spectral analysis of non-stationary ARMA processes are established. For this purpose the Fourier Transform is extended to the field of fractions of polynomials. Then, the Extended Fourier Transform pair *pseudo-covariance generating function / pseudo-spectrum*, analogous to the Fourier Transform pair *covariance generating function / spectrum*, is defined. The new transform pair is well defined for stationary and non-stationary ARMA processes. This new approach can be viewed as an extension of the classical spectral analysis. It is shown that the frequency domain has some additional algebraic advantages over the time domain.

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1 Introduction

Time series literature provides two alternatives to represent non-stationary stochastic processes in the frequency domain. The first one is based on the same mathematical concepts used in the case of stationary processes, i.e., the Hilbert spaces l_2 and L_2 and the Fourier transform $\mathfrak{F}$. In this case the property of finite variances is required and then, we are limited to analyzing stochastic processes of finite duration. Since, in practice, we are facing non-stationary processes, the *covariance generating functions* depend on the time origin and, therefore, their corresponding *pseudo-spectra* are, equally, time dependent (see Hatanaka and Suzuki, 1967). Also included within this approach we can find some processes whose parameters change slowly with time so that we can analyze the evolution of its *pseudo-spectra*. Priestley (1981) has coined them *evolutionary spectra*.

The second approach deals with non-stationary and infinite processes with infinite variance, and its *pseudo-spectrum* functional form is obtained from the structural model of the time series processes. This approach has been extensively used in the literature (e.g., Harvey (1989), Young, Pedregal, and Tych (1999)). Within this approach the *pseudo-spectra* are similar to the spectra as regards to its functional structure and simplicity. There is, however, one drawback. Contrary to the spectra, the *pseudo-spectra* are not mathematically well founded since they are outside of the classical Fourier analysis. Our goal in this paper is to fill out this gap by generalizing the spectral theory to the non-stationary AutoRegressive Moving Average (ARMA) processes. To do that, we will extend the Fourier transform outside the l_2 space.

Priestley's (1981) evolutionary spectra are local, i.e., they describe the variance at each instant in time. Equally local are the Hatanaka and Suzuki's (1967) *pseudo-spectra* since its estimation is based on the use of samples of finite length. In our case, however, the *pseudo-spectrum* describe the variance for the whole process in a similar way as the spectra of the stationary processes. Besides, they are also time independent if the parameters of the ARMA model remain unchanged.

This paper is organized as follows: the *pseudo-covariance* is defined in Section 2, and the *pseudo-covariance generating function* in Section 3. In Section 4 we provide a new definition for the *pseudo-spectra* based on the *pseudo-covariance generating function*. In Section 5 we show that the frequency domain has some additional algebraic advantages over the time domain. Finally, in Section 6 we underline some conclusions. Some notation and the algebraic results used in the paper appear in the Appendix: in Section A.1 the algebraic structure needed to deal with the *pseudo-spectra* is exposed; in Section A.2 we extend the Fourier transform outside the Hilbert space l_2.

2 The pseudo-covariance

Let $(\Omega, \mathscr{B}, P)$ be a probability space. For a given wide-sense stationary stochastic process of uncorrelated random variables with zero mean and variance equal to one, $\boldsymbol{\epsilon} = \{\epsilon_j; j \in \mathbb{Z}\}$ on $(\Omega, \mathscr{B}, P)$, we may consider the set $\mathbf{H}$ of all finite linear combinations:

$$L(\boldsymbol{\epsilon}) = \{f : \Omega \to \mathbb{R}; \text{ such that } f = \sum_{j=n}^{m} \lambda_j \epsilon_j, \ -\infty < n \leq m < \infty\}.$$

$L(\boldsymbol{\epsilon})$ is a subspace of the Hilbert space, $\mathbf{H}^{\epsilon}$, generated by ϵ, with scalar product of two vectors $x, y \in \mathbf{H}^{\epsilon}$:

$$\langle x|y \rangle = E\left[xy\right] = \langle \sum_{j=n}^{m} a_j \epsilon_j \mid \sum_{j=n'}^{m'} b_j \epsilon_j \rangle = \sum_{j=n''}^{m''} a_j b_j, \qquad (1)$$

where $x = \sum_{j=n}^{m} a_j \epsilon_j$, and $y = \sum_{j=n'}^{m'} b_j \epsilon_j$. This scalar product is the *covariance* of x and y. The vector space $\mathbf{H}^{\epsilon}$ is a subspace of the well known Hilbert space $L_2(\Omega, \mathscr{B}, P)$[1]. To define the *pseudo-covariance* we need a wider framework.

2.1 The dual space $L(\boldsymbol{\epsilon})^*$

Let $L(\boldsymbol{\epsilon})^*$ be the dual space of $L(\boldsymbol{\epsilon})$, i.e., the set of all linear functionals defined on $L(\boldsymbol{\epsilon})$:

$$L(\boldsymbol{\epsilon})^* = \{f : L(\boldsymbol{\epsilon}) \to \mathbb{R}; \text{ such that } f \text{ is linear}\}.$$

Because all the elements of $L(\boldsymbol{\epsilon})^*$ are linear functionals it is possible to describe each functional f using the value of $f(\cdot)$ for each ϵ_j, i.e., using the values $f(\epsilon_j) = a_j$. Then we know that $f(\sum_{j=n}^{m} \lambda_j \epsilon_j) = \sum_{j=n}^{m} \lambda_j a_j$. It follows that each element of the dual space, $f \in L(\boldsymbol{\epsilon})^*$, is associated to a sequence

[1](see Caines, 1988, Chapter one.)

$\{a_j\}_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ [2]. Is easy to check that the functionals of $L(\boldsymbol{\epsilon})^*$ are of the form $f(\,\cdot\,) \equiv \langle\, \sum_{j\in\mathbb{Z}} a_j\epsilon_j \mid \cdot\,\rangle$, and therefore

$$f\Big(\sum_{j=n}^{m} \lambda_j\epsilon_j\Big) \equiv \langle\, \sum_{j\in\mathbb{Z}} a_j\epsilon_j \mid \sum_{j=n}^{m} \lambda_j\epsilon_j \,\rangle = \sum_{j=n}^{m} \lambda_j a_j.$$

We let $f \equiv \sum_{j\in\mathbb{Z}} a_j\epsilon_j$ denote the elements $f \in L(\boldsymbol{\epsilon})^*$, and let B denote the usual backward operator: $Bx_t = x_{t-1}$.

Definition 2.1 (Filtered process). Let $\{u_t\}_{t=-\infty}^{\infty}$ be an stochastic process. We define a *linear filter* (or simply, *filter*) as a linear operator $b(B) = \sum_{j\in\mathbb{Z}} b_jB^j$, where the weights are characterized by the sequence $b \equiv \{b_j\}_{j\in\mathbb{Z}}$. The *filtered process*, $\{b(B)u_t\}_{t=-\infty}^{\infty}$ will be then expressed as

$$b(B)u_t = \cdots + b_1u_{t-1} + b_0u_t + b_{-1}u_{t+1} + \cdots = \sum_{r+s=t} b_ru_s \equiv (b*u)_t;$$

where the elements of the new stochastic process $\{b(B)u_t\}_{t=-\infty}^{\infty} = b*u$, are weighted sums of the random variables of the original stochastic process $\{u_t\}_{t=-\infty}^{\infty}$. Therefore, the *filtered process* is a sequence of functionals of the dual space $L(u)^*$, associated to the sequence b.

2.2 Covariance and pseudo-covariance

In this subsection we define *covariance* and the *pseudo-covariance* using three subspaces of the dual space $L(\boldsymbol{\epsilon})^*$. The first subspace is the set, E_{l_2}, of all functionals $f \equiv \sum_{j\in\mathbb{Z}} a_j\epsilon_j$ whose associated sequence $\{a_j\}_{j\in\mathbb{Z}}$ are square summable[3].

$$\mathrm{E}_{l_2} = \{f \in L(\boldsymbol{\epsilon})^*;\ \text{such that } a \in l_2\}.$$

The subspaces E_{l_2} and $\mathbf{H}^{\epsilon} \subseteq L_2(\Omega, \mathscr{B}, P)$ are *isometrically isomorphic.*[4] From this, it is straight forward to see that the stationary solution[5] of any ARMA models is a sequence of functionals $y_t = \sum_{j=\infty}^{\infty} \varphi_j\epsilon_{t-j}$, where each $y_t \in \mathrm{E}_{l_2}$. We can now redefine the *covariance* as a scalar product of functionals in E_{l_2}:

[2] see definition of $\mathbb{R}^{\mathbb{Z}}$ in Section A.1 in the Appendix.

[3] see definition of l_2 at the end of Section A.1.

[4] In fact, if ϵ is a base of $L_2(\Omega, \mathscr{B}, P)$, then $\mathbf{H}^{\epsilon} = L_2(\Omega, \mathscr{B}, P)$; and then E_{l_2} and $L_2(\Omega, \mathscr{B}, P)$ are *isometrically isomorphic.*

[5] (see Brockwell and Davis, 1987, Theorem 3.1.3)

Definition 2.2 (Covariance). The *covariance* of $x, y \in \mathrm{E}_{l_2}$ is the following bilinear form

$$\begin{aligned} \operatorname{cov} : \mathrm{E}_{l_2} \times \mathrm{E}_{l_2} & \longrightarrow \mathbb{R} \\ \operatorname{cov}(x,y) \equiv E\,[xy] & \longrightarrow \langle\, \textstyle\sum_{j\in\mathbb{Z}} a_j \epsilon_j \mid \sum_{j\in\mathbb{Z}} b_j \epsilon_j \,\rangle = \sum_{j\in\mathbb{Z}} a_j b_j , \end{aligned} \tag{2}$$

where $x = \sum_{j\in\mathbb{Z}} a_j \epsilon_j,\ y = \sum_{j\in\mathbb{Z}} b_j \epsilon_j$.

The only difference with (1) is that here, x and y are elements of $L(\boldsymbol{\epsilon})^*$; but since a and b are squared summable sequences, the functionals $x = \sum_{j\in\mathbb{Z}} a_j \epsilon_j$ and $y = \sum_{j\in\mathbb{Z}} b_j \epsilon_j$, are also random variables with finite first and second moments. Therefore, this new definition is operationally indistinguishable from (1).

Clearly, from the former definition, the *covariance* is related to sequences that belong to l_2. The *pseudo-covariance* will be related to *left finite* sequences ($\{a_j\},\ n < j < \infty$), and *right finite* sequences ($\{b_j\},\ -\infty < j < m$); where n and m are scalars[6]. With the two following subspaces of $L(\boldsymbol{\epsilon})^*$ we will define the *pseudo-covariance*. Let $\mathrm{E}_{\blacktriangleright}$ and $\mathrm{E}_{\triangleleft}$ be the subspaces:[7]

$$\begin{aligned} \mathrm{E}_{\blacktriangleright} &= \{f \in L(\boldsymbol{\epsilon})^*; \text{ such that } a \text{ is left finite}\} \\ \mathrm{E}_{\triangleleft} &= \{f \in L(\boldsymbol{\epsilon})^*; \text{ such that } a \text{ is right finite}\} \end{aligned} .$$

Definition 2.3 (Pseudo-covariance). The *pseudo-covariance* of $x \in \mathrm{E}_{\blacktriangleright}$ and $y \in \mathrm{E}_{\triangleleft}$ is the following bilinear form:

$$\begin{aligned} \text{pseudo} - \operatorname{cov} : \mathrm{E}_{\blacktriangleright} \times \mathrm{E}_{\triangleleft} & \longrightarrow \mathbb{R} \\ \text{pseudo-cov}(x,y) \equiv E\,[x,y] & \longrightarrow \langle\, \textstyle\sum_{j\in\mathbb{Z}} a_j \epsilon_j \mid \sum_{j\in\mathbb{Z}} b_j \epsilon_j \,\rangle = \sum_{j\in\mathbb{Z}} a_j b_j . \end{aligned}$$

where $x = \sum_{j\in\mathbb{Z}} a_j \epsilon_j$, and $y = \sum_{j\in\mathbb{Z}} b_j \epsilon_j$.

Note that, since a_j is a left finite sequence and b_j is a right finite sequence, only a finite number of products $a_j b_j$ are non-zero.

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[6] This sequences belong to the sets $\mathbb{R}(z)$ or $\mathfrak{R}\,[\mathbb{R}(z)]$ defined in Notation A.1.2 in the Appendix.

[7] The subspace E_{l_2} is *isomorphic* to the subspace, M_{l_2}, of sequences $a \in \mathbb{R}^{\mathbb{Z}}$ associated to the functionals $f \in \mathrm{E}_{l_2}$. Therefore M_{l_2} is a subspace of l_2. In the same manner, the subspaces $\mathrm{E}_{\blacktriangleright}$ and $\mathrm{E}_{\triangleleft}$ are, respectively, *isomorphic* to the subspaces $M_{\mathbb{R}(z)} \subset \mathbb{R}(z)$ and $M_{\mathfrak{R}[\mathbb{R}(z)]} \subset \mathfrak{R}\,[\mathbb{R}(z)]$, of sequences $a \in \mathbb{R}^{\mathbb{Z}}$ associated to the functionals $f \in \mathrm{E}_{\blacktriangleright}$ and $f \in \mathrm{E}_{\triangleleft}$.

3 The pseudo-covariance generating function

From now on let $\{\xi_t\} \sim$ w.n. $N(0, \sigma^2_\xi)$ denote a Gaussian stochastic process that satisfies $E(\xi_t) = 0$, $E(\xi_t^2) = \sigma^2$, and $E(\xi_t \xi_{t-k}) = 0$ when $k \neq 0$. In the next subsection we deal with the three inverses of polynomials defined in equations (15), (11), and (12); i.e., the inverses that belong to l_1 *(the set of absolutely summable sequences)*; or belong to $\mathbb{R}((z))$ *(the set of left finite sequences)*; or belong to $\mathfrak{R}\left[\mathbb{R}((z))\right]$ *(the set of right finite sequences)*[8].

3.1 Three solutions of ARMA models

In this paper we restrict ourselves on ARMA(p, q) stochastic process, i.e., stochastic process $\{y_t\}_{t=-\infty}^{\infty}$ that satisfy the ARMA(p, q) model

$$\phi(B) y_t = \theta(B) \xi_t, \qquad \{\xi_t\} \sim \text{w.n. } N(0, \sigma^2_\xi) \tag{3}$$

where $\phi(B) = \sum_{j=0}^{p} \phi_j B^j$ is the AutoRegressive (AR) polynomial and $\theta(B) = \sum_{j=0}^{q} \theta_j B^j$ is the Moving Average (MA) polynomial. We assume that $\phi_0 = \theta_0 = 1$, and $\phi(\cdot)$ and $\theta(\cdot)$ have no common zeros. $\{\xi_t\}_{t=-\infty}^{\infty}$ denotes the innovation in the process $\{y_t\}_{t=-\infty}^{\infty}$.

If the AR polynomial does not have roots with unit modulus, there is a unique stationary solution of (3) (see Brockwell and Davis, 1987); this stationary solution is a filtered process whose associated sequence belongs to l_1. To find the *stationary solution* we should use the inverse of the AR polynomial that belongs to l_1 (see (15)):

$$y_t = \frac{\theta(B)}{\phi(B)}{}_{(l_1)} \xi_t, \quad \{\xi_t\} \sim \text{w.n. } N(0, \sigma^2_\xi).$$

Then, the weighted sum of the $\{\xi_t\}_{t=-\infty}^{\infty}$ random variables generates a random variable y_t with finite first and second moments; and therefore, the random variables $\{y_t\}_{t=-\infty}^{\infty}$ belong to $\mathbf{H}^\epsilon$.

For any AR polynomial we can define two more solutions. The causal or *backward solution* is related to the *left finite* inverse sequence (see (11)):

$$y_t = \frac{\theta(B)}{\phi(B)}\blacktriangleright \xi_j, \quad \{\xi_t\} \sim \text{w.n. } N(0, \sigma^2_\xi).$$

[8]A formal definition of the sets $\mathbb{R}((z))$, and $\mathfrak{R}\left[\mathbb{R}((z))\right]$ appear in Godement (1974) or in Notation A.1.1 in the Appendix.

This solution is a weighted sum of the *past* values of $\{\xi_t\}_{t=-\infty}^{\infty}$. When all the roots of $\phi(z)$ lie outside the unit circle the left finite inverse sequence is absultely summable; and therefore this solution is the stationary one. We can also define the *forward solution* using the *right finite* inverse sequence (see (12)):

$$y_t = \triangleleft\frac{\theta(B)}{\phi(B)}\xi_j, \quad \{\xi_t\} \sim \text{w.n. } N(0, \sigma_\xi^2),$$

where the solution is expressed as a weighted sum of *future* values of $\{\xi_t\}_{t=-\infty}^{\infty}$. When all the roots of $\phi(z)$ lie inside the unit circle the right finite inverse sequence is absultely summable; and therefore this solution is the stationary one.

3.2 The covariance generating function

Let us consider an ARMA model whose AR polynomial has no roots with unit modulus. If we restrict ourselves to its *stationary solution*

$$y_t = \frac{\theta(B)}{\phi(B)}(l_1)\ \xi_t = \varphi(B)\xi_t, \quad \{\xi_t\} \sim \text{w.n. } N(0, \sigma_\xi^2); \tag{4}$$

using Definition 2.2, and considering the (reversing) map $\mathfrak{R}[\cdot]$ such that $\mathfrak{R}[a_j] = a_{-j}$, the covariances are

$$\begin{aligned} \gamma_{k,t} &\equiv \operatorname{cov}(y_t y_{t-k}) = E[y_t y_{t-k}] = \\ &= E\left[\begin{array}{c} \left(\cdots + \varphi_{-1}\xi_{t+1} + \varphi_0\xi_t + \varphi_1\xi_{t-1} + \cdots\right) \\ \left(\cdots + \varphi_{-1}\xi_{t-k+1} + \varphi_0\xi_{t-k} + \varphi_1\xi_{t-k-1} + \cdots\right) \end{array}\right] = \\ &= \sigma_\xi^2 \sum\nolimits_{j\in\mathbb{Z}} \varphi_{j+k}\varphi_j = \sigma_\xi^2 \sum\nolimits_{j\in\mathbb{Z}} \varphi_{j+k}\mathfrak{R}[\varphi_{-j}] = \\ &= \sigma_\xi^2 \left(\varphi(z) * \mathfrak{R}[\varphi(z)]\right)_k = \sigma_\xi^2 \left(\varphi(z) * \varphi(z^{-1})\right)_k, \end{aligned} \tag{5}$$

where $\varphi(z^{-1}) \equiv \mathfrak{R}[\varphi(z)]$, and $[\varphi(z) * \varphi(z^{-1})]_k$ is the k-th element of the sequence $\varphi(z)*\varphi(z^{-1})$. This expression is independent of t, and hence $\gamma_{k,t} = \gamma_k$. When the roots of $\phi(B)$ are outside the unit circle $(\phi(B))^{-1(l_1)} = (\phi(B))^{-1\blacktriangleright}$, and then (4) becomes the *backward solution*; and it is said that the process is causal stationary. The *covariance generating function* is

$$\begin{aligned} \Gamma_y(z) &= \sigma_\xi^2 \cdot \frac{\theta(z)}{\phi(z)}(l_1) * \mathfrak{R}\left[\frac{\theta(z)}{\phi(z)}(l_1)\right] = \sigma_\xi^2 \cdot \frac{\theta(z)}{\phi(z)}\blacktriangleright * \triangleleft\frac{\theta(z^{-1})}{\phi(z^{-1})} \\ &= \sigma_\xi^2 \cdot \frac{\theta(z) * \theta(z^{-1})}{\phi(z) * \phi(z^{-1})}(l_1) = \sum_{k=-\infty}^{\infty} \gamma_k z^k \end{aligned}. \tag{6}$$

This sequence is summable, symmetric, and both left and right infinite.

3.3 The Pseudo-covariance generating function

Let us consider an ARMA(p,q) with AR roots with modulus that could be *greater, equal or smaller than one*. If one of them has unit modulus the inverse $(\phi(B))^{-1_{(l_1)}}$ is not defined and therefore there is no *stationary solution*. Neither there is a *covariance generating function*. Nevertheless, it is always possible to define the *pseudo-covariance generating function* using the forward and backward solutions. Let $y_t^{\blacktriangleright} = \varphi(B)\xi_t$, and $y_t^{\triangleleft} = \upsilon(B)\xi_t$, denote respectively the backward and forward solutions; where $\varphi(B) = (\phi(B))^{-1_{\blacktriangleright}} * \theta(B)$ and $\upsilon(B) = (\phi(B))^{-1_{\triangleleft}} * \theta(B)$. Using Definition 2.3, the pseudo-covariances, $\lambda_{k,t}$, of the process are

$$\begin{aligned} \lambda_{k,t} &= E\left[y_t^{\blacktriangleright} y_{t-k}^{\triangleleft}\right] = E\left[\varphi(B)\xi_t \cdot \upsilon(B)\xi_{t-k}\right] \\ &= E\left[\begin{array}{l} \left(\varphi_0\xi_t + \varphi_1\xi_{t-1} + \varphi_2\xi_{t-2} + \cdots\right) \\ \left(\cdots + \upsilon_{-1}\xi_{t-k+1} + \upsilon_0\xi_{t-k} + \cdots + \upsilon_{q-p}\xi_{t-k-(q-p)}\right) \end{array}\right] \\ &= \sigma_\xi^2 \sum_{r+s=k}^{\infty} \varphi_r \upsilon_{-s} = \sigma_\xi^2 \sum_{r+s=k}^{\infty} \varphi_r \mathfrak{R}\left[\upsilon\right]_s = \sigma_\xi^2 \left(\varphi(z) * \mathfrak{R}\left[\upsilon(z)\right]\right)_k . \end{aligned} \tag{7}$$

The convolution product is well defined since both $\varphi(z), \mathfrak{R}\left[\upsilon(z)\right]$ are left finite. This expression *does not depend* on t, therefore, $\lambda_{k,t} = \lambda_k$.

The *pseudo-covariance generating function*, $\Lambda_y(z)$, is the whole sequence of pseudo-covariances

$$\begin{aligned} \Lambda_y(z) &= \sigma_\xi^2 \cdot \frac{\theta(z)}{\phi(z)}{}_{\blacktriangleright} * \mathfrak{R}\left[{}_{\triangleleft}\frac{\theta(z)}{\phi(z)}\right] = \sigma_\xi^2 \cdot \frac{\theta(z)}{\phi(z)}{}_{\blacktriangleright} * \frac{\mathfrak{R}\left[\theta(z)\right]}{\mathfrak{R}\left[\phi(z)\right]}{}_{\blacktriangleright} \\ &= \sigma_\xi^2 \cdot \frac{\theta(z) * \mathfrak{R}\left[\theta(z)\right]}{\phi(z) * \mathfrak{R}\left[\phi(z)\right]}{}_{\blacktriangleright} = \sigma_\xi^2 \cdot \frac{\theta(z) * \theta(z^{-1})}{\phi(z) * \phi(z^{-1})}{}_{\blacktriangleright} = \sum_{k=\mathrm{q-p}}^{\infty} \lambda_k z^k . \end{aligned} \tag{8}$$

This sequence is left finite and starts in the lag (q−p). The *pseudo-covariance generating function* is defined for any ARMA(p,q) process.

4 The pseudo-spectrum

From the previous *pseudo-covariance generating function* we can now define the *pseudo-spectrum* of an ARMA(p,q) process as:

Definition 4.1 (Pseudo-spectrum). Let $\{y_t\}_{t=-\infty}^{\infty}$ be a stochastic ARMA(p,q) process such as

$$\phi(B)y_t = \theta(B)\xi_t, \qquad \{\xi_t\} \sim \text{w.n. } N(0, \sigma_\xi^2)$$

where $\phi(\cdot)$ could have roots with modulus greater, equal or smaller than one. We define the pseudo-spectrum of $\{y_t\}_{t=-\infty}^{\infty}$ as the extended Fourier transform[9] of its *pseudo-covariance generating function*

$$f_y(\omega) = \mathcal{F}\mathcal{E}\left(\Lambda_y(z)\right) = \sigma_\xi^2 \frac{\mathcal{F}\left(\theta(z) * \theta(z^{-1})\right)}{\mathcal{F}\left(\phi(z) * \phi(z^{-1})\right)} = \sigma_\xi^2 \frac{\theta(e^{-i\omega})\theta(e^{i\omega})}{\phi(e^{-i\omega})\phi(e^{i\omega})} = \Lambda_y(e^{-i\omega}). \tag{9}$$

Note that, as a result of Theorem A.9, when the ARMA(p,q) process is stationary the spectrum *and the* pseudo-spectrum *coincide.*

Example 4.1. Let $\{y_t\}_{t=-\infty}^{\infty}$ be a random walk process

$$y_t(1-B) = \xi_t, \quad \{\xi_t\} \sim \text{w.n. } N(0, \sigma_\xi^2);$$

following Equation (8), its *pseudo-covariance generating function* is

$$\Lambda_y(z) = \sigma_\xi^2 \cdot \frac{1}{1-z}{\blacktriangleright} * \Re\left[{\triangleleft}\frac{1}{1-z}\right] = \frac{\sigma_\xi^2}{(1-z)*(1-z^{-1})}{\blacktriangleright}. \tag{10}$$

On one hand $\frac{1}{1-z}{\blacktriangleright} = 1+z+z^2+z^3+\cdots$. On the other, from Equation (12)

$$\begin{aligned} \Re\left[(1-z)^{-1\triangleleft}\right] &= \Re\left[\Re\left[(\Re\left[1-z\right])^{-1\blacktriangleright}\right]\right] = (1-z^{-1})^{-1\blacktriangleright} \\ &= -z^{-1} - 1 - z - z^2 - \cdots . \end{aligned}$$

Hence, the *pseudo-covariance generating function* is

$$\Lambda_y(z) = (1+z+z^2+\cdots) * (-z^{-1} - 1 - z - \cdots) = -\sum_{j=-1}^{\infty}(j+2)z,$$

that is, $\lambda_j = 0$ for all $j \leq -2$, and $\lambda_{-1} = -1,\ \lambda_0 = -2,\ \lambda_1 = -3,\ \ldots$.

Finally, from Equation (10), we know that the *pseudo-spectrum* is

$$f_y(\omega) = \mathcal{F}\mathcal{E}\left(\frac{\sigma_\xi^2}{(1-z)*(1-z^{-1})}{\blacktriangleright}\right) = \frac{\sigma_\xi^2}{(1-e^{-i\omega})(1-e^{i\omega})} = \frac{\sigma_\xi^2}{2-2\cos(\omega)}.$$

[9] see Definition A.2 in the appendix.

5 Frequency domain *versus* time domain

Looking at Figure 1 we may observe that both sides have a Hilbert space. Nevertheless, in the left side (the time domain) there are three rings that are not simultaneously embedable in any ring, because the convolution product is not associative, whereas the whole right side (the frequency domain) $\left(\mathbb{C}^{[-\pi,\pi]}/\sim,+,\cdot\right)$[10] is a ring. Furthermore, the extended Fourier transform simplifies the algebraic structure because both fields, $\mathbb{R}(z)$ and $\mathfrak{R}\left[\mathbb{R}(z)\right]$, have a common image, the field $\mathcal{Q}$. There is an important consequence: *only when the* pseudo-covariance generating functions *belong to* l_1 *we have guarantee that it is equivalent to deal with ARMA(p,q) models in the time domain or in the frequency domain.* The reason is that, whereas the product of the *pseudo-spectra* is associative, the convolution product of sequences as a partial operation in $\mathbb{R}^{\mathbb{Z}}$ is not[11], or even it is not defined for some sequences[12]. Therefore, the frequency domain has clear operational advantages when the ARMA(p,q) processes are non-stationary. Some problems can be solved using the *pseudo-spectra* but not using the *pseudo-covariance generating functions*. For example, Bujosa, García-Ferrer, and Young (2002) try to fit by Ordinary Least Squares (OLS) a linear combination of *pseudo-spectra* of non-stationary unobserved components to the *pseudo-spectrum* of a seasonal non-stationary observed time series. But, since those *pseudo-spectra* are outside the Hilbert space L_2, OLS are not applicable. However, the problem can be shift to L_2 by products inside the ring $\left(\mathbb{C}^{[-\pi,\pi]}/\sim,+,\cdot\right)$. This is not feasible in the time domain.

6 Conclusions

The *pseudo-spectrum* has some interesting properties. The first one is that the *pseudo-covariance generating function* and the *pseudo-spectrum* are defined for any (stationary or non-stationary) ARMA(p,q) process. Secondly, we have shown that the frequency domain has some additional algebraic advantages over the time domain. Finally, the *pseudo-spectrum* keeps its

[10] See Section A.2.

[11] For example: $\left[(1-\alpha z)^{-1\blacktriangleright} * (1-\alpha z)\right] * (1-\alpha z)^{-1\triangleleft} = (1-\alpha z)^{-1\triangleleft}$; but $(1-\alpha z)^{-1\blacktriangleright} * \left[(1-\alpha z) * (1-\alpha z)^{-1\triangleleft}\right] = (1-\alpha z)^{-1\blacktriangleright}$.

[12] For example: $(1-z)^{-1\blacktriangleright} * \left(1-z^{-1}\right)^{-1\triangleleft}$ is not defined.

physical interpretation as the power distribution among different frequencies. When the ARMA(p,q) process is stationary, the *spectrum* and the *pseudo-spectrum* coincide. It follows that both sequences $\Gamma(z)$, and $\Lambda(z)$ have the information about the power distribution of the process. When facing roots with unit modulus in the AR side, $\Gamma(z)$ and the spectrum are no longer defined. But, $\Lambda(z)$ is defined; and its extended Fourier transform (the *pseudo-spectrum*) shows poles in the frequencies associated with the roots with unit modulus. This means that there are infinite contributions to the variance at those frequencies. In those cases the *pseudo-spectra* are non-integrable and therefore, the process has infinite variance. However, the *pseudo-spectra* are time invariant, such that although the variance is infinite, it's distribution among the frequencies is time invariant.

A Appendix

A.1 Algebraic foundations

Let $\mathbb{R}^{\mathbb{Z}}$ consist of all the sequences of real numbers $\{a_j\}_{j=-\infty}^{\infty}$. This set together with the two usual operations $(+,\cdot)$ is a vector space. Let

$$b \equiv \{b_j\}_{j\in\mathbb{Z}} \equiv \sum\nolimits_{j\in\mathbb{Z}} b_j z^j,$$

denote the elements of $\mathbb{R}^{\mathbb{Z}}$, where the coefficient on z^j is the jth element of b. In the above expression the Greek letter $\sum$ does not indicate summation, since $\sum b_j z^j$ is a mere notation and can also denote a non-convergent sequence.

Let *codegree* denote the bigger lowest index of a non-null sequence that verifies that for each $j < \text{codegree}\,(a)\,,\ a_j = 0$. For the null sequence, 0, we set $\text{codegree}\,(0) = -\infty$.

Notation A.1.1. Let $\mathbb{R}((z))$ be the subset of $\mathbb{R}^{\mathbb{Z}}$ of all the sequences with finite codegree and the null sequence *(the set of left finite sequences).* If we define the convolution product $*$ of two sequences $a, b \in \mathbb{R}((z))$ as:

$$(a * b)_j = \sum\nolimits_{r+s=j} a_r b_s,$$

it is easy to prove that $(\mathbb{R}((z)), +, *)$ is a field[13], and that $\text{codegree}\,(a * b) =$

[13] $(\mathbb{R}((z)), +, *)$ is the *field of fractions of formal series* (see Godement, 1974).

codegree (a) + codegree (b). The inverse sequence of $a \in \mathbb{R}((z))$, $a \neq 0$, is:

$$(a)^{-1\blacktriangleright} \equiv \frac{1}{a}\blacktriangleright = \begin{cases} 0 & \text{when} \quad j < -k \\ \frac{1}{a_k} & \text{when} \quad j = -k \\ \frac{-1}{a_k}\sum_{r=-k}^{j-1} b_r a_{j+k-r} & \text{when} \quad j > -k \end{cases}, \quad k = \text{codegree}\,(a)\,. \tag{11}$$

Let us consider the *auto-morphism* $\mathfrak{R} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ such that $\mathfrak{R}\,[a_j] = a_{-j}$. The image $\mathfrak{R}\,[\mathbb{R}((z))]$ with the sum and convolution product operations form a field *isomorphic* to $(\mathbb{R}((z)), +, *)$. The *degree* of a non-null sequence is the highest integer index that verifies that for each $j >$ degree (a), $a_j = 0$. For the null sequence, 0, we say degree $(0) = \infty$. Therefore $\mathfrak{R}\,[\mathbb{R}((z))]$ is the set of all sequences with degree *(the set of right finite sequences)*; and degree $(a * b) =$ degree (a) + degree (b). The inverse sequence of a in $(\mathfrak{R}\,[\mathbb{R}((z))]\,, +, *)$ is the sequence $(a)^{-1\triangleleft}$ that verifies

$$(a)^{-1\triangleleft} \equiv \triangleleft\frac{1}{a} = \triangleleft\frac{1}{\mathfrak{R}\,[\mathfrak{R}\,[a]]} = \mathfrak{R}\left[\frac{1}{\mathfrak{R}\,[a]}\blacktriangleright\right]. \tag{12}$$

Notation A.1.2. We define the set of all polynomials $\mathbb{R}[z]$ as the set of sequences with codegree equal or greater than zero that also have degree. The triple $(\mathbb{R}[z], +, *)$ is a sub-ring of the aforementioned two fields. The smallest field $\mathbb{K} \subset \mathbb{R}((z))$ containing $\mathbb{R}[z]$ is isomorphic to the field of fractions of polynomials $\mathbb{R}(z)$. This set is characterized by

$$\mathbb{R}(z) = \{p * (q)^{-1\blacktriangleright} \equiv \frac{p}{q}\blacktriangleright \in \mathbb{R}((z));\ \text{such that } p, q \in \mathbb{R}[z] \text{ and } q \neq 0\}. \tag{13}$$

Because $\mathfrak{R}$ is an isomorphism, there is an isomorphic field, $\mathfrak{R}\,[\mathbb{R}(z)]$, of $\mathbb{R}(z)$ in $\mathfrak{R}\,[\mathbb{R}((z))]$.

It follows that the polynomials have, at least, two inverse sequences; one in $\mathbb{R}(z)$, and another in $\mathfrak{R}\,[\mathbb{R}(z)]$ (equations (11) and (12)).

The space l_1 consist of all absolutely summable sequences, or simply all summable sequences in Schwartz's (1961) sense[14], that is, the sequences that verify $\sum_{j\in\mathbb{Z}} a_j < \infty$. The convolution product of two sequences in l_1 belongs

[14]If I is an index set, and if $(u_i)_{i\in I}$ is a family of real numbers defined by the index set I, the sequence $\sum_{i\in I} u_i$ is *summable* and $\sum_{i\in I} u_i = S$ if, for every $\epsilon > 0$, there is a finite index subset $J \subset I$ such that, for any index subset $K \subset J$, it is verified that $|S - S_K| \leq \epsilon$, where $S_K = \sum_{i\in K} u_i$.

to l_1. The inverse sequence of the polynomial $(1-az)$ in $\mathbb{R}(z)$ is $(1+az+a^2z^2+\cdots)$, and the inverse sequence of $(1-az^{-1})$ in $\mathfrak{R}\left[\mathbb{R}(z)\right]$ is $(\cdots+a^2z^{-2}+az^{-1}+1)$. Given that every polynomial b with degree $(n+r),\ n,r\geq 0$ can be expressed as

$$b = \text{constant}\cdot\left[z^n * \prod_{j=1}^{r}(1-\beta_j z)\right], \tag{14}$$

it is easy to prove that $(b)^{-1\blacktriangleright}$ belongs to l_1 if all the roots of b lie outside the unit circle, and that $(b)^{-1\triangleleft}$ belongs to l_1 if all the roots of b lie inside the unit circle. This fact suggests another inverse sequence for the polynomials φ with roots of modulus different from one. We can write φ as $\varphi = \phi * \Phi$, where ϕ is the polynomial with all its roots outside the unit circle, and Φ is the polynomial with all its roots inside the unit circle. Then, the sequence

$$(\varphi)^{-1(l_1)} \equiv \frac{1}{\varphi}^{(l_1)} = \frac{1}{\phi}^{\blacktriangleright} * {}^{\triangleleft}\frac{1}{\Phi}, \tag{15}$$

is an inverse sequence of φ that belongs to l_1 because $*$ is associative for $\varphi,\ (\phi)^{-1\blacktriangleright},\ (\Phi)^{-1\triangleleft} \in l_1$.

The vector space $\left(l_2,+,\cdot\right)$ consists of all square summable sequences, that is, $\sum_{j\in\mathbb{Z}}|a_j|^2 < \infty$. This vector space is a Hilbert space when the inner product is $\langle a|b\rangle_{l_2} = \sum_{j\in\mathbb{Z}} a_j b_j$. The set l_2 contains the set l_1.

A.2 The Extended Fourier Transform $\mathcal{FE}$

Let $\mathbb{C}^{[-\pi,\pi]}$ denote the set of all real valued functions $f(\omega)$ on the closed interval $[-\pi,\pi]$. We say that f is equivalent to g, and write $f\sim g$, if and only if $\mu\{\omega|f(\omega)\neq g(\omega)\}=0$, where $\mu\{\omega\}$ is the Lebesgue measure of the set $\{\omega\}$. The set $\mathbb{C}^{[-\pi,\pi]}$ with the equivalence relation $\sim$ is a vector space. Indeed, $\left(\mathbb{C}^{[-\pi,\pi]}/\sim,+,\cdot\right)$ is an unitary ring, where the equivalence class $[f]=\{g|g\sim f\}$ is invertible if and only if $\mu\{\omega|f(\omega)=0\}=0$.

The space $L_2[-\pi,\pi]$ consist of all equivalence classes of real valued functions $[f]\in\mathbb{C}^{[-\pi,\pi]}$ such that $\int_{-\pi}^{\pi} f^2 \mathrm{d}(\omega) < \infty$, where $\int_{-\pi}^{\pi} f^2\mathrm{d}(\omega)$ is the Lebesgue integral (Luenberger, 1968). We now let the function f denote the equivalence class of functions $[f]\in\mathbb{C}^{[-\pi,\pi]}$. The vector space $L_2[-\pi,\pi]$ with the inner product $\langle f|g\rangle_{L_2} = \int_{-\pi}^{\pi}(f\cdot\overline{g})\,\mathrm{d}(\omega)$ is a Hilbert space.

Definition A.1. We define the Fourier Transform $\mathcal{F}$, to be the bijective mapping $\mathcal{F}:\ l_2 \longrightarrow L_2$ such that

$$a=\sum_{j=-\infty}^{\infty} a_j z^j \quad\longrightarrow\quad f_a(\omega)=\sum_{j=-\infty}^{\infty} a_j e^{-i\omega j}. \tag{16}$$

$\mathcal{F}$ is an isometric isomorphism from l_2 to $L_2[-\pi,\pi]$; and $\mathcal{F}(l_1) \subset L_2$. The Fourier Transform verifies the well known convolution property:

Proposition A.1 (Convolution property 1). *If $p, q \in l_1$, then $\mathcal{F}(p)(\omega) \cdot \mathcal{F}(q)(\omega) = \mathcal{F}(p * q)(\omega)$, for all $\omega \in [-\pi, \pi]$.*

Therefore, $(\mathcal{F}(l_1), +, \cdot)$ is a sub-ring of $\mathbb{C}^{[-\pi,\pi]}$ isomorphic to l_1. In particular, $\mathcal{F}$ is defined in the ring $\mathbb{R}[z]$. We may extend the Fourier Transform $\mathcal{F}$[15] to the whole field $\mathbb{R}(z)$, in such a way that the new mapping is an isomorphism of fields. To do so, we need the following proposition:

Proposition A.2. *For all $p \in \{\mathbb{R}[z] - \{0\}\}$, $\mathcal{F}(p)$ is invertible.*

Proof. We need to prove that there is a finite number of points where $\mathcal{F}(p)$ is zero. This is straightforward from Equation (14), since $e^{i\omega}$ has no zeros, and $1 - \alpha e^{i\omega}$ has one zero or none, depending on $|\alpha|$. □

Let $(\mathcal{Q}, +, \cdot)$ be the smallest field containing $\mathcal{F}(\mathbb{R}[z])$. Then $\mathcal{Q}$ is set of fractions $\mathcal{Q} = \{f/g;$ such that $f, g \in \mathcal{F}(\mathbb{R}[z]), g \neq 0\}$; and we can extend $\mathcal{F}$ defining the following new transform $\mathcal{F}_\blacktriangleright$:

$$\mathcal{F}_\blacktriangleright : \mathbb{R}(z) \longrightarrow \mathcal{Q}; \quad \text{such that} \quad \frac{p}{q}_\blacktriangleright \longrightarrow \frac{\mathcal{F}(p)}{\mathcal{F}(q)}. \tag{17}$$

Proposition A.3. *$\mathcal{F}_\blacktriangleright$ is well defined, bijective, and it is an isomorphism from the field $\mathbb{R}(z)$ to the field $\mathcal{Q}$.*

Proof. 1. $\mathcal{F}_\blacktriangleright$ is well defined since

$$\begin{aligned} p * (q)^{-1_\blacktriangleright} = p' * (q')^{-1_\blacktriangleright} &\Rightarrow p * q' = q * p' \\ &\Rightarrow \mathcal{F}(p) \cdot \mathcal{F}(q') = \mathcal{F}(q) \cdot \mathcal{F}(p') \\ &\Rightarrow \tfrac{\mathcal{F}(p)}{\mathcal{F}(q)} = \tfrac{\mathcal{F}(p')}{\mathcal{F}(q')} \\ &\Rightarrow \mathcal{F}_\blacktriangleright(p * (q)^{-1_\blacktriangleright}) = \mathcal{F}_\blacktriangleright(p' * (q')^{-1_\blacktriangleright}). \end{aligned}$$

2. $\mathcal{F}_\blacktriangleright$ is an homomorphism of fields:

$$\mathcal{F}_\blacktriangleright\left(\frac{p}{q}_\blacktriangleright + \frac{p'}{q'}_\blacktriangleright\right) = \mathcal{F}_\blacktriangleright\left(\frac{p * q' + p' * q}{q * q'}_\blacktriangleright\right) = \tfrac{\mathcal{F}(p)\cdot\mathcal{F}(q') + \mathcal{F}(p')\cdot\mathcal{F}(q)}{\mathcal{F}(q)\cdot\mathcal{F}(q')}$$
$$= \tfrac{\mathcal{F}(p)}{\mathcal{F}(q)} + \tfrac{\mathcal{F}(p')}{\mathcal{F}(q')} = \mathcal{F}_\blacktriangleright(p * (q)^{-1_\blacktriangleright}) + \mathcal{F}_\blacktriangleright(p' * (q')^{-1_\blacktriangleright}); \text{ and}$$

$$\mathcal{F}_\blacktriangleright\left(\frac{p}{q}_\blacktriangleright * \frac{p'}{q'}_\blacktriangleright\right) = \tfrac{\mathcal{F}(p*p')}{\mathcal{F}(q*q')} = \tfrac{\mathcal{F}(p)\cdot\mathcal{F}(p')}{\mathcal{F}(q)\cdot\mathcal{F}(q')} = \mathcal{F}_\blacktriangleright\left(\frac{p}{q}_\blacktriangleright\right) \cdot \mathcal{F}_\blacktriangleright\left(\frac{p'}{q'}_\blacktriangleright\right).$$

[15] The restriction of $\mathcal{F}$ on $\mathbb{R}[z]$

3. Finally, $\mathcal{F}_{\blacktriangleright}(1) = \mathcal{F}_{\blacktriangleright}(1 * (1)^{-1_{\blacktriangleright}}) = \frac{\mathcal{F}(1)}{\mathcal{F}(1)} = 1$

$\square$

The following lemma shows that $\mathcal{F}_{\blacktriangleright}$ extends the Fourier Transform to $\mathbb{R}(z)$.

Lemma A.4. *For all $a \in l_2 \cap \mathbb{R}(z)$, $\mathcal{F}(a) = \mathcal{F}_{\blacktriangleright}(a)$.*

Proof. The result is trivial when a is a polynomial. In the other case, let $a = p*(q)^{-1_{\blacktriangleright}} \in l_2 \cap \mathbb{R}(z)$, where $p, q \in \mathbb{R}[z]$ and $q \neq 0$, then, due to $(p*(q)^{-1_{\blacktriangleright}})*q = p \in \mathbb{R}[z]$, it follows that $\mathcal{F}((p*(q)^{-1_{\blacktriangleright}})*q) = \mathcal{F}(p*(q)^{-1_{\blacktriangleright}}) \cdot \mathcal{F}(q) = \mathcal{F}(p)$, and $\mathcal{F}_{\blacktriangleright}((p*(q)^{-1_{\blacktriangleright}})*q) = \mathcal{F}_{\blacktriangleright}(p*(q)^{-1_{\blacktriangleright}}) \cdot \mathcal{F}(q) = \mathcal{F}(p)$. Since $q \neq 0$, $\mathcal{F}(q) \in \mathcal{Q} - \{0\}$ is invertible on $\mathbb{C}^{[-\pi,\pi]}$, then $\mathcal{F}_{\blacktriangleright}(p*(q)^{-1_{\blacktriangleright}}) \cdot \mathcal{F}(q) = \mathcal{F}(p*(q)^{-1_{\blacktriangleright}}) \cdot \mathcal{F}(q)$ implies $\mathcal{F}_{\blacktriangleright}(p*(q)^{-1_{\blacktriangleright}}) = \mathcal{F}(p*(q)^{-1_{\blacktriangleright}})$. $\square$

In the same way we define the transform $\mathcal{F}_{\triangleleft}$ to be

$$\mathcal{F}_{\triangleleft} : \ \mathfrak{R}[\mathbb{R}(z)] \longrightarrow \mathcal{Q}; \quad \text{such that} \quad {}_{\triangleleft}\frac{p}{q} \longrightarrow \frac{\mathcal{F}(p)}{\mathcal{F}(q)}. \tag{18}$$

The transform $\mathcal{F}_{\triangleleft}$ has similar properties, that is, it is well defined, bijective, it is an isomorphism from the field $\mathfrak{R}[\mathbb{R}(z)]$ to the field $\mathcal{Q}$. Furthermore, it extends the Fourier Transform from l_2 to $\mathfrak{R}[\mathbb{R}(z)]$.

Note that if $a \in \mathbb{R}(z) \cap \mathfrak{R}[\mathbb{R}(z)]$ then $a \in l_2 \cap \mathbb{R}(z) \cap \mathfrak{R}[\mathbb{R}(z)]$ because only finitely many of the a_j are non-zero. Hence, from Lemma A.4, (and the analogue proposition for $a \in l_2 \cap \mathfrak{R}[\mathbb{R}(z)]$), we have $\mathcal{F}(a) = \mathcal{F}_{\blacktriangleright}(a) = \mathcal{F}_{\triangleleft}(a)$. Now we come to the Extended Fourier Transform $\mathcal{F}\mathcal{E}$.

Definition A.2 (Extended Fourier Transform, $\mathcal{F}\mathcal{E}$). We define the Extended Fourier Transform, $\mathcal{F}\mathcal{E}$, to be the mapping:

$$\begin{aligned} \mathcal{F}\mathcal{E} : \ l_2 \cup \mathbb{R}(z) \cup \mathfrak{R}[\mathbb{R}(z)] \ &\rightarrow \ L_2 \cup \mathcal{Q} \\ a \ &\rightarrow \ \mathcal{F}\mathcal{E}(a) = \begin{cases} \mathcal{F}(a) & \text{if } a \in l_2 \\ \mathcal{F}_{\blacktriangleright}(a) & \text{if } a \in \mathbb{R}(z) \\ \mathcal{F}_{\triangleleft}(a) & \text{if } a \in \mathfrak{R}[\mathbb{R}(z)] \end{cases} . \end{aligned} \tag{19}$$

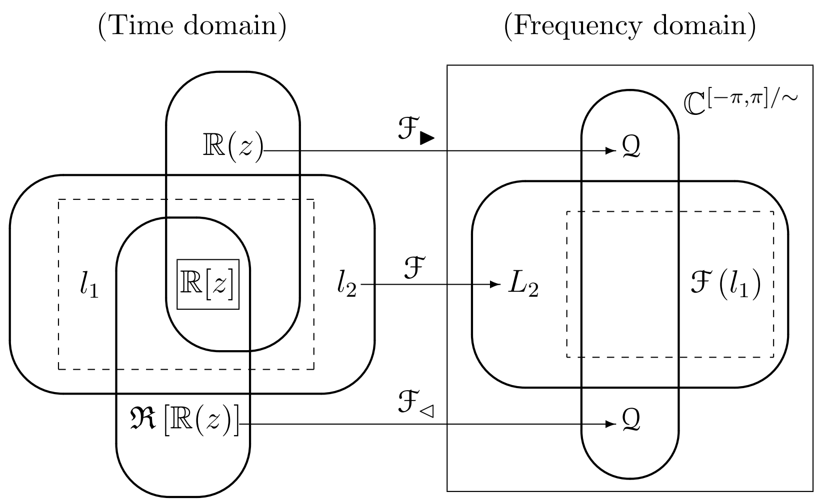

Figure 1: Extended Fourier Transform Structure

A.2.1 Extended Fourier Transform Properties

The *Extended Fourier Transform* verifies the following properties (in addition to Proposition A.1):

Lemma A.5 (Convolution property 2). *Let* $p * (q)^{-1\blacktriangleright} \in \mathbb{R}(z)$, *let* $b \in l_2 \cup \mathbb{R}(z) \cup \mathfrak{R}[\mathbb{R}(z)]$, *and let* $(p * (q)^{-1\blacktriangleright}) * b \in l_2 \cup \mathbb{R}(z) \cup \mathfrak{R}[\mathbb{R}(z)]$, *then*

$$\mathcal{F}\mathcal{E}(p * (q)^{-1\blacktriangleright} * b) = \mathcal{F}\mathcal{E}(p * (q)^{-1\blacktriangleright}) \cdot \mathcal{F}\mathcal{E}(b) .$$

Proposition A.6. *If* $p \in \mathbb{R}[z],\ q \in l_2$, *then* $\mathcal{F}(p * q) = \mathcal{F}(p) \cdot \mathcal{F}(q)$

Proof. We know that

$$\begin{aligned} \mathcal{F}(z^r) \cdot \mathcal{F}(q) &= e^{-ir\omega} \cdot \textstyle\sum_{s=-\infty}^{\infty} q_s e^{-i\omega s} = \sum_{s=-\infty}^{\infty} q_s e^{-i\omega s} e^{-ir\omega} \\ &= \mathcal{F}\left(\textstyle\sum_{s=-\infty}^{\infty} q_s z^{r+s}\right) = \mathcal{F}\left(z^r * \sum_{s=-\infty}^{\infty} q_s z^s\right) \qquad = \mathcal{F}(z^r * q) . \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{F}(p * q) &= \mathcal{F}(p_0 q + p_1 z * q + p_2 z^2 * q + \cdots + p_n z^n * q) \\ &= p_0 \mathcal{F}(q) + p_1 \mathcal{F}(z * q) + \cdots + p_n \mathcal{F}(z^n * q) \\ &= p_0 \cdot \mathcal{F}(q) + \mathcal{F}(p_1 z) \cdot \mathcal{F}(q) + \cdots + \mathcal{F}(p_n z^n) \cdot \mathcal{F}(q) \quad = \mathcal{F}(p) \cdot \mathcal{F}(q) . \end{aligned}$$

□

Lemma A.7. *If* p *is a polynomial, and* $q * a$ *is defined,* $p * (q * a) = (p * q) * a$.

Proof. $((z^j * q) * a)_n = \sum_{r+s=n} (z^j * q)_r a_s = \sum_{r+s=n} q_{r-j} a_s$. If we substitute r' by $r-j$, then $\sum_{r'+s=n-j} q_{r'} a_s = (q * a)_{n-j} = (z^j * (q * a))_n$. Therefore

$$\begin{aligned}(p * q) * a = \left(p_0 * q + p_1 z * q + p_2 z^2 * q + \cdots + p_m z^m * q\right) * a &= \\ (p_0 * q) * a + (p_1 z * q) * a + \cdots + (p_m z^m * q) * a &= \\ p_0 (q * a) + p_1 z * (q * a) + \cdots + p_m z^m * (q * a) &= p * (q * a)\end{aligned}$$

□

Proof of Lemma A.5 (The convolution property 2). On the one hand, the extended Fourier transform of $q * (p * (q)^{-1\blacktriangleright} * a)$ is

$$\begin{aligned}\mathcal{FE}(q * (p * (q)^{-1\blacktriangleright} * a)) &= \mathcal{FE}(q) \cdot \mathcal{FE}(p * (q)^{-1\blacktriangleright} * a) \\ &= \mathcal{F}(q) \cdot \mathcal{FE}(p * (q)^{-1\blacktriangleright} * a)\end{aligned},$$

because q is a polynomial (Proposition A.6 and Point 2 of the proof of Proposition A.3). On the other, from Lemma A.7 it follows that $q*(p*(q)^{-1\blacktriangleright}*a) = (q * p * (q)^{-1\blacktriangleright}) * a = p * a$, and its extended Fourier transform is

$$\mathcal{FE}(q * (p * (q)^{-1\blacktriangleright} * a)) = \mathcal{FE}(p * a) = \mathcal{FE}(p) \cdot \mathcal{FE}(a) = \mathcal{F}(p) \cdot \mathcal{FE}(a),$$

because p is a polynomial. Then

$$\begin{aligned}\mathcal{F}(q) \cdot \mathcal{FE}(p * (q)^{-1\blacktriangleright} * a) &= \mathcal{F}(p) \cdot \mathcal{FE}(a) \\ \Rightarrow \mathcal{FE}(p * (q)^{-1\blacktriangleright} * a) &= \tfrac{\mathcal{F}(p)}{\mathcal{F}(q)} \cdot \mathcal{FE}(a) \\ \Rightarrow \mathcal{FE}(p * (q)^{-1\blacktriangleright} * a) &= \mathcal{FE}(p * (q)^{-1\blacktriangleright}) \cdot \mathcal{FE}(a),\end{aligned}$$

given that $q \neq 0$; and therefore, $\mathcal{F}(q) \in \mathcal{Q} - \{0\}$ is invertible on $\mathbb{C}^{[-\pi,\pi]}$. □

Lemma A.8 (Convolution property 3). *Let* $p * (q)^{-1\triangleleft} \in \mathfrak{R}[\mathbb{R}(z)]$*, let* $b \in l_2 \cup \mathbb{R}(z) \cup \mathfrak{R}[\mathbb{R}(z)]$*, and let* $(p * (q)^{-1\triangleleft}) * b \in l_2 \cup \mathbb{R}(z) \cup \mathfrak{R}[\mathbb{R}(z)]$*, then*

$$\mathcal{FE}(p * (q)^{-1\triangleleft} * b) = \mathcal{FE}(p * (q)^{-1\triangleleft}) \cdot \mathcal{FE}(b).$$

The proof is similar to the former.

The following theorem shows a most remarkable property. We have seen that the polynomials have different inverse sequences, in contrast, *the image of all of then in* $\mathcal{Q}$ *is common.*

Theorem A.9 (Common image of the inverse sequences). *Let* $p \in \mathbb{R}(z)$*, and let* $q, q' \in l_2 \cup \mathbb{R}(z) \cup \mathfrak{R}[\mathbb{R}(z)]$ *such that* $(p * q) = 1$ *y* $(p * q') = 1$*, then* $\mathcal{FE}(q) = \mathcal{FE}(q')$.

Proof. $\mathcal{FE}(p * q) = \mathcal{FE}(p) \cdot \mathcal{FE}(q) = 1 = \mathcal{FE}(p) \cdot \mathcal{FE}(q') = \mathcal{FE}(p * q')$. Since $q \neq 0$, $\mathcal{F}(q) \in \mathcal{Q} - \{0\}$ is invertible on $\mathbb{C}^{[-\pi,\pi]}$, it follows that $\mathcal{FE}(q) = \mathcal{FE}(q')$. □

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