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ON EXPLICIT PROBABILITY LAWS FOR CLASSES OF SCALAR DIFFUSIONS

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Abstract. This paper uses Lie symmetry group methods to obtain transition probability densities for scalar diffusions, where the diffusion coefficient is given by a power law. We will show that if the drift of the diffusion satisfies a certain family of Riccati equations, then it is possible to compute a generalized Laplace transform of the transition density for the process. Various explicit examples are provided. We also obtain fundamental solutions of the Kolmogorov forward equation for diffusions, which do not correspond to transition probability densities.

1. Introduction

In the literature there exists a relatively small class of processes which have explicitly known transition probability densities. Related to the Gaussian density are transformations of the Wiener process, such as Geometric Brownian motion and Ornstein-Uhlenbeck type processes. Sums of squares of Wiener processes and the Bessel type processes that relate to them have also been extensively investigated. The references Revuz and Yor [18] and Borodin and Salminen [3] contain detailed discussions of many of the properties of these diffusions.

Fundamental to the understanding of any process is its transition probability density. For this reason, we investigate a new technique which allows for the explicit construction of transition probability densities for large classes of Markov processes. We study some new processes which arise from our methods and conjecture that deep properties similar to those known for the already existing classes may be discovered. This paper aims to encourage research in this direction by

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presenting illustrative examples to indicate how the new methods may be exploited.

We begin by onsidering a scalar diffusion $X = \{X_t : t \geq 0\}$ solving the stochastic differential equation (SDE)

$$dX_t = f(X_t)dt + \sqrt{2\sigma X_t^\gamma}dW_t, \quad X_0 = x > 0, \tag{1.1}$$

where $W = \{W_t : t \geq 0\}$ is a standard one dimensional Brownian motion. A distribution, or generalized function, which solves the associated Cauchy problem for the partial differential equation (PDE)

$$\begin{aligned} u_t &= \sigma x^\gamma u_{xx} + f(x)u_x, \quad x \geq 0, \ \sigma > 0 \\ u(x,0) &= \delta(x-y), \end{aligned} \tag{1.2}$$

where δ is the Dirac delta function, is known as a fundamental solution of the PDE. This PDE is known as the Fokker-Planck or Kolmogorov forward equation. If the fundamental solution satisfies certain extra conditions, then it gives the transition probability density for the process X.

Cauchy problems of the form (1.2) have been the subject of considerable research and there exist numerous techniques for their solution. Integral transforms, eigenfunction expansions and methods based upon Lie symmetries have all proved to be effective for wide classes of problems. An exhaustive survey of the literature is not possible here, but [7], [15] and [20] are useful references.

In this paper we examine solvable models which arise from Lie symmetry analysis. We will be primarily interested in obtaining fundamental solutions which give a transition probability density for X. Fundamental solutions are not however unique, unless we impose further conditions, in particular at the boundaries, and not every fundamental solution of (1.2) is a probability density. Our methods allow us to exhibit multiple fundamental solutions for the same PDE. One may give a probability density while the others usually do not. However, we will also present examples where we have two fundamental solutions, both of which integrate to one.

The use of Lie symmetries to obtain fundamental solutions has a long history. Indeed there are several different approaches to the construction of fundamental solutions of the problem (1.2) using Lie theory. For example, Bluman and Cole studied a class of Fokker-Planck equations, for which they calculated fundamental solutions using group invariance methods. This work is reproduced in, for example, [2]. The so called method of reduction to canonical form has also been used to obtain fundamental solutions; see for example Goard's extensive study in [8].

However, these and some other existing methods have certain drawbacks, which are best illustrated by an example. Lie proved that any PDE of the form $u_t = A(x,t)u_{xx}+B(x,t)u_x+C(x,t)u$, which has a four dimensional Lie algebra of symmetries can be reduced to an equation of the form

$$u_t = u_{xx} - \frac{A}{x^2}u, \tag{1.3}$$

where A is a constant. This PDE has a well known fundamental solution

$$K(x,y,t) = \frac{\sqrt{xy}}{2t}\exp\left(-\frac{x^2+y^2}{4t}\right) I_{\frac{1}{2}\sqrt{1+4A}}\left(\frac{xy}{2t}\right). \tag{1.4}$$

Here $I_\nu(z)$ is the usual modified Bessel function, given by formula (9.6.10) of [1]. Now consider the diffusion $X = \{X_t : t \geq 0\}$, satisfying the SDE

$$dX_t = \frac{2aX_t}{2+aX_t}dt + \sqrt{2X_t}dW_t, \; X_0 = x > 0, \quad a > 0. \tag{1.5}$$

We seek the transition density and hence we require a fundamental solution of the PDE $u_t = xu_{xx} + \frac{2ax}{2+ax}u_x$. This PDE can be reduced to (1.3) with $A = \frac{3}{4}$ by letting $x \to \sqrt{x}$ and $t \to \frac{1}{4}t$, then defining $u(x,t) = e^{\psi(x)}\tilde{u}(x,t)$ for $\psi(x) = \ln\left(\frac{\sqrt{x}}{2+ax^2}\right)$. From the change of variables and the fundamental solution (1.4) we deduce that

$$q(x,y,t) = \frac{1}{t}\frac{2+ay}{(2+ax)}\sqrt{\frac{x}{y}}e^{-\frac{(x+y)}{t}} I_1\left(\frac{2\sqrt{xy}}{t}\right), \tag{1.6}$$

is a fundamental solution of the Kolmogorov equation for the diffusion. The conclusion is correct, but (1.6) is not the transition density for the diffusion, since $l(x,t) = \int_0^\infty q(x,y,t)dy = 1 - \frac{e^{-\frac{x}{2t}}}{2+ax} \neq 1$. Thus q is not a probability density. We will show later that the transition density is actually

$$p(t,x,y) = \frac{e^{-\frac{(x+y)}{t}}}{(2+ax)t}\left[\sqrt{\frac{x}{y}}(2+ay)I_1\left(\frac{2\sqrt{xy}}{t}\right) + t\delta(y)\right]. \tag{1.7}$$

The term involving the delta function arises naturally from our method and does not need to be imposed using other considerations.

Thus reduction to canonical form can be relied upon to provide a fundamental solution, but there is no reason a priori to believe that this fundamental solution will also be a probability density. Often the true probability density must contain terms involving generalized functions, such as the Dirac delta in (1.7), which the reduction method does not produce. The same problem arises in the method of group invariant solutions, as well as certain other techniques. See [6] for a discussion of this problem.

By contrast, the method presented in this paper, which was first introduced in [7] and further developed in [5] and [4], will always produce a fundamental solution which is a probability density. This method is based upon the fact that if the Lie symmetry group of (1.2) is at least four dimensional, then it is always possible to construct, up to a change of variables, a Laplace or Fourier transform of a fundamental solution, by applying a single symmetry to a time independent solution. In this paper, we will investigate the Laplace transforms arising from equations of the form (1.2) and present some applications.

2. Lie Symmetries and Generalized Laplace Transforms

A *symmetry* of a differential equation is a transformation which maps solutions of the equation to other solutions. See Olver's book, [16] for a modern account of Lie's theory of symmetry groups. Lie's algorithm requires that we look for vector fields of the form $\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u$, where $\partial_x = \frac{\partial}{\partial x}$ etc. These vector fields are called *infinitesimal symmetries*. The group transformation is obtained by *exponentiating* the infinitesimal symmetry $\mathbf{v}$ which is carried out by solving the system of equations

$$\frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x},\tilde{t},\tilde{u}), \quad \frac{d\tilde{t}}{d\epsilon} = \tau(\tilde{x},\tilde{t},\tilde{u}), \quad \frac{d\tilde{u}}{d\epsilon} = \phi(\tilde{x},\tilde{t},\tilde{u}), \tag{2.1}$$

subject to $\tilde{x}(0) = x, \tilde{t}(0) = t, \tilde{u}(0) = u$. These transformations map solutions to solutions, allowing us to construct complex solutions from simple ones. It turns out that we can often construct a fundamental solution from a time independent solution in this manner.

Example 2.1. Consider the diffusion $X = \{X_t : t \geq 0\}$ satisfying the Itô SDE

$$dX_t = \left(\frac{a}{X_t} + bX_t\right)dt + \sqrt{2}dW_t, \quad X_0 = x. \tag{2.2}$$

This is a radial Ornstein-Uhlenbeck process which has been studied by Göing-Jaeschke and Yor in [9] and others. We restrict attention to the case when $a > -\frac{1}{2}$. To obtain the transition density we want a positive fundamental solution of the PDE

$$u_t = u_{xx} + \left(\frac{a}{x} + bx\right)u_x, \tag{2.3}$$

which integrates to one. We begin by computing the Lie symmetry group. Omitting the calculations, we find that the Lie symmetry algebra for the PDE has a basis

$$\mathbf{v}_1 = \partial_t, \mathbf{v}_2 = bxe^{2bt}\partial_x + e^{2bt}\partial_t - (b^2x^2 + b(1+a))e^{2bt}u\partial_u$$
$$\mathbf{v}_3 = -bxe^{-2bt}\partial_x + e^{-2bt}\partial_t, \ \mathbf{v}_4 = u\partial_u, \mathbf{v}_\beta = \beta(x,t)\partial_u.$$

Here β is an arbitrary solution of (2.3). The presence of these vector fields reflects the linearity of the equation and the superposition of solutions.

We exponentiate the infinitesimal symmetry $\mathbf{v}_2$ to obtain the following group symmetry of the PDE. We find that if $u(x,t)$ solves (2.3), then for ϵ sufficiently small, so does

$$u_\epsilon(t,x) = \frac{1}{(1+2b\epsilon e^{2bt})^\nu} \exp\left\{-\frac{b^2\epsilon e^{2bt}x^2}{1+2b\epsilon e^{2bt}}\right\} \times u\left(\frac{x}{\sqrt{1+2b\epsilon e^{2bt}}}, \frac{1}{2b}\ln\left(\frac{e^{2bt}}{1+2b\epsilon e^{2bt}}\right)\right). \tag{2.4}$$

Here $\nu = \frac{1}{2}(a+1)$. Since $u=1$ is a solution of the PDE, then by applying (2.4) we see that for $\epsilon > 0$

$$U_\epsilon(t,x) = \frac{1}{(1+2b\epsilon e^{2bt})^\nu} \exp\left\{-\frac{b^2\epsilon e^{2bt}x^2}{1+2b\epsilon e^{2bt}}\right\}, \tag{2.5}$$

is also a solution. Now observe that $U_\epsilon(0,x) = \frac{1}{(1+2b\epsilon)^\nu} e^{-\frac{b^2\epsilon x^2}{1+2b\epsilon}}$. If $p(t,x,y)$ is the transition density, then we should have

$$\int_0^\infty U_\epsilon(0,y)p(t,x,y)dy = U_\epsilon(t,x). \tag{2.6}$$

With the change of parameters $\lambda = \frac{b^2\epsilon}{1+2b\epsilon}$, the preceding integral becomes

$$\int_0^\infty e^{-\lambda y^2} p(t,x,y)dy = \frac{b^{2\nu}}{(b^2+2b\lambda(e^{2bt}-1))^\nu} \exp\left\{\frac{-b\lambda e^{2bt}x^2}{b+2\lambda(e^{2bt}-1)}\right\}.$$

Letting $y^2 \to z$ converts this into a Laplace transform. To invert the transform we use $\mathcal{L}^{-1}\left[\frac{1}{\lambda^\nu}e^{\frac{a}{\lambda}}\right] = \left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} I_{\nu-1}(2\sqrt{az}),\ \nu > 0$.

Carrying out the inversion and replacing the original variables we obtain

$$p(t,x,y) = \frac{ybe^{b(1-\nu)t}}{e^{2bt}-1}\left(\frac{y}{x}\right)^{\nu-1} \exp\left\{\frac{-b(e^{2bt}x^2+y^2)}{2(e^{2bt}-1)}\right\} I_{\nu-1}\left(\frac{be^{bt}xy}{e^{2bt}-1}\right).$$

This is the transition density for a radial Ornstein-Uhlenbeck process.

Although the previous example has been treated in an ad hoc manner, we can perform exactly the same analysis for many PDEs. It turns out that if a PDE of the form (1.2) has at least a four dimensional Lie symmetry algebra, then the symmetry group contains a generalized Laplace transform of a fundamental solution. We now present some results that we will employ later. The first result can be found in [6].

Proposition 2.1. *The PDE*

$$u_t = \sigma x^\gamma u_{xx} + f(x)u_x - g(x)u,\ \gamma \neq 2 \tag{2.7}$$

has a Lie algebra of symmetries which is at least four dimensional if and only if, for a given g, the function $h(x) = x^{1-\gamma}f(x)$ is a solution *of one of the following families of Riccati equations*

$$\sigma x h' - \sigma h + \frac{1}{2}h^2 + 2\sigma x^{2-\gamma} g(x) = 2\sigma A x^{2-\gamma} + B$$

$$\sigma x h' - \sigma h + \frac{1}{2}h^2 + 2\sigma x^{2-\gamma} g(x) = \frac{A x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B x^{2-\gamma}}{2-\gamma} + C,$$

$$\sigma x h' - \sigma h + \frac{1}{2}h^2 + 2\sigma x^{2-\gamma} g(x) = \frac{A x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B x^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{C x^{2-\gamma}}{2-\gamma} - \kappa,$$

with $\kappa = \frac{\gamma}{8}(\gamma-4)\sigma^2$ *and* A, B *and* C *arbitrary constants.*

Remark 2.2. The first Riccati equation is a special case of the second, but is treated separately. The factors multiplying A, B and C are included as a notational convenience. A similar, but slightly more complicated result holds when $\gamma = 2$; see [6].

Craddock and Lennox proved in [6] the following result.

Theorem 2.3. *Suppose that* $\gamma \neq 2$ *and for a given* g, $h(x) = x^{1-\gamma} f(x)$ *is a solution of the Riccati equation*

$$\sigma x h' - \sigma h + \frac{1}{2}h^2 + 2\sigma x^{2-\gamma} g(x) = 2\sigma A x^{2-\gamma} + B. \tag{2.8}$$

Then the PDE

$$u_t = \sigma x^\gamma u_{xx} + f(x) u_x - g(x) u, \ x \geq 0 \tag{2.9}$$

has a symmetry of the form

$$\overline{U}_\epsilon(t,x) = \frac{1}{(1+4\epsilon t)^{\frac{1-\gamma}{2-\gamma}}} \exp\left\{ \frac{-4\epsilon(x^{2-\gamma} + A\sigma(2-\gamma)^2 t^2)}{\sigma(2-\gamma)^2(1+4\epsilon t)} \right\} \times$$
$$\exp\left\{ \frac{1}{2\sigma}\left(F\left(\frac{x}{(1+4\epsilon t)^{\frac{2}{2-\gamma}}} \right) - F(x) \right) \right\} u\left(\frac{t}{1+4\epsilon t}, \frac{x}{(1+4\epsilon t)^{\frac{2}{2-\gamma}}} \right),$$

where $F'(x) = f(x)/x^\gamma$ *and* u *is a solution of(2.9). That is, for* ϵ *sufficiently small,* U_ϵ *is a solution of (2.9) whenever* u *is. If* $u(t,x) = u_0(x)$ *with* u_0 *an analytic, stationary solution, then there is a fundamental solution* $p(t,x,y)$ *of (2.9) such that*

$$\int_0^\infty e^{-\lambda y^{2-\gamma}} u_0(y) p(t,x,y) dy = U_\lambda(t,x). \tag{2.10}$$

Here $U_\lambda(t,x) = \overline{U}_{\frac{1}{4}\sigma(2-\gamma)^2\lambda}(t,x)$. *Further, if* $g = 0$, *then we may take* $u_0 = 1$, *and the fundamental solution arising from this choice satisfies* $\int_0^\infty p(t,x,y)dy = 1$.

Remark 2.4. The change of variables $z = y^{2-\gamma}$ converts (2.10) into a Laplace transform. The $\gamma = 2$ case is treated in [6].

The second and third Riccati equations provide the richest class of diffusions, as they have solutions in terms of confluent hypergeometric functions. Craddock proved the following two results in [4].

Theorem 2.5. *Suppose that $\gamma \neq 2$ and that for a given g, the drift f in the PDE (2.9) is such that $h(x) = x^{1-\gamma}f(x)$ satisfies the Riccati equation*

$$\sigma x h' - \sigma h + \frac{1}{2}h^2 + 2\sigma x^{2-\gamma}g(x) = \frac{A}{2(2-\gamma)^2}x^{4-2\gamma} + \frac{B}{2-\gamma}x^{2-\gamma} + C,$$

where $A > 0, B$ and C are arbitrary constants. Let u_0 be an analytic, stationary solution of (2.9). Then for ϵ sufficiently small (2.9) has a solution

$$\begin{aligned}
&\overline{U}_\epsilon(t,x) = (1 + 2\epsilon^2(\cosh(\sqrt{A}t) - 1) + 2\epsilon\sinh(\sqrt{A}t))^{-c} \\
&\times \left|\frac{\cosh(\frac{\sqrt{A}t}{2}) + (1+2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}{\cosh(\frac{\sqrt{A}t}{2}) - (1-2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}\right|^{\frac{B}{2\sigma\sqrt{A}(2-\gamma)}} e^{-\frac{1}{2\sigma}F(x) - \frac{Bt}{\sigma(2-\gamma)}} \\
&\times \exp\left\{\frac{-\sqrt{A}\epsilon x^{2-\gamma}(\cosh(\sqrt{A}t) + \epsilon\sinh(\sqrt{A}t))}{\sigma(2-\gamma)^2(1 + 2\epsilon^2(\cosh(\sqrt{A}t) - 1) + 2\epsilon\sinh(\sqrt{A}t)}\right\} \\
&\times \exp\left\{\frac{1}{2\sigma}F\left(\frac{x}{(1 + 2\epsilon^2(\cosh(\sqrt{A}t) - 1) + 2\epsilon\sinh(\sqrt{A}t))^{\frac{1}{2-\gamma}}}\right)\right\} \\
&\times u_0\left(\frac{x}{(1 + 2\epsilon^2(\cosh(\sqrt{A}t) - 1) + 2\epsilon\sinh(\sqrt{A}t))^{\frac{1}{2-\gamma}}}\right), \qquad (2.11)
\end{aligned}$$

where $F'(x) = \frac{f(x)}{x^\gamma}$ and $c = \frac{(1-\gamma)}{2-\gamma}$. Furthermore, there exists a fundamental solution $p(t,x,y)$ of (2.9) such that

$$\int_0^\infty e^{-\lambda y^{2-\gamma}} u_0(y) p(t,x,y)dy = U_\lambda(t,x) \qquad (2.12)$$

in which $U_\lambda(t,x) = \overline{U}_{\frac{\sigma(2-\gamma)^2\lambda}{\sqrt{A}}}(t,x)$. If $g = 0$, then we may take $u_0 = 1$, and the fundamental solution satisfies $\int_0^\infty p(t,x,y)dy = 1$.

Remark 2.6. The extension of these results to the case $A < 0$ is obtained by replacing $\cosh(\sqrt{A}t)$ with $\cos(\sqrt{|A|}t)$ etc.

Theorem 2.7. *Suppose that $\gamma \neq 2$ and that for a given g, $h(x) = x^{1-\gamma}f(x)$ is a solution of the Riccati equation*

$$\sigma x h' - \sigma h + \frac{1}{2}h^2 + 2\sigma x^{2-\gamma}g(x) = \frac{Ax^{4-2\gamma}}{2(2-\gamma)^2} + \frac{Bx^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{Cx^{2-\gamma}}{2-\gamma} - \kappa,$$

where $\kappa = \frac{\gamma}{8}(\gamma-4)\sigma^2, \gamma \neq 2$ and $A > 0$. Let u_0 be an analytic stationary solution of the PDE (2.9). Define the following constants: $a = \frac{C}{2\sigma(2-\gamma)}, b = \frac{(1-\gamma)\sqrt{A}}{2(2-\gamma)}, k = \frac{2(2-\gamma)B}{3\sqrt{A}}, d = \frac{B^2}{9A\sigma},\ l = \frac{B\gamma}{3Ak}$ and $s =$

$\frac{a+d}{\sqrt{A}} - \frac{\sqrt{A}k^2}{2\sigma(2-\gamma)^2}$. *Let ϵ be sufficiently small and*

$$X(\epsilon,x,t) = \left(\frac{x^{1-\frac{\gamma}{2}} + k}{\sqrt{1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t)}} - k \right)^{\frac{2}{2-\gamma}},$$

and $F'(x) = \frac{f(x)}{x^\gamma}$. Then equation (2.9) has a solution

$$\overline{U}_\epsilon(t,x) = \frac{x^l(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))^{-\frac{2b}{\sqrt{A}}}}{(k+kx^{\frac{\gamma}{2}}(1-\sqrt{1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t)}))^l}$$

$$\times \left| \frac{\cosh(\frac{\sqrt{A}t}{2}) + (1+2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}{\cosh(\frac{\sqrt{A}t}{2}) - (1-2\epsilon)\sinh(\frac{\sqrt{A}t}{2})} \right|^s e^{\frac{\sqrt{A}k^2}{\sigma(2-\gamma)^2} - 2s\sqrt{A}t}$$
$$\times \exp\left\{ \frac{-\sqrt{A}\epsilon(x^{1-\frac{\gamma}{2}}+k)^2(\cosh(\sqrt{A}t)+\epsilon\sinh(\sqrt{A}t))}{\sigma(2-\gamma)^2(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t)} \right\}$$
$$\times \exp\left\{ \frac{1}{2\sigma}\left(F\left(X(\epsilon,x,t) - F(x)\right)\right) \right\} u_0\left(X(\epsilon,x,t)\right).$$

Further, (2.9) has a fundamental solution $p(t,x,y)$ such that

$$\int_0^\infty e^{-\lambda(y^{2-\gamma}+2ky^{1-\frac{\gamma}{2}})} u_0(y)p(t,x,y)dy = U_\lambda(t,x), \tag{2.13}$$

in which $U_\lambda(t,x) = \overline{U}_{\frac{\sigma(2-\gamma)^2\lambda}{\sqrt{A}}}(t,x)$. If $g=0$, then we may take $u_0 = 1$, and $\int_0^\infty p(t,x,y)dy = 1$ for the fundamental solution arising from this choice.

Remark 2.8. The transform of a locally integrable function ϕ defined by $\Phi(\lambda) = \int_0^\infty e^{-\lambda(y^{2-\gamma}+2ky^{1-\frac{\gamma}{2}})}\phi(y)dy$ reduces to a Laplace transform when we make the substitution $z = y^{2-\gamma} + 2ky^{1-\frac{\gamma}{2}}$. When $A < 0$ a similar result holds, with $\cosh(\sqrt{A}t)$ replaced by $\cos(\sqrt{|A|}t)$ etc.

We also mention a previous result of Craddock and Platen [7].

Theorem 2.9. *Let f be a solution of the Riccati equation,*

$$\sigma x f' - f + \frac{1}{2}f^2 = Ax^{\frac{3}{2}} + Cx - \frac{3}{8}\sigma^2. \tag{2.14}$$

Let $u_0(x)$ be a stationary solution of the PDE (1.2), with $\gamma = 1$. Let $F'(x) = \frac{f(x)}{x}$, $H(\lambda,x,t) = \frac{(12(1+\lambda\sigma t)\sqrt{x} - A\lambda(\sigma t)^3)^2}{144(1+\lambda\sigma t)^4}$ and $G(\lambda,x,\frac{t}{\sigma}) = -\frac{\lambda(x+\frac{1}{2}Ct^2)}{1+\lambda t} - \frac{\frac{2}{3}At^2\sqrt{x}(3+\lambda t)}{(1+\lambda t)^2} + \frac{A^2t^4(2\lambda t(3+\frac{1}{2}\lambda t)-3)}{108(1+\lambda t)^3}$. Then there is a fundamental solution $p(t,x,y)$ of the PDE (2.9) such that

$$\int_0^\infty u_0(y)e^{-\lambda y}p(t,x,y)dy = U_\lambda(t,x) \tag{2.15}$$

where $U_\lambda(x,t) =$

$$\sqrt{\frac{\sqrt{x}(1+\lambda\sigma t)}{\sqrt{x}(1+\lambda\sigma t)-\frac{A\lambda}{12}(\sigma t)^3}}e^{G(\lambda,x,t)-\frac{1}{2\sigma}(F(x)-F(H(\lambda,x,t)))}u_0(H(\lambda,x,t))$$

If $u_0 = 1$, *then* $\int_0^\infty p(t,x,y)dy = 1$.

As we are here interested in obtaining probability densities, we focus on the case when $g = 0$. Taking g nonzero allows us to compute a wide variety of functionals via the Feynman-Kac formula, a topic discussed extensively in [6]. An important feature of these theorems is the fact that taking $u_0 = 1$ will always give a fundamental solution which is a probability density. We have the following result.

Corollary 2.10. *Let* $X = \{X_t : t \geq 0\}$ *be the unique strong solution of the Itô SDE*

$$dX_t = f(X_t)dt + \sqrt{2\sigma X_t^\gamma}dW_t, \quad X_0 = x,\ \gamma \neq 2$$

and suppose that $h(x) = x^{1-\gamma}f(x)$ *is a solution of either of the Riccati equations in Theorems 2.3, 2.5, 2.7 or 2.9. Then*

$$\mathbb{E}_x\left[e^{-\lambda X_t}\right] = U_\lambda(t,x) \tag{2.16}$$

where the value of $U_\lambda(t,x)$ *is given by taking the symmetry solution in the respective theorem, with* $u_0(x) = 1$.

3. Solutions of the Riccati Equations

One of the principle aims of this paper is to begin the task of analysing and exploiting the processes which arise via the symmetry methods introduced in the previous section. The first task is to determine the types of drifts which are possible. We concentrate on the case $\gamma = 1$. The methods for other values of γ are essentially the same.

3.1. The First Riccati Equation.

Taking $\gamma = 1$ we are lead to the following equation for the drift functions f handled by the first theorem:

$$\sigma x f' - \sigma f + \frac{1}{2}f^2 = Ax + B. \tag{3.1}$$

3.1.1. *The case* $A = 0$. The change of variables $f = 2\sigma x\frac{y'}{y}$ produces the linear equation $2\sigma^2x^2y'' - By = 0$. This leads to the basic solutions

$$f(x) = \begin{cases} \alpha & B = \frac{1}{2}\alpha^2 - \sigma\alpha \\ \sigma + 2\sigma\mu\left(\frac{Cx^{2\mu}-1}{Cx^{2\mu}+1}\right) & 2B+\sigma^2 > 0 \\ \sigma + \frac{2\sigma}{\ln|x|+C} & 2B+\sigma^2 = 0 \\ \sigma - 2\sigma\kappa\tan(\kappa\ln|x|+C) & 2B+\sigma^2 < 0 \\ \frac{2a\sigma x}{b+ax} & 2B+\sigma^2 = \sigma^2, \end{cases} \tag{3.2}$$

where $\kappa = \frac{1}{2\sigma}\sqrt{|2B+\sigma^2|}$, $\mu = \frac{1}{2\sigma}\sqrt{2B+\sigma^2}$ and a, b and C are constants. These constants may all be chosen to be real or complex. We will only consider real constants and leave consideration of the complex valued diffusions which arise from this method to a later study. Note that the last case is a special case of the second.

The drift functions listed above give rise to the following formal SDEs.

$$dX_t = \alpha dt + \sqrt{2\sigma X_t}dW_t \tag{3.3}$$

$$dX_t = \left(\sigma + 2\sigma\mu\left(\frac{CX_t^{2\mu}-1}{CX_t^{2\mu}+1}\right)\right)dt + \sqrt{2\sigma X_t}dW_t \tag{3.4}$$

$$dX_t = \left(\sigma + \frac{2\sigma}{\ln|X_t|+C}\right)dt + \sqrt{2\sigma X_t}dW_t \tag{3.5}$$

$$dX_t = (\sigma - 2\sigma\kappa\tan(\kappa\ln|X_t|+C))\,dt + \sqrt{2\sigma X_t}dW_t \tag{3.6}$$

$$dX_t = \frac{2a\sigma X_t}{b+aX_t}dt + \sqrt{2\sigma X_t}dW_t. \tag{3.7}$$

3.1.2. *Interpreting the Fundamental Solutions.* Interpreting the fundamental solutions which arise from Theorems 2.3, 2.5 and 2.7 as transition probability densities for Itô diffusions is not always straightforward. To see why, consider the PDE $u_t = 2xu_{xx} - au_x, x \geq 0, a > 0$. A fundamental solution for this PDE with $a = 2$ will be obtained in Example 4.2, and this fundamental solution satisfies $\int_0^\infty p(t,x,y)dy = 1$. From the PDE point of view, this is quite standard. It also follows from a theorem of Kolmogorov, (see for example Theorem 1, p246 of [19]), that there exists a Markov process which has $F(z) = \int_0^z p(t,x,y)dy$ as its probability distribution function. However, it is not clear that this will necessarily be an Itô process. We would like to argue that p is the density for $X = \{X_t : t \geq 0\}$ satisfying the SDE $dX_t = -adt + 2\sqrt{X_t}dW_t,\ X_0 = x$. However, this interpretation has difficulties, since for such a process, $\tau_0 = \inf\{t : X_t = 0\}$ is almost surely finite and when X hits zero, the negative drift will guarantee that the process becomes complex valued.

Because of the negative drift, it would be better to write the SDE as $dX_t = -adt + 2\sqrt{|X_t|}dW_t$, which is the SDE for a squared Bessel process of dimension $-a$. The transition densities for these processes were computed by Göing-Jaeschke and Yor in [9]. However in this case we would have to consider the PDE $u_t = 2|x|u_{xx} - au_x,\ x \in \mathbb{R}$, which we do not consider here.

There are many examples with this type of feature. A second issue it that our methods also yield more than one fundamental solution for the same PDE. Therefore we will treat our SDEs and PDEs in a purely formal manner in this paper and defer a detailed discussion of their interpretation to another publication.

3.2. **The Fundamental Solutions.** Here we will examine the fundamental solutions that arise from the stationary solution $u_0(x) = 1$.

3.2.1. *The First SDE.* Suppose that $\alpha > 0$. In this case the first drift is obviously that for a scaled version of a squared Bessel process. We take $\sigma = 2$ for simplicity. The stationary solution $u_0(x) = 1$ yields the Laplace transform

$$U_\lambda(t,x) = \frac{1}{(1+2\lambda t)^{\frac{\alpha}{2}}} \exp\left(-\frac{\lambda x}{1+2\lambda t}\right). \tag{3.8}$$

Inversion of the Laplace transform gives

$$p(t,x,y) = \frac{1}{2t}\left(\frac{x}{y}\right)^{\frac{2-\alpha}{4}} \exp\left(-\frac{x+y}{2t}\right) I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{xy}}{t}\right), \tag{3.9}$$

which is the well known transition density of a squared Bessel process of dimension α. Taking the second stationary solution $u_0(x) = x^{1-\frac{\alpha}{2}}, \alpha \neq 2$ or $u_0(x) = \ln x$ when $\alpha = 2$ produces a second fundamental solution that is not in general a probability density. However if $\alpha = 2n$, $n = 2, 3, 4, \ldots$, then the second fundamental solution does integrate to one. We will discuss this feature later in Proposition 4.3.

3.2.2. *The second SDE.* By Theorem 2.3, for the diffusion satisfying the SDE $dX_t = \left(\sigma + 2\sigma\mu\left(\frac{CX_t^{2\mu}-1}{CX_t^{2\mu}+1}\right)\right)dt + \sqrt{2\sigma X_t}dW_t$ we compute $F(x) = \int \frac{f(x)}{x}dx = \sigma \ln x + 2\sigma \ln|Cx^\mu + x^{-\mu}|$. After some simplification we are led to the Laplace transform

$$\int_0^\infty e^{-\lambda y} p(t,x,y)dy = \frac{Cx^{2\mu}}{Cx^{2\mu}+1}\frac{1}{(1+\lambda\sigma t)^{2\mu+1}} e^{-\frac{x}{\sigma t}+\frac{x/(\sigma t)^2}{\lambda+\frac{1}{\sigma t}}}$$
$$+ \frac{1}{Cx^{2\mu}+1}\frac{1}{(1+\lambda\sigma t)^{1-2\mu}} e^{-\frac{x}{\sigma t}+\frac{x/(\sigma t)^2}{\lambda+\frac{1}{\sigma t}}}. \tag{3.10}$$

If $0 \leq \mu \leq \frac{1}{2}$, then the Laplace transform may easily be inverted giving the density

$$p(t,x,y) = \frac{\exp\left(-\frac{x+y}{\sigma t}\right)}{\sigma t(Cx^{2\mu}+1)}\left[C(xy)^\mu I_{2\mu}\left(\frac{2\sqrt{xy}}{\sigma t}\right) + \left(\frac{y}{x}\right)^\mu I_{-2\mu}\left(\frac{2\sqrt{xy}}{\sigma t}\right)\right].$$

If $\mu \geq \frac{1}{2}$, then

$$p(t,x,y) = \frac{\exp\left(-\frac{x+y}{\sigma t}\right)}{\sigma t(Cx^{2\mu}+1)}\left[C(xy)^\mu I_{2\mu}\left(\frac{2\sqrt{xy}}{\sigma t}\right) + (\sigma t)^{2\mu}\mathcal{L}^{-1}[\frac{e^{\frac{x}{(\sigma t)^2\lambda}}}{\lambda^{1-2\mu}}]\right].$$

For $a \leq 0$ the inverse Laplace transform $\mathcal{L}^{-1}[\lambda^{-a}e^{k/\lambda}]$ exists as a distribution. We will discuss the inversion of these Laplace transforms in Section 4.

3.2.3. *The Third SDE.* For the formal SDE

$$dX_t = \left(\sigma + \frac{2\sigma}{\ln|X_t| + C}\right) dt + \sqrt{2\sigma X_t} dW_t,$$

we use the stationary solution $u_0(x) = 1$ of the Kolmogorov forward equation to obtain

$$U_\lambda(t,x) = \frac{\left|\ln\left(\frac{x}{(1+\lambda\sigma t)^2}\right) + C\right|}{|\ln x + C|(1+\lambda\sigma t)} \exp\left(-\frac{\lambda x}{1+2\lambda t}\right). \tag{3.11}$$

This Laplace transform can be inverted explicitly for any values of x and t. In the inversion, we require the inverse Laplace transform of $e^{k/\lambda}$. We will discuss the issue of inverting such a Laplace transform later. Here we simply quote the result, which is

$$\mathcal{L}^{-1}[e^{k/\lambda}] = \delta(y) + \sqrt{\frac{k}{y}} I_1(2\sqrt{ky}), \tag{3.12}$$

where $\delta(y)$ is the Dirac delta function. We can therefore write the fundamental solution of the Kolmogorov equation as

$$p(t,x,y) = \frac{e^{-\frac{x+y}{\sigma t}}}{|\ln x + C|} \int_0^y \left(\delta(y) + \frac{1}{\sigma t}\sqrt{\frac{x}{\xi}} I_1\left(\frac{2\sqrt{xy}}{\sigma t}\right)\right) \eta(t,x,y-\xi) d\xi, \tag{3.13}$$

where $\eta(t,x,y) = \mathcal{L}^{-1}\left[\frac{|\ln x + C - 2\ln(\sigma t) - 2\ln\lambda|}{\lambda}\right]$. For any choices of x, t and C, this inverse Laplace transform can be explicitly evaluated using $\mathcal{L}^{-1}[\frac{1}{\lambda}\ln\lambda] = \Gamma'(1) - \ln y$, where Γ' is the derivative of the Gamma function.

3.2.4. *The Fourth SDE.* For $dX_t = (\sigma - 2\sigma\kappa\tan(\kappa\ln|X_t| + C))\, dt + \sqrt{2\sigma X_t} dW_t$ we have

$$U_\lambda(t,x) = \frac{|\cos(\kappa\ln\left(\frac{x}{(1+\lambda\sigma t)^2}\right) + C)|}{|\cos(\kappa\ln x + C)|(1+\lambda\sigma t)} \exp\left(-\frac{\lambda x}{1+\lambda\sigma t}\right). \tag{3.14}$$

For any choice of x, t and C it is possible to explicitly invert this Laplace transform. However, giving a general formula for the inverse transform leads to a rather complicated expression. An explicit example is given in [7].

3.2.5. *The Final SDE.* For the diffusion following the SDE

$$dX_t = \frac{2a\sigma X_t}{b + aX_t} dt + \sqrt{2\sigma X_t} dW_t,$$

with $a, b > 0$, the choice $u_0 = 1$ leads to

$$U_\lambda(t,x) = \frac{b + \frac{ax}{(1+\lambda\sigma t)^2}}{b + ax} \exp\left(-\frac{\lambda x}{1+\lambda\sigma t}\right). \tag{3.15}$$

The density is

$$p(t,x,y) = e^{-\frac{x+y}{\sigma t}}\left[\frac{1}{\sigma t}\frac{b+ay}{b+ax}\sqrt{\frac{x}{y}}I_1\left(\frac{2\sqrt{xy}}{\sigma t}\right)+\frac{b\delta(y)}{b+ax}\right]. \tag{3.16}$$

The presence of the Dirac delta function in the density is needed to guarantee that $\int_0^\infty p(t,x,y)dy = 1$. Observe that the drift is Lipschitz continuous for positive a and b and $\sqrt{x}$ is Hölder continuous, with Hölder constant $\frac{1}{2}$. Thus by the Yamada-Watanabe Theorem, (Theorem 5.5 of [14]), the SDE has a unique strong solution. Furthermore, if $X_0 = 0$, then $X_t = 0,\ t \geq 0$ is a solution and by uniqueness it is the only solution. So by the Markov property, if $\tau_0 = \inf\{t \geq 0; X_t = 0\}$, then $X_t = 0$ for all $t \geq \tau$. This leads us to interpret the delta function term in the density as being related to the probability of absorption at zero.

Note that if we discard the term involving the Delta function, then the result is

$$q(t,x,y) = e^{-\frac{x+y}{\sigma t}}\left[\frac{1}{\sigma t}\frac{b+ay}{b+ax}\sqrt{\frac{x}{y}}I_1\left(\frac{2\sqrt{xy}}{\sigma t}\right)\right]. \tag{3.17}$$

This was the fundamental solution obtained by the reduction to canonical form, when $b = 2$.

3.3. **The case** $A \neq 0$**.** If A is nonzero, we again set $f = 2\sigma x\frac{y'}{y}$ and find that $2\sigma^2x^2y'' - (Ax+B)y = 0$, which has solutions

$$y(x) = \begin{cases} c_1\sqrt{x}I_\mu(\frac{\sqrt{2Ax}}{\sigma}) + c_2\sqrt{x}K_\mu(\frac{\sqrt{2Ax}}{\sigma}), & A > 0, \\ c_1\sqrt{x}J_\mu(\frac{\sqrt{2Ax}}{\sigma}) + c_2\sqrt{x}Y_\mu(\frac{\sqrt{2Ax}}{\sigma}), & A < 0, \end{cases} \tag{3.18}$$

in which $\mu = \frac{1}{\sigma}\sqrt{2B+\sigma^2}$ and J_μ, I_μ, Y_μ and $K_\mu(z)$ are the Bessel functions as defined in Chapter Nine of [1]. In general we have

$$f(x) = \begin{cases} \sigma(1+\mu) + \sqrt{2Ax}\frac{c_1I_{1+\mu}(\frac{\sqrt{2Ax}}{\sigma}) - c_2K_{1+\mu}(\frac{\sqrt{2Ax}}{\sigma})}{c_1I_\mu(\frac{\sqrt{2Ax}}{\sigma}) + c_2K_\mu(\frac{\sqrt{2Ax}}{\sigma})}, & A > 0, \\ \sigma(1+\mu) - \sqrt{2|A|x}\frac{c_1J_{1+\mu}(\frac{\sqrt{2|A|x}}{\sigma}) + c_2Y_{1+\mu}(\frac{\sqrt{2|A|x}}{\sigma})}{c_1J_\mu(\frac{\sqrt{2|A|x}}{\sigma}) + c_2Y_\mu(\frac{\sqrt{2|A|x}}{\sigma})}, & A > 0. \end{cases}$$

These drifts correspond to the purely formal Itô SDEs

$$dX_t = \left(\nu + \sqrt{2AX_t}\frac{c_1I_{1+\mu}(\frac{\sqrt{2AX_t}}{\sigma}) - c_2\sqrt{X_t}K_{1+\mu}(\frac{\sqrt{2AX_t}}{\sigma})}{c_1I_\mu(\frac{\sqrt{2AX_t}}{\sigma}) + c_2K_\mu(\frac{\sqrt{2AX_t}}{\sigma})}\right)dt$$
$$+ \sqrt{2\sigma X_t}dW_t,\ A > 0,$$
$$dX_t = \left(\nu - \sqrt{2|A|X_t}\frac{c_1J_{1+\mu}(\frac{\sqrt{2|A|X_t}}{\sigma}) + c_2Y_{1+\mu}(\frac{\sqrt{2|A|X_t}}{\sigma})}{c_1J_\mu(\frac{\sqrt{2|A|X_t}}{\sigma}) + c_2Y_\mu(\frac{\sqrt{2|A|X_t}}{\sigma})}\right)dt$$
$$+ \sqrt{2\sigma X_t}dW_t,\ A < 0,$$

with $\nu = \sigma(1+\mu)$. For different choices of A, B, c_1, c_2 these functions can represent a very wide range of drifts.

If $A > 0$ and $|\mu| < 1$, then we may recover a very general form of the fundamental solution.

Theorem 3.1. *Consider a diffusion process $X = \{X_t : t \geq 0\}$ which satisfies the SDE*

$$dX_t = f(X_t)dt + \sqrt{2\sigma X_t}dW_t \quad X_0 = x, \tag{3.19}$$

and for which the drift function is a solution of the Riccati equation (3.1), with $A > 0$. Let $\mu = \sqrt{1 + 2B/\sigma^2}$ and suppose further that $|\mu| < 1$. Then the transition probability density for X is given by

$$p(t,x,y) = \frac{e^{-\frac{x+y}{\sigma t} - \frac{At}{2\sigma}}}{\sigma t} \frac{c_1 I_\mu\left(\frac{2\sqrt{xy}}{\sigma t}\right) I_\mu\left(\frac{\sqrt{2Ay}}{\sigma}\right) + c_2 I_{-\mu}\left(\frac{2\sqrt{xy}}{\sigma t}\right) I_{-\mu}\left(\frac{\sqrt{2Ay}}{\sigma}\right)}{c_1 I_\mu\left(\frac{\sqrt{2Ax}}{\sigma}\right) + c_2 I_{-\mu}\left(\frac{\sqrt{2Ax}}{\sigma}\right)}.$$

Proof. For $\mu > -1$, $\mathcal{L}^{-1}[\frac{1}{\lambda} e^{\frac{a^2+b^2}{\lambda}} I_\mu(\frac{2ab}{\lambda})] = I_\mu(2a\sqrt{y}) I_\mu(2b\sqrt{y})$. We replace $K_\mu(z)$ with $I_\mu(z)$ for μ not an integer. Let $a = \frac{\sqrt{x}}{\sigma t}, b = \frac{\sqrt{A}}{\sqrt{2\sigma}}$. Then we have

$$U_\lambda(t,x) = \frac{1}{\sigma t} e^{-\frac{x}{\sigma t} - \frac{At}{2\sigma}} \frac{c_1 \frac{1}{\lambda} e^{\frac{a^2+b^2}{\lambda + 1/\sigma t}} I_\mu(\frac{2ab}{\lambda + 1/\sigma t}) + c_2 \frac{1}{\lambda} e^{\frac{a^2+b^2}{\lambda+1/\sigma t}} I_{-\mu}(\frac{2ab}{\lambda+1/\sigma t})}{c_1 I_\mu\left(\frac{\sqrt{2Ax}}{\sigma}\right) + c_2 I_{-\mu}\left(\frac{\sqrt{2Ax}}{\sigma}\right)}.$$

Using the given inverse Laplace transform yields the result of the theorem. □

Proving a corresponding result for $A < 0$ appears to be difficult. However, individual examples can be handled easily. All the transforms which arise from these drifts can be exactly inverted by the methods of this paper.

3.4. **The Second Riccati Equation.** We turn now to the problem of solving $\sigma x f' - \sigma f + \frac{1}{2} f^2 = \frac{1}{2} A x^2 + Bx + C$. The change of variables $f = \frac{2\sigma x y'}{y}$ converts this Riccati equation into the linear ordinary differential equation (ODE)

$$2\sigma^2 x^2 y'' - (\frac{1}{2} A x^2 + Bx + C) y = 0. \tag{3.20}$$

Solutions of this equation are given by

$$y(x) = x^\beta e^{-\frac{\sqrt{A}x}{2\sigma}} \left(c_1 {}_1F_1(\alpha, 2\beta, \frac{\sqrt{A}x}{\sigma}) + c_2 U(\alpha, 2\beta, \frac{\sqrt{A}x}{\sigma}) \right) \tag{3.21}$$

where $\alpha = \frac{\frac{B}{\sqrt{A}} + \sigma + \sigma\sqrt{\frac{2C}{\sigma^2} + 1}}{2\sigma}$, $\beta = \frac{1}{2}\left(\sqrt{\frac{2C}{\sigma^2} + 1} + 1\right)$, ${}_1F_1(a,b,z)$ is Kummer's confluent hypergeometric function and $U(a,b,z)$ is Tricomi's confluent hypergeometric function, given by (13.2.1) and (13.2.5) of [1], respectively.

Since the class of drifts encompassed by these solutions is considerably richer than the first class, providing an exhaustive list of the possible SDEs which arise from it, is not practical. Many important examples, such as the square root process, (see [17]), are included in the second class of Riccati equations. The third class of Riccati equations can also be solved in terms of confluent hypergeometric equations. An explicit description of all the fundamental solutions arising from the resulting PDEs is possible, but it is quite difficult and will be discussed elsewhere.

3.5. **Some Explicit Drifts and Associated Transition Densities.** Riccati equations possess a nonlinear superposition principle that was discovered by Lie and independently by Eduard Weyr in 1875; see for example [12]. Suppose that f_1, f_2, f_3 satisfy any scalar Riccati equation. Let a be a constant. Then

$$f_4 = \frac{f_1(f_3 - f_2) + af_2(f_1 - f_3)}{f_3 - f_2 + a(f_1 - f_3)}, \tag{3.22}$$

is also a solution of the same Riccati equation. In this way we can produce chains of solutions of Riccati equations and introduce arbitrary parameters into the solutions. In fact, if we know any three solutions of a Riccati equation, we can construct the general solution, a subject discussed in [12]. For $B = -\frac{3}{8}, A = \frac{1}{2}b^2$ three solutions of the Riccati equation $xf' - f + \frac{1}{2}f^2 = Ax + B$ are

$$f(x) = \frac{1}{2}+b\sqrt{x},\ g(x) = \frac{1}{2}+b\sqrt{x}\tanh(b\sqrt{x}),\ h(x) = \frac{1}{2}+b\sqrt{x}\coth(b\sqrt{x}).$$

Using these solutions and the nonlinear superposition principle we easily generate the fourth solution $f_2(x) =$

$$\frac{2b\sqrt{x}(a - \coth(b\sqrt{x}) - a + \mu\coth(b\sqrt{x}) + (1 - 2\mu b\sqrt{x})\ \tanh(b\sqrt{x})}{2\ (-a + \mu\ \coth(b\sqrt{x}) + \tanh(b\sqrt{x}))},$$

with $\mu = a - 1$. Using f_2 and two of f, g, h we can generate solutions f_3, f_4, f_5 etc.

For convenience we will take $\sigma = b = 1$ in the following examples.

Example 3.1. Consider the diffusion $X = \{X_t : t \geq 0\}$ satisfying the SDE $dX_t = \left(\frac{1}{2} + \sqrt{X_t}\right)dt + \sqrt{2X_t}dW_t$. We take $u_0(x) = 1$ and obtain from Theorem 2.3 the Laplace transform

$$U_\lambda(x,t) = \frac{\cosh\left(\frac{\sqrt{x}}{1+\lambda t}\right)}{\cosh(\sqrt{x})\sqrt{1+\lambda t}}\exp\left\{-\frac{\lambda(x+\frac{1}{4}t^2)}{1+\lambda t}\right\}. \tag{3.23}$$

Inversion of the Laplace transform gives the transition density

$$p(t,x,y) = \frac{e^{-\sqrt{x}}}{\sqrt{\pi y t}}\cosh\left(\frac{(t+2\sqrt{x})\sqrt{y}}{t}\right)\exp\left\{-\frac{(x+y)}{t} - \frac{1}{4}t\right\}.$$

By a similar calculation we can show that the transition density for the diffusion $X = \{X_t : t \geq 0\}$ which is the solution of the SDE

$$dX_t = \left(\frac{1}{2} + \sqrt{X_t}\coth\left(\sqrt{X_t}\right)\right)dt + \sqrt{2X_t}dW_t,\ X_0 = x, \tag{3.24}$$

is given by

$$p(t,x,y) = e^{-\frac{1}{4}t}\frac{\sinh\left(\frac{2\sqrt{xy}}{t}\right)}{\sqrt{\pi y t}}\frac{\sinh(\sqrt{y})}{\sinh(\sqrt{x})}\exp\left\{-\frac{x+y}{t}\right\}. \tag{3.25}$$

For the diffusion satisfying the SDE

$$dX_t = \left(\frac{1}{2} + \sqrt{X_t}\tanh\left(\sqrt{X_t}\right)\right)dt + \sqrt{2X_t}dW_t,\ X_0 = x,$$

the transition density is

$$p(t,x,y) = e^{-\frac{1}{4}t}\frac{\cosh\left(\frac{2\sqrt{xy}}{t}\right)}{\sqrt{\pi y t}}\frac{\cosh(\sqrt{y})}{\cosh(\sqrt{x})}\exp\left\{-\frac{x+y}{t}\right\}. \tag{3.26}$$

The process of finding densities via Theorem 2.5 is essentially identical. We present two examples.

Example 3.2. Consider the drift function $f(x) = x\coth\left(\frac{x}{2}\right)$. Here $xf' - f + \frac{1}{2}f^2 = \frac{1}{2}x^2$, so to find the transition density of $X = \{X_t : t \geq 0\}$, where

$$dX_t = X_t\coth\left(\frac{X_t}{2}\right)dt + \sqrt{2X_t}dW_t \tag{3.27}$$

we have to use Theorem 2.5. It is straightforward to show that the theorem implies that

$$p(t,x,y) = \frac{\sinh\left(\frac{y}{2}\right)}{\sinh\left(\frac{x}{2}\right)}\mathcal{L}^{-1}\left(\exp\left\{\frac{-(2\lambda(1+e^t)+e^t-1)x}{2((2\lambda+1)e^t-(2\lambda-1))}\right\}\right)$$
$$= \frac{\sinh(\frac{y}{2})}{\sinh(\frac{x}{2})}\exp\left\{-\frac{(x+y)}{2\tanh(\frac{t}{2})}\right\}\left[\frac{1}{2\sinh(\frac{t}{2})}\sqrt{\frac{x}{y}}I_1\left(\frac{\sqrt{xy}}{\sinh(\frac{t}{2})}\right) + \delta(y)\right].$$

Note that the Dirac delta term arises naturally from the inverse Laplace transform.

Example 3.3. For the square root process $X = \{X_t : t \geq 0\}$ where

$$dX_t = (a - bX_t)dt + \sqrt{2\sigma X_t}dW_t,\ a > 0,\ X_0 = x \tag{3.28}$$

and $a > \sigma$, we have $\sigma xf' - \sigma f + \frac{1}{2}f^2 = \frac{1}{2}Ax^2 + Bx + C$ with $A = b^2$, $B = -ab, C = \frac{1}{2}a^2 - a\sigma$. We require a fundamental solution of

$$u_t = \sigma x u_{xx} + (a - bx)u_x, \tag{3.29}$$

Using Theorem 2.5, with $u_0(x) = 1, F(x) = a \ln x - bx$ and after some cancellations, we arrive at

$$\int_0^\infty e^{-\lambda y} p(t,x,y)dy = \frac{b^{\frac{a}{\sigma}} e^{\frac{ab}{\sigma}t}}{(\lambda\sigma(e^{bt}-1)+be^{bt})^{\frac{a}{\sigma}}} \exp\left\{\frac{-\lambda bx}{\lambda\sigma(e^{bt}-1)+be^{bt}}\right\}.$$

The inverse Laplace transform is

$$p(t,x,y) = \frac{be^{b(\frac{a}{\sigma}+1)t}}{\sigma(e^{bt}-1)}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \exp\left\{\frac{-b(x+e^{bt}y)}{\sigma(e^{bt}-1)}\right\} I_\nu\left(\frac{b\sqrt{xy}}{\sigma\sinh\left(\frac{bt}{2}\right)}\right),$$

with $\nu = \frac{a}{\sigma} - 1 > 0$. This is the transition probability density for the square root process. If $\nu < 0$, then we must consider whether the process is reflected or absorbed at zero, but we will not discuss this here.

4. The Laplace Transform as a Distribution

In general the Laplace transforms we obtain from our results are the Laplace transforms of distributions. In this section we review the basic facts about the distributional Laplace transform. For a more detailed study of the Laplace transform for right sided distributions, see the book by Zemanian, [21].

For the Dirac delta function $\delta(y)$, $\mathcal{L}(\delta) = \int_0^\infty \delta(y)e^{-sy}dy = 1$, from which we conclude that $\mathcal{L}^{-1}(1) = \delta(y)$. One also shows that $\mathcal{L}(\delta^{(n)}(y)) = s^n$.

A more difficult problem is to define the Laplace transform of a function such as $f(y) = y^{-3/2}$. The Laplace integral for this function diverges near zero and so we cannot take the classical Laplace transform.

Suppose however, that ϕ is a compactly supported C^∞ test function. For $f(y) = y^{-3/2}$ we wish to define a distribution Λ_f which acts on functions ϕ. The approach we take here is due to Hadamard [11]. We would like to define

$$\Lambda_f(\phi) = \int_0^\infty y^{-3/2}\phi(y)dy, \tag{4.1}$$

but the integral (4.1) will diverge at the origin. However, we can represent ϕ as $\phi(y) = \phi(0) + y\psi(y)$, where ψ is finite at the origin and has compact support. From which

$$\Lambda_f(\phi) = \int_0^\infty y^{-3/2}\phi(0)dy + \int_0^\infty \frac{\psi(y)}{y^{1/2}}dy. \tag{4.2}$$

The first of these integrals will in general be divergent, but the second will be finite. Hadamard's approach is to discard the first integral. This amounts to setting

$$\Lambda_f(\phi) = \int_0^\infty y^{-3/2}(\phi(y) - \phi(0))dy, \tag{4.3}$$

and this does define a distribution; see [21] for details. The Laplace transform of $y^{-3/2}$ as a distribution is then defined by

$$\Lambda_f(e^{-sy}) = \int_0^\infty y^{-3/2}(e^{-sy} - 1)dy = -2\sqrt{\pi}\sqrt{s}. \tag{4.4}$$

We denote the distribution Λ_f by $\mathrm{Pf}(y^{-3/2}1_+(y))$. Here $1_+(y)$ is the Heaviside function and Pf stands for pseudo function. We have the following useful result.

Proposition 4.1. *The Laplace transform of the right sided distributions* $\mathrm{Pf}(y^a 1_+(y))$ *satisfies*

$$\mathcal{L}[(\mathrm{Pf}(y^a 1_+(y))](s) = \frac{\Gamma(a+1)}{s^{a+1}}, \ \Re(s) > 0, a \neq -1, -2, -3, ... \tag{4.5}$$

For $a = -1$,

$$\mathcal{L}[(\mathrm{Pf}(y^{-1} 1_+(y))](s) = -\ln s - \gamma, \Re(s) > 0. \tag{4.6}$$

In general, for $k = 2, 3, 4, ...$

$$\mathcal{L}[(\mathrm{Pf}(y^{-k} 1_+(y))](s) = -\frac{(-s)^{k-1}}{(k-1)!}\left(\ln s + \gamma - \sum_{n=1}^{k-1}\frac{1}{n}\right), \Re(s) > 0. \tag{4.7}$$

Here γ *is Euler's constant. For* $k = 0, 1, 2, ...$

$$\mathcal{L}[\delta^{(k)}(y)] = y^k. \tag{4.8}$$

Proof. See Chapter 8 of [21]. □

We will make use of this result to invert some Laplace transforms which arise from Theorem 2.3. The most common Laplace transforms we need to invert are of the form $\mathcal{L}^{-1}[\lambda^a e^{k/\lambda}]$.

Proposition 4.2. *The following Laplace transform inversion formula holds when* n *is a non-negative integer:*

$$\mathcal{L}^{-1}[\lambda^n e^{\frac{k}{\lambda}}] = \sum_{l=0}^{n}\frac{k^l}{l!}\delta^{(n-l)}(y) + \left(\frac{k}{y}\right)^{\frac{n+1}{2}} I_{n+1}\left(2\sqrt{ky}\right). \tag{4.9}$$

If $n - 1 < \mu < n$ *then*

$$\mathcal{L}^{-1}[\lambda^\mu e^{\frac{k}{\lambda}}] = \left(\frac{k}{y}\right)^{\frac{\mu+1}{2}} I_{-\mu-1}(2\sqrt{ky}) + \sum_{l=0}^{n-1}\frac{k^l}{l!}\frac{\mathrm{Pf}(y^{-\mu-1-l}1_+(y))}{\Gamma(-\mu-l)} - \sum_{j=0}^{n-1}\frac{k^j y^{j-\mu-1}}{j!\Gamma(j-\mu)}.$$

Proof. We have when n is a positive integer

$$\lambda^n e^{\frac{k}{\lambda}} = \lambda^n + k\lambda^{n-1} + \cdots + \frac{k^n}{n!} + \sum_{j=0}^{\infty}\frac{k^{n+j+1}}{(n+j+1)!\lambda^{j+1}}.$$

Taking the inverse Laplace transform termwise gives

$$\begin{aligned}\mathcal{L}^{-1}[\lambda^n e^{\frac{k}{\lambda}}] &= \sum_{l=0}^{n} \frac{k^l}{l!}\delta^{(n-l)}(y) + \sum_{j=0}^{\infty} \frac{k^{n+j+1}y^j}{j!(n+j+1)!} \\ &= \sum_{l=0}^{n} \frac{k^l}{l!}\delta^{(n-l)}(y) + \left(\frac{k}{y}\right)^{\frac{n+1}{2}} I_{n+1}\left(2\sqrt{ky}\right).\end{aligned}$$

The proof of the second case is similar. □

4.1. **Families of Fundamental Solutions.** If we take the constant stationary solution of (1.2), then we will always obtain a fundamental solution which integrates to one. If we take a second non-constant stationary solution, we will obtain a second fundamental solution, which is not in general a probability density. Now suppose that $u = u_1(x)$ is a nonconstant stationary solution, then from the stationary solution $u_{a,b}(x) = a + bu_1(x)$ we obtain a family of fundamental solutions, when at least one of a or b is nonzero. This family will contain the probability density. The other fundamental solutions may be related to the behaviour of the diffusion at zero. The range of possibilities is quite substantial and an exhaustive treatment is not possible here. Let us discuss a typical family of fundamental solutions where such issues arise.

4.1.1. *Squared Bessel Processes.* The most general stationary solution of the PDE $u_t = 2xu_{xx} + \alpha u_x$ is $u_{a,b} = a + bx^{1-\frac{\alpha}{2}}$ for $\alpha \neq 2$ and $u_{a,b}(x) = a + b\ln x$ for $\alpha = 2$. The stationary solution $u_{a,0}(x)$ will give the transition density for a squared Bessel process in the case when $\alpha \geq 0$. For $0 < \alpha < 2$ we have from Theorem 2.3 the Laplace transform

$$\int_0^\infty e^{-\lambda y}(a + by^{1-\frac{\alpha}{2}})p_{a,b}(t,x,y)dy = U_\lambda(t,x), \tag{4.10}$$

where

$$U_\lambda(t,x) = \frac{1}{(1+2\lambda t)^{\frac{\alpha}{2}}} \exp\left(-\frac{\lambda x}{1+2\lambda t}\right)\left(a + \frac{bx^{1-\frac{\alpha}{2}}}{(1+2\lambda t)^{2-\alpha}}\right). \tag{4.11}$$

Inverting the Laplace transform we obtain the family of fundamental solutions

$$p_{a,b}(t,x,y) = \frac{\left(\frac{x}{y}\right)^{\frac{2-\alpha}{4}} e^{-\frac{x+y}{2t}}}{2(a+by^{1-\frac{\alpha}{2}})t}\left[aI_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{xy}}{t}\right) + by^{1-\frac{\alpha}{2}}I_{1-\frac{\alpha}{2}}\left(\frac{\sqrt{xy}}{t}\right)\right],$$

where at least one of a, b is nonzero. The transition density is given by $p_{a,0}(t,x,y)$, whereas $p_{0,b}(t,x,y)$ is not a density. We have

$$\int_0^\infty p_{0,b}(t,x,y)dy = 1 - \frac{\Gamma\left(1-\frac{\alpha}{2}, \frac{x}{2t}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}. \tag{4.12}$$

Here $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$. Note that for $x > 0$, $\int_0^\infty p_{0,b}(t,x,y)dy \to 0$ as $t \to \infty$.

The probabilistic interpretation of the fundamental solutions that are not transition densities is a problem that has to be resolved for the particular case being studied. The boundary conditions for the diffusion must play a crucial role in any interpretation, and we must proceed on a case by case basis. In the current situation, the fundamental solution $p_{0,b}(t,x,y)$ may be interpreted as giving the probability that $X_t \le y$ given that X has not yet hit zero. This follows from a construction due to Hulley and Platen [13]. Let $\tau_0 = \inf\{t \ge 0 : X_t = 0\}$. Then

$$\mathbb{P}\left(X_t \le y\right) = \mathbb{P}\left(X_t \le y : \tau_0 > t\right) + \mathbb{P}\left(X_t \le y : \tau_0 \le t\right). \tag{4.13}$$

If $\overline{p}_0(t,x)$ is the transition probability density of the process hitting zero, starting at x, for $0 < \alpha < 2$, then by the Markov property

$$\mathbb{P}\left(X_t \le y\right) = \mathbb{P}\left(X_t :\le y : \tau_0 > t\right) + \int_0^y \int_0^t \overline{p}_0(s,x) p_{a,0}(t-s,0,y')dsdy'. \tag{4.14}$$

Göing-Jaeschke and Yor proved in [9] that

$$\overline{p}_0(t,x) = \frac{1}{t\Gamma(1-\frac{\alpha}{2})}\left(\frac{x}{2t}\right)^{1-\frac{\alpha}{2}} e^{-\frac{x}{2t}}, \tag{4.15}$$

from which Hulley and Platen established the identity

$$p_{a,0}(t,x,y) = p_{0,b}(t,x,y) + \int_0^t \overline{p}_0(s,x)p_{a,0}(t-s,0,y)ds. \tag{4.16}$$

From which it follows that

$$\mathbb{P}\left(X_t \le y : \tau_0 > t\right) = \int_0^y p_{0,b}(t,x,y')dy'. \tag{4.17}$$

It is interesting to note that for certain squared Bessel processes of higher dimensions, the second fundamental solution can also be a probability density.

Example 4.1. Consider a squared Bessel process of dimension four. Take the stationary solution $u_0(x) = a + \frac{b}{x}$. Then

$$\int_0^\infty e^{-\lambda y}(a + \frac{b}{y})p_{a,b}(t,x,y)dy = \exp\left(-\frac{\lambda x}{1+2\lambda t}\right)\left(\frac{a}{(1+2\lambda t)^2} + \frac{b}{x}\right),$$

and inversion of this Laplace transform gives

$$p_{a,b}(t,x,y) = \frac{1}{2t}\sqrt{\frac{y}{x}}e^{-\frac{x+y}{2t}} I_1\left(\frac{\sqrt{xy}}{t}\right) + \frac{by}{x(ay+b)}e^{-\frac{x+y}{2t}}\delta(y). \tag{4.18}$$

Once more $p_{a,0}(t,x,y)$ gives the classical transition density. Interestingly however, for any nonzero a, b we have

$$\int_0^\infty p_{a,b}(t,x,y)dy = 1. \tag{4.19}$$

This family of fundamental solutions is also positive, but the distributional term is divergent as $x \to 0$. However, a squared Bessel process of dimension four, which starts at $x > 0$, never reaches zero. The process can start at zero, in which case it is instantaneously reflected from the origin and never returns to zero. To interpret the fundamental solution $p_{a,b}$, we need to consider the dynamics of the actual process, which in this case will allow us to exclude the term involving the Dirac delta.

This family of fundamental solutions has the following interesting feature. Suppose that we denote expectation with respect to $p_{a,b}$ by $\mathbb{E}_{p_{a,b}}$. Then

$$\mathbb{E}_{p_{a,b}}\left[a + \frac{b}{X_t}\bigg| X_0 = x\right] = \int_0^\infty (a + \frac{b}{y})p_{a,b}(t,x,y)dy = a + \frac{b}{x}. \tag{4.20}$$

The process $a + \frac{b}{X_t}$ is a strict local martingale. However, the distributional term in $p_{a,b}$ provides the "missing probability mass" necessary to turn it into a true martingale under a change of measure. In some way this can be interpreted as "formally" using a measure change, where the Radon-Nikodym derivative is not a martingale. Such measure change does not relate equivalent probability measures. It only links measures. Such a situation arises, for instance, when interpreting Platen's minimal market model under a putative risk neutral measure, see [17].

Suppose further that $\psi(y) = \frac{1}{y}\phi(y)$ where ϕ has suitable decay at infinity and $\phi(0) \neq 0$. Then ψ is in the domain of the integral operators defined by integration against $p_{a,b}$ and

$$u(x,t) = \int_0^\infty \psi(y)p_{a,b}(t,x,y)dy = u_2(x,t) + \frac{\phi(0)}{x}e^{-\frac{x}{2t}}, \tag{4.21}$$

where $u_2(x,t) = \int_0^\infty \psi(y)p_{a,0}(t,x,y)dy$. Both u_2 and u solve the Cauchy problem for $u_t = 2xu_{xx} + 4u_x$, $u(x,0) = \psi(x)$ on $x > 0$. However, u_2 is not continuous at the origin. It is usual to require the transition density to yield continuous solutions on the whole of $[0,\infty)$ for continuous initial data. From this we conclude that $p_{a,b}, b \neq 0$ is not the transition probability density. This does not however exclude the possibility that in some probabilistic dynamics, in which different boundary behaviour is desired, this new family of fundamental solutions may play a role.

The fact that the second stationary solution yields another density is not unique to the $n = 4$ case.

Proposition 4.3. *Consider the PDE*

$$u_t = 2xu_{xx} + 2nu_x. \tag{4.22}$$

If $n = 2, 3, 4, \ldots.,$then using the stationary solution $u_0(x) = a + bx^{1-n}$ in Theorem 2.3 produces a family of fundamental solutions $p_{a,b}(t, x, y)$ with $\int_0^\infty p_{a,b}(t, x, y)dy = 1$, whenever at least one of a, b is nonzero. Moreover, if $\mathbb{E}_{p_{a,b}}$ denotes expectation taken with respect to this fundamental solution, then the process $Y = \{a + bX_t^{1-n} : t \geq 0\}$, where X is the squared Bessel process of dimension $2n$, is a true martingale with respect to this fundamental solution. That is

$$\mathbb{E}_q \left[a + bX_t^{1-n} \middle| X_0 = x \right] = \int_0^\infty u(y)p_{a,b}(t, x, y)dy = a + bx^{1-n}.$$

Proof. If $u_0(x) = a + bx^{1-n}$ then Theorem 2.3 gives

$$U_\lambda(t, x) = e^{-\frac{\lambda x}{1+2\lambda t}} \left(\frac{a}{(1 + 2\lambda t)^n} + bx^{1-n} (1 + 2\lambda t)^{n-2} \right). \tag{4.23}$$

The fundamental solution obtained from this is

$$p_{a,b}(t, x, y) = q(t, x, y) + \frac{b(2t)^{n-2}x^{1-n}}{a + by^{1-n}} e^{-\frac{x+y}{2t}} \sum_{l=0}^{n-2} \frac{x^l \delta^{(n-2-l)}(y)}{(2t)^{2l} l!},$$

where $q(t, x, y) = \frac{1}{2t} \left(\frac{y}{x}\right)^{\frac{n-1}{2}} e^{-\frac{x+y}{2t}} I_{n-1} \left(\frac{2\sqrt{xy}}{t}\right)$. From the definition of $\delta^{(k)}$ it is easy to check that $\int_0^\infty p_{a,b}(t, x, y)dy = 1$. To complete the proof we note that $U_0(t, x) = u_0(x)$ and

$$U_\lambda(t, x) = \int_0^\infty e^{-\lambda y} u_0(y)p_{a,b}(t, x, y)dy. \tag{4.24}$$

Which implies that $\int_0^\infty u_0(y)p_{a,b}(t, x, y)dy = u_0(x)$. Since Y is integrable with respect to $p_{a,b}$, then Y is a martingale under $p_{a,b}$. $\square$

If the dimension α of the squared Bessel process is not an even integer, then the situation is different. For example, if $\alpha = 5$ the general stationary solution $u_0(x) = a + bx^{-3/2}$ of $u_t = 2xu_{xx} + 5u_x$ yields a family of fundamental solutions with

$$p_{0,b}(t, x, y) = e^{-\frac{x+y}{2\,t}} \frac{\sqrt{xy}\sinh(\frac{\sqrt{xy}}{t}) - t\,\cosh(\frac{\sqrt{xy}}{t})}{\sqrt{2\,\pi t}\; x^{\frac{3}{2}}}. \tag{4.25}$$

Here

$$\int_0^\infty p_{0,b}(t, x, y)dy = 1 - (t - x)\sqrt{\frac{2t}{\pi x^3}} e^{-\frac{x}{2t}} - \operatorname{erfc}\left(\sqrt{\frac{x}{2t}}\right). \tag{4.26}$$

So $p_{0,b}$ is not a probability density, nor is $p_{0,b}$ everywhere positive.

We can also handle the case where α in the PDE $u_t = 2u_{xx} + \alpha u_x$ is negative.

Example 4.2. We study the PDE $u_t = 2xu_{xx} - 2u_x, x \geq 0$. Taking the stationary solution $u_0(x) = 1$ gives the Laplace transform

$$U_\lambda(x,t) = (1+2\lambda t)\exp\left(-\frac{\lambda x}{1+2\lambda t}\right), \tag{4.27}$$

and this may be inverted to produce the fundamental solution

$$p(t,x,y) = \frac{1}{2t}e^{-\frac{x+y}{2t}}\frac{x}{y}I_2\left(\frac{\sqrt{xy}}{t}\right) + \frac{x}{2t}e^{-\frac{x+y}{2t}}\delta(y) + 2te^{-\frac{x+y}{2t}}\delta'(y).$$

If ϕ is a test function with the property that both $\phi(0)$ and $\phi'(0)$ exist, then

$$\begin{aligned}\int_0^\infty \phi(y)p(t,x,y)dy &= \int_0^\infty \frac{1}{2t}e^{-\frac{x+y}{2t}}\frac{x}{y}I_2\left(\frac{\sqrt{xy}}{t}\right)\phi(y)dy \\ &+ \left(1+\frac{x}{2t}\right)e^{-\frac{x}{2t}}\phi(0) - 2te^{-\frac{x+y}{2t}}\phi'(0).\end{aligned} \tag{4.28}$$

Notice that if $\phi'(0) = 0$, then $u(t,x) = \int_0^\infty \phi(y)p(t,x,y)dy$ satisfies the boundary condition $u_x(t,0) = 0$. Now we consider $q(t,x,y) = p(t,x,y) - 2te^{-\frac{x+y}{2t}}\delta'(y)$. This is also a fundamental solution and one can easily show that $\int_0^\infty q(t,x,y)dy = 1$. For the solution $u(t,x) = \int_0^\infty \phi(y)q(t,x,y)dy$, we also have $u_x(t,0) = 0$. This suggests that these densities are associated with the -2 dimensional squared Bessel process, reflected at zero.

Next we consider what happens if we take the second stationary solution $u_0(x) = x^2$. This gives

$$U_\lambda(x,t) = \frac{x^2}{(1+2\lambda t)^3}\exp\left(-\frac{\lambda x}{1+2\lambda t}\right). \tag{4.29}$$

From which we obtain a third fundamental solution

$$p_2(t,x,y) = \frac{1}{2t}\frac{x}{y}e^{-\frac{x+y}{2t}}I_2\left(\frac{\sqrt{xy}}{t}\right). \tag{4.30}$$

Observe that $p_2(t,y,x)$ is the transition probability density for a squared Bessel process of dimension six. This is consistent with a well known identity relating the transition densities of squared Bessel processes of different dimensions, [18].

Squared Bessel processes are studied in considerable depth in [18], and many properties of these processes can be understood in terms of the above families of fundamental solutions. This paper provides the means to construct these types of families of fundamental solutions for many different processes. It is very likely that the presence of these families will reveal deeper probabilistic properties of the associated processes. We hope that our work stimulates research in this direction.

Finally we present some examples of the complex types of PDEs which can arise with distributional fundamental solutions. The following two cases are all solutions of the first Riccati equation. We do not attempt here to interpret these solutions probabilistically.

Example 4.3. Consider the following drift in the case $A=\frac{1}{2}, B=\frac{5}{8}$. A solution of the Riccati equation is $f(x)=-\frac{(1+2x+\sqrt{x})}{2(1+\sqrt{x})}$. An application of Theorem 2.3 gives us the Laplace transform

$$U_\lambda(t,x)=\frac{1}{1+\sqrt{x}}\exp\left\{-\frac{\lambda(\sqrt{x}-\frac{1}{2}t)^2)}{1+\lambda t}\right\}\left[\sqrt{1+\lambda t}+\frac{\sqrt{x}}{\sqrt{1+\lambda t}}\right].$$

Let $k=(\sqrt{x}-\frac{1}{2}t)^2/t^2$, then $k\geq 0$. To compute the inverse Laplace transform we use $\mathcal{L}^{-1}[\frac{1}{\sqrt{\lambda}}e^{k/\lambda}]=\frac{\cosh(2\sqrt{ky})}{\sqrt{\pi y}}$, and

$$\mathcal{L}^{-1}[\sqrt{\lambda}e^{k/\lambda}]=\frac{-\mathrm{Pf}(y^{-3/2}1_+(y))}{2\sqrt{\pi}}+\frac{2\sqrt{ky}\sinh(2\sqrt{ky})-\cosh(2\sqrt{ky})+1}{2\sqrt{\pi y^3}},$$

to obtain

$$p(t,x,y)=\frac{\sqrt{x}e^{\sqrt{x}}}{\sqrt{t}(1+\sqrt{x})}e^{-\frac{(x+y)}{t}-\frac{1}{4}t}\frac{\cosh(2\sqrt{ky})}{\sqrt{\pi y}}+\frac{\sqrt{t}e^{\sqrt{x}}}{(1+\sqrt{x})}e^{-\frac{(x+y)}{t}-\frac{1}{4}t}\times$$
$$\left[\frac{-\mathrm{Pf}(y^{-3/2}1_+(y))}{2\sqrt{\pi}}+\frac{2\sqrt{ky}\sinh(2\sqrt{ky})-\cosh(2\sqrt{ky})+1}{2\sqrt{\pi y^3}}\right]. \quad (4.31)$$

Example 4.4. A drift function in which we have a free parameter can be obtained by taking $B=\frac{5}{8}$ and letting $A>0$ remain arbitrary. This produces $f(x)=\frac{2Ax}{\sqrt{2Ax}\tanh(\sqrt{2Ax})-1}-\frac{1}{2}$. Theorem 2.3 immediately gives

$$U_\lambda(t,x)=\frac{\sqrt{1+\lambda t}\left(\frac{\sqrt{2Ax}\sinh(\frac{\sqrt{2Ax}}{1+\lambda t})}{1+\lambda t}-\cosh(\frac{\sqrt{2Ax}}{1+\lambda t})\right)}{\sqrt{2Ax}\sinh(\sqrt{2Ax})-\cosh(\sqrt{2Ax})}e^{-\frac{x+\frac{At^2}{2}\lambda}{1+\lambda t}}. \quad (4.32)$$

Setting $k_1=\frac{(\sqrt{x}+\frac{A}{2}t)^2}{t^2}$, $k_2=\frac{(\sqrt{x}-\frac{A}{2}t)^2}{t^2}$ and it is clear that $k_1>0, k_2>0$. We then have

$$p(t,x,y)=\frac{e^{-\frac{(x+y)}{t}-\frac{At}{2}}}{\sqrt{2Ax}\sinh\sqrt{2Ax}-\cosh(\sqrt{2Ax})}\times$$
$$\left\{\sqrt{\frac{2Ax}{t}}\frac{\cosh(2\sqrt{k_1y})-\cosh(2\sqrt{k_2y})}{2\sqrt{\pi y}}+\sqrt{\frac{t}{\pi}}\mathrm{Pf}(y^{-3/2}1_+(y))-\right.$$
$$\frac{(\sqrt{k_1y}\sinh(2\sqrt{k_1y})+\sqrt{k_2y}\sinh(2\sqrt{k_2y}))+1}{2\sqrt{\pi y^3}}-$$
$$\left.\frac{\cosh(2\sqrt{k_1y})-\cosh(2\sqrt{k_2y})}{4\sqrt{\pi y^3}}\right\}.$$

5. Some Applications

Using the results we have obtained, we can compute the Laplace transforms of a number of interesting processes. We show how to obtain the Laplace transform of the transition density of the process $Y = 1/X$ when $X = \{X_t : t \geq 0\}$ satisfies $dX_t = f(X_t)dt + \sqrt{2\sigma X_t}dW_t$ and $\sigma x f' - \sigma f + \frac{1}{2}f^2 = Ax + B$. We could, of course, use the Ito formula, which will ultimately yield a generalized Lalplace transform, but the following approach is illuminating and avoids stochastic calculus. The key is the following lemma.

Lemma 5.1. *Let $x > 0$, $\mu > 0$ and $J_1(y)$ be a Bessel function of the first kind. Then*

$$\exp\left(-\frac{\mu}{x}\right) = 1 - \int_0^\infty J_1(y)\exp\left(-\frac{y^2}{4\mu}x\right)dy. \tag{5.1}$$

Proof. We have the Taylor series expansion

$$\exp\left(-\frac{\mu}{x}\right) = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{\mu^n}{n!x^n}, \quad x \neq 0. \tag{5.2}$$

We also have

$$J_1(y) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{y}{2}\right)^{2n-1}}{n!(n-1)!}. \tag{5.3}$$

Using the result $\int_0^\infty y^{2n-1}e^{-\beta y^2} = \frac{(n-1)!}{2\beta^n}$ we find that

$$\int_0^\infty J_1(y)\exp\left(-\frac{y^2}{4\mu}x\right)dy = -\sum_{n=1}^{\infty}(-1)^n\frac{\mu^n}{n!x^n}, \tag{5.4}$$

and the lemma follows. □

Corollary 5.2. *Let $X = \{X_t : t \geq 0\}$ be a diffusion satisfying the SDE*

$$dX_t = f(X_t)dt + \sqrt{2\sigma X_t}dW_t, \ X_0 = x > 0.$$

Let $\tau = \inf\{t : X_t = 0\}$. Suppose that $\sigma x f' - \sigma f + \frac{1}{2}f^2 = Ax + B$. Then for $t < \tau$, $\mu > 0$

$$\mathbb{E}_x\left[e^{-\frac{\mu}{X_t}}\right] = 1 - \int_0^\infty J_1(y)e^{-\frac{y^2(x+\frac{1}{2}At^2)}{4\mu+y^2\sigma t}+\frac{1}{2\sigma}\left(F\left(\frac{16\mu^2 x}{(4\mu+y^2\sigma t)^2}\right)-F(x)\right)}dy, \tag{5.5}$$

provided the integral converges. As usual $F'(x) = \frac{f(x)}{x}$.

Proof. The proof is a consequence of the previous lemma and Corollary 2.10. We have

$$\exp\left(-\frac{\mu}{X_t}\right) = 1 - \int_0^\infty J_1(y)\exp\left(-\frac{y^2}{4\mu}X_t\right)dy. \tag{5.6}$$

Given that the integral converges we use Fubini's theorem to obtain

$$\mathbb{E}_x\left[\exp\left(-\frac{\mu}{X_t}\right)\right] = 1 - \int_0^\infty J_1(y)\mathbb{E}_x\left[\exp\left(-\frac{y^2}{4\mu}X_t\right)\right]dy. \qquad (5.7)$$

Now we use the given expression for $E[e^{-\lambda X_t}]$. □

It is possible to choose drifts which satisfy the Riccati equation for which the integral (5.5) diverges. However for many examples the integrals are convergent. Clearly we may obtain several other similar results using Corollary 2.10 and the results giving generalized Laplace transforms.

Example 5.1. Let $dX_t = \frac{aX_t}{1+\frac{1}{2}aX_t}dt + \sqrt{2X_t}dW_t,\ X_0 > 0$. Then the integral (5.5) is convergent and for $t < \tau$,

$$\mathbb{E}_x\left[e^{-\frac{\mu}{X_t}}\right] = 1 - \frac{1}{2+ax}\int_0^\infty \left(2 + \frac{16\mu^2 ax}{(4\mu+y^2t)^2}\right)J_1(y)e^{-\frac{y^2x}{4\mu+y^2t}}dy.$$

We may establish other such results. For example, the calculation of certain moments for our processes is straightforward.

Corollary 5.3. *Let $X = \{X_t : t \geq 0\}$ be a diffusion which satisfies the SDE $dX_t = \sqrt{2\sigma X_t}dW_t + f(X_t)dt, X_0 = x > 0$, with*

$$\sigma x f' - f + 1/2 f^2 = Ax + B.$$

If the following integrals converge, then for $t < \tau = \inf\{t : X_t = 0\}$ and $\alpha > 0$ we have

$$\mathbb{E}_x\left[(X_t)^{-\alpha}e^{-\lambda X_t}\right] = \frac{e^{-\frac{1}{2\sigma}F(x)}}{\Gamma(\alpha)}\times$$
$$\int_0^\infty y^{\alpha-1}\exp\left\{-\frac{(\lambda+y)(x+\frac{1}{2}At^2)}{1+(\lambda+y)\sigma t} + \frac{1}{2}F\left(\frac{x}{(1+(\lambda+y)\sigma t)^2}\right)\right\}dy$$

and

$$\mathbb{E}_x\left[U(a,b,X_t)e^{-\lambda X_t}\right] = \frac{e^{-\frac{1}{2\sigma}F(x)}}{\Gamma(\alpha)}\int_0^\infty y^{a-1}(1+y)^{b-a+1}\times$$
$$\exp\left\{-\frac{(\lambda+y)(x+\frac{1}{2}At^2)}{1+(\lambda+y)\sigma t} + \frac{1}{2}F\left(\frac{x}{(1+(\lambda+y)\sigma t)^2}\right)\right\}dy,$$

where U is a confluent hypergeometric function of the second kind,$F'(x) = \frac{f(x)}{x}$ and $\lambda \geq 0$.

If f is a solution of the Riccati equation $\sigma x f' - \sigma f + \frac{1}{2}f^2 = \frac{2}{3}Ax^{3/2} + Cx - \frac{3}{8}$, then

$$\mathbb{E}_x\left[(X_t)^{-\alpha}e^{-\lambda X_t}\right] = \frac{e^{-\frac{1}{2\sigma}F(x)}}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}G(\lambda+y,x,\frac{t}{\sigma})dy$$

and

$$\mathbb{E}_x\left[U(a,b,X_t)e^{-\lambda X_t}\right] = \frac{e^{-\frac{1}{2\sigma}F(x)}}{\Gamma(\alpha)}\int_0^\infty y^{a-1}(1+y)^{b-a+1}G(\lambda+y,x,\frac{t}{\sigma})dy,$$

where $G(\lambda, x, \frac{t}{\sigma})$ is given by (2.15). Again the result is valid whenever the integrals are convergent.

Proof. We use the representation $\frac{1}{x^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-xy}dy$. We therefore have

$$\mathbb{E}_x\left[(X_t)^{-\alpha}e^{-\lambda X_t}\right] = \frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}E_x[e^{-(\lambda+y)X_t}]dy. \tag{5.8}$$

We obtain $\mathbb{E}_x[e^{-(\lambda+y)X_t}]$ from Corollary 2.10 and (2.15). For the relationship involving the hypergeometric functions, we use the representation $U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-yx}y^{a-1}(1+y)^{b-a+1}dy$. □

Again other results of this nature may be established from Theorems 2.5 and 2.7.

5.1. **First Passage Times.** We define the first passage time to z for the process X_t by

$$\tau_z = \inf\{t : X_t = z\}. \tag{5.9}$$

We wish to characterize τ_z by computing its Laplace transform. Let $p(t,x,y)$ be the density for X_t. Then we define

$$G_\alpha(x,y) = \int_0^\infty e^{-\alpha t}p(t,x,y)dt. \tag{5.10}$$

We have

$$\mathbb{E}_x\left[e^{-\alpha\tau_z}\right] = \frac{G_\alpha(x,y)}{G_\alpha(z,y)}, \tag{5.11}$$

see [3]. Using the transition probability densities we may easily obtain the Laplace transforms of the first hitting times of the processes.

Example 5.2. Consider the SDE $dX_t = \left(\frac{1}{2}+\sqrt{X_t}\right)dt + \sqrt{2X_t}dW_t$. The transition density for X is

$$p(t,x,y) = \frac{e^{-\sqrt{x}}}{\sqrt{\pi y t}}\cosh\left(\frac{(t+2\sqrt{x})\sqrt{y}}{t}\right)\exp\left\{-\frac{(x+y)}{t} - \frac{1}{4}t\right\}.$$

From this we find

$$\begin{aligned} G_\alpha(x,y) &= \int_0^\infty e^{-\alpha t}p(t,x,y)dt \\ &= \frac{e^{-\sqrt{x}}\left(e^{-\sqrt{y}-\sqrt{1+4\alpha}(\sqrt{x}+\sqrt{y})} + e^{\sqrt{y}-\sqrt{1+4\alpha}|\sqrt{x}-\sqrt{y}|}\right)}{\sqrt{y}\sqrt{1+4\alpha}}. \end{aligned} \tag{5.12}$$

So, if $z \geq x$ then we find that $E_x\left[e^{-\alpha\tau_z}\right] = e^{\left(\sqrt{x}-\sqrt{z}\right)\left(\sqrt{4\alpha+1}-1\right)}$. Inverting the Laplace transform we find the transition probability density for $\tau_z, z \geq x$ to be

$$p_t(x,z) = \frac{e^{-\frac{(t+2\sqrt{x}-2\sqrt{z})^2}{4t}}\left(\sqrt{z}-\sqrt{x}\right)}{\sqrt{\pi}t^{3/2}}. \tag{5.13}$$

If $|\sqrt{1+2B/\sigma^2}| < 1$, we have the general result

$$p(t,x,y) = \frac{e^{-\frac{x+y}{\sigma t}-\frac{At}{2\sigma}}}{\sigma t}\frac{c_1 I_\mu\left(\frac{2\sqrt{xy}}{\sigma t}\right)I_\mu\left(\frac{\sqrt{2Ay}}{\sigma}\right)+c_2 I_{-\mu}\left(\frac{2\sqrt{xy}}{\sigma t}\right)I_{-\mu}\left(\frac{\sqrt{2Ay}}{\sigma}\right)}{c_1 I_\mu\left(\frac{\sqrt{2Ax}}{\sigma}\right)+c_2 I_{-\mu}\left(\frac{\sqrt{2Ax}}{\sigma}\right)}.$$

The fact that

$$G_\alpha(x,y) = \int_0^\infty e^{-\alpha t}\frac{1}{t}\exp\left(-\frac{x+y}{\sigma t}\right)I_\mu\left(\frac{2\sqrt{xy}}{\sigma t}\right)dt = 2I_\mu(z_1)K_\mu(z_2)$$

where $z_1 = 2\sqrt{\frac{\alpha}{\sigma}}\min(\sqrt{x},\sqrt{y})$ and $z_2 = 2\sqrt{\frac{\alpha}{\sigma}}\max(\sqrt{x},\sqrt{y})$, can be deduced from formula (6.635.3) of [10]. From this approach, a large number of Laplace transforms of first passage times may be obtained.

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