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Research Paper 274 April 2010

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ISSN 1441-8010 www.qfrc.uts.edu.au

LIE SYMMETRY METHODS FOR MULTIDIMENSIONAL LINEAR, PARABOLIC PDES AND DIFFUSIONS

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Abstract. In this paper we introduce methods based upon Lie symmetry analysis for the construction of explicit fundamental solutions of multidimensional parabolic PDEs. We give applications to the problem of finding transition probability densities for multidimensional diffusions and to representation theory.

1. Introduction

Lie symmetry methods have proven to be very effective for the computation of fundamental solutions for linear PDEs. Though symmetry analysis can be used to obtain fundamental solutions for elliptic and parabolic problems, they are particularly useful for parabolic PDEs on the real line. In this situation, a number of different methods are available, each of which has its own strengths and limitations.

For parabolic problems in one space dimension, the papers [8] and [13] show how one may construct classical integral transforms of the desired fundamental solutions, using an $SL(2, \mathbb{R})$ or Heisenberg group symmetry. In this framework, the group parameter is regarded as a transform parameter. In one dimension there is no problem with this. There is only one space variable, so a suitable one parameter subgroup of the symmetry group can yield an appropriate integral transform.

Moving to higher dimensions, we would also like to be able to compute fundamental solutions by symmetry. There is however a major obstacle. To put it simply, there is usually not enough symmetry available to compute a fundamental solution by the established symmetry methods, except in certain special cases. For example, the paper [21] shows

Date: April 2010, *Note:* Some results in this paper are taken from Lennox's forthcoming PhD thesis.

Key words and phrases. Lie Symmetry Groups, Fundamental Solutions, Transition Densities, Representation Theory.

that group invariant solution techniques will yield fundamental solutions only for certain classes of problems, where the symmetry group has dimension greater than four. Even in those cases where group invariance works, the fundamental solutions obtained could easily be constructed as products of one dimensional fundamental solutions. This is because for these cases the PDE is of the form $u_t = (L_x + L_y)u$, with $[L_x, L_y] = 0$, so that $e^{t(L_x+L_y)} = e^{tL_x}e^{tL_y}$.

Unfortunately, equations of the form $u_t = \Delta u + A(x)u$ on $\mathbb{R}^n$ for $n \geq 2$ typically only have $SL(2,\mathbb{R})$ as the symmetry group. So we only have three one parameter subgroups to work with. Our aim then is to answer the following question: *Can we obtain a fundamental solution by Lie symmetry analysis if the symmetry group at least contains $SL(2,\mathbb{R})$?* We will show that the answer is yes, if we also use the linearity of the PDE. In two space dimensions the problem can be regarded as completely solved and we can obtain very useful expressions in n dimensions.

In the next section we discuss the symmetries that we need and the background material. In Section 3 we consider some Fourier transform results in the case when the PDE also has Heisenberg group symmetries. In Sections 4 and 5 we discuss methods for finding fundamental solutions in the case when we only have $SL(2,\mathbb{R})$ symmetries and Section 6 contains applications to representation theory. Examples are given throughout to illustrate the results.

The authors wish to thank the Australian Research Council, St George's Bank Ltd and Professor Eckhard Platen.

2. Symmetries of the PDE $u_t = \Delta u + A(x)u$

We are interested in this paper in PDES of the form

$$u_t = \Delta u + 2\nabla\phi \cdot \nabla u + B(x)u, \ x \in \Omega \subseteq \mathbb{R}^n. \tag{2.1}$$

The coefficients $2\phi_{x_1}, 2\phi_{x_2}$ etcetera are called the drift functions. Equations of this form are of importance in the theory of stochastic processes, since the transition probability density for the processes governed by the stochastic differential equations (SDE)

$$dX_t^i = 2\phi_{x_i}(X_t^1, ..., X_t^n)dt + \sqrt{2}dW_t^i, i = 1, ..., n \tag{2.2}$$

is a fundamental solution of the Kolmogorov forward equation (2.1) for $B = 0$.

Our starting point will be the equation

$$u_t = \Delta u + A(x)u, \ x \in \Omega \subseteq \mathbb{R}^n. \tag{2.3}$$

For $n = 2$ the Lie symmetries were calculated by Finkel, [16]. An equation of the form (2.3) can be transformed into one of the form (2.1) by letting $u = e^{\phi}v$. This change of variables leads to the equation

$$v_t = \Delta v + 2\nabla\phi \cdot \nabla v + (\Delta\phi + |\nabla\phi|^2 + A(x))v. \tag{2.4}$$

If we suppose that

$$\Delta\phi + |\nabla\phi|^2 + A(x) = B(x), \tag{2.5}$$

then we are lead to the PDE $v_t = \Delta v + 2\nabla\phi \cdot \nabla v + B(x)v$.

The quasilinear equation (2.5) can obviously be linearised by letting $w = e^{\phi}$. The result is that $\Delta w + A(x)w = B(x)w$. Thus finding stationary solutions of (2.3) is equivalent to finding drift functions for (2.1) with $B = 0$. We also observe that any equation of the form (2.1) with $B = 0$ and $\Delta\phi + |\nabla\phi|^2 = 0$, can be transformed to the n dimensional heat equation.

The particular classes of functions A that we are interested in are of the form

$$A(x) = \frac{1}{x_1^2}k\left(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}\right) + \sum_{i=1}^{n}(c_i x_i^2 + a_i x_i) + E, \tag{2.6}$$

where k is an arbitrary continuous function and $c_1, ..., c_n, , a_1, ..., a_n$ and E are arbitrary constants. Actually we can take $E = 0$ without loss of generality, by letting $v = e^{Et}u$ in the equation. If A is of this form then (2.3) has nontrivial symmetries. In fact we have the following simple result.

Proposition 2.1. *Let u be a solution of (2.3) and let*

$$A(x) = \frac{1}{x_1^2}k\left(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}\right),$$

for some function k. Then for ϵ sufficiently small, so is

$$\tilde{u}_\epsilon(x_1, ..., x_n, t) = \frac{\exp\left(-\frac{\epsilon\|x\|^2}{1+4\epsilon t}\right)}{(1+4\epsilon t)^{\frac{n}{2}}} u\left(\frac{x_1}{1+4\epsilon t}, ..., \frac{x_n}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right).$$

Proof. Applying Lie's algorithm to the PDE shows that it has an infinitesimal symmetry $\mathbf{v} = \sum_{i=1}^n 4x_i t\partial_{x_i} + 4t^2\partial_t - (\|x\|^2 + 2nt)u\partial_u$. Exponentiating this symmetry completes the proof. $\square$

The next result is immediate.

Corollary 2.2. *Suppose that $\Delta\phi + |\nabla\phi|^2 + \frac{1}{x_1^2}k\left(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}\right) = 0$ and u satisfies $u_t = \Delta u + 2\nabla\phi \cdot \nabla u$. Then for ϵ sufficiently small*

$$\tilde{u}_\epsilon(x_1, ..., x_n, t) = \frac{1}{(1+4\epsilon t)^{\frac{n}{2}}} e^{-\frac{\epsilon\|x\|^2}{1+4\epsilon t} + \phi\left(\frac{x_1}{1+4\epsilon t}, ..., \frac{x_n}{1+4\epsilon t}\right) - \phi(x_1, ..., x_n)}$$
$$\times\, u\left(\frac{x_1}{1+4\epsilon t}, ..., \frac{x_n}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right),$$

is also a solution.

The symmetries when the constants c_i, a_i are nonzero are slightly more complicated and we will present the ones we need later.

2.0.1. *Finding Drifts.* For applications, we will need to be able to compute drift functions. There are a number of practical ways in which this may be done. We will illustrate the two dimensional situation. In n dimensions similar techniques can be used.

Suppose we wish to solve the equation

$$\phi_{xx} + \phi_{yy} + \phi_x^2 + \phi_y^2 + \frac{1}{x^2} k\left(\frac{y}{x}\right) = 0. \tag{2.7}$$

Linearisation is one method, but this leads to a second order PDE. A simpler approach if all we wish to find are explicit solutions, is to let $r = y/x$. Then $\phi_x = -\frac{y}{x^2}\phi_r, \phi_y = \frac{1}{x}\phi_r, \phi_{xx} = \frac{y^2}{x^4}\phi_{rr} + 2\frac{y}{x^3}\phi_r$ and $\phi_{yy} = \frac{1}{x^2}\phi_{rr}$. So that $(1+r^2)\phi_{rr} + 2r\phi_r + (1+r^2)\phi_r^2 + k(r) = 0$.

We then let $F(r) = \phi_r$ and obtain the Riccati equation

$$(1+r^2)F' + 2rF + (1+r^2)F^2 + k(r) = 0. \tag{2.8}$$

Every Riccati equation can be linearised, and in this case we set $F = G'/G$. This gives $(1+r^2)G'' + 2rG' + k(r)G = 0$. We then have the drifts given by

$$\phi_x = -\frac{y}{x^3}\frac{G'(y/x)}{G(y/x)}, \ \phi_y = \frac{1}{x}\frac{G'(y/x)}{G(y/x)}. \tag{2.9}$$

For example, $k(r) = -\frac{2}{r^2}$ leads to $G(r) = \frac{c_1}{r} + c_2\frac{\left(r - \tan^{-1}(r)\right)}{r}$, and so

$$\phi_x = \frac{\left(c_1x^2 + c_2yx + c_1y^2 - c_2\left(x^2+y^2\right)\tan^{-1}\left(\frac{y}{x}\right)\right)}{x\left(x^2+y^2\right)\left(c_1x + c_2y - c_2x\tan^{-1}\left(\frac{y}{x}\right)\right)} \tag{2.10}$$

$$\phi_y = -\frac{x\left(c_1x^2 + c_2yx + c_1y^2 - c_2\left(x^2+y^2\right)\tan^{-1}\left(\frac{y}{x}\right)\right)}{y\left(x^2+y^2\right)\left(c_1x + c_2y - c_2x\tan^{-1}\left(\frac{y}{x}\right)\right)}. \tag{2.11}$$

For drift equations of the form $\phi_{xx} + \phi_{yy} + \phi_x^2 + \phi_y^2 = C(x^2+y^2)$ the change of variables $r = x^2 + y^2$ will also lead to a Riccati equation.

3. Multi-dimensional Integral Transforms

Though there are many ways of obtaining fundamental solutions, the integral transform method has very attractive features. See [8] for an extensive discussion of the method for parabolic equations on the line.

In one dimension the situation is simple. In [9] it was proved that if the PDE $u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u$ has at least a four dimensional symmetry group, we can always find a point symmetry which maps a nonzero solution to a Fourier or Laplace transform of a fundamental solution. Typically we use stationary solutions, if they are available. We can obtain transition probability densities for Itô diffusions, as well as multiple fundamental solutions very efficiently via this technique. See [13] for examples.

In higher dimensions, the stationary solutions will be solutions of elliptic PDEs and hence there will now be infinitely many linearly independent stationary solutions. So we can find potentially infinitely many linearly independent fundamental solutions, which could be of great interest in studying properties of these equations.

There are cases where we can actually find integral transforms of fundamental solutions via symmetry. In some cases we are able to use the $SL(2,\mathbb{R})$ symmetries to obtain the desired transforms. Let us consider an interesting example involving a multidimensional process that appears to be new.

Example 3.1 (A Generalisation of Bessel Processes). Let $(B_t^1, ..., B_t^n)$ be a standard n dimensional Brownian motion, and consider the distance from the origin, $Z_t = \sqrt{(B_t^1)^2 + \cdots + (B_t^n)^2}$. Z_t is an n dimensional Bessel process. A standard reference for such processes is Revuz and Yor [24]. The Itô formula shows that

$$dZ_t = \frac{n-1}{2Z_t}dt + dW_t,$$

where W is a standard Brownian motion. Our aim is to consider a generalization of this process, specifically the multidimensional process $X = (X_t^1, ..., X_t^n)$, where

$$dX_t^i = \frac{2a_i}{\sum_{i=1}^n a_i X_t^i}dt + \sqrt{2}dW_t^i,\ X_t > 0,\ i = 1, 2, 3, ..., \tag{3.1}$$

with $a_i > 0, i = 1, 2, ..., n$. We begin with the $n = 2$ case, introducing the function $\phi(x, y) = \ln(ax+by)$ where a and b are constants, which we assume to be real and positive. One easily checks that the Kolmogorov forward equation

$$u_t = u_{xx} + u_{yy} + \frac{2a}{ax+by}u_x + \frac{2b}{ax+by}u_y, \tag{3.2}$$

has a symmetry

$$U_\epsilon(x, y, t) = \frac{e^{-\frac{\epsilon(x^2+y^2)}{1+4\epsilon t} + \phi\left(\frac{x}{1+4\epsilon t}, \frac{y}{1+4\epsilon t}\right) - \phi(x,y)}}{1+4\epsilon t} u\left(\frac{x}{1+4\epsilon t}, \frac{y}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right).$$

We wish to extract a fundamental solution of the PDE from this. Since there is only one group parameter, we cannot immediately realise this as an integral transform. However, if we take $u = 1$, then this symmetry can be split into the product and sum of two transformations. If we let

$$\bar{U}_{\epsilon,\delta}(x, y, t) \quad = \frac{e^{-\frac{y^2\,\delta}{1+4\,t\,\delta} - \frac{x^2\,\epsilon}{1+4\,t\,\epsilon}}\ (a\ (x + 4\,t\,x\,\delta) + b\,y\ (1 + 4\,t\,\epsilon))}{(a\,x + b\,y)\ ((1 + 4\,t\,\delta)\ (1 + 4\,t\,\epsilon))^{\frac{3}{2}}},$$

this is actually a solution of (3.2), and $\bar{U}_{\epsilon,\epsilon}(x, y, t) = U_\epsilon(x, y, t)$, with $u = 1$. It satisfies the initial condition $U_{\epsilon,\delta}(x, y, 0) = e^{-\epsilon x^2 - \delta y^2}$. So we look for a fundamental solution $p(t,x,y,\xi,\eta)$ such that

$$\int_0^\infty \int_0^\infty e^{-\epsilon\xi^2-\delta\eta^2} p(t,x,y,\xi,\eta)d\xi d\eta = U_{\epsilon,\delta}(x,y,t). \tag{3.3}$$

This is a generalised Laplace transform. We convert it to a Laplace transform by letting $\xi^2 = z, \eta^2 = w$. So that

$$\int_0^\infty \int_0^\infty e^{-\epsilon z-\delta w} p(t,x,y,\sqrt{z},\sqrt{w})\frac{dzdw}{4\sqrt{zw}} = U_{\epsilon,\delta}(x,y,t). \tag{3.4}$$

This means that

$$p(t,x,y,\sqrt{z},\sqrt{w})\frac{1}{4\sqrt{zw}} = \mathcal{L}^{-1}[U_{\epsilon,\delta}(x,y,t)], \tag{3.5}$$

where $\mathcal{L}$ is the two dimensional Laplace transform. This can be written as the sum of two Laplace transforms, each of which is a product of one dimensional Laplace transforms. The individual inversions are elementary with the aid of tables and we have

$$\mathcal{L}^{-1}[U_{\epsilon,\delta}(x,y,t)] = \frac{1}{4\pi t(ax+by)}e^{-\frac{x^2+y^2+z+w}{4t}} \times$$
$$\left[\frac{a}{\sqrt{w}}\sinh(\frac{x\sqrt{z}}{2t})\cosh(\frac{y\sqrt{w}}{2t}) + \frac{b}{\sqrt{z}}\cosh(\frac{x\sqrt{z}}{2t})\sinh(\frac{y\sqrt{w}}{2t})\right].$$

Now $z = \xi^2$ and $w = \eta^2$, so

$$p(t,x,y,\xi,\eta) = \frac{1}{\pi t(ax+by)}e^{-\frac{x^2+y^2+\xi^2+\eta^2}{4t}} \times$$
$$\left[a\xi\sinh(\frac{x\xi}{2t})\cosh(\frac{y\eta}{2t}) + b\eta\cosh(\frac{x\xi}{2t})\sinh(\frac{y\eta}{2t})\right].$$

This is a fundamental solution of the original PDE. Notice that because

$$\int_0^\infty \int_0^\infty e^{-\epsilon\xi^2-\delta\eta^2} p(t,x,y,\xi,\eta)d\xi d\eta = U_{\epsilon,\delta}(x,y,t), \tag{3.6}$$

and $U_{0,0}(x,y,t) = 1$, it follows that $\int_0^\infty \int_0^\infty p(t,x,y,\xi,\eta)d\xi d\eta = 1$. Thus this fundamental solution is also a probability density. Using the Itô formula one can now check that it in fact is the transition probability density for the two dimensional process

$$dX_t = \frac{2a}{aX_t+bY_t}dt + \sqrt{2}dW_t^1, \; dY_t = \frac{2b}{aX_t+bY_t}dt + \sqrt{2}dW_t^2, \tag{3.7}$$

where both $a, b > 0$.

Corollary 3.1. *The joint density of a two dimensional Bessel process and a one dimensional reflected standard Brownian motion is*

$$p(t,x,y,\xi,\eta) = \frac{2}{\pi tx}e^{-\frac{x^2+y^2+\xi^2+\eta^2}{2t}}\xi\sinh(\frac{x\xi}{t})\cosh(\frac{y\eta}{t}).$$

Proof. Take $b \to 0$ and let $t \to \frac{1}{2}t$. $\square$

Remark 3.2. The PDE (3.2) can be reduced to the heat equation by setting $u(x,y,t) = e^{-2\ln(ax+by)}v(x,y,t)$. From this we may obtain a fundamental solution

$$q(t,x,y,\xi,\eta) = \frac{1}{4\pi t}\left(\frac{ax+by}{a\xi+b\eta}\right)^2 e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}}. \tag{3.8}$$

But this is not the transition probability density we found. To obtain the desired transition density, we need to consider fundamental solutions of the heat equation valid on the first quadrant. Our method is more efficient.

By a similar calculation we can find the transition probability density for

$$dX_t^i = \frac{2a_i}{\sum_{i=1}^n a_i X_t^i}dt + \sqrt{2}dW_t^i,\ X_0^i = x^i, i = 1,...,n, \tag{3.9}$$

with $a_i > 0, i = 1, 2, ...$

Proposition 3.3. *The associated transition density for the n dimensional process $X_t = (X_t^1, ..., X_t^n)$ satisfying (3.9), may be found by inverting the n dimensional generalized Laplace transform*

$$U_{\epsilon_1,...,\epsilon_n}(x_1, ..., x_n,t) = \frac{1}{\sqrt{1+4\epsilon_1 t}\cdots\sqrt{1+4\epsilon_n t}} \exp\left(-\sum_{i=1}^n \frac{\epsilon_i x_i^2}{1+4\epsilon_1 t}\right)$$
$$\times \exp\left(\phi\left(\frac{x_1}{1+4\epsilon_1 t}, ..., \frac{x_n}{1+4\epsilon_n t}\right) - \phi(x_1, ..., x_n)\right),$$

where $\phi(x_1, x_2, ..., x_n) = \log(a_1x_1 + \cdots + a_nx_n)$.

Proof. One easily checks that the given function $U_{\epsilon_1,...,\epsilon_n}(x_1, ..., x_n, t)$ is a Laplace transform and a solution of the Kolmogorov forward equation for the diffusion. That the inverse transform integrates to one can be readily checked and we also may easily show, (see the proof of Theorem 3.6), that the inverse Laplace transform is a fundamental solution. □

Remark 3.4. We can compute the joint density of a two dimensional Bessel process and $n-1$ independent reflected Wiener processes by taking $a_2 = \cdots = a_n = 0$.

There is another connection between this n dimensional process and the standard Bessel process.

Lemma 3.5. *If $Z_t = \sqrt{\sum_{i=1}^n (X_t^i)^2}$, where*

$$dX_t^i = \frac{2a_i}{\sum_{i=1}^n a_i X_t^i}dt + \sqrt{2}dW_t^i,\ X_0^i = x^i, i = 1,...,n, \tag{3.10}$$

then Z_t is an n dimensional Bessel process.

Proof. This is just an application of the Itô formula. □

The problem of obtaining integral transforms when we only have access to $SL(2,\mathbb{R})$ symmetries is not yet fully resolved. There are some interesting approaches available, but we will not discuss them here. When we have Heisenberg group symmetries, the situation is easier.

We illustrate with the two dimensional heat equation, which has symmetries of the form $\mathbf{v}_1 = 2t\partial_x - xu\partial_u$, $\mathbf{v}_2 = 2t\partial_y - yu\partial_u$. If we exponentiate these symmetries, we see that if $u(x,y,t)$ is a solution of $u_t = u_{xx} + u_{yy}$ then so are

$$\rho(\exp\epsilon\mathbf{v}_1)u(x,y,t) = e^{-\epsilon x+\epsilon^2 t}u(x-2\epsilon t, y, t) \tag{3.11}$$

$$\rho(\exp\delta\mathbf{v}_2)u(x,y,t) = e^{-\delta y+\delta^2 t}u(x, y-2\delta t, t). \tag{3.12}$$

If we apply these symmetries one after the other we obtain that

$$U_{\epsilon,\delta}(x,y,t) = e^{-\epsilon x-\delta y+(\delta^2+\epsilon^2)t}u(x-2\epsilon t, y-2\delta t, t) \tag{3.13}$$

is a solution. Now we use the right hand side of (3.13) and let $u = u_0$ be a stationary solution. Let $p(t,x,y,\xi,\eta)$ be a fundamental solution of the two dimensional heat equation. Using the basic principle behind the construction of our integral transforms, (see [11]) we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_{\epsilon,\delta}(\xi,\eta,0)p(t,x,y,\xi,\eta)d\xi d\eta = U_{\epsilon,\delta}(x,y,t) \tag{3.14}$$

which is the same as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\epsilon\xi-\delta\eta}u_0(\xi,\eta)p(t,x,y,\xi,\eta)d\xi d\eta$$
$$= e^{-\epsilon x-\delta y+(\delta^2+\epsilon^2)t}u_0(x-2\epsilon t, y-2\delta t). \tag{3.15}$$

But (3.15) is nothing more than the two dimensional, two sided Laplace transform of p. Suppose we take $u_0 = 1$ and replace ϵ with $i\epsilon$ and δ with $i\delta$. Then we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\epsilon\xi-i\delta\eta}p(t,x,y,\xi,\eta)d\xi d\eta = e^{-i\epsilon x-i\delta y-(\delta^2+\epsilon^2)t}. \tag{3.16}$$

We have thus obtained the two dimensional Fourier transform of the heat kernel. Let us consider another example of this approach to finding fundamental solutions.

Example 3.2. We will compute a fundamental solution of the PDE

$$u_t = u_{xx} + u_{yy} + (ax+by+c)u, \ (x,y) \in \mathbb{R}^2. \tag{3.17}$$

We use the exponential solution $u(x,y,t) = e^{\frac{1}{3}t\left(\left(a^2+b^2\right)t^2+3c+3(ax+by)\right)}$. The PDE has a symmetry

$$\tilde{u}_{\epsilon,\delta}(x,y,t) = e^{\frac{(a^2+b^2)t^3}{3}+ct+b(y-it\delta)t+a(x-it\epsilon)t-\left(\delta^2+\epsilon^2\right)t-iy\delta-ix\epsilon}. \tag{3.18}$$

Using a similar argument to the case of the heat equation above, we have a fundamental solution

$$p(t,x,y,\xi,\eta) = \frac{e^{\frac{(a^2+b^2)t^3}{3}+ct}}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\xi\epsilon+i\eta\delta}e^{b(y-it\delta)t+a(x-it\epsilon)t-\left(\delta^2+\epsilon^2\right)t-iy\delta-ix\epsilon}d\epsilon d\delta$$

$$= \frac{e^{ct}}{4\pi t}e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}+\frac{1}{2}t(a(x+\xi)+b(y+\eta))+\frac{1}{12}(a^2+b^2)t^3}.$$

Similarly, one can show that the n dimensional PDE

$$u_t = \Delta u + (\sum_{i=1}^{n} a_i x_i + c)u, \tag{3.19}$$

has a fundamental solution

$$p(t,x,y) = \frac{e^{ct}}{(4\pi t)^{\frac{n}{2}}}\exp\left(\frac{1}{12}\sum_{i=1}^{n} a_i^2 t^3 - \frac{\|x-y\|^2}{4t} + \frac{t}{2}\sum_{i=1}^{n} a_i(x_i+y_i)\right).$$

It is possible to compute these types of Fourier transforms whenever there is a Heisenberg group of symmetries. There are two cases where this occurs. The first result is the following.

Theorem 3.6. *We consider the PDE*

$$u_t = \Delta u + 2\nabla\phi\cdot\nabla u + B(x)u, \quad x\in\mathbb{R}^n, \tag{3.20}$$

where ϕ is a solution of the quasi-linear PDE

$$\Delta\phi + |\nabla\phi|^2 + A(x) = B(x),$$

and $A(x) = \sum_{i=1}^{n} a_i x_i + a_{n+1}$. Suppose also that u_0 is a stationary solution such that the function

$$K(t,x,\epsilon) = e^{-i\sum_{k=1}^{n}\epsilon_k(x_k-a_k t^2)-\sum_{k=1}^{n}\epsilon_k^2 t+z(x,\epsilon)}u_0(x_1-2i\epsilon_1 t,...,x_n-2i\epsilon_n t)$$

where, $z(x,\epsilon) = \phi(x_1-2i\epsilon_1 t,...,x_n-2i\epsilon_n t) - \phi(x_1,...,x_n)$is positive definite. Then there is a fundamental solution $p(t,x,y)$ of (3.20) such that

$$\int_{\mathbb{R}^n} e^{-i\sum_{k=1}^{n}\epsilon_k y_k}p(t,x,y)u_0(y)dy = K(t,x,\epsilon). \tag{3.21}$$

Proof. By Bochner's Theorem, (c.f. Theorem 3.2.3 of [5]), $K(t,x,\epsilon)$ is a Fourier transform. Now the PDE (3.20) has a Lie symmetry, given by

$$\rho(\exp(i\epsilon_1\mathbf{v}_1))\cdots\rho(\exp(i\epsilon_n\mathbf{v}_n))u(x,t) = e^{-i\sum_{k=1}^{n}\epsilon_k(x_k-a_k t^2)-\sum_{k=1}^{n}\epsilon_k^2 t+z(x,\epsilon)} \times u(x_1-2i\epsilon_1 t,...,x_n-2i\epsilon_n t,t)$$

and so $K(t,x,\epsilon)$ is a solution of the PDE. By Bochner's Theorem, there exists $p(t,x,y)u_0(y)$ such that (3.21) holds. We prove that $p(t,x,y)$ is a fundamental solution. To do this we observe first that

$$K(0,x,\epsilon) = e^{-i\sum_{k=1}^{n}\epsilon_k x_k}u_0(x).$$

Now we let $u(x,t) = \int_{\mathbb{R}^n} \varphi(\epsilon)K(t,x,\epsilon)d\epsilon$, where φ is a test function of suitably rapid decay. Notice that

$$\begin{aligned} u(x,0) &= \int_{\mathbb{R}^n} \varphi(\epsilon)e^{-i\sum_{k=1}^n \epsilon_k x_k} u_0(x)d\epsilon \\ &= u_0(x)\Phi(x), \end{aligned}$$

where Φ is the Fourier transform of φ.

An application of Fubini's Theorem then shows that

$$\begin{aligned} \int_{\mathbb{R}^n} u_0(y)\Phi(y)p(t,x,y)dy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_0(y)\varphi(\epsilon)p(t,x,y)e^{-i\sum_{k=1}^n \epsilon_k y_k} d\epsilon dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_0(y)\varphi(\epsilon)p(t,x,y)e^{-i\sum_{k=1}^n \epsilon_k y_k} dy d\epsilon \\ &= \int_{\mathbb{R}^n} \varphi(\epsilon)K(t,x,\epsilon)d\epsilon = u(x,t). \end{aligned}$$

Thus integrating $u_0\Phi$ against p produces a solution with $u(x,0) = u_0\Phi$. Hence p is a fundamental solution. □

There is a second case when we can extract fundamental solutions by Fourier inversion.

Theorem 3.7. *We consider the PDE*

$$u_t = \Delta u + 2\nabla\phi \cdot \nabla u + B(x)u, \quad x \in \mathbb{R}^n, \tag{3.22}$$

where ϕ is a solution of the quasi-linear PDE

$$\Delta\phi + |\nabla\phi|^2 + A(x) = B(x),$$

and $A(x) = -\frac{1}{4}\sum_{k=1}^n c_k x_k^2,\ c_k > 0$. Suppose also that u_0 is a stationary solution such that the function

$$\begin{aligned} K(t,x,\epsilon) &= e^{-i\sum_{k=1}^n \epsilon_k x_k \cosh(\sqrt{c_k}t) - \sum_{k=1}^n \epsilon_k^2 \frac{\sinh(2\sqrt{c_k}t)}{2\sqrt{c_k}} + z(x,\epsilon)} \\ &\quad \times u_0\Big(x_1 - \frac{2i\epsilon_1 \sinh(\sqrt{c_1}t)}{\sqrt{c_1}}, ..., x_n - \frac{2i\epsilon_n \sinh(\sqrt{c_n}t)}{\sqrt{c_n}}\Big) \end{aligned}$$

where, $z(x,\epsilon) = \phi(x_1 - \frac{2i\epsilon_1 \sinh(\sqrt{c_1}t)}{\sqrt{c_2}}, ..., x_n - \frac{2i\epsilon_n \sinh(\sqrt{c_n}t)}{\sqrt{c_n}}) - \phi(x_1, ..., x_n)$ is positive definite. Then there is a fundamental solution $p(t,x,y)$ of (3.22) such that

$$\int_{\mathbb{R}^n} e^{-i\sum_{k=1}^n \epsilon_k y_k} p(t,x,y)u_0(y)dy = K(t,x,\epsilon). \tag{3.23}$$

Proof. Using Lie's algorithm, we can show that there is a symmetry of the PDE of the form

$$\begin{aligned} \Pi_{k=1}^n \rho(\exp(i\epsilon_k \mathbf{v}_k)u(x,t) &= e^{-i\sum_{k=1}^n \epsilon_k x_k \cosh(\sqrt{c_1}t) - \sum_{k=1}^n \frac{\sinh(2\sqrt{c_k}t)}{2\sqrt{c_k}} \epsilon_k^2 + z(x,\epsilon)} \\ &\quad \times u\Big(x_1 - \frac{2i\epsilon_1 \sinh(\sqrt{c_1}t)}{\sqrt{c_1}}, ..., x_n - \frac{2i\epsilon_n \sinh(\sqrt{c_n}t)}{\sqrt{c_n}}, t\Big). \end{aligned}$$

The remainder of the proof proceeds along the same lines as the previous theorem. □

Remark 3.8. The case where $c_k < 0$ can be handled by replacing cosh with cos etc. The case when $A(x) = \sum_{k=1}^n(-\frac{1}{4}a_kx_k^2 + b_kx_k + c)$ can be handled by completing the square and making a change of variables of the form $x \to x - \alpha$ in the Fourier transform.

We may thus recover fundamental solutions by inverting Fourier transforms. Different stationary solutions will typically yield different fundamental solutions. This is a major advantage over other methods, such as group invariant solutions. To obtain a transition probability density we have the following easy result.

Corollary 3.9. *If $B = 0, u_0 = 1$ in Theorems 3.6 and 3.7, the resulting fundamental solution has the property that $\int_{\mathbb{R}^n} p(t, x, y)dy = 1$.*

Proof. From Theorem 3.6, $\int_{\mathbb{R}^n} e^{-i\sum_{k=1}^n \epsilon_k y_k} p(t, x, y)dy = K(t, x, \epsilon)$, so that $\int_{\mathbb{R}^n} p(t, x, y)dy = K(t, x, 0)$. Now $K(t, x, 0) = 1$ provided $u_0 = 1$. Similarly for Theorem 3.7. □

4. Expansions of Fundamental Solutions via Lie Symmetries

Obtaining integral transforms of fundamental solutions for other PDEs of our class is difficult because often there will not be enough symmetry. Except for some special cases, the Lie point symmetry group for a PDE of the form

$$u_t = \Delta u + \frac{1}{x_1^2}k\left(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}\right)u, \tag{4.1}$$

will always be $SL(2, \mathbb{R}) \times \mathbb{R}$ independent of n. In some cases we may extract an n dimensional integral transform from the $SL(2, \mathbb{R})$ symmetries, as Example 3.1 shows, but typically we do not have enough one parameter subgroups. Nor can the fundamental solutions be obtained as a product of one dimensional solutions.

However, even though the size of the Lie point symmetry group remains the same, there is an important difference between the one dimensional problem and the n dimensional problem. In the case of a PDE on the line, such as $u_t = u_{xx} + A(x)u$, there are only two linearly independent stationary solutions. But for $n \geq 2$, the PDE (4.1) has *infinitely many* stationary solutions. The idea is to combine the superposition of solutions and the integration of an $SL(2, \mathbb{R})$ symmetry to produce an explicit fundamental solution.

The best results available are for $n = 2$ and we consider this situation first, looking at higher dimensions later. A PDE of the form

$$u_t = u_{xx} + u_{yy} + \frac{1}{x^2}k\left(\frac{y}{x}\right)u, \tag{4.2}$$

always has a Lie point symmetry of the form

$$\tilde{u}_\epsilon(x,y,t) = \frac{1}{(1+4\epsilon t)}\exp\left(-\frac{\epsilon(x^2+y^2)}{1+4\epsilon t}\right)u\left(\frac{x}{1+4\epsilon t},\frac{y}{1+4\epsilon t},\frac{t}{1+4\epsilon t}\right).$$

Let us integrate the symmetry against a test function φ. This produces the solution

$$U(x,y,t) = \int_0^\infty \varphi(\epsilon)\tilde{u}_\epsilon(x,y,t)d\epsilon. \tag{4.3}$$

This solves the Cauchy problem for (4.2) with initial data of the form $U(x,y,0) = u(x,y,0)\Phi(x^2+y^2)$ and Φ is the laplace transform of φ. Since there are infinitely many stationary solutions, we can, by taking linear combinations, solve an initial value problem of the form

$$U(x,y,0) = \sum_{k=1}^\infty u_k(x,y)\Phi_k(x^2+y^2). \tag{4.4}$$

Here each u_k is a stationary solution, and Φ_k is the Laplace transform of a test function φ_k. If the stationary solutions are sufficiently rich, we may recover essentially any initial condition. This is the basis for the next result. It turns out that the problem is best treated in polar coordinates.

Theorem 4.1. *Suppose that K is continuous and that the Sturm-Liouville problem*

$$L''(\theta) + (K(\theta)+\lambda)L(\theta) = 0 \tag{4.5}$$

$$\alpha_1 L(a) + \alpha_2 L'(a) = 0 \tag{4.6}$$

$$\beta_1 L(b) + \beta_2 L'(b) = 0, \tag{4.7}$$

has a complete set of eigenfunctions and eigenvalues, and that the eigenvalues are all positive. Consider the initial and boundary value problem

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{K(\theta)}{r^2}u, \tag{4.8}$$

$$r>0,\ a\le\theta\le b,\ a,b\in[0,2\pi].$$

$$u(r,\theta,0) = f(r,\theta),\ f\in\mathcal{S}(\mathbb{R}^2),$$

$$\alpha_1 u(r,a,t) + \alpha_2 u_\theta(r,a,t) = 0$$

$$\beta_1 u(r,b,t) + \beta_2 u_\theta(r,b,t) = 0.$$

Then there is a solution of the form

$$u(r,\theta,t) = \int_0^\infty\int_a^b f(\rho,\phi)p(t,r,\theta,\rho,\phi)\rho d\phi d\rho \tag{4.9}$$

where

$$p(t,r,\theta,\rho,\phi) = \frac{1}{2t}e^{-\frac{r^2+\rho^2}{4t}}\sum_n \overline{L_n(\phi)}L_n(\theta)I_{\sqrt{\lambda_n}}\left(\frac{r\rho}{2t}\right), \tag{4.10}$$

in which $L_n(\theta), \lambda_n, n = 1, 2, 3...$ are the normalised eigenfunctions and corresponding eigenvalues for the given Sturm-Liouville problem.

Proof. The key result is the fact that the PDE has a Lie group symmetry which in polar coordinates is given by

$$\tilde{u}_\epsilon(r, \theta, t) = \frac{1}{1 + 4\epsilon t} e^{-\frac{\epsilon r^2}{1+4\epsilon t}} u\left(\frac{r}{1 + 4\epsilon t}, \theta, \frac{t}{1 + 4\epsilon t}\right), \tag{4.11}$$

valid for ϵ sufficiently small. We will integrate this symmetry and use the linearity of the equations in order to construct a solution which satisfies the specified conditions.

We use superposition and symmetry integration to produce a solution of the form

$$u(r, \theta, t) = \int_0^\infty \sum_n \varphi_n(\epsilon) \frac{1}{1 + 4\epsilon t} e^{-\frac{\epsilon r^2}{1+4\epsilon t}} u_n\left(\frac{r}{1 + 4\epsilon t}, \theta\right) d\epsilon, \tag{4.12}$$

in which each u_n is a stationary solution of (4.8). The functions φ_n are chosen to guarantee that the integrals and sums are uniformly convergent. The stationary solutions we choose will be separable, that is

$$u_n(r, \theta) = R_n(r)\Theta_n(\theta).$$

Substitution into (4.8) shows that we require

$$r^2 R_n'' + rR_n' - \lambda R_n = 0 \tag{4.13}$$

$$\Theta_n''(\theta) + (K(\theta) + \lambda)\Theta_n(\theta) = 0. \tag{4.14}$$

We choose λ_n and $\Theta_n(\theta) = L_n(\theta)$ to be the eigenvalues and normalised eigenfunctions of the given Sturm-Liouville problem. We also choose $R_n(r) = r^{\sqrt{\lambda_n}}$. Thus our stationary solution is

$$u_n(r, \theta) = r^{\sqrt{\lambda_n}} L_n(\theta).$$

Now from the solution (4.12), we have with this choice of stationary solution

$$u(r, \theta, 0) = \sum_n \Phi_n(r^2) r^{\sqrt{\lambda_n}} L_n(\theta). \tag{4.15}$$

We require $u(r, \theta, 0) = f(r, \theta)$. Thus we must have

$$f(r, \theta) = \sum_n \Phi_n(r^2) r^{\sqrt{\lambda_n}} L_n(\theta). \tag{4.16}$$

Now since the eigenfunctions L_n are complete, if $f \in \mathcal{S}(\mathbb{R}^2)$, then we may write

$$f(r, \theta) = \sum_n c_n(r) L_n(\theta), \tag{4.17}$$

where

$$c_n(r) = \int_a^b f(r, \phi)\overline{L_n(\phi)} d\phi. \tag{4.18}$$

This implies that we must choose

$$\Phi_n(r^2) = \frac{1}{r^{\sqrt{\lambda_n}}} \int_a^b f(r,\phi)\overline{L_n(\phi)}d\phi. \qquad (4.19)$$

Moreover, by general Sturm-Liouville theory, (see [3]), this sum converges uniformly to f for each fixed r.

The solution we are working with has the form

$$u(r,\theta,t) = \int_0^\infty \sum_n \psi_n(\epsilon) \frac{r^{\sqrt{\lambda_n}}}{(1+4\epsilon t)^{1+\sqrt{\lambda_n}}} e^{-\frac{\epsilon r^2}{1+4\epsilon t}} L_n(\theta)d\epsilon. \qquad (4.20)$$

We wish to rewrite this expression using the Laplace transform.

So we observe that

$$\frac{r^{\sqrt{\lambda_n}}}{(1+4\epsilon t)^{1+\sqrt{\lambda_n}}} e^{-\frac{\epsilon r^2}{1+4\epsilon t}} = \int_0^\infty \frac{z^{\frac{1}{2}\sqrt{\lambda_n}}}{4t} e^{-\frac{r^2+z}{4t}} I_{\sqrt{\lambda_n}}\left(\frac{r\sqrt{z}}{2t}\right) e^{-\epsilon z}dz. \quad (4.21)$$

Using this Laplace transform, we find that

$$\begin{aligned} u(r,\theta,t) &= \int_0^\infty \int_0^\infty \sum_n \varphi_n(\epsilon) \frac{z^{\frac{1}{2}\sqrt{\lambda_n}}}{4t} e^{-\frac{r^2+z}{4t}} I_{\sqrt{\lambda_n}}\left(\frac{r\sqrt{z}}{2t}\right) e^{-\epsilon z} L_n(\theta)dzd\epsilon \\ &= \int_0^\infty \sum_n \Phi_n(z) \frac{z^{\frac{1}{2}\sqrt{\lambda_n}}}{4t} e^{-\frac{r^2+z}{4t}} I_{\sqrt{\lambda_n}}\left(\frac{r\sqrt{z}}{2t}\right) L_n(\theta)dz, \qquad (4.22) \end{aligned}$$

where we have reversed the order of integration and evaluated the ϵ integral. Then we set $z = \rho^2$ to obtain

$$\begin{aligned} u(r,\theta,t) &= \int_0^\infty \frac{\rho}{2t} e^{-\frac{r^2+\rho^2}{4t}} \sum_n \Phi_n(\rho^2)\rho^{\sqrt{\lambda_n}} L_n(\theta) I_{\sqrt{\lambda_n}}\left(\frac{r\rho}{2t}\right) d\rho \\ &= \int_0^\infty \int_a^b f(\rho,\phi)\frac{\rho}{2t} e^{-\frac{r^2+\rho^2}{4t}} \sum_n L_n(\theta)\overline{L_n(\phi)} I_{\sqrt{\lambda_n}}\left(\frac{r\rho}{2t}\right) d\phi d\rho. \end{aligned}$$

Here we have replaced $\Phi_n(\rho^2)$ with the value given by (4.19). By construction this function satisfies both the initial and boundary conditions and by symmetry it is a solution of the PDE. This completes the proof.

□

Remark 4.2. The change of variables $t \to it$ allows us to compute fundamental solutions of the Schrödinger equation $iu_t = \Delta u + \frac{1}{x^2}k\left(\frac{y}{x}\right)u$.

Consequently, from linearity and a single $SL(2,\mathbb{R})$ symmetry, we can recover an exact, explicit fundamental solution. We still have to solve a Sturm-Liouville problem, but the second order ODE can, at least in principle, be solved by power series methods. We will briefly address the practical implementation of this result below. Now we consider some examples.

Let us recover the two dimensional heat kernel from the given theorem. In this example we wish to solve $L''(\theta) + \lambda L(\theta) = 0$, with $L(0) - L(2\pi) = 0$. The eigenvalues are $\lambda = n^2, n = 0, \pm 1, \pm 2, ...$ and the eigenfunctions are $L_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$.

This gives us the following representation of the solution.

$$u(r, \theta, t) = \int_0^\infty \int_0^{2\pi} f(\rho, \phi) \frac{\rho}{4\pi t} e^{-\frac{r^2+\rho^2}{4t}} \sum_{n \in \mathbb{Z}} e^{in(\theta-\phi)} I_{|n|} \left(\frac{r\rho}{2t}\right) d\phi d\rho.$$

Now the identity $e^{a \cos y} = I_0(a) + 2\sum_{n=1}^\infty I_n(a) \cos(ny)$, is on page 376 of [2]. Further $e^{in(\theta-\phi)} + e^{-in(\theta-\phi)} = 2\cos(n(\theta - \phi))$. So

$$\sum_{n \in \mathbb{Z}} e^{in(\theta-\phi)} I_{|n|} \left(\frac{r\rho}{2t}\right) = I_0 \left(\frac{r\rho}{2t}\right) + 2\sum_{n=1}^{\infty} \cos(n(\theta - \phi)) I_n \left(\frac{r\rho}{2t}\right)$$
$$= e^{\frac{r\rho}{2t} \cos(\theta-\phi)}. \tag{4.23}$$

From this we conclude that

$$u(r, \theta, t) = \int_0^\infty \int_0^{2\pi} f(\rho, \phi) \frac{\rho}{4\pi t} e^{-\frac{r^2+\rho^2}{4t}} e^{\frac{r\rho}{2t} \cos(\theta-\phi)} d\phi d\rho. \tag{4.24}$$

We can convert this back to Cartesian coordinates, setting $x = r \cos \theta, y = r \sin \theta, \xi = \rho \cos \phi$ and $\eta = \rho \sin \phi$. This gives the solution in Cartesian coordinates as

$$U(x, y, t) = \int_{\mathbb{R}^2} \tilde{f}(\xi, \eta) \frac{1}{4\pi t} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}} d\xi d\eta. \tag{4.25}$$

Here U is the solution in Cartesian coordinates and the initial value of the solution becomes $\tilde{f}(\xi, \eta) = f(\sqrt{\xi^2 + \eta^2}, \tan^{-1} \frac{\xi}{\eta})$. We have thus recovered the two dimensional heat kernel from a single $SL(2, \mathbb{R})$ symmetry.

We can also obtain fundamental solutions restricted to different domains, with some different boundary conditions. Suppose that we want a fundamental solution restricted to the first quadrant, which solves the problem $u_t = \Delta u$ subject to $u(r, \theta, 0) = f(r, \theta)$ and $u(r, 0, t) = 0$.

To obtain this solution, we restrict the range of the θ variable to $[0, \frac{\pi}{2}]$. Then we choose stationary solutions of the heat equation of the form $u_n(r, \theta) = \frac{2}{\sqrt{\pi}} r^{2|n|} \sin(2n\theta)$. Hence we establish that

$$u(r, \theta, t) = \int_0^\infty \int_0^{\frac{\pi}{2}} f(\rho, \phi) \frac{2}{\pi t} e^{-\frac{r^2+\rho^2}{4t}} \sum_{n \in \mathbb{Z}} \sin(2n\theta) \sin(2n\phi) I_{2|n|} \left(\frac{r\rho}{2t}\right) d\Omega,$$

is a solution of the heat equation on $0 \leq r < \infty, 0 \leq \theta \leq \frac{\pi}{2}$ which satisfies $u(r, \theta, 0) = f(r, \theta)$ for all $\theta \in [0, \frac{\pi}{2}]$ and moreover $u(r, 0, t) = 0$. Here $d\Omega = \rho d\phi d\rho$. A solution of the initial value problem with $u_\theta(r, 0, t) = 0$ may be found by using the eigenfunctions $\cos(2n\theta)$ in place of the sine eigenfunctions used here.

Example 4.1. We will solve the equation

$$u_t = u_{xx} + u_{yy} - \frac{A}{x^2+y^2}u, \; (x,y) \in \mathbb{R}^2 - \{0,0\}, \; A > 0, \tag{4.26}$$

subject to the initial condition $u(x,y,0) = f(x,y)$, $u(r,0,t) = u(r,2\pi,t)$. In polar coordinates the equation is

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} - \frac{A}{r^2}u, \; A > 0. \tag{4.27}$$

We have to solve the eigenvalue problem $L'' + (-A+\lambda)L = 0$, with $L(0) - L(2\pi) = 0$. The eigenvalues are $\lambda_n = n^2 + A$ and we again take $L_n(\theta) = \frac{1}{\sqrt{2\pi}}e^{in\theta}$.

We can conclude that equation (4.27) has a fundamental solution

$$p(r,\theta,t,\rho,\phi) = \frac{1}{4\pi t}e^{-\frac{r^2+\rho^2}{4t}} \sum_{n\in\mathbb{Z}} e^{in(\theta-\phi)} I_{\sqrt{n^2+A}}\left(\frac{r\rho}{2t}\right).$$

It does not seem possible to obtain a closed form expression for this series. We also note that we can obtain different fundamental solutions by using different boundary conditions.

Example 4.2. We now consider the PDE

$$u_t = u_{xx} + u_{yy} - \frac{\mu}{\tan^{-1}\left(\frac{y}{x}\right)^2 (x^2+y^2)}u, \quad \mu > 0. \tag{4.28}$$

In polar coordinates this becomes

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} - \frac{\mu}{r^2\theta^2}u, \tag{4.29}$$

and we suppose that $\theta \in (0, 2\pi]$ and impose the two boundary conditions that $u(r,2\pi,t) = 0$ and $u(r,0^+,t)$ is finite. The Sturm-Liouville problem in this case is

$$L'' + (-\frac{\mu}{\theta^2} + \lambda)L = 0, \tag{4.30}$$

$L(0^+)$ finite and $L(2\pi) = 0$. The general solution of (4.30) is

$$L(\theta) = c_1\sqrt{\theta}J_{\frac{1}{2}\sqrt{1+4\mu}}(\sqrt{\lambda}\theta) + c_2\sqrt{\theta}Y_{\frac{1}{2}\sqrt{1+4\mu}}(\sqrt{\lambda}\theta).$$

To satisfy the finiteness condition, we set $c_2 = 0$ and choose λ_n so that $2\pi\sqrt{\lambda_n} = \alpha_n$ is the nth positive zero of $J_{\frac{1}{2}\sqrt{1+4\mu}}(\theta)$. So if α_n is the nth zero of $J_{\frac{1}{2}\sqrt{1+4\mu}}(\theta)$, then $\lambda_n = \alpha_n^2/4\pi^2$ and we take

$$L_n(\theta) = \sqrt{\theta}J_{\frac{1}{2}\sqrt{1+4\mu}}\left(\frac{\alpha_n}{2\pi}\theta\right).$$

This leads to the fundamental solution

$$p(r,\theta,t,\rho,\phi) = \frac{1}{2t}e^{-\frac{r^2+\rho^2}{4t}} \sum_{n=1}^{\infty} \frac{\sqrt{\theta\phi}}{c_n} J_k\left(\frac{\alpha_n}{2\pi}\theta\right) J_k\left(\frac{\alpha_n}{2\pi}\phi\right) I_{\sqrt{\frac{\alpha_n}{2\pi}}}\left(\frac{r\rho}{2t}\right),$$

with $k = \frac{1}{2}\sqrt{1+4\mu}$ and $c_n = \int_0^{2\pi} \theta J_k\left(\frac{\alpha_n}{2\pi}\theta\right)^2 d\theta$. Converting this back to Cartesian coordinates is straightforward.

Example 4.3. For the equation $u_t = \Delta u - \frac{1}{4}\frac{(\tan^{-1}(\frac{y}{x}))^2}{x^2+y^2}u$, we solve the equation $L''(\theta) + (-\frac{1}{4}\theta^2 + \lambda)L(\theta) = 0$, subject to $L(0) = L(2\pi) = 0$. With ${}_1F_1$ Kummer's confluent hypergeometric function, the eigenfunctions are $L_n(\theta, \lambda) = \theta e^{-\frac{\theta^2}{4}}{}_1F_1\left(\frac{3}{4} - \frac{\lambda_n}{2}; \frac{3}{2}; \frac{\theta^2}{2}\right)$, and the first few eigenvalues are $\lambda_1 = 1.5000005$, $\lambda_2 = 3.500093883$, $\lambda_3 = 5.50402734375$, $\lambda_4 = 7.556616211$ etc. We then find $\int_0^{2\pi} |L_1(\theta, \lambda_1)|^2 d\theta = 1.25331$, $\int_0^{2\pi} |L_2(\theta, \lambda_2)|^2 d\theta = 0.835359$, etc. From this we obtain our fundamental solution.

Example 4.4. Here we look at the equation $u_t = \Delta u - \frac{1}{x^2}\frac{x^2+\mu y^2}{x^2+y^2}u$, where $x, y, \mu > 0$ and $\mu \neq 1$. We solve $L'' + (\lambda - 1 - \mu\tan^2\theta)L = 0$ subject to $L(0) = 0$ and $L(\frac{\pi}{2}^+)$ finite. The eigenvalues are $\lambda_n = 1 - \mu + \left(2n + \frac{3}{2} + \sqrt{\mu + 1/4}\right)^2$, $n = 0, 1, 2, 3, \ldots$. The corresponding eigenfunctions are $L_n(\theta) = \cos^\alpha\theta\, {}_2F_1\left(-n - \frac{1}{2}, \beta; \gamma; \cos^2\theta\right)$, where $\alpha = \sqrt{\mu + 1/4} + \frac{1}{2}, \beta = n + 1 + \sqrt{\mu + 1/4}, \gamma = 1 + \sqrt{\mu + 1/4}$. Converting to Cartesian coordinates we obtain the fundamental solution

$$p(t, x, y, \xi, \eta) = \frac{1}{2t}e^{-\frac{x^2+y^2+\xi^2+\eta^2}{4t}}\frac{(x\xi)^\alpha}{((x^2+y^2)(\xi^2+\eta^2))^{\frac{\alpha}{2}}}$$
$$\times\sum_{n=0}^\infty c_n\, {}_2F_1\left(-n - \frac{1}{2}, \beta; \gamma; \frac{x^2}{r^2}\right){}_2F_1\left(-n - \frac{1}{2}, \beta; \gamma; \frac{\xi^2}{\rho^2}\right)I_{\sqrt{\lambda_n}}\left(\frac{r\rho}{2t}\right),$$

with $r = \sqrt{x^2 + y^2}, \rho = \sqrt{\xi^2 + \eta^2}$. $\frac{1}{c_n^2} = \int_0^{\frac{\pi}{2}} L_n(\theta)^2 d\theta$.

An important application is to the calculation of transition densities.

Example 4.5. Consider the two dimensional Itô process

$$dX_t = \frac{X_t}{X_t^2 + Y_t^2}dt + \sqrt{2}dW_t,\ dY_t = \frac{Y_t}{X_t^2 + Y_t^2}dt + \sqrt{2}dB_t, \tag{4.31}$$

where W and B are independent Wiener processes. The transition density is

$$p(t, x, y, \xi, \eta) = \frac{1}{4\pi t}\sqrt{\frac{\xi^2 + \eta^2}{x^2 + y^2}}e^{-\frac{x^2+y^2+\xi^2+\eta^2}{4t}}\sum_{n=-\infty}^\infty\left(\frac{(x + iy)(\xi - i\eta)}{(x - iy)(\xi + i\eta)}\right)^{\frac{n}{2}}$$
$$\times I_{\sqrt{n^2+1}}\left(\frac{\sqrt{x^2 + y^2}\sqrt{\xi^2 + \eta^2}}{2t}\right). \tag{4.32}$$

To see this, take $A = 1$ in (4.26) and let $u = e^{\frac{1}{2}\ln(x^2+y^2)}v$ to convert to the Kolmogorov forward equation $u_t = \Delta u + \frac{x}{x^2+y^2}u_x + \frac{y}{x^2+y^2}u_y$. This shows that p is a fundamental solution. That it is real valued is trivial. We need only to check that it is a density. This is easiest in polar coordinates. Because $\int_0^{2\pi} e^{in\theta}d\theta = 0$ for $n \neq 0$, there is only one term in the series that contributes to the integral. Now $\int_0^\infty \frac{\rho^2}{2tr} e^{-\frac{r^2+\rho^2}{4t}} I_1\left(\frac{r\rho}{2t}\right) d\rho = 1$, so (4.32) is a probability density.

Example 4.6. As a slightly more complex example, consider the two dimensional Itô process

$$dX_t = \left(\frac{X_t(c_1(X_t^2+Y_t^2)^2-c_2)}{(X_t^2+Y_t^2)(c_1(X_t^2+Y_t^2)+c_2)}\right)dt + \sqrt{2}dW_t, \tag{4.33}$$

$$dY_t = \left(\frac{Y_t(c_1(X_t^2+Y_t^2)^2-c_2)}{(X_t^2+Y_t^2)(c_1(X_t^2+Y_t^2)+c_2)}\right)dt + \sqrt{2}dB_t, \tag{4.34}$$

where W_t and B_t are independent Wiener processes and c_1, c_2 are constants, at least one of which is nonzero. We require a fundamental solution of the PDE

$$u_t = \Delta u + \frac{x(c_1(x^2+y^2)^2-c_2)u_x}{(x^2+y^2)(c_1(x^2+y^2)+c_2)} + \frac{y(c_1(x^2+y^2)^2-c_2)u_y}{(x^2+y^2)(c_1(x^2+y^2)+c_2)}.$$

This PDE reduces to (4.26) and we see that the transition density is

$$\begin{aligned} p(t,x,y,\xi,\eta) &= \frac{1}{4\pi t}\frac{(c_1(\xi^2+\eta^2)+c_2)\sqrt{x^2+y^2}}{(c_1(x^2+y^2)+c_2)\sqrt{\xi^2+\eta^2}} e^{-\frac{x^2+y^2+\xi^2+\eta^2}{4t}} \\ &\times \sum_{n=-\infty}^{\infty}\left(\frac{(x+iy)(\xi-i\eta)}{(x-iy)(\xi+i\eta)}\right)^{\frac{n}{2}} I_{\sqrt{n^2+1}}\left(\frac{\sqrt{x^2+y^2}\sqrt{\xi^2+\eta^2}}{2t}\right). \end{aligned}$$

The proof is similar to the previous example.

Example 4.7. We present a further example which is cautionary in nature. Consider the PDE

$$u_t = \Delta u + \frac{2y(ay-bx)}{(ax+by)(x^2+y^2)}u_x + \frac{2x(bx-ay)}{(ax+by)(x^2+y^2)}u_y, \tag{4.35}$$

with a, b constants. We can reduce this to (4.26) with $A = -1$. Even though one of the eigenvalues is now negative, we can still obtain a fundamental solution using the same analysis and this is

$$\begin{aligned} p(t,x,y,\xi,\eta) &= \frac{1}{4\pi t}\frac{(a\xi+b\eta)\sqrt{x^2+y^2}}{(ax+by)\sqrt{\xi^2+\eta^2}} e^{-\frac{x^2+y^2+\xi^2+\eta^2}{4t}} \\ &\times \sum_{n=-\infty}^{\infty}\left(\frac{(x+iy)(\xi-i\eta)}{(x-iy)(\xi+i\eta)}\right)^{\frac{n}{2}} I_{\sqrt{n^2-1}}\left(\frac{\sqrt{x^2+y^2}\sqrt{\xi^2+\eta^2}}{2t}\right). \end{aligned}$$

It is tempting to identify this as the transition probability density for the Itô process whose generator is given by the right hand side of (4.35), but this fundamental solution is not real valued, so it is not a probability density. This shows that we must be careful when we try to obtain transition probability densities by making changes of variables in the Kolmogorov equation. We will produce a real valued fundamental solution in Example 4.10 below.

We can establish another result in the two dimensional case, for a somewhat different type of problem.

Theorem 4.3. *Suppose that the Sturm-Liouville problem*

$$L''(\theta) + (K(\theta) + \lambda)L(\theta) = 0 \tag{4.36}$$

$$\alpha_1 L(a) + \alpha_2 L'(a) = 0 \tag{4.37}$$

$$\beta_1 L(b) + \beta_2 L'(b) = 0, \tag{4.38}$$

has a complete set of eigenfunctions and eigenvalues, and that the eigenvalues are all positive. Consider the initial and boundary value problem[1]

$$\begin{aligned} u_t &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{K(\theta)}{r^2}u - \frac{1}{4}cr^2u, \\ &r > 0,\ a \le \theta \le b,\ a, b \in [0, 2\pi],\ c > 0. \\ u(r, \theta, 0) &= f(r, \theta),\ f \in \mathcal{S}(\mathbb{R}^2), \\ &\alpha_1 u(r, a, t) + \alpha_2 u_\theta(r, a, t) = 0 \\ &\beta_1 u(r, b, t) + \beta_2 u_\theta(r, b, t) = 0. \end{aligned}$$

Then there is a solution of the form

$$u(r, \theta, t) = \int_0^\infty \int_a^b f(\rho, \phi)p(t, r, \theta, \rho, \phi)\rho d\phi d\rho \tag{4.39}$$

where

$$\begin{aligned} p(t, r, \theta, \rho, \phi) = &\frac{\sqrt{c}}{\sqrt{2(\cosh(2\sqrt{c}t) - 1)}} e^{-\frac{\sqrt{c}(r^2+\rho^2)\sinh(2\sqrt{c}t)}{4(\cosh(2\sqrt{c}t)-1)}} \times \\ &\sum_n \frac{1}{I_{\frac{\sqrt{\lambda_n}}{2}}\left(\frac{\sqrt{c}\rho^2}{4}\right)} \Gamma_n(r, \rho, t)\overline{L_n(\phi)}L_n(\theta), \end{aligned} \tag{4.40}$$

and

$$\Gamma_n(r, \rho, t) = \mathcal{L}^{-1}\left(\frac{1}{\sqrt{\varepsilon^2 - \frac{c}{16}}} e^{\left(\frac{a\varepsilon r^2}{\varepsilon^2 - \frac{c}{16}}\right)} I_{\frac{\sqrt{\lambda_n}}{2}}\left(\frac{br^2}{\varepsilon^2 - \frac{c}{16}}\right)\right),$$

$a = \frac{c}{8(\cosh(2\sqrt{c}t)-1)}$, $b = \frac{c^{3/2}}{32(\cosh(2\sqrt{c}t)-1)}$ *and* $L_n(\theta)$ *and* $\lambda_n, n = 1, 2, 3, ...$ *are the normalised eigenfunctions and corresponding eigenvalues for the Sturm-Liouville problem.*

Proof. The details of the proof are similar to those for Theorem 4.1. Symmetries of these types of equations involve exponentials in t and the idea is to find a symmetry with the property that $U_\epsilon(r, \theta, 0) =$

[1]Note that the $\frac{1}{4}$ multiplying c in the PDE is purely for notational convenience.

$e^{-\epsilon r^2}u(r,\theta,0)$. The necessary symmetry turns out to be

$$U_\epsilon(r,\theta,t) = \frac{e^{-\frac{\epsilon c r^2(\cosh^2(\sqrt{c}t)+\sinh^2(\sqrt{c}t)+2\sqrt{c}\epsilon\sinh(2\sqrt{c}t)}{1+16c\epsilon^2\sinh^2(\sqrt{c}t)+4\sqrt{c}\epsilon\sin(2\sqrt{c}t)}}}{\sqrt{1+8c\epsilon^2(\cosh(2\sqrt{c}t)-1)+4\sqrt{c}\epsilon\sinh(2\sqrt{c}t)}}$$
$$\times v\left(\frac{r}{\sqrt{1+8c\epsilon^2(\cosh(2\sqrt{c}t)-1)+4\sqrt{c}\epsilon\sinh(2\sqrt{c}t)}}, \theta, T(\epsilon,t)\right),$$

where $T(\epsilon,t) = \frac{\coth^{-1}(4\sqrt{c}\epsilon+\coth(\sqrt{c}t))}{\sqrt{c}}$. We now let $\epsilon c \to \epsilon$.

Again we build a solution by superposition and integration to obtain

$$u(r,\theta,t) = \int_0^\infty \sum_n \varphi_n(\epsilon)U_\epsilon^n(r,\theta,t)d\epsilon, \tag{4.41}$$

where the stationary solutions are of the form $v_n(r,\theta) = R(r)L(\theta)$ and

$$r^2R'' + rR' - (\frac{1}{4}cr^4 + \lambda)R = 0, \;\; L'' + (K(\theta)+\lambda)L = 0, \tag{4.42}$$

and L satisfies the given boundary conditions. We will let

$$v_n(r,\theta) = I_{\frac{\sqrt{\lambda_n}}{2}}\left(\frac{\sqrt{c}r^2}{4}\right)L_n(\theta). \tag{4.43}$$

where L_n is the nth eigenfunction and λ_n the corresponding eigenvalue. For the solution (4.41) we have

$$u(r,\theta,0) = \sum_n \Phi_n(r^2)I_{\frac{\sqrt{\lambda_n}}{2}}\left(\frac{\sqrt{c}r^2}{4}\right)L_n(\theta).$$

and Φ_n is the Laplace transform of φ_n. Expanding f as a series of eigenfunctions we have

$$f(r,\theta) = \sum_n \widehat{f}(r,n)L_n(\theta) \tag{4.44}$$

where as before $\widehat{f}(r,n) = \int_a^b f(r,\xi)\overline{L_n(\phi)}d\phi$, so that

$$\Phi_n(r^2) = \frac{1}{I_{\frac{\sqrt{\lambda_n}}{2}}(r^2)}\int_a^b f(r,\phi)\overline{L_n(r,\phi)}d\phi. \tag{4.45}$$

The Laplace transform identity we need here is more difficult to obtain, but one can show that

$$\int_0^\infty \frac{\sqrt{c}\Gamma_n(r,z,t)}{\sqrt{8(\cosh(2\sqrt{c}t)-1)}}e^{-\frac{\sqrt{c}(r^2+z)\sinh(2\sqrt{c}t)}{4(\cosh(2\sqrt{c}t)-1)}}e^{-\epsilon z}dz$$
$$= \frac{e^{-\frac{\epsilon c r^2(\cosh^2(\sqrt{c}t)+\sinh^2(\sqrt{c}t)+2\sqrt{c}\epsilon\sinh(2\sqrt{c}t)}{1+16c\epsilon^2\sinh^2(\sqrt{c}t)+4\sqrt{c}\epsilon\sin(2\sqrt{c}t)}}}{\sqrt{1+8c\epsilon^2(\cosh(2\sqrt{c}t)-1)+4\sqrt{c}\epsilon\sinh(2\sqrt{c}t)}}$$
$$\times I_{\frac{\sqrt{\lambda_n}}{2}}\left(\frac{cr^2}{4(1+8c\epsilon^2(\cosh(2\sqrt{c}t)-1)+4\sqrt{c}\epsilon\sinh(2\sqrt{c}t))}\right).$$

Consequently we can write (4.41) as

$$u(r,\theta,t) = \int_0^\infty \int_0^\infty \sum_n \frac{\varphi_n(\epsilon)\sqrt{c}L_n(\theta)\Gamma_n(r,z,t)}{\sqrt{8(\cosh(2\sqrt{c}t)-1)}} e^{-\frac{\sqrt{c}(r^2+z)\sinh(2\sqrt{c}t)}{4(\cosh(2\sqrt{c}t)-1)}} e^{-\epsilon z} d\mu$$

$$= \int_0^\infty \sum_n \Phi_n(z) \frac{\sqrt{c}L_n(\theta)\Gamma_n(r,z,t)}{\sqrt{8(\cosh(2\sqrt{c}t)-1)}} e^{-\frac{\sqrt{c}(r^2+z)\sinh(2\sqrt{c}t)}{4(\cosh(2\sqrt{c}t)-1)}} e^{-\epsilon z} d\epsilon.$$

Here $d\mu = dzd\epsilon$. Letting $z \to \rho^2$ and replacing $\Phi_n(\rho^2)$ with (4.45) completes the proof. □

We note that the inverse Laplace transform of the theorem does not seem to be known analytically. It can however be computed numerically. See [1] and [10] for the numerical inversion of Laplace transforms.

Because the eigenfunction equations are the same in Theorems 4.1 and 4.3, we may readily adapt the exact examples given previously to our second class of equations.

Example 4.8. Consider the PDE $u_t = \Delta u - \frac{A}{x^2+y^2}u - \frac{1}{4}c(x^2+y^2)u$. Applying the previous theorem and using the eigenfunction calculation from Example 4.26, we find

$$p(t,r,\theta,\rho,\phi) = \frac{\sqrt{c}e^{-\frac{\sqrt{c}(r^2+\rho^2)\sinh(2\sqrt{c}t)}{4(\cosh(2\sqrt{c}t)-1)}}}{2\pi\sqrt{2(\cosh(2\sqrt{c}t)-1)}} \sum_{n\in\mathbb{Z}} \frac{\Gamma_n(r,\rho,t)e^{in(\theta-\phi)}}{I_{\frac{\sqrt{n^2+A}}{2}}\left(\frac{\sqrt{c}\rho^2}{4}\right)},$$

where $r = \sqrt{x^2+y^2}, \rho = \sqrt{\xi^2+\eta^2}$, etc.

4.1. **Practical Implementation of the 2D Expansion Theorems.**

Our discussion of this subject will be brief, giving only the basic ideas. Given a function K we need to solve the Sturm-Liouville problem

$$L''(\theta) + (K(\theta)+\lambda)L(\theta) = 0 \tag{4.46}$$

$$\alpha_1 L(a) + \alpha_2 L'(a) = 0 \tag{4.47}$$

$$\beta_1 L(b) + \beta_2 L'(b) = 0, \tag{4.48}$$

in order to write down a fundamental solution. There are many problems for which we may write down the eigenvalues and eigenfunctions explicitly. Bailey, Everitt and Zettl maintain a database of such problems, [22]. Unfortunately for most choices of K, closed form solutions are not available.

There are two different approaches. We can obtain analytical approximations to the eigenfunctions and eigenvalues. We can also compute them numerically. There are different techniques that can be used for each approach.

To illustrate the first approach, suppose that $a = 0$. We then have the following easy result.

Proposition 4.4. *Suppose that* $K(\theta) = \sum_{k=0}^{\infty} A_k \theta^k$ *and that* L *is a solution of* $L''(\theta) + (K(\theta) + \lambda)L(\theta) = 0,\ L(0) = 0, L(b) = 0$. *Then we may write* $L(\theta) = \sum_{n=0}^{\infty} c_n \theta^n$, *where* $c_0 = 0$ *and*

$$c_{n+2} = -\frac{(B_n + \lambda)c_n}{(n+2)(n+1)},$$

$n = 0, 1, 2, 3, ...$ *and* $B_n = \sum_{j=0}^{n} c_j A_{n-j}$.

Proof. The ODE has only ordinary points, so the general theory of ODES (e.g. Rabenstein [23]), tells us that $L(\theta) = \sum_{n=0}^{\infty} c_n \theta^n$. It is then easy to establish that $\sum_{n=2}^{\infty} n(n-1)c_n\theta^{n-2} + \sum_{n=0}^{\infty}(B_n + \lambda c_n)\theta^n = 0$, so that $\sum_{n=0}^{\infty}((n+2)(n+1)c_{n+2} + B_n + \lambda c_n)\,\theta^n = 0$. Now $L(\theta) = c_0 + c_1\theta + \cdots$, hence $L(0) = c_0 = 0$. □

We may then approximate the eigenvalues as zeroes of the partial sums of the series expansion. Knowing the eigenvalues, we then easily obtain the eigenfunctions. Packages such as Mathematica will greatly simply the process. This method will often be effective, but Taylor series sometimes do not converge rapidly enough to be computationally useful.

Alternatively, the eigenvalues may be computed numerically and much work has been done on this. Suppose that the boundary conditions are $L(a) = L(b) = 0$. A simple method is to use the finite difference approximation

$$L''(\theta_i) \approx \frac{L_{i-1} - 2L_i + L_{i+1}}{h^2}, \tag{4.49}$$

where $L_i = L(\theta_i)$ and $\theta_i = a + hi$, $i = 1, ..., n-1$ and $h = \frac{b-a}{n}$. Then we have the $(n-1) \times (n-1)$ system

$$\frac{L_{i-1} - 2L_i + L_{i+1}}{h^2} + K(\theta_i)L_i + \lambda L_i = 0, \tag{4.50}$$

$L_0 = L_n = 0$, $i = 1, ..., n-1$. We thus consider the $(n-1)\times(n-1)$ matrix M, where $M_{ii} = K(\theta_i) - \frac{2}{h^2}$, $M_{ii+1} = M_{i+1i} = \frac{1}{h^2}$ and all other entries are zero. The negative of the eigenvalues of the tridiagonal matrix M will provide approximations to the eigenvalues of the Sturm-Liouville problem.

To illustrate, consider the test problem of finding the eigenvalues of $y'' + \lambda y = 0$, with $y(0) = y(1) = 0$. We know that the eigenvalues are $\lambda = n^2\pi^2$ and the eigenfunctions are $C\sin(n\pi x)$ for arbitrary constants C. If we take $n = 5$ in (4.50) and solve the resulting eigenvalue problem, we obtain the approximate first eigenvalue $\bar{\lambda}_1 = 9.54915$ which compares reasonably well with the true value of $\lambda_1 = 9.8696$. For the second eigenvalue however, we find the approximate eigenvalue of $\bar{\lambda}_2 = 34.5492$ compared to the true value of $\lambda_2 = 39.4784$.

Taking $n = 100$ gives $\bar{\lambda}_1 = 9.86879$ and $\bar{\lambda}_2 = 39.4654$, which are good approximations. In fact one can get good approximations to the first six eigenvalues with this step size, but the accuracy decreases considerably as n increases. This simple minded approach is therefore not recommended if we need a large number of eigenvalues. For large numbers of eigenvalues, more sophisticated methods, such as those developed in [19], must be used. Good error estimates for the eigenvalues are also available, see for example [25]. See also the SLEIGN project and its more recent extensions, [4].

Given that we have computed the eigenvalues, we can then calculate the corresponding eigenfunctions. These can be obtained in a number of different ways. A simple approach is to compute the eigenvectors of the system (4.50) corresponding to each eigenvalue, then interpolate. For our test problem, using $n = 10$, we find $\lambda_1 = 9.789$ and using the Eigenvector command of Mathematica 6, we obtain the corresponding eigenvector, with entries rounded to three decimal places

$$e_1 = (1, 1.902, 2.618, 3.078, 3.236, 3.078, 2.618, 1.902, 1).$$

The real eigenfunctions are $C\sin(\pi x)$ and it is not hard to see that the entries of this eigenvector are approximately equal to $3.078 \sin\left(\frac{n\pi}{10}\right)$, $n = 1, ..., 9$.

Using the InterpolatingPolynomial command in Mathematica 6, we find the first approximate eigenfunction to be

$$\begin{aligned}L_1(x) &= -0.0801455x^{10} + 0.400727x^9 - 0.140713x^8 - 1.84151x^7 \\ &- 0.0464004x^6 + 8.26755x^5 - 0.00325058x^4 - 16.7226x^3 \\ &- 0.0000338842x^2 + 10.1664x.\end{aligned}$$

Now we easily compute $\int_0^1 L_1(x)^2 dx = 0.5$ which matches the true value of $\int_0^1 \sin^2(\pi x)dx$. L_1 provides an excellent approximation to the true eigenfunction. For example $\sin(\frac{\pi}{20}) = 0.156434 = L_1(\frac{1}{20})$, $\sin(\frac{3\pi}{20}) = 0.45399 = L_2(\frac{3}{20})$ etc. Other forms of interpolation can also be used, such as approximation by rational functions, but we will not consider this here.

Example 4.9. Suppose that $L(0) = L(2\pi) = 0$ and $K(\theta) = -\sqrt{1 + \theta/10}$. We implemented the previous algorithm in Mathematic 6. Using $n = 10$ we compute the first eigenvalues to be $\lambda_1 = 1.389, \lambda_2 = 2.112, \lambda_3 = 3.233, ...$ The first eigenfunction is approximated by

$$L_1(\theta) = 0.008\theta^{11} - 0.285\theta^{10} + 4.343\theta^9 - 37.824\theta^8 + 208.078\theta^7 - 752.207\theta^6 + 1799.99\theta^5 - 2796.18\theta^4 + 2674.44\theta^3 - 1404.6\theta^2 + 302.233\theta,$$

and $\int_0^{2\pi} L_1^2(\theta)d\theta \approx 181.19$. We can compute as many eigenfunctions as we need and from this and Theorem 4.1, we can write down an analytic expression for a fundamental solution of the PDE $u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} - \frac{\sqrt{1+\frac{1}{10}\theta}}{r^2}u$, to whatever degree of accuracy we desire.

More can be said about implementation of these expansion results, but the basic ideas are straightforward.

4.2. **Fundamental Solutions with Distributional Terms.** We have seen in Example 4.7 that finding a transition probability density by reduction of the Kolmogorov equation to another PDE is not always straightforward. This is to be expected from the one dimensional case.

As an example, the PDE $u_t = u_{xx} - \frac{A}{x^2}u$ has a known fundamental solution given by $K(t,x,y) = \frac{\sqrt{xy}}{2t}\exp\left(-\frac{x^2+y^2}{4t}\right) I_{\frac{1}{2}\sqrt{1+4A}}\left(\frac{xy}{2t}\right)$. Now in [8], the diffusion $X = \{X_t : t \geq 0\}$, satisfying the SDE

$$dX_t = \frac{2aX_t}{2+aX_t}dt + \sqrt{2X_t}dW_t, \ X_0 = x > 0, \quad a > 0, \tag{4.51}$$

was considered. The PDE $u_t = xu_{xx} + \frac{2ax}{2+ax}u_x$ can be reduced to $u_t = u_{xx} - \frac{3}{4x^2}u$ by letting $x \to \sqrt{x}$ and $t \to \frac{1}{4}t$, then defining $u(x,t) = e^{\psi(x)}\tilde{u}(x,t)$ for $\psi(x) = \ln\left(\frac{\sqrt{x}}{2+ax^2}\right)$. Using $K(t,x,y)$ we find the fundamental solution $q(t,x,y) = \frac{1}{t}\frac{2+ay}{(2+ax)}\sqrt{\frac{x}{y}}e^{-\frac{(x+y)}{t}}I_1\left(\frac{2\sqrt{xy}}{t}\right)$, but q is not a probability density. The transition density for (4.51) is actually

$$p(t,x,y) = \frac{e^{-\frac{(x+y)}{t}}}{(2+ax)t}\left[\sqrt{\frac{x}{y}}(2+ay)I_1\left(\frac{2\sqrt{xy}}{t}\right) + t\delta(y)\right]. \tag{4.52}$$

The problem is that q was constructed from the 'wrong' fundamental solution. The reduced PDE has another fundamental solution

$$\overline{K}(t,x,y) = 2\, e^{\frac{-(x^2+y^2)}{4\,t}}\, y\sqrt{\frac{y}{x}}\left(\frac{xI_1(\frac{x\,y}{2\,t})}{4\,t\,y} + \delta(y^2)\right), \tag{4.53}$$

and using the change of variables with (4.53) will produce the desired transition density.

In higher dimensions, similar phenomena can occur. Consider the PDE

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} - \frac{A}{r^2}u, \tag{4.54}$$

for which we found a fundamental solution earlier. Let $A = 1$, but now use the stationary solutions $u_n(r,\theta) = r^{-\sqrt{n^2+1}}e^{in\theta}$. We then form the solution

$$u(r,\theta,t) = \int_0^\infty \sum_{n=-\infty}^{\infty} \psi_n(\epsilon)e^{in\theta}\frac{(1+4\epsilon t)^{\sqrt{n^2+1}-1}}{r^{\sqrt{n^2+1}}}e^{-\frac{\epsilon r^2}{1+4\epsilon t}}d\epsilon. \tag{4.55}$$

This leads to

$$f(r,\theta) = \sum_{n=-\infty}^{\infty}\Phi_n(r^2)\frac{1}{r^{\sqrt{n^2+1}}}e^{in\theta} = \sum_{n=-\infty}^{\infty}\widehat{f}(r,n)e^{in\theta}.$$

So that $\Phi_n(r^2) = r^{\sqrt{n^2+1}}\int_0^{2\pi} f(r,\phi)e^{-in\phi}d\phi$.

In order to carry out the analysis used in Theorem 4.1, we need to be able to invert Laplace transforms of the form $F(\lambda) = \lambda^a e^{k/\lambda}$, where $a \geq 0$. The following result is proved in [14].

Proposition 4.5. *The following Laplace transform inversion formula holds when n is a non-negative integer.*

$$\mathcal{L}^{-1}[\lambda^n e^{\frac{k}{\lambda}}] = \sum_{l=0}^{n} \frac{k^l}{l!}\delta^{(n-l)}(y) + \left(\frac{k}{y}\right)^{\frac{n+1}{2}} I_{n+1}\left(2\sqrt{ky}\right). \tag{4.56}$$

If $\mu > 0$ is not an integer then

$$\mathcal{L}^{-1}[\lambda^\mu e^{\frac{k}{\lambda}}] = \left(\frac{k}{y}\right)^{\frac{\mu+1}{2}} I_{-\mu-1}(2\sqrt{ky}).$$

Here $\delta^{(l)}$ is the lth derivative of the Dirac delta function. These inverse Laplace transforms are to be treated as distributions.

Consult the book [26] to see how these distributions are to be evaluated. From Theorem 4.5 we have

$$\int_0^\infty e^{-\frac{r^2+z}{4t}} \left(\frac{1}{4\sqrt{z}t} I_1\left(\frac{r\sqrt{z}}{2t}\right) + \frac{1}{r}\delta(z)\right) e^{-\epsilon z} dz = \frac{1}{r} e^{-\frac{\epsilon r^2}{1+4\epsilon t}}. \tag{4.57}$$

We also have

$$\mathcal{L}^{-1}\left(\frac{(1+4\epsilon t)^{\sqrt{n^2+1}-1}}{r^{\sqrt{n^2+1}}} e^{-\frac{\epsilon r^2}{1+4\epsilon t}}\right) = \frac{1}{4tz^{\sqrt{n^2+1}}} e^{-\frac{r^2+z}{4t}} I_{-\sqrt{n^2+1}}\left(\frac{r\sqrt{z}}{2t}\right).$$

This leads to the solution

$$u(r,\theta,t) = \int_0^\infty \int_0^{2\pi} f(\rho,\phi) p(t,r,\theta,\rho,\phi)\rho d\phi d\rho, \tag{4.58}$$

in which $p(t,r,\theta,\rho,\phi) =$

$$\frac{1}{2t} e^{-\frac{r^2+\rho^2}{4t}} \left(I_1\left(\frac{r\rho}{2t}\right) + \sum_{n\neq 0} e^{in(\theta-\phi)} I_{-\sqrt{n^2+1}}\left(\frac{r\rho}{2t}\right) + \frac{2\rho t}{r}\delta(\rho^2)\right).$$

So we have constructed a second fundamental solution involving distributions. Such fundamental solutions can be constructed for all the PDEs covered by the expansion theorems that we have constructed so far. For example, one may prove the following.

Theorem 4.6. *Suppose that K is continuous and that the Sturm-Liouville problem*

$$L''(\theta) + (K(\theta) + \lambda)L(\theta) = 0 \tag{4.59}$$

$$\alpha_1 L(a) + \alpha_2 L'(a) = 0 \tag{4.60}$$

$$\beta_1 L(b) + \beta_2 L'(b) = 0, \tag{4.61}$$

has a complete set of eigenfunctions and eigenvalues, and that λ_n is never a perfect square. Consider the initial and boundary value problem

$$
\begin{aligned}
u_t &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{K(\theta)}{r^2}u, \qquad (4.62)\\
&r>0,\ a \le \theta \le b,\ a,b \in [0,2\pi].\\
u(r,\theta,0) &= f(r,\theta),\ f \in \mathcal{S}(\mathbb{R}^2),\\
&\alpha_1 u(r,a,t) + \alpha_2 u_\theta(r,a,t) = 0\\
&\beta_1 u(r,b,t) + \beta_2 u_\theta(r,b,t) = 0,
\end{aligned}
$$

Then there is a solution of the form

$$u(r,\theta,t) = \int_0^\infty \int_a^b f(\rho,\varphi)p(t,r,\theta,\rho,\varphi)\rho d\rho d\varphi \tag{4.63}$$

where

$$p(t,r,\theta,\rho,\varphi) = \frac{1}{2t}\sum_n e^{-\frac{r^2+\rho^2}{4t}} L_n(\theta)\overline{L_n(\varphi)} I_{-\sqrt{\lambda_n}}\left(\frac{r\rho}{2t}\right), \tag{4.64}$$

in which $L_n(\theta)$ and $\lambda_n, n = 1,2,3...$ are the normalised eigenfunctions and corresponding eigenvalues for the given Sturm-Liouville problem.

Other results are possible, but we will not attempt to give an exhaustive list. Let us now revisit Example 4.7.

Example 4.10. This example shows why it is crucial that we be able to obtain more than one fundamental solution for certain problems. We consider equation (4.35) again. The average of two fundamental solutions is a fundamental solution. So using Theorem 4.6 and Example 4.7, we obtain the real valued fundamental solution

$$
\begin{aligned}
p(t,x,y,\xi,\eta) &= \frac{1}{8\pi t}\frac{(a\xi+b\eta)\sqrt{x^2+y^2}}{(ax+by)\sqrt{\xi^2+\eta^2}} e^{-\frac{x^2+y^2+\xi^2+\eta^2}{4t}}\\
&\times \sum_{n=-\infty}^{\infty}\left(\frac{(x+iy)(\xi-i\eta)}{(x-iy)(\xi+i\eta)}\right)^{\frac{n}{2}}\left(I_{\sqrt{n^2-1}}\left(\frac{r\rho}{2t}\right) + I_{-\sqrt{n^2-1}}\left(\frac{r\rho}{2t}\right)\right),
\end{aligned}
$$

where $r = \sqrt{x^2+y^2}, \rho = \sqrt{\xi^2+\eta^2}$. This is also a probability density.

5. Expansions in Higher Dimensions

We next consider the problem of extending the results to higher dimensions. We illustrate with the three dimensional case and then present the general results. Consider the PDE

$$u_t = \Delta u + \frac{1}{x^2}k\left(\frac{y}{x},\frac{z}{x}\right)u, \tag{5.1}$$

where k is an arbitrary continuous function. Introducing spherical coordinates, $x = r\cos\theta\sin\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\phi$, with $r \ge$

$0,\ \theta \in [0, 2\pi], \phi \in [0, \pi]$ the equation becomes

$$u_t = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(\frac{1}{\sin^2\phi}u_{\theta\theta} + \cot\phi u_\phi + u_{\phi\phi} + G(\theta, \phi)u), \tag{5.2}$$

with $G(\theta, \phi) = \frac{k(\tan\theta, \cot\phi\sec\theta)}{\cos^2\theta\sin^2\phi}$. The PDE in spherical coordinates has a symmetry

$$U_\epsilon(r, \theta, \phi, t) = \frac{1}{(1+4\epsilon t)^{3/2}}e^{-\frac{\epsilon r^2}{1+4\epsilon t}}u\left(\frac{r}{1+4\epsilon t}, \theta, \phi, \frac{t}{1+4\epsilon t}\right). \tag{5.3}$$

As in the two dimensional case we form a solution

$$U(r, \theta, \phi, t) = \int_0^\infty \varphi(\epsilon)\frac{1}{(1+4\epsilon t)^{3/2}}e^{-\frac{\epsilon r^2}{1+4\epsilon t}}u\left(\frac{r}{1+4\epsilon t}, \theta, \phi, \frac{t}{1+4\epsilon t}\right)d\epsilon,$$

and we will let the solutions $u(r, \theta, \phi, t)$ be stationary solutions. We let $u(r, \theta, \phi) = R(r)\Psi(\theta, \phi)$ and we require

$$\Psi(R'' + \frac{2}{r}R') + \frac{R}{r^2}(\frac{1}{\sin^2\phi}\Psi_{\theta\theta} + \cot\phi\Psi_\phi + \Psi_{\phi\phi} + G(\theta, \phi)\Psi) = 0.$$

So we have

$$\frac{1}{\sin^2\phi}\Psi_{\theta\theta} + \cot\phi\Psi_\phi + \Psi_{\phi\phi} + (G(\theta, \phi) + \lambda)\Psi = (\Delta_{S^2} + (G + \lambda))\Psi = 0, \tag{5.4}$$

and $r^2R'' + 2rR' - \lambda R = 0$. This gives

$$R(r) = c_1 r^{\frac{1}{2}(-1+\sqrt{1+4\lambda})} + c_2 r^{\frac{1}{2}(-1-\sqrt{1+4\lambda})}.$$

We will take $R_n(r) = r^{\frac{1}{2}(-1+\sqrt{1+4\lambda})}$ and choose the eigenfunctions $L^n_\lambda(\theta, \phi)$ of (5.4) to form an orthonormal basis for $L^2(S^2)$ where S^2 is the two dimensional unit sphere. In fact, we can always choose suitable boundary conditions such that the eigenfunctions of $\Delta_{S^{n-1}} + G$, where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on the $n-1$ dimensional sphere and G is continuous, form an orthonormal basis for $L^2(S^{n-1})$, (see Theorem 15.2 of Itô [20]). Then we may expand arbitrary $f \in L^2(S^2)$ as

$$f(\theta, \phi) = \sum_n c_n L_{\lambda_n}(\theta, \phi),$$

where $c_n = \int_0^\pi \int_0^{2\pi} f(\xi, \eta)\overline{L_{\lambda_n}(\xi, \eta)}\sin\eta d\xi d\eta$. Let $u_n = R_n(r)L^n_\lambda(\theta, \phi)$ and

$$u(r, \theta, \phi, t) = \int_0^\infty \sum_n \varphi_n(\epsilon)\frac{e^{-\frac{\epsilon r^2}{1+4\epsilon t}}}{(1+4\epsilon t)^{3/2}}u_n\left(\frac{r}{1+4\epsilon t}, \theta, \phi, \frac{t}{1+4\epsilon t}\right)d\epsilon,$$

where Φ_n is the Laplace transform of φ_n. So that we require

$$u(r, \theta, \phi, 0) = \sum_n \Phi_n(r^2)r^{\frac{1}{2}(-1+\sqrt{1+4\lambda_n})}L_{\lambda_n}(\theta, \phi) = f(r, \theta, \phi). \tag{5.5}$$

From this we find that

$$\Phi(r^2)=\frac{1}{r^{\frac{1}{2}(-1+\sqrt{1+4\lambda_n})}}\int_0^{\pi}\int_0^{2\pi}f(r,\xi,\eta)\overline{L_{\lambda_n}(\xi,\eta)}\sin\eta d\xi d\eta. \tag{5.6}$$

Let $l=\frac{1}{2}(-1+\sqrt{1+4\lambda_n})$, then our solution is

$$u(r,\theta,\phi,t)=\int_0^\infty\sum_n\frac{\varphi_n(\epsilon)r^l}{(1+4\epsilon t)^{\frac{3}{2}+l}}L_{\lambda_n}(\theta,\phi)e^{-\frac{\epsilon r^2}{1+4\epsilon t}}d\epsilon.$$

We use the Laplace transform identity

$$\frac{1}{(1+4\epsilon t)^{\frac{3}{2}+l}}e^{-\frac{\epsilon r^2}{1+4\epsilon t}}=\frac{1}{4t}\int_0^\infty\left(\frac{z}{r^2}\right)^{\frac{l}{2}+\frac{1}{4}}e^{-\frac{r^2+z}{4t}}I_{l+\frac{1}{2}}\left(\frac{r\sqrt{z}}{2t}\right)e^{-\epsilon z}dz, \tag{5.7}$$

and rewrite our solution as

$$\begin{aligned}u(r,\theta,\phi,t)&=\int_0^\infty\int_0^\infty\sum_n e^{-\frac{r^2+z}{4t}}\frac{L_{\lambda_n}(\theta,\phi)}{4tr^{\frac{1}{2}}}z^{\frac{l}{2}+\frac{1}{4}}I_{l+\frac{1}{2}}\left(\frac{r\sqrt{z}}{2t}\right)\varphi_n(\epsilon)e^{-\epsilon z}d\mu\\&=\frac{1}{4t}\int_0^\infty\sum_n\Phi_n(z)e^{-\frac{r^2+z}{4t}}r^{-\frac{1}{2}}z^{\frac{l}{2}+\frac{1}{4}}L_{\lambda_n}(\theta,\phi)I_{l+\frac{1}{2}}\left(\frac{r\sqrt{z}}{2t}\right)dz,\end{aligned}$$

with $d\mu=dzd\epsilon$. Making the change of variables $z\to\rho^2$ gives

$$u(r,\theta,\phi,t)=\int_0^\infty\int_0^{2\pi}\int_0^{\pi}f(\rho,\xi,\eta)p(t,r,\theta,\phi,\xi,\eta)\rho d\eta d\xi d\rho, \tag{5.8}$$

where

$$p(t,r,\theta,\phi,\xi,\eta)=\frac{1}{2t}\sqrt{\frac{\rho}{r}}e^{-\frac{r^2+\rho^2}{4t}}\sin\eta\sum_n L_{\lambda_n}(\theta,\phi)\overline{L_{\lambda_n}(\xi,\eta)}I_{l+\frac{1}{2}}\left(\frac{r\rho}{2t}\right),$$

is a fundamental solution of (5.2). The sum is taken over all the eigenvalues.

Actually the same calculation works in arbitrary dimensions and by the same method we can prove the following result:

Theorem 5.1. *Consider the equation*

$$\begin{aligned}u_t&=u_{rr}+\frac{(n-1)}{r}u_r+\frac{1}{r^2}(\Delta_{S^{n-1}}+G(\boldsymbol{\Theta}))u,\\u(r,\boldsymbol{\Theta},0)&=f(r,\boldsymbol{\Theta}),\end{aligned}$$

and $\alpha(\boldsymbol{\Theta})\Psi(\boldsymbol{\Theta})+(1-\alpha(\boldsymbol{\Theta}))\frac{\partial\Psi}{\partial n}=0$, *with* α *a continuous function and* $\frac{\partial\Psi}{\partial n}$ *the normal derivative on the surface of the unit sphere* S^{n-1} *of dimension* $n-1$. *Here* $\Delta_{S^{n-1}}$ *is the Laplace-Beltrami operator on* S^{n-1} *and* $f\in\mathcal{S}(\mathbb{R}^n)$. *Let* $\boldsymbol{\Theta}=(\theta,\phi_1,...,\phi_{m-2})$. *Then there is a solution of the form*

$$U(r,\boldsymbol{\Theta},t)=\int_0^\infty\int_{S^{n-1}}f(\rho,\xi)p(t,r,\boldsymbol{\Theta},\rho,\xi)\rho d\xi d\rho, \tag{5.9}$$

where for $n \geq 2$,

$$p(t, r, \mathbf{\Theta}, \rho, \xi) = \frac{1}{2t} \left(\frac{\rho}{r}\right)^{\frac{n}{2}-1} e^{-\frac{r^2+\rho^2}{4t}} \sum_{\lambda_m} L_m(\mathbf{\Theta}) \overline{L_m(\xi)} I_{\mu_m} \left(\frac{r\rho}{2t}\right), \tag{5.10}$$

where $\mu_m = \frac{1}{2}\sqrt{4\lambda_m + (n-2)^2}$ and $L_m(\mathbf{\Theta})$ are normalised eigenfunctions of the problem $\Delta_S L + (\lambda + G)L = 0$ and λ_m are the eigenvalues.

Proof. The calculations are essentially the same as in three dimensions. The boundary condition on the unit sphere guarantees the orthonormality and completeness of the eigenfunctions L_{λ_n}. □

Remark 5.2. It is also possible to numerically compute the desired eigenvalues and eigenfunctions, but this is a more difficult problem and we will not consider it here.

Example 5.1. If $P_l(\mathbf{\Theta})$ denotes the lth spherical harmonic on the sphere S^{n-1}, then it is well known that $\Delta_{S^{n-1}} P_l = l(l+n-2)P_l$, see [18]. So that the fundamental solution of $u_t = \Delta u - \frac{A}{r^2}u, A \geq 0$ is

$$p(t, r, \mathbf{\Theta}, \rho, \xi) = \frac{1}{2t} \left(\frac{\rho}{r}\right)^{\frac{n}{2}-1} e^{-\frac{r^2+\rho^2}{4t}} \sum_{l=0}^{\infty} P_l(\mathbf{\Theta}) \overline{P_l(\xi)} I_{\mu_l} \left(\frac{r\rho}{2t}\right), \tag{5.11}$$

where $\mu_l = \sqrt{4l(l+n-2) + (n-2)^2 + 4A}$. Taking $A = 0$ will give the expansion for the heat kernel on $\mathbb{R}^n$.

On S^2 $\Psi_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\phi) e^{im\theta}$, $l = 0, 1, 2, 3, \ldots$ and $-l \leq m \leq l$, are the normalised spherical harmonics. Here $P_l^m(x)$ is the usual Legendre function, see [17]. Thus $u_t = \Delta u - \frac{A}{x^2+y^2+z^2}u$, $A \geq 0$, has the fundamental solution

$$\begin{aligned} p(t, r, \theta, \phi, \rho, \xi, \eta) = \frac{1}{2t}\sqrt{\frac{\rho}{r}} e^{-\frac{r^2+\rho^2}{4t}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(2l+1)(l-m)!}{4\pi(l+m)!} P_l^m(\cos\phi) \\ \times P_l^m(\cos\eta) e^{im(\theta-\xi)} I_{\sqrt{l^2+l+A+\frac{1}{4}}} \left(\frac{r\rho}{2t}\right). \end{aligned}$$

Setting $A = 0$ gives the expansion for the three dimensional heat kernel.

Comparison of (5.11) for $A = 0$ with the heat kernel leads to an interesting summation, that may be new. We present the $n = 3$ version, and the reader will easily see how it extends to arbitrary n.

Corollary 5.3. *The following summation formula holds.*

$$\begin{aligned} &\sqrt{\frac{\rho}{r}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(2l+1)(l-m)!}{(l+m)!} P_l^m(\cos\phi) P_l^m(\cos\eta) e^{im(\theta-\xi)} I_{l+\frac{1}{2}} \left(\frac{r\rho}{2t}\right) \\ &= \frac{\exp\left\{\frac{r\rho(\cos\eta\cos\phi + \cos\theta\cos\xi\sin\eta\sin\phi + \sin\eta\sin\theta\sin\xi\sin\phi)}{2t}\right\}}{\sqrt{\pi t}}. \end{aligned}$$

Proof. Compare the series expansion with the usual heat kernel and note that the heat kernel is unique up to terms involving distributions, which do not arise from the series. □

From this example we can easily compute transition densities.

Example 5.2. Consider the n dimensional Itô process

$$dX_t^i = X_t^i \frac{c_1\alpha_n \|\mathbf{X}_t\|^{\sqrt{(n-2)^2+4A}} - c_2\beta_n}{2\|\mathbf{X}_t\|^2\left(c_1\|\mathbf{X}_t\|^{\sqrt{(n-2)^2+4A}}+c_2\right)} + \sqrt{2}dW_t^i,\ A>0, \quad (5.12)$$
$$i=1,2,3,...,n,$$

where $\|\mathbf{X}_t\|^2 = (X_t^1)^2+\cdots+(X_t^n)^2$, $\alpha_n = \left(2-n+\sqrt{(n-2)^2+4A}\right)$ and $\beta_n = \left(n-2+\sqrt{(n-2)^2+4A}\right)$. The transition probability density is

$$p(t,x,y) = \frac{1}{2t}\frac{\left(c_1+c_2\|y\|^{\sqrt{(n-2)^2+4A}}\right)}{\left(c_1+c_2\|x\|^{\sqrt{(n-2)^2+4A}}\right)}\left(\frac{\|y\|}{\|x\|}\right)^{\frac{n}{2}-1-\frac{1}{2}\beta_n}$$
$$\times e^{-\frac{\|x\|^2+\|y\|^2}{4t}}\sum_{l=0}^{\infty}P_l(x)\overline{P_l(y)}I_{\mu_l}\left(\frac{\|x\|\|y\|}{2t}\right),$$

$x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$ and P_l is the lth harmonic polynomial restricted to the unit sphere. Note that again only the $l=0$ term contributes to the evaluation of the probability distribution.

The second class of expansion theorems generalizes to the following:

Theorem 5.4. *Consider the equation*

$$u_t = u_{rr} + \frac{(n-1)}{r}u_r + \frac{1}{r^2}(\Delta_{S^{n-1}}+G(\mathbf{\Theta}))u - \frac{1}{4}cr^2u,\ c>0$$
$$u(r,\mathbf{\Theta},0) = f(r,\mathbf{\Theta}),$$

and $\alpha(\mathbf{\Theta})\Psi(\mathbf{\Theta}) + (1-\alpha(\mathbf{\Theta}))\frac{\partial\Psi}{\partial n} = 0$, *with* α *a continuous function and* $\frac{\partial\Psi}{\partial n}$ *the normal derivative on the surface of the unit sphere* S^{n-1} *of dimension* $n-1$. *Here* $\Delta_{S^{n-1}}$ *is the Laplace-Beltrami operator on* S^{n-1} *and* $f\in\mathcal{S}(\mathbb{R}^n)$. *Let* $\mathbf{\Theta}=(\theta,\phi_1,...,\phi_{m-2})$. *Then there is a solution of the form*

$$U(r,\mathbf{\Theta},t) = \int_0^\infty\int_{S^{n-1}} f(\rho,\xi)p(t,r,\mathbf{\Theta},\rho,\xi)\rho d\xi d\rho, \quad (5.13)$$

where for $n\geq 2$,

$$p(t,r,\mathbf{\Theta},\rho,\xi) = \frac{2\sqrt{c}}{r^{(n-2)/2}}\left(8(\cosh(2\sqrt{c}t)-1)\right)^{-\frac{n}{4}}\sum_{\lambda_m}\frac{1}{I_{\frac{\sqrt{(n-2)^2+4\lambda_m}}{4}}\left(\frac{\sqrt{c}\rho^2}{4}\right)}$$
$$\times e^{-\frac{\sqrt{c}(r^2+\rho^2)\sinh(2\sqrt{c}t)}{4(\cosh(2\sqrt{c}t)-1)}}\Gamma_m(r,\rho,t)\overline{L_m(\xi)}L_m(\mathbf{\Theta}),$$

$$\Gamma_m(r,\rho,t) = \mathcal{L}^{-1}\left(\frac{1}{(\varepsilon^2-\frac{c}{16})^{\frac{n}{4}}} e^{\left(\frac{a\varepsilon r^2}{\varepsilon^2-\frac{c}{16}}\right)} I_{\frac{\sqrt{(n-2)^2+4\lambda_m}}{4}}\left(\frac{br^2}{\varepsilon^2-\frac{c}{16}}\right)\right),$$

and $a = \frac{c}{8(\cosh(2\sqrt{c}t)-1)}$, $b = \frac{c^{3/2}}{32(\cosh(2\sqrt{c}t)-1)}$, *and* $L_m(\mathbf{\Theta})$ *are normalised eigenfunctions of the problem* $\Delta_{S^{n-1}}L + (\lambda+G)L = 0$ *and* λ_m *are the eigenvalues.*

The examples we have previously considered can obviously be extended to Theorem 5.4. Numerous new examples of densities and functionals can be obtained by these results. A full discussion of this will be given elsewhere, [12].

6. Applications to Representation Theory

Lie symmetries are a priori only locally defined transformations, a fact which has been seen as placing them outside the realm of representation theory. However Craddock has shown that the Lie symmetries of many important PDEs such as the heat equation, are actually equivalent to global representations of the underlying symmetry groups; see [6], [7]. Recently Craddock and Dooley extended this work to some important classes of multi-dimensional PDEs, [9]. In this section, we extend these results to the $SL(2,\mathbb{R})$ symmetries of any PDE of the form $iu_t = \Delta u + A(x)u$.

For simplicity, we consider the unitary case for equations in two space variables of the form

$$iu_t = \Delta u + \frac{1}{x^2}k\left(\frac{y}{x}\right)u. \tag{6.1}$$

The extension to the n dimensional case is easy. In [9] the following irreducible projective representation of $SL(2,\mathbb{R})$ was introduced. See that paper for more details.

Definition 6.1. For $\Re(\nu) > -2$, $\lambda \in \mathbb{R}^*$ and $f \in L^2(\mathbb{R}^+)$ we define the modified Segal-Shale-Weil representation by

$$R_\lambda^\nu \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f(z) = e^{-i\lambda b z^2} f(z) \tag{6.2}$$

$$R_\lambda^\nu \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f(z) = \sqrt{|a|} f(az) \tag{6.3}$$

$$R_\lambda^\nu \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f(z) = \sqrt{|\lambda|}\tilde{f}_\nu(\lambda z). \tag{6.4}$$

Here $\tilde{f}_\nu(y) = \int_0^\infty f(x)\sqrt{xy}J_\nu(xy)dy$, is the Hankel transform of f. The function J_ν is a Bessel function of the first kind.

We introduce the operator

$$\begin{aligned}\mathcal{A}f(r,\rho,t) &= \int_0^\infty \int_0^{2\pi} \frac{\sqrt{\rho}}{4\pi i t} f(\rho) L_n(\theta) L_n(\phi) e^{-\frac{i(r^2+\rho^2)}{4t}} J_\nu\left(\frac{r\rho}{2t}\right) d\phi d\rho \\ &= \int_{\mathbb{R}^2} \frac{\sqrt{\rho}}{4\pi i t} f(\rho) l_n(\theta,\phi) e^{-\frac{i(r^2+\rho^2)}{4t}} J_\nu\left(\frac{r\rho}{2t}\right) d\mu,\end{aligned}$$

where $l_n(\theta,\phi) = L_n(\theta)L_n(\phi)$, $d\mu = d\phi d\rho$, $\nu = \sqrt{\lambda_n}$, with λ_n the nth eigenvalues of the Sturm-Liouville problem in Theorem 4.1, $f \in L^2(\mathbb{R}^+)$, be a solution of (6.1) in polar coordinates. This is constructed by taking one term from the expansion for the fundamental solution. The following result is an elementary consequence of Theorem 4.1.

Lemma 6.2. *Let $u(r,\rho,t) = \mathcal{A}f$, for $f \in L^2(\mathbb{R}^+)$. Then u is a solution of the equation $iu_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{K(\theta)}{r^2}u$, with $K(\theta) = \frac{k(\tan\theta)}{\cos^2\theta}$.*

Notice that our intertwining operator is not a fundamental solution, as is the case for most of the cases treated in [9]. The operator $\mathcal{A}$ is nevertheless sufficient for our purposes.

Theorem 6.3. *The PDE $iu_t = \Delta u + \frac{1}{x^2}k(\frac{y}{x})u$ has Lie symmetry group $SL(2,\mathbb{R}) \times \mathbb{R}$. Moreover if σ represents the Lie symmetry operator and R_1 represents the modified Segal-Shale-Weil projective representation of $SL(2,\mathbb{R})$, then for all $g \in SL(2,\mathbb{R})$ and $f \in L^2(\mathbb{R}^+)$*

$$(\sigma(g)Af)(x,y,t) = (AR_1^\nu(g)f)(x,y,t).$$

Proof. It is sufficient to prove the result in polar coordinates. That is, we prove the equivalence for the PDE $iu_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{K(\theta)}{r^2}u$.

The symmetries in polar coordinates are

$$\sigma(\exp(\epsilon\mathbf{v}_1))u(r,\theta,t) = u(x,y,t-\epsilon),$$

$$\sigma(\exp(\epsilon\mathbf{v}_2))u(r,\theta,t) = e^{-\epsilon}u(e^\epsilon r,\theta,e^{2\epsilon}t),$$

$$\sigma(\exp(\epsilon\mathbf{v}_3))u(x,y,t) = \frac{1}{1+4\epsilon t}\exp\left(-\frac{i\epsilon r^2}{1+4\epsilon t}\right)u\left(\frac{r}{1+4\epsilon t},\theta,\frac{t}{1+4\epsilon t}\right),$$

$$\sigma(\exp(\epsilon\mathbf{v}_4)u(r,\theta,t) = e^{i\epsilon}u(r,\theta,t).$$

Then we need to show that for $k = 1,2,3$,

$$\sigma(\exp(\epsilon\mathbf{v}_k))Af)(r,\theta,t) = AR_1^\nu(\exp(\epsilon X_k)f)(r,\theta,t), \tag{6.5}$$

where $X_2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $X_3 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $X_1 = \begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}$, is a basis for the Lie algebra $\mathfrak{sl}_2$. The result for $\mathbf{v}_4$ is trivial. We can then use the fact that any element of $SL(2,\mathbb{R})$ can be written as a product of exponentials of these basis vectors. We exponentiate X_2 to get $\exp(\epsilon X_2) = \begin{pmatrix}e^\epsilon & 0\\ 0 & e^{-\epsilon}\end{pmatrix}$. Thus we have

$$R_1^\nu(\exp(\epsilon X_2)f)(\rho) = e^{\frac{1}{2}\epsilon}f(e^\epsilon\rho).$$

Now for the equivalence calculation.

$$AR_1^\nu(\exp(\epsilon X_2)f)(r,\theta,t) = \int_{\mathbb{R}^2} e^{\frac{1}{2}\epsilon} f(e^\epsilon \rho)\frac{\sqrt{\rho}}{4\pi t} l_n(\theta,\phi) e^{-\frac{i(r^2+\rho^2)}{4t}} J_\nu\left(\frac{r\rho}{2t}\right) d\mu$$

$$= \int_{\mathbb{R}^2} e^{\frac{1}{2}\epsilon} f(\rho)\frac{\sqrt{e^{-\epsilon}\rho}}{4\pi t} l_n(\theta,\phi) e^{-i(\frac{r^2+e^{-2\epsilon}\rho^2)}{4t}} J_\nu\left(\frac{re^{-2\epsilon}\rho}{2t}\right) e^{-\epsilon} d\mu$$

$$= \int_{\mathbb{R}^2} e^{-\epsilon} f(\rho)\frac{\sqrt{\rho}}{4\pi t} l_n(\theta,\phi) e^{-\frac{i(((e^\epsilon r)^2+\rho^2)}{4e^{2\epsilon}t}} J_\nu\left(\frac{re^{\epsilon}\rho}{2e^{2\epsilon}t}\right) d\mu$$

$$= e^{-\epsilon} u(e^\epsilon r,\theta, e^{2\epsilon}t) = \sigma(\exp(\epsilon \mathbf{v}_2)u)(r,\theta,t).$$

For the $\mathbf{v}_3$ calculation, apply the symmetry

$$\sigma(\exp(\epsilon \mathbf{v}_3))u(x,y,t) = \frac{1}{1+4\epsilon t}\exp\left(-\frac{i\epsilon r^2}{1+4\epsilon t}\right) u\left(\frac{r}{1+4\epsilon t},\theta,\frac{t}{1+4\epsilon t}\right)$$

to

$$p(r,\theta,\rho,\phi,t) = \frac{\sqrt{\rho}}{4\pi i t} L_n(\theta) e^{-\frac{i(r^2+\rho^2)}{4t}} L_n(\phi) J_\nu\left(\frac{r\rho}{2t}\right).$$

Now the terms involving J_ν and $1/(4\pi i t)$ are unchanged by the symmetry. Plainly $\exp\left(-i\frac{(\frac{r^2}{1+4\epsilon t}+\rho^2)}{\frac{4t}{1+4\epsilon t}}\right) = e^{-i\epsilon\rho^2}\exp\left(\frac{-i(r^2+\rho^2)}{4t}\right)$. Thus

$$(\sigma(\exp(\epsilon \mathbf{v}_3))Af)(r,\theta,t) = \int_{\mathbb{R}^2} e^{-i\epsilon\rho^2} f(\rho)\frac{\sqrt{\rho}}{4\pi i t} e^{-i\frac{r^2+\rho^2}{4t}} l_n(\theta,\phi) J_\nu\left(\frac{r\rho}{2t}\right) d\mu$$

$$= \int_{\mathbb{R}^2} R_1^\nu(\exp(\epsilon X_1)) f(\rho)\frac{\sqrt{\rho}}{4\pi i t} e^{-i\frac{r^2+\rho^2}{4t}} l_n(\theta,\phi) J_\nu\left(\frac{r\rho}{2t}\right) d\mu.$$

This establishes the second equivalence.

Finally, $\mathfrak{H}_\nu\left(\frac{\sqrt{\rho}}{2it} e^{\frac{-i(r^2+\rho^2)}{4t}} J_\nu\left(\frac{r\rho}{2t}\right)\right)(z) = e^{-\frac{i\nu\pi}{2}}\sqrt{z}e^{itz^2}J_\nu(rz)$, (see [15]). An elementary calculation, detailed in [9], shows that

$$\mathcal{A}R_1^\nu(\exp(\epsilon X_3))f)(\rho) = \mathfrak{H}_\nu\left(e^{i\epsilon\rho^2}\tilde{f}_\nu\right)(\rho).$$

Now using $\int_0^\infty \tilde{f}_\nu(\rho)g(\rho)d\rho = \int_0^\infty f(\rho)\tilde{g}_\nu(\rho)d\rho$ we have

$$(\mathcal{A}R_1^\nu(\exp(\epsilon X_3))f)(r,\theta,t) = \int_{\mathbb{R}^2} \mathfrak{H}_\nu\left(e^{i\epsilon\rho^2}\tilde{f}_\nu\right)(\rho) p(r,\theta,\phi,t) d\mu$$

$$= \int_{\mathbb{R}^2} e^{i\epsilon z^2}\tilde{f}_\nu(z) l_n(\theta,\phi)\frac{1}{2\pi} e^{-\frac{i\nu\pi}{2}}\sqrt{z}e^{itz^2}J_\nu(rz)dzd\phi$$

$$= \int_{\mathbb{R}^2} \tilde{f}_\nu(z) l_n(\theta,\phi)\frac{1}{2\pi} e^{-\frac{i\nu\pi}{2}}\sqrt{z}e^{i(t-\epsilon)z^2}J_\nu(rz)dzd\phi$$

$$= \int_{\mathbb{R}^2} f(\rho) l_n(\theta,\phi)\frac{\sqrt{\rho}}{4\pi i(t-\epsilon)} e^{-\frac{i(r^2+\rho^2)}{4(t-\epsilon)}} J_\nu\left(\frac{r\rho}{2(t-\epsilon)}\right) d\mu$$

$$= u(x,t-\epsilon).$$

Here $p(r,\theta,\phi,t) = l_n(\theta,\phi)\frac{\sqrt{\rho}}{4\pi i t} e^{-\frac{i(r^2+\rho^2)}{4t}} J_\nu\left(\frac{r\rho}{2t}\right)$. This completes the proof. $\square$

This theorem may easily be extended to the PDE $iu_t = \Delta u + \frac{1}{x_1^2}k\left(\frac{x_2}{x_1},...,\frac{x_n}{x_1}\right)u$. We work in the n dimensional form of polar coordinates and we use the intertwining operator

$$(\mathcal{A}f)(r,\mathbf{\Theta},t) = \int_0^\infty \int_{S^{n-1}} f(\rho)\psi_{\lambda_k}(\mathbf{\Theta},\xi)\frac{\rho^{\frac{1}{2}}}{2tr^{\frac{n-2}{2}}}e^{-\frac{r^2+\rho^2}{4t}}I_l\left(\frac{r\rho}{2t}\right)d\xi d\rho, \tag{6.6}$$

where $l = \frac{1}{2}\sqrt{4\lambda_k + (n-2)^2}$ and $\psi_{\lambda_k} = L_k(\mathbf{\Theta})L_k(\xi)$ and $L_k(\mathbf{\Theta})$ is the kth eigenfunction of $\Delta_{S^{n-1}}u + (G+\lambda)u = 0$, with λ_k the corresponding eigenvalue. The calculations are essentially identical to the previous result. This leads to the following theorem.

Theorem 6.4. *The PDE*

$$iu_t = \Delta u + 2\nabla\phi \cdot \nabla u + B(x)u, x \in \mathbb{R}^n, \tag{6.7}$$

where $\Delta\phi + |\nabla\phi|^2 + \frac{1}{x_1^2}k\left(\frac{x_2}{x_1},...,\frac{x_n}{x_1}\right) = B(x)$ *has* $SL(2,\mathbb{R})$ *as a global group of Lie point symmetries and if* σ *represents the Lie symmetry operator and* R_1 *represents the modified Segal-Shale-Weil projective representation of* $SL(2,\mathbb{R})$*, then for all* $g \in SL(2,\mathbb{R})$ *and* $f \in L^2(\mathbb{R}^+)$

$$(\sigma(g)\mathcal{A}f)(x,y,t) = (\mathcal{A}R_1^\nu(g)f)(x,y,t),$$

with $\mathcal{A}$ *given by (6.6).*

Proof. Equation (6.7) is equivalent to $iu_t = \Delta u + \frac{1}{x_1^2}k\left(\frac{x_2}{x_1},...,\frac{x_n}{x_1}\right)u$ and so they have isomorphic symmetry groups. □

Analagous results can be established for PDEs of the form $iu_t = \Delta u + \frac{1}{x_1^2}k\left(\frac{x_2}{x_1},...,\frac{x_n}{x_1}\right)u - \frac{1}{4}c(x_1^2 + \cdots + x_n^2)$ and also for equations of the form $u_t = \Delta u + A(x)u$. The details will appear in Lennox's thesis.

References

1. J. Abate and W. Whitt, *Numerical Inversion of Laplace Transforms of Probability Distributions*, ORSA J. on Comp **7** (1995), no. 1, 36–53.
2. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables, 10th edition*, Dover, New York, 1972.
3. M.A. Al-Gwaiz, *Sturm-Liouville Theory and its Applications*, Springer Undergraduate Mathematics Series, Springer, Berlin, 2008.
4. P.B. Bailey, *SLEIGN:An eigenvalue-eigenfunction code for Sturm-Liouville problems*, Tech. Report Sand77-2044, Sandia National Laboratory, New Mexico, USE, 1978.
5. S. Bochner, *Harmonic Analysis and the Theory of Probability*, University of California Press, Berkley and Los Angeles, 1955.
6. M. Craddock, *The Symmetry Groups of Linear Partial Differential Equations and Representation Theory I*, J. Differential Equations **116** (1995), no. 1, 202–247.

7. ______, *The Symmetry Groups of Linear Partial Differential Equations and Representation Theory : The Laplace and Axially Symmetric Wave Equations*, J. Differential Equations **166** (2000), no. 1, 107–131.
8. ______, *Fundamental Solutions, Transition Densities and the Integration of Lie Symmetries*, J. Differential Equations (2009), 2538–2560.
9. M. Craddock and A.H. Dooley, *On the Equivalence of Lie Symmetries and Group Representations*, To Appear: J. Differential Equations, 2010.
10. M. Craddock, D. Heath, and E. Platen, *Numerical Inversion of Laplace Transforms: A Survey of Techniques with Applications to Derivative Pricing*, J. Computational Finance **4** (2000), no. 1, 57–81.
11. M. Craddock and K. A. Lennox, *Lie Group Symmetries as Integral Transforms of Fundamental Solutions*, J. Differenital Equations **232** (2007), 652–674.
12. M. Craddock and K.A. Lennox, *Calculating Transition Densities and Functionals for Multi-Dimensional Diffusions by Lie Symmetry Methods*, In preparation.
13. ______, *The Calculation of Expectations for Classes of Diffusion Processes by Lie Symmetry Methods*, Ann. Appl. Prob. **19** (2009), no. 1, 127–157.
14. M. Craddock and E. Platen, *On Explicit Probability Laws for Classes of Scalar Diffusions*, Submitted, March 2009.
15. A. Erdélyi et al., *Tables of Integral Transforms*, vol. 2, McGraw Hill, Inc., New York, Toronto, London, 1954.
16. F. Finkel, *Symmetries of the Fokker-Planck equation with a constant diffusion matrix in 2+1 dimensions*, J. Phys. A: Math. Gen **32** (1999), 2671–2684.
17. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, 6 ed., Academic Press, 2000.
18. D. Gurarie, *Symmetries and Laplacians. Introduction to Harmonic Analysis, Group Representations and Applications*, North Holland Mathematics Study Series, vol. 174, North Holland, 1992.
19. D.L. Harrar II and M.R. Osborne, *Computing eigenvalues of ordinary differential equations*, Anziam J E **44** (2003), 313–334.
20. S. Itô, *Diffusion Equations*, Translations of Mathematical Monographs, vol. 114, American Mathematical Society, Providence R.I., 1992, Translated by S. Itô.
21. P. Laurence and T.H. Wang, *Closed Form Solutions for Quadratic and Inverse Quadratic Term Structure Models*, Preprint, 2005.
22. W.N. Everitt P.B. Bailey and A.Zettl, *The SLEIGN2 database*, Web site: http//www.math.niu.edu/˜zettl/SL2.
23. A. L. Rabenstein, *Introduction to Ordinary Differential Equations. Second Enlarged Edition with Applications*, Academic Press, New York, 1972.
24. D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, 3rd ed., Grundleheren der mathematischen Wissenschaften, vol. 293, Springer, Berlin, 1998.
25. E. Suli and D. Mayers, *An Introduction to Numerical Analysis*, Cambridge University Press, 2003.
26. A. H. Zemanian, *Distribution Theory and Transform Analysis. An Introduction to Generalized Functions with Applications*, Dover Books, 1965.