QUANTITATIVE FINANCE
RESEARCH CENTRE

UNIVERSITY OF
TECHNOLOGY SYDNEY

THINK.CHANGE.DO

QUANTITATIVE FINANCE RESEARCH CENTRE

Research Paper 272 February 2010

Option Valuation in Multivariate SABR Models

Jörg Kienitz and Manuel Wittke

ISSN 1441-8010 www.qfrc.uts.edu.au

Option Valuation in Multivariate SABR Models

- with an Application to the CMS Spread -

Jörg Kienitz*and Manuel Wittke†

February 17, 2010

Abstract

We consider the joint dynamic of a basket of n-assets where each asset itself follows a SABR stochastic volatility model. Using the Markovian Projection methodology we approximate a univariate displaced diffusion SABR dynamic for the basket to price caps and floors in closed form. This enables us to consider not only the asset correlation but also the skew, the cross-skew and the decorrelation in our approximation. The latter is not possible in alternative approximations to price e.g. spread options. We illustrate the method by considering the example where the underlyings are two constant maturity swap (CMS) rates. Here we examine the influence of the swaption volatility cube on CMS spread options and compare our approximation formulae to results obtained by Monte Carlo simulation and a copula approach.

JEL Classification: C63, G12, G13

Keywords: SABR, CMS Spread, Displaced Diffusion, Markovian Projection, Gyöngy Lemma

*Deutsche Postbank AG, Head of Quantitative Analysis, Friedrich-Ebert-Allee 114-126, 53113 Bonn, joerg.kienitz@postbank.de

†University of Bonn, BWL 3 - banking and finance, Adenaueralle 24-32, 53113 Bonn, manuel.wittke@uni-bonn.de. Many thanks to the University of Technology, Sydney and especially to Erik Schlögl, for the support to complete this work.

1 Introduction and Objectives

To value financial instruments taking into account the whole volatility cube can be done by applying a model with stochastic volatility. This approach gained popularity over the last years. One popular model for forward price processes and therefore heavily used in the fixed income market is the SABR model of Hagan et al. [2003]. This model assumes that the forward price process of an asset evolves under a stochastic volatility process correlated with the forward price process. One of the major advantages of the SABR model in comparison to other models with stochastic volatility is, that an approximation of a strike and time to maturity dependent volatility function exists. This approximation can be plugged into the well-known Black [1976] formula to calculate an arbitrage-free price.

In the setting of a basket of forward price processes, an option on the basket can only be valued analytically by the formula of Margrabe [1978] in the case of two assets and a zero strike. For higher dimensions the arbitrage-free price needs to be computed numerically. One numerical method suited to these kind of problem is the Monte Carlo simulation. But in the case of stochastic volatility, this procedure can be very time consuming. This is acceptable if only an arbitrage-free price is be computed, but it is a major problem if the concern is the calibration of a model to market prices. Therefore, approximation formulae for the contracts to be calibrated to should be available.

The Markovian Projection is a method introduced to quantitative finance by Piterbarg [2006] which applies the results by Gyoengy [1986]. The approximation is in the sense of the terminal distribution a basket of diffusions by a univariate diffusion. This method is capable to incorporate stochastic volatility models with a correlation structure between all stochastic variables and has been applied by Antonov and Misirpashaev [2009] to project the spread of two Heston diffusions. Using the case of multivariate SABR diffusions we show, how the basket can be approximated by a displaced diffusion model of Rubinstein [1983] with a SABR style stochastic volatility. Given the approximated SDE, caps/floors on a basket of n-assets can be valued in closed form taking into account the volatility cube and a full correlation structure. As a special case we consider the Geometric Brownian Motion and the Constant Elasticity of Variance.

A liquid financial instrument in the fixed income market that depends on two correlated forward price processes is the CMS spread option. The contracts payoff depends on the spread of two CMS rates with different tenors. A CMS rate is a swaprate paid in one installment. Its name origins from constant maturity swaps.
Regarding the valuation of spread options with nonzero strike several approximations and simulations are discussed in the related literature. Using deterministic volatility the valua-

tion can be done by a semi-analytic conditioning technique, see Belomestny et al. [2008] or in a swap market model or a displaced diffusion swap market model by Monte Carlo simulation as shown by Leon [2006] and Joshi and Yang [2009]. Solutions for stochastic volatility models are given by Dempster and Hong [2000] who extended the FFT method to spread options, Antonov and Arneguy [2009] and Lutz and Kiesel [2010] who consider a stochastic volatility LIBOR Market Model and approximations to the CMS rate as well as numerical integration methods.
One approach in a SABR framework is to use a Gaussian copula with the margins being SABR processes as shown by Berrahoui [2004] and Benhamou and Croissant [2007]. The advantage of our proposed method using the Markovian Projection is that we can include a rich correlation structure and derive a closed form solution which can be extended to the n-asset case.

Concerning the valuation of products dependent on CMS rates, the expected value of a CMS rate under a forward measure is its forward starting value and a convexity correction independent of the chosen pricing model. This convexity correction can be computed by an analytical approximation as discussed in Lu and Neftci [2003] or by using a replication portfolio of European swaptions as proposed by Hagan [2003]. In the case of a Markovian projected spread diffusion the convexity correction can be approximated by the difference of the original CMS convexity corrections under a so-called spread measure.

Numerical results for CMS spread options show, that the Markovian Projection of multivariate SABR diffusions is a good approximation which for example can be used for volatility and correlation calibration. For a liquid range of strike prices from 0 to 100 bp the model prices lie close to the results obtained by Monte Carlo simulation and even outperform a copula approach. But there are parameter sets for which the approximation is less accurate. This is for instance the case for a large time to maturity, which rarely occurs in practical applications.
Concerning the properties of a CMS spread option, the numerical studies show a significant influence of the swaption volatility cube and the correlation between the stochastic correlation parameters on the options price. The final issue cannot be modeled by previous mentioned approximations.

The paper is structured as follows. In Section 2 we first describe the multivariate SABR and the SABR style displaced diffusion model. In a second step the approximated Markovian Projection is computed for the general case of a n-dimensional basket. Section 3 applies the results to the special case of a CMS spread option, where also the convexity correction of CMS rates and a copula approach are presented. The accuracy of the suggested approximations and the properties of CMS spread options are illustrated in Section 4 by numerical examples. Section 5 is the conclusion.

2 Model

One problem encountered when modeling derivatives like swaptions in a Swap Market Model and therefore using the Black [1976] formula is, that the market prices for swaptions cannot be obtained with a constant volatility parameter as the model demands. Instead the volatility tends to rise if the option is out of the money. This results in the so called volatility smile describing the fact that implied Black volatility is strike depended. The problem with implied volatility is that it needs to be interpolated from market data and more important the assumption of a different model for each strike. With this in mind Dupire [1994] proposed the local volatility model. The advantage of this approach is that the model perfectly replicates the current market situation. But the approach behaves poorly in forecasting future dynamics and option pricing is not possible in closed form. An alternative suggested by Hagan et al. [2003] is the so called SABR model where a forward price process is modeled under its forward measure using a correlated stochastic volatility process. Assuming the usual conditions, the diffusion of a forward price $S(t)$ is given by:

$$\begin{aligned}
dS(t) &= \alpha(t)S(t)^{\beta}dW(t) \\
d\alpha(t) &= \nu\alpha(t)dZ(t) \\
S(0) &= s \\
\alpha(0) &= \alpha_0 \\
\langle dW(t), dZ(t)\rangle &= \gamma_{W,Z}dt
\end{aligned}$$

with $\alpha(t)$ the stochastic volatility, ν the volatility of the volatility and $W(t)$ and $Z(t)$ correlated Brownian Motions. $\gamma_{W,Z}$ is the correlation of the forward price and volatility process under an appropriate forward measure. β can be chosen to further specify the distribution of the forward price process. For example $\beta = 1$ constitutes a lognormal distribution and $\beta = 0$ a normal distribution under the assumption of a deterministic volatility and is also called the backbone of the diffusion process.

For a fixed maturity the parameters can be calibrated to all strikes where market data of option volatilities is available. This is the so called volatility cube as shown in figure (1) for the swaption market. One major advantage of the model is that there exists an approximation formula to implied Black [1976] volatility using the SABR parameters. Therefore option prices can be calculated using the well known pricing framework but taking into account the volatility cube using a strike dependent volatility function. Today the SABR model has become one standard model in the financial industry because of the described properties and

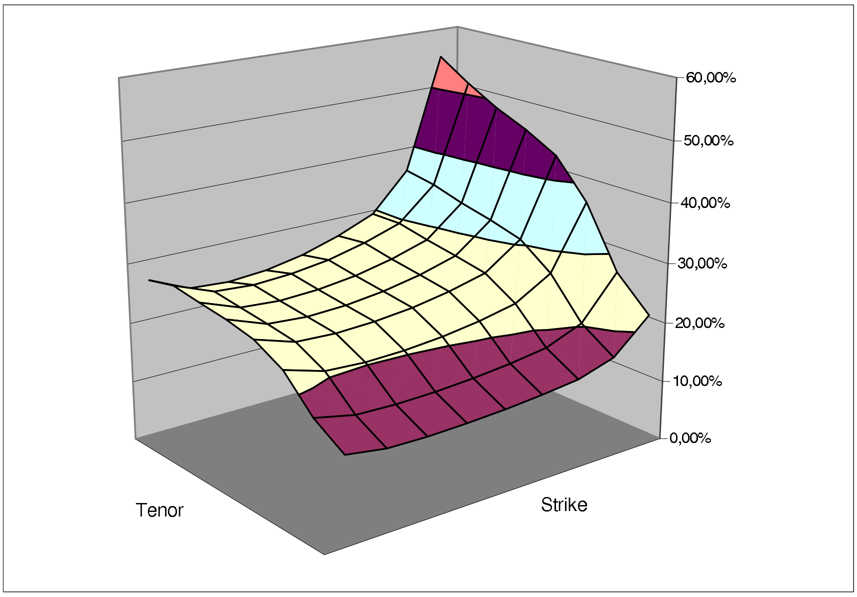

Figure 1: Implied 10y Swaption Volatilities of 18.09.2009.

easy application.

Basket options are options where the underlying is a basket of assets. Let N be the number of different correlated assets, denoting the weights by ϵ_i, $i = 1, \ldots, N$. For instance $N = 2$, $\epsilon_1 = 1$ and $\epsilon_2 = -1$ constitutes a spread. To compute the arbitrage-free price of a basket option with the underlying

$$\sum_{i=1}^{N} \epsilon_i S_i$$

we propose to use a multidimensional SABR model.

Definition 2.1 *A multidimensional SABR diffusion is given as follows. For each asset $S_i(t)$ with $i = \{1, \ldots, N\}$ let:*

$$\begin{aligned}
dS_i(t) &= \alpha_i(t) S_i(t)^{\beta_i} dW_i(t) \\
d\alpha_i(t) &= \nu_i \alpha_i(t) dZ_i(t) \\
S_i(0) &= s_i^0 \\
\alpha_i(0) &= \alpha_i^0 \\
\langle dW_i(t), dW_j(t) \rangle &= \rho_{ij} dt \\
\langle dW_i(t), dZ_j(t) \rangle &= \gamma_{ij} dt \\
\langle dZ_i(t), dZ_j(t) \rangle &= \xi_{ij} dt.
\end{aligned} \tag{1}$$

where ρ_{ij} is the correlation between the Brownian Motions driving the asset price processes, γ_{ij} the cross-skew and ξ_{ij} the so called decorrelation between the stochastic volatilites.

The multidimensional SABR process models the dependency between all factors, which will be further examined in section (4).

A major problem when valuing basket options is that only for $\beta_i = 0 \; i = 1, \ldots, N$, the case of a normal distributed asset and deterministic volatility $\nu_i = 0 \; i = 1, \ldots, N$, the distribution of the basket is known and option prices can be computed in closed form. For the special case of two assets with $\beta_{1,2} = 1$ and a zero strike a solution is given by the Margrabe [1978] formula. But for nonzero strikes and more than two assets under a SABR stochastic volatility only numerical methods and semi-analytic approximation formulae are known. In the following, we extend the framework by a projected multivariate SABR diffusion which can applied to the n-assets case.

2.1 Markovian Projection

An approximation method introduced to quantitative finance by Piterbarg [2006] is the Markovian Projection. It applies the results of Gyoengy [1986] to project multidimensional processes onto a reasonable simple process. Using this methodology we project a multidimensional SABR diffusion process onto a one-dimensional displaced diffusion SABR model. Formally, we approximate the diffusion of the basket with a displaced diffusion with stochastic volatility. The latter results using the Markovian Projection imply $\beta = 1$ and therefore we restrict ourselves to this special case.

Definition 2.2 *A displaced SABR diffusion for $\beta = 1$ is given by:*

$$\begin{aligned} dS(t) &= \alpha(t)F(S(t))dW(t) \\ d\alpha(t) &= \nu\alpha(t)dZ(t) \\ \langle dW(t), dZ(t)\rangle &= \gamma dt \\ \textit{with } F(S(t)) &= p + q(S(t) - S(0)) \\ p &= F(S(0)) \\ q &= F'(S(0)) \end{aligned} \tag{2}$$

where γ denotes the correlation between the forward price and the volatility process.

A displaced diffusion is a reasonable choice, since in case of spread options negative realizations of the spread must have positive probabilities.

The key result to approximate the multidimensional model of Eq. (1) using a single SABR like diffusion, Eq. (2), is the following result derived by Gyoengy [1986].

Lemma 2.1 *Let $X(t)$ be given by*

$$\textit{Let} \quad dX(t) = \alpha(t)dt + \beta(t)dW(t), \tag{3}$$

where $\alpha(.), \beta(.)$ are adapted bounded stochastic processes such that Eq. (3) admits a unique solution. Define $a(t,x), b(t,x)$ by

$$\begin{aligned} a(t,x) &= E[\alpha(t)|X(t)=x] \\ b^2(t,x) &= E[\beta^2(t)|X(t)=x] \end{aligned}$$

Then, the SDE

$$\begin{aligned} dY(t) &= a(t,Y(t))dt + b(t,Y(t))dW(t), \\ Y(0) &= X(0), \end{aligned}$$

admits a weak solution $Y(t)$ that has the same one-dimensional distribution as $X(t)$.

Using Lemma 2.1, the multidimensional model of Eq. (1) is projected onto the displaced SABR diffusion of Eq. (2). The computations involve approximations, which we explain in detail in the proof of the following Theorem.

Theorem 2.1 *The dynamics of a basket of assets following a multivariate SABR model, Eq. (1), is approximated by:*

$$\begin{aligned} dS(t) &= u(t)F(S(t))dW(t) \\ du(t) &= \eta u(t)dZ(t) \\ S(0) &= s_0 \\ u(0) &= 1 \\ \langle dW(t), dZ(t)\rangle &= \gamma dt \\ F(S(0)) &= p \\ F'(S(0)) &= q. \end{aligned}$$

Proof

The approximation is computed in several steps. First, we rewrite the original SABR diffusion of Eq. (1) as a single diffusion with stochastic volatility driven by a Brownian Motion. To preserve the starting values of the process we rescale the volatility of Eq. (1) by:

$$\begin{aligned} u_i(t) &= \frac{\alpha_i(t)}{\alpha_i(0)} \\ \varphi(S_i(t)) &= \alpha_i(0)S_i(t)^{\beta_i} \\ \Rightarrow dS_i(t) &= u_i(t)\varphi(S_i(t))dW_i(t). \end{aligned}$$

Furthermore, we assume $\beta_i = \beta$ and introduce the notation:

$$\begin{aligned} \varphi(S_i(0)) = p_i &= \alpha_i(0)S_i(0)^{\beta} \\ \varphi'(S_i(0)) = q_i &= \alpha_i(0)\beta_i S_i(0)^{\beta-1}. \end{aligned}$$

In the SABR setting we thus choose the local volatility to be $f(x) = x^\beta$ but other choices are possible. Then, we replace the latter expressions by $p_i = f(S_i(0))$ and $q_i = f'(S_i(0))$.

First, using the SDE for the individual assets, we find:

$$\begin{aligned} dS(t) &= \sum_{i=1}^{N} \epsilon_i dS_i(t) \\ &= \sum_{i=1}^{N} \epsilon_i u_i(t) \varphi(S_i(t)) dW_i(t). \end{aligned}$$

Choosing the Brownian Motion such that:

$$dW(t) = \sigma^{-1}(t) \sum_{i=1}^{N} \epsilon_i u_i \varphi(S_i(t)) dW_i(t)$$

we have the representation:

$$dS(t) = \sigma(t) dW(t)$$

with $\epsilon_{ij} = \epsilon_i \cdot \epsilon_j$ and $\sigma(t)$ given by:

$$\sigma^2(t) = \sum_{i=1}^{N} \epsilon_i u_i^2 \varphi^2(S_i(t)) + 2 \sum_{i<j}^{N} \epsilon_{ij} \rho_{ij} u_i u_j \varphi(S_i(t)) \varphi(S_j(t)).$$

Under this specification, the Levy characterization gives that $W(t)$ is a Brownian Motion.

To apply the result of Gyoengy [1986] we need to compute the variance of Eq. (2) on which Eq. (1) is to be projected. We compute $u^2(t)$ as:

$$u^2(t) = \frac{1}{p^2} \left(2 \sum_{i<j}^{N} p_i p_j u_i(t) u_j(t) \rho_{ij} \epsilon_{ij} + \sum_{i=1}^{N} p_i^2 u_i(t)^2 \right) \tag{4}$$

$$\text{with } p = \sqrt{\sum_{i=1}^{N} p_i^2 + 2 \sum_{i<j}^{N} \rho_{ij} p_i p_j \epsilon_{ij}}. \tag{5}$$

The factor $1/p$ is necessary to ensure $u(0) = 1$. For $t = 0$ we find $\sigma(0) = p$.

Now, we are in a position to apply the result of Gyöngy. With the notation of Lemma 2.1 we set $b(t,x) = \mathbb{E}[\sigma^2(t)|S(t) = x]$ and on the other hand $b(t,x) = \mathbb{E}[u^2(t)|S(t) = x] \cdot F^2(x)$. Thus, we have

$$F^2(x) = \frac{\mathbb{E}[\sigma^2(t)|S(t) = x]}{\mathbb{E}[u^2(t)|S(t) = x]}. \tag{6}$$

To compute the conditional expectations of the nominator and the denominator we observe that $\sigma^2(t)$ and $u^2(t)$ are linear combinations of the form:

$$\begin{aligned} f_{ij}(t) &= \varphi(S_i) \varphi(S_j) u_i(t) u_j(t) \\ g_{ij}(t) &= \frac{p_i p_j u_i(t) u_j(t)}{p^2} \end{aligned} \tag{7}$$

and can be represented as follows:

$$
\begin{aligned}
\sigma^2(t) &= \sum_{i=1}^{N} f_{ii}(t) + 2\sum_{i<j}^{N} f_{ij}(t)\rho_{ij}\epsilon_{ij} \\
u^2(t) &= \sum_{i=1}^{N} g_{ii}(t) + 2\sum_{i<j}^{N} g_{ij}(t)\rho_{ij}\epsilon_{ij}.
\end{aligned}
$$

To compute the conditional expectation, a first order Taylor expansion leads to

$$
f_{ij} \approx p_i p_j \left(1 + \frac{q_i}{p_i}(S_i(t) - S_i(0)) + \frac{q_j}{p_j}(S_j(t) - S_j(0)) + (u_i(t) - 1) + (u_j(t) - 1)\right) \tag{8}
$$

and

$$
g_{ij} \approx \frac{p_i p_j}{p^2}\left(1 + (u_i(t) - 1) + (u_j(t) - 1)\right). \tag{9}
$$

Thus, to compute the conditional expectations of Eq. (6) we need simple expressions for

$$
\begin{aligned}
&\mathbb{E}[S_i(t) - S_i(0)|S(t) = x] \\
&\mathbb{E}[u_i(t) - 1|S(t) = x].
\end{aligned} \tag{10}
$$

To find a simple formula we apply a Gaussian approximation to compute the expected values. The Gaussian approximation is a simple but reasonable approximation and is given by:

$$
\begin{aligned}
dS_i(t) &\approx d\bar{S}_i(t) = p_i dW_i(t), \\
du_i(t) &\approx d\bar{u}_i(t) = \nu_i dZ(t), \\
dS(t) &\approx d\bar{S}(t) = p d\bar{W}(t), \\
d\bar{W} &= p^{-1}\sum_{i=1}^{N} p_i \epsilon_i dW_i(t).
\end{aligned}
$$

We have the correlation structure:

$$
\begin{aligned}
\langle d\bar{W}(t), dW_i(t)\rangle &= p^{-1}\sum_{j=1}^{N} p_j \epsilon_j \rho_{ji} dt = \rho_i dt \\
\langle d\bar{W}(t), dZ_i(t)\rangle &= p^{-1}\sum_{j=1}^{N} p_j \epsilon_j \gamma_{ji} \rho_{j+N} dt = \rho_{i+N} dt.
\end{aligned}
$$

The expected values can now be computed. We have:

$$
\mathbb{E}[\bar{S}_i(t) - S_i(0)|\bar{S}(t) = x] = \frac{\langle \bar{S}_i(t), \bar{S}(t)\rangle}{\langle \bar{S}(t), \bar{S}(t)\rangle}(x - S(0)) = p_i \rho_i \frac{x - S(0)}{p}
$$

and

$$
\mathbb{E}[\bar{u}_i(t) - 1|\bar{S}(t) = x] = \nu_i \rho_{i+N} \frac{x - S(0)}{p}.
$$

Using these expressions we compute $F(t,x)$. Denoting the coefficient appearing in the denominator by A_d and the numerator by A_u we find:

$$F^2(x) = \frac{\mathbb{E}[\sigma^2(t)|S(t)=x]}{\mathbb{E}[u^2(t)|S(t)=x]} \approx \frac{p^2 + A_u(x - S(0))}{1 + A_d(x - S(0))} \tag{11}$$

with:

$$\begin{aligned} A_u &= \frac{2}{p}\left\{\sum_{i=1}^{N} p_i^2(q_i\rho_i + \nu_i\rho_{i+N}) \right. \\ &\quad \left. + \sum_{i<j}^{N} \epsilon_{ij}\rho_{ij}p_ip_j(q_i\rho_i + q_j\rho_j + \nu_i\rho_{i+N} + \nu_j\rho_{j+N})\right\}, \\ A_d &= \frac{2}{p^3}\left\{\sum_{i=1}^{N} p_i^2\nu_i\rho_{i+N} \sum_{i<j}^{N} \epsilon_{ij}\rho_{ij}p_ip_j(\nu_i\rho_{i+N} + \nu_j\rho_{j+N})\right\}. \end{aligned}$$

Given these solutions $F(S(0))$ and $F'(S(0))$ are given by:

$$F(S(0)) = p \qquad\qquad F'(S(0)) = q$$

with p given by Eq.(5) and q given by:

$$q = \frac{\sum_{i=1}^{N}\left(p_i^2 q_i\rho_i + \sum_{i\neq j}^{N} p_iq_i\rho_{ij}\rho_i\epsilon_{ij}p_j\right)}{p^2}.$$

Finally, we need to derive a SABR like diffusion for the stochastic volatility and apply the Itô formula to derive the SDE for $u(t)$. Only using first order approximations and replacing the quotients $\frac{u_i(t)u_j(t)}{u^2(t)}$ with the expected value,

$$\mathbb{E}\left[\frac{u_i^2(t)}{u^2(t)}\right] = \mathbb{E}\left[\frac{u_i(t)u_j(t)}{u^2(t)}\right] = 1$$

we find:

$$\begin{aligned} \frac{du(t)}{u(t)} &= \frac{1}{p^2}\sum_{i=1}^{N}\left(p_i^2\frac{u_i^2}{u^2} + \sum_{i\neq j}^{N}\epsilon_{ij}\rho_{ij}p_ip_j\frac{u_iu_j}{u^2}\right)\nu_i dZ_i(t) \\ &= \frac{1}{p^2}\sum_{i=1}^{N}\left(p_i^2 + \sum_{i\neq j}^{N}\epsilon_{ij}p_ip_j\right)\nu_i dZ_i(t). \end{aligned} \tag{12}$$

For more accurate approximations we may keep the higher order terms. This results in a more complex expression and drift terms. In this case we can apply the results for the λ-SABR model, see Labordere [2005].

Thus, by computing the (simple) approximation we obtain a SDE for $u(t)$ which we denote by:

$$du(t) = \eta u(t) dZ(t).$$

For the Brownian Motion $Z(t)$ we have

$$\begin{aligned} dZ(t) &= \frac{\sum_{i=1}^{N} \left(p_i^2 \nu_i + \sum_{i \neq j}^{N} \epsilon_{ij} \rho_{ij} p_i p_j \nu_i \right) dZ_i(t)}{\eta p^2}, \\ \eta^2 &= Var\left(\frac{\sum_{i=1}^{N} \left(p_i^2 \nu_i + \sum_{i \neq j}^{N} \epsilon_{ij} \rho_{ij} p_i p_j \nu_i \right) dZ_i(t)}{p^2} \right) \end{aligned}$$

with η such that $Z(t)$ scales to $\langle Z(t) \rangle = t$. We determine the correlation between the dynamics of the forward price process and the stochastic volatility as:

$$\begin{aligned} \gamma &= \frac{\langle dW(t), dZ(t) \rangle}{dt} \\ &\approx \frac{\langle d\bar{W}(t), dZ(t) \rangle}{dt} \\ &= \frac{1}{\eta p^3} \sum_{i=1}^{N} \sum_{k=1}^{N} \left(p_i^2 \nu_i + \sum_{i \neq j}^{N} \epsilon_{ij} \rho_{ij} p_i p_j \nu_i \right) p_k \epsilon_k \gamma_{ik}. \end{aligned} \tag{13}$$

End of Proof

For $\nu_i = 0,\ i = 1, \ldots, N$ we end up with the projection of CEV diffusions since all stochastic volatility and cross correlation terms in the calculation of q cancel out. For $\nu = 0$ and $\beta_i = 1,\ i = 1, \ldots, N$ the basket of SABR diffusion even simplifies to a basket of Geometric Brownian Motions.

In Figure (2) the densities for the spread of Geometric Brownian Motions are plotted. We compare the application of Markovian Projection and of Monte Carlo simulation. The negative values, especially for a maturity of 10 years, are modeled appropriately. The influence of the difference between the Markovian Projection and the Monte Carlo simulation on the price of options in the case of SABR diffusions is discussed in Section (4).

2.2 Pricing

We now apply our method to the valuation of CMS caplets resp. floorlets. In the above setting we linearize $F(S(t))$ as:

$$\begin{aligned} F(S(t)) &= (S(t) + a)q \\ \text{with } a &= \frac{p}{q} - S(0). \end{aligned}$$

This is to rewrite the projected SDE as a displaced diffusion. Using the implied SABR volatility function σ_{SABR}, the solution of the projected SDE can be written as an asset in a Black [1976] framework and therefore the closed form displaced diffusion formula can be

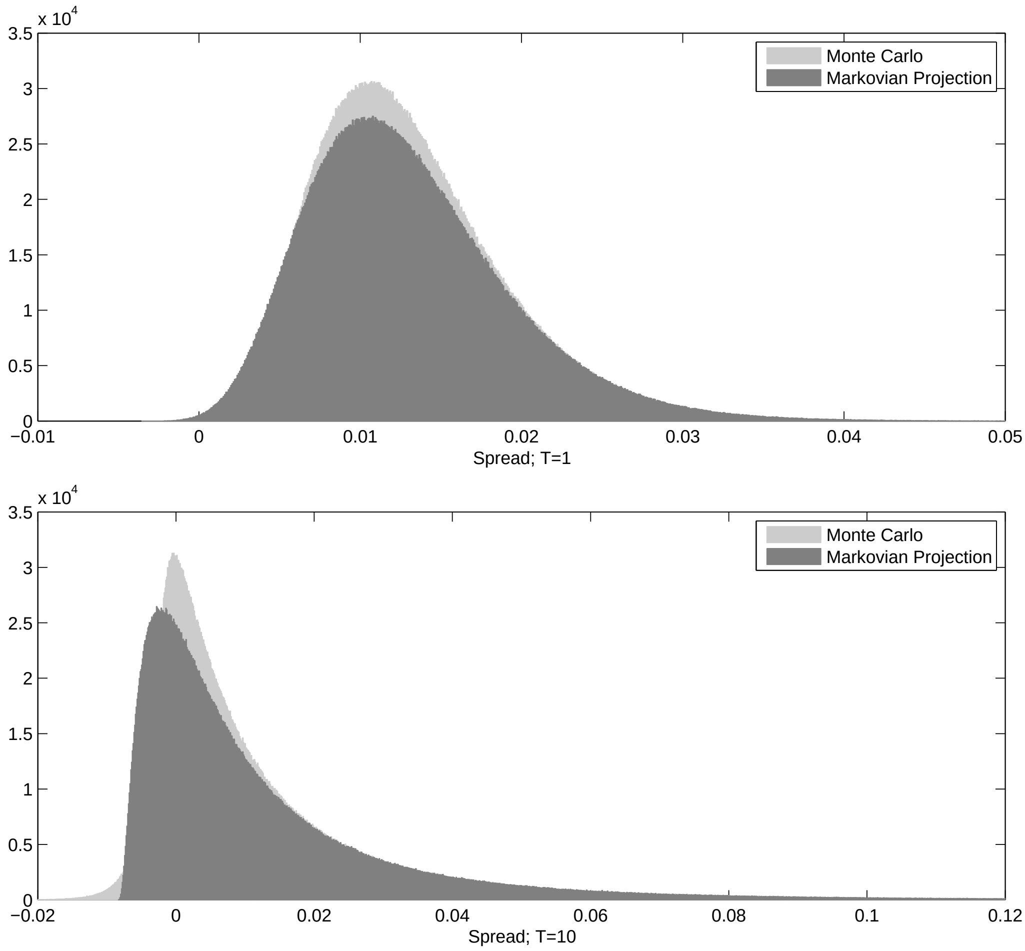

Figure 2: Density of the spread of two Geometric Brownian Motions using a Monte Carlo simulation and a Markovian Projection onto a displaced diffusion.

used. But the expectation of the payoff has to be taken under the T forward measure and since solution of the asset is lognormal distributed we can formulate the pricing equation as:

$$\begin{aligned}
\text{CMScaplet} &= B(0,T)E_T[S(T)-K]^+ \\
&= B(0,T)E_T[S(T)+a-(K+a)]^+ \\
&= B(0,T)\{(E_{P^T}[S(T)]+a)N(d1)-(K+a)N(d2)\} \\
\text{with} \qquad & d_{1/2} = \frac{\ln\left(\frac{E_{P^T}[S(T)]+a}{K+a}\right) \pm \frac{1}{2}\sigma_{SABR}^2(\tau-t)}{\sigma_{SABR}\sqrt{(\tau-t)}}
\end{aligned}$$

The volatility function is given by:

$$
\begin{aligned}
\sigma_{SABR} \quad &= \quad \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}log^2 f/K+\frac{(1-\beta)^4}{1920}log^4 f/K\right\}}\left(\frac{z}{x(z)}\right) \\
& \qquad \left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]t\right\} \\
\text{with} \qquad & z=\frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K, \\
& f=E_{P^T}[S(T)], \\
& \alpha=q \\
\text{and} \qquad & x(z)=\log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}.
\end{aligned}
$$

Therefore, the pricing depends on the chosen measure of the projected SDE. If the SDE is given with respect to the same forward measure, the pricing equation is solved by setting $E_{P^T}[S(T)] = S(0)$. This is in general not the case if different swaprates are projected onto a SDE. The solution in this case is its forward value and a convexity correction as discussed in Section (3). This is the case for a CMS spread option.

3 Application to CMS

Constant maturity swaps (CMS) are interest rate swaps where the fixed leg pays a swaprate with a constant time to maturity at every payment date. These are liquid financial instruments that allow to take positions on future long term rates due to the constant maturity of the fixed leg payments. The underlying swaprate are also an important building block of structured products in todays fixed income markets. Such products incorporate a CMS structure with payment dates similar to a swap but use the constant maturity swaprates as an underlying for embedded options. Common CMS payments in fixed income structured products are

- Capped / Floored CMS Coupons, (CMS_N),
- Capped / Floored CMS Spread Coupons, $(CMS_M - CMS_N)$ and
- Capped / Floored CMS Swing Coupons, $(CMS_N - CMS_O) - (CMS_M - CMS_N)$.

The subscripts indicate that the underlying CMS yields are for different time to maturity with $M > N > O$. The structure of CMS spread and swing options allow to express views on future changes of the shape of the yield curve. Particularly, steepening or flattening is traded using spread options and the curvature of the yield curve using swing options. Therefore, such options can be used as hedges of interest rate correlation risk.

Concerning the pricing of CMS options, regardless of the chosen model, an expectation of a CMS rate at the maturity of the option needs to be computed. Since the CMS rate at maturity τ given by $y_N(\tau) = \frac{1-B(\tau,\tau+N)}{\sum_{i=1}^{N} B(\tau,\tau+i)}$ is payed in one installment and not as a standard swap we can not choose the annuity as a numeraire. Instead we choose a forward risk adjusted measure which coincides with the options maturity and the CMS rate is not a martingale under this measure. To compute the expectation of the CMS rate one has to incorporate a convexity adjustment to the forward swaprate. For a Geometric Brownian Motion this convexity adjustment can be approximated analytically, see Lu and Neftci [2003].
Since we assume stochastic volatility we have to incorporate this into computation of the convexity adjustment. One method proposed by Hagan [2003] is the replication of CMS caplet by a portfolio of European swaptions in the SABR model

$$\begin{aligned} \text{CMS caplet} &= \frac{B(t,\tau)}{A(t)} \left\{ [1+f'(K)]C(K) + \int_K^\infty C(x) f''(x) dx \right\} \\ \text{with} \quad & A(t) = \text{Annuity}, \\ & f(x) = \text{Weightfunction}, \\ & C(x) = \text{Payer Swaption with } K = x. \end{aligned} \tag{14}$$

Therefore, replicating CMS caplets uses all market information by a static hedge portfolio consisting of plain-vanilla swaptions. Using the cap-floor parity

$$\tilde{y}_N(t) = y_N(t,\tau) + \underbrace{\text{CMS caplet} - \text{CMS floorlet}}_{\text{convexity correction}} \tag{15}$$

the convexity adjusted CMS rate $\tilde{y}_N(t)$ can be computed by the forward starting swaprate and a portfolio of payer and receiver swaptions.

3.1 CMS Spread Options

To give comparable numerical results for CMS options priced by the Markovian Projetion approach in a multidimensional SABR model we restrict the implementation to the case of a CMS spread option. The payoff at maturity τ is as follows:

$$\max\{y_M(\tau) - y_N(\tau) - K, 0\}$$

with $M \geq N$.
The special case of zero strike options, $K = 0$, can be solved analytically using the formula for exchange options, see Margrabe [1978]. For $K \neq 0$ an analytical solution is only feasible if the spread is modeled as a normal distributed random variable

$$\begin{aligned} y_M(\tau) - y_N(\tau) &= \bar{y}(\tau) \\ \text{with } d\bar{y}(t) &= \bar{\sigma} d\bar{W}(t). \end{aligned}$$

This framework is too simple to consistently price CMS Spread Options since implicitly a perfect correlation is assumed. And it is also not taking into account the smile and the skew effects. The market quotes spread options by their implied normal volatilities such as swaptions are quoted by their implied Black volatility.

In the following we present the copula approach of Berrahoui [2004] and Benhamou and Croissant [2007] and show how to project the spread onto a displaced diffusion using a SABR model. Then, we price a CMS spread option using both approaches.

3.2 Approximation of the Correlation Structure

One way to approximate spreads in a SABR framework is the copula approach which we review in the following. The idea is that the payoff of spread options with two correlated price processes can be decomposed into a portfolio of digital options and is given as:

$$\max\{y_M(\tau) - y_N(\tau) - K, 0\} = \int_0^\infty 1_{[y_M(\tau)>x+K]} 1_{[y_N(\tau)<x]} dx.$$

Now taking the discounted expectation under the risk adjusted measure P^τ, the fair value can be computed by using numerical integration:

$$\text{CMS}^{spread}_{caplet} = B(t,\tau) \int_0^\infty P^\tau(\tilde{y}_M(\tau) > x + K, \tilde{y}_N(\tau) < x) dx. \tag{16}$$

The joint probability function $P^\tau(\ldots)$ can be computed using a Gaussian copula with the SABR margins. The procedure consists of two steps. First, we have to compute the margins of the SABR distributions and then have to map the quantiles them onto a lognormal distribution as shown in Figure (3). The second step uses the Gaussian copula to obtain the joint probability function. As stated above, we first need the margins $P_{SABR}(\tilde{y}_i(\tau) > x_i)$

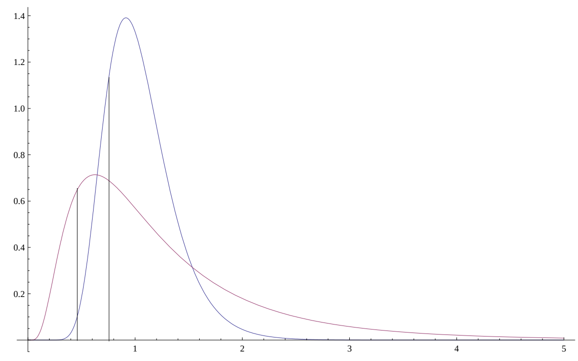

Figure 3: Quantile mapping of a SABR distribution onto a lognormal distribution

which can be computed numerically or replicated using digital options. To map the SABR distribution onto a lognormal for a given SABR quantile we compute the equivalent Black quantile and solve for the value $\bar{x}_i$:

$$P_{SABR}(\tilde{y}_i(\tau) > x_i) = P_{Black}(\tilde{y}_i(\tau) > \bar{x}_i | \sigma = \sigma_{SABR}).$$

The joint probability distribution is computed using a Gaussian copula and correlation $\langle dW_N(t), dW_M(t)\rangle = \rho dt$. It is given by:

$$\begin{aligned} & P(\tilde{y}_M(\tau) > x + K, \tilde{y}_N(\tau) < x) \approx N(d_2) - N_2(d_1, d_2, \rho) \\ & \text{with } d_i = \frac{1}{\sigma_{SABR}\sqrt{\tau}}\left(ln\left(\frac{\bar{x}_i}{\tilde{y}_{M/N}(t)}\right) + \frac{1}{2}\sigma^2_{SABR}\tau\right). \end{aligned} \tag{17}$$

To compute the approximated arbitrage-free price of the CMS spread option we need to apply Eq. (17) and substitute it into Eq. (16). Finally we use a numerical integration method.

We can alter the correlation structure using a different copula, for instance the t-copula with heavier tail dependence. As will be shown in Section (4) the copula approach prices the CMS Spread Options fairly accurate, but there are still some drawbacks of this method:

- The copula method is static and we have no process of the spread dynamic.
- The numerical integration is time consuming.
- The decorrelation and cross skews are assumed to be uncorrelated.
- The methodology cannot be extended to CMS options with more than two CMS rates.

3.3 Approximation of the Spread Diffusion

In this subsection we apply the general results obtained in the previous section to the case of a CMS spread option. The guiding idea is to compute a SDE for the spread dynamics which approximates the joint SABR dynamics at maturity using the full correlation structure including decorrelation and cross skew. It also captures the volatility smile as it can be seen in Figure (4) for some given parameters.

Theorem 3.1 *The dynamics of the spread can be approximated by*

$$\begin{aligned} dS(t) &= u(t)F(S(t))dW(t) \\ du(t) &= \eta u(t)dZ(t) \end{aligned}$$

with $u(t)$ and $p(t)$ given by Eq. (4) with $N = 2$ and the function $F(.)$ satisfying:

$$F(S(0)) = p \qquad F'(S(0)) = q$$

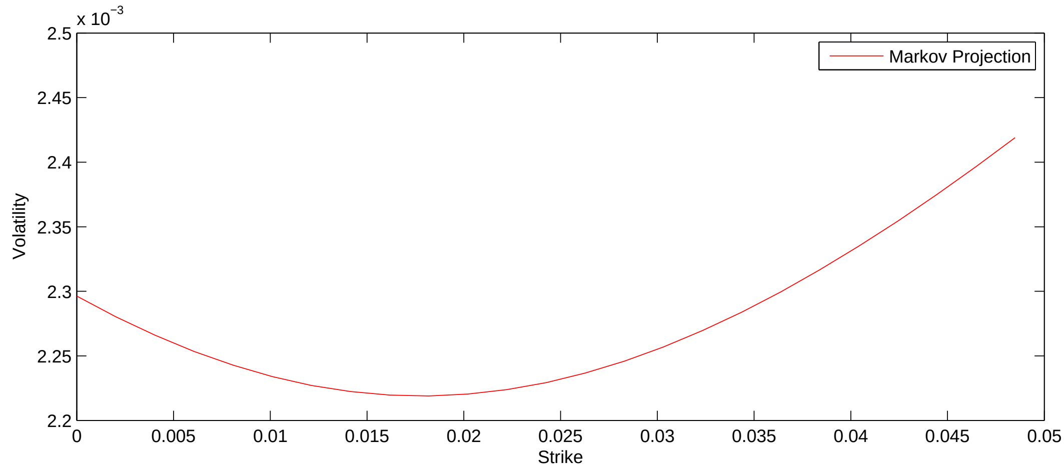

Figure 4: Strike dependent implied CMS spread call volatilities of prices obtained by a Markov Projection with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$, $\nu_2 = 0.4$ and $T = 10$.

with

$$q = \frac{p_1 q_1 \rho_1^2 - p_2 q_2 \rho_2^2}{p},$$

$$\eta = \sqrt{\frac{1}{p^2}\left[(p_1\nu_1\rho_1)^2 + (p_2\nu_2\rho_2)^2 - 2\xi_{12}p_1\nu_1\rho_1p_2\nu_2\rho_2\right]}.$$

and

$$\gamma = \frac{1}{\eta p^2}\left(p_1^2\nu_1\rho_1\gamma_{11} + p_2^2\nu_2\rho_2\gamma_{22} - p_1p_2\nu_2\rho_2\gamma_{21} - p_1p_2\nu_1\rho_1\gamma_{12}\right).$$

Proof

We consider the dynamics of Eq. (1) and compute the diffusion for the spread taking $N = 2, \epsilon_1 = 1$ and $\epsilon_2 = -1$. Since we model swaprates with different tenor structures, we cannot model them as driftless processes under the same forward measure P^T since they obtain a drift term μ_i. In fact, they are driftless under their own annuity measure P^{A_i}. Therefore, we change both measures to a so-called spread measure P^S under which their spread SDE is driftless and given by:

$$\begin{aligned} dS(t) &= dS_1(t) - dS_2(t) \\ &= (\mu_1 dt + u_1(t)\varphi(S_1(t))dW_1^T(t)) - (\mu_2 dt + u_2(t)\varphi(S_2(t))dW_2^T(t)) \\ &\overset{!}{=} \sigma(t)dW^S(t) \end{aligned}$$

with:

$$\begin{aligned} dW^S(t) &= \frac{1}{\sigma(t)}\left(u_1(t)\varphi(S_1(t))dW_1^S - u_1(t)\varphi(S_2(t))dW_2^S\right) \\ \sigma^2(t) &= u_1^2(t)\varphi(S_1(t))^2 + u_2^2(t)\varphi(S_2(t))^2 \\ &- 2\rho_{12}u_1(t)u_2(t)\varphi(S_1(t))\varphi(S_2(t)). \end{aligned} \tag{18}$$

In a second step we compute the variance of the approximating SDE as given by Eq. (18):

$$\begin{aligned} u^2(t) &= \frac{1}{p^2}\Big(p_1^2 u_1^2(t) + p_2^2 u_2^2(t) - 2\rho_{12} p_1 p_2 u_1(t) u_2(t)\Big) \\ \text{with } p = \sigma(0) &= \sqrt{p_1^2 + p_2^2 - 2\rho_{12} p_1 p_2}. \end{aligned}$$

At this point we have two representations for the spread SDE:

$$dS(t) = \sigma(t) dW^S(t) \quad \text{and} \quad dS(t) = u(t) F(S(t)) dW^S(t).$$

With the first equation being the original spread SDE and the second one the approximating SDE under the spread measure. We now have to compute the parameters of the approximating SDE that mimic the terminal one-dimensional distribution of the original SDE. Applying the Gyoengy [1986] result, we have to choose $F^2(t,x)$ such that:

$$F^2(x) = \frac{\mathbb{E}\left[\sigma^2(t) | S(t) = x\right]}{\mathbb{E}\left[u^2(t) | S(t) = x\right]}. \tag{19}$$

To proceed, we use Eq. (7) to further simplify the notation. Then, we can compute the volatilities:

$$\begin{aligned} \sigma^2(t) &= f_{11}(t) + f_{22}(t) - 2\rho_{12} f_{12}(t) \\ u^2(t) &= g_{11}(t) + g_{22}(t) - 2\rho_{12} g_{12}(t). \end{aligned}$$

To be able to compute the conditional expectations, we use the first order Taylor approximation as in Eq. (8) and Eq. (9). This reduces the problem to the computation of the conditional expectations for $S_i(t)$ and $u_i(t)$, Eq. (10). To make the calculations more explicit we apply a Gaussian approximation. Using the approximation we can simplify the conditional expectations as follows:

$$\mathbb{E}[\bar{S}_i(t) - S_i(0) | \bar{S}(t) = x] = p_i \rho_i \frac{(x - S(0))}{p}$$

and

$$\mathbb{E}[\bar{u}_i(t) - 1 | \bar{S}(t) = x] = \nu_i \rho_{i+2} \frac{(x - S(0))}{p}.$$

This leads to a simple expression for the numerator and the denominator of equation (19):

$$\begin{aligned} \mathbb{E}\left[\sigma^2(t) | S(t) = x\right] &\approx p^2 + (x - S(0)) A_u \\ \text{with } A_u &= \frac{2}{p}\Big(p_1^2(q_1\rho_1 + \nu_1\rho_3) + p_2^2(q_2\rho_2 + \nu_2\rho_4) \\ &\quad - p_1 p_2 \rho_{12}\big(q_1\rho_1 + q_2\rho_2 + \nu_1\rho_3 + \nu_2\rho_4\big)\Big), \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left[u^2(t) | S(t) = x\right] &\approx 1 + (x - S(0)) A_d \\ \text{with } A_d &= \frac{2}{p^3}\Big(\nu_1 p_1 (p_1 - p_2\rho_{12})\rho_3 + \nu_2 (p_2 - p_1\rho_{12})\rho_3\Big). \end{aligned}$$

We can compute the approximating SDE for $S(t)$. To compute the dynamics of $u(t)$, we apply Eq. (12) to get:

$$\begin{aligned} du(t) &= \frac{1}{p^2}\Big(p_1^2\nu_1\frac{u_1^2}{u^2} - \rho_{12}p_1p_2\nu_1\frac{u_1u_2}{u^2}\Big)u(t)dZ_1(t) \\ &+ \frac{1}{p^2}\Big(p_2^2\nu_2\frac{u_2^2}{u^2} - \rho_{12}p_1p_2\nu_2\frac{u_1u_2}{u^2}\Big)u(t)dZ_2(t) \\ \Leftrightarrow \frac{du(t)}{u(t)} &= \left(\frac{p_1\nu_1\rho_1}{p}dZ_1(t) + \frac{p_2\nu_2\rho_2}{p}dZ_2(t)\right) \end{aligned}$$

and we have the SDE by setting:

$$\begin{aligned} Z(t) &= \frac{1}{\eta p}\Big(p_1\nu_1\rho_1 dZ_1 - \rho_2\nu_2\rho_2 dZ_2\Big) \\ \eta^2 &= \frac{1}{p^2}\left[(p_1\nu_1\rho_1)^2 + (p_2\nu_2\rho_2)^2 - 2\xi_{12}p_1\nu_1\rho_1p_2\nu_2\rho_2\right]. \end{aligned}$$

The correlation between the projected forward price process $S(t)$ and its stochastic volatility process $u(t)$ can be computed using Eq. (13):

$$\gamma = \frac{1}{\eta p^2}\left(p_1^2\nu_1\rho_1\gamma_{11} + p_2^2\nu_2\rho_2\gamma_{22} - p_1p_2\nu_2\rho_2\gamma_{21} - p_1p_2\nu_1\rho_1\gamma_{12}\right).$$

End of Proof

To compute the price of an option using the displaced diffusion model, the expectation of the approximated spread at maturity needs to be computed. But the expectation is under the forward measure, while the approximated spread is under the spread measure. This can be solved by changes of measure and using the convexity adjustment, Eq. (15). Denoting by $A_i(t)$ the numeraire of the annuity measure P^{A_i} and by $SN(t)$ the numeraire of the spread measure the expectation can be computed:

$$\begin{aligned} & E_{P^T}\left[S(T)\right] \\ &= E_{P^S}\left[S(T)\frac{B(T,T)}{SN(T)}\frac{SN(0)}{B(0,T)}\right] \\ &= S(0) + E_{P^{A_1}}\left[S_1(T)\left(\frac{B(T,T)}{A_1(T)}\frac{A_1(0)}{B(0,T)} - 1\right)\right] - E_{P^{A_1}}\left[S_1(T)\left(\frac{SN(T)}{A_1(T)}\frac{A_1(0)}{SN(0)} - 1\right)\right] \\ &\quad - E_{P^{A_2}}\left[S_2(T)\left(\frac{B(T,T)}{A_2(T)}\frac{A_2(0)}{B(0,T)} - 1\right)\right] + E_{P^{A_2}}\left[S_2(T)\left(\frac{SN(T)}{A_2(T)}\frac{A_2(0)}{SN(0)} - 1\right)\right] \\ &\approx \{S_1(0) - S_2(0)\} + \{\text{convexity correction}(S_1) - \text{convexity correction}(S_2)\}. \end{aligned}$$

The terms $E_{P^{A_i}}\left[S_i(T)\left(\frac{B(T,T)}{A_i(T)}\frac{A_i(0)}{B(0,T)} - 1\right)\right]$ denote the convexity correction, see for instance Hagan [2003], of the swapyield i which can be conducted by a replication portfolio within a SABR framework. The difference:

$$\left\{E_{P^{A_1}}\left[S_1(T)\left(\frac{SN(T)}{A_1(T)}\frac{A_1(0)}{SN(0)} - 1\right)\right] - E_{P^{A_2}}\left[S_2(T)\left(\frac{SN(T)}{A_2(T)}\frac{A_2(0)}{SN(0)} - 1\right)\right]\right\} \approx 0$$

is approximated with a zero value, since the corrections due to the mismatch of the annuity measures and the spread measure can be assumed to be close to zero with nearly identical values for both expectations. Using convexity corrected swaprates the valuation of a CMS spread caplet or floorlet is now possible.

4 Numerical Results

To illustrate the approximation in the case of a basket option using the Copula and the Markovian Projection approach, we apply the results obtained in Section (3) for valuation of spread options in a SABR model. Since the copula approach is as discussed only valid for two underlying diffusions. As a benchmark we apply a Monte Carlo simulation for the multivariate SABR model.
In the following we consider the parameters: $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$, $\nu_2 = 0.4$ and $T = 10$.

First, we study the effect of changing the time to maturity and strike prices on the option prices. To this end we price CMS spread calls and change the time to maturity and the strike prices. In Figure (5) the numerical results of the Copula approach, the Markovian Projection approach and a Monte Carlo simulation are plotted.
It can be seen that the fit of the Copula approach and the Markovian Projection approach is reasonable good for five years to maturity. For ten years to maturity the goodness of the approximations is still good but the reference prices of the Monte Carlo simulation are clearly not in line with them. Both prices lie strictly below the Monte Carlo simulation but the Markovian Projection outperforms the Copula approach. As a result the approximations depend on the time to maturity and therefore should for longer times to maturity only be used with care for the calibration to market prices.

To examine the influence of the swaption volatility cube on the prices of CMS spread options we consider the strike dependent prices in Figure (6). For a SABR model calibrated to market data of a given strike range and a GBM using the ATM volatility we consider their price differences. It can be seen that the influence is significant. Therefore, it must be incorporated for longer times to maturity.

One advantage of the Markovian Projection in comparison to the Copula approach is that the cross skew and the decorrelation are incorporated into the pricing. The influence of these parameters on the arbitrage-free price is significant as shown in Figure (8). There, arbitrage-free prices are plotted in dependence of the strike prices for different parameter

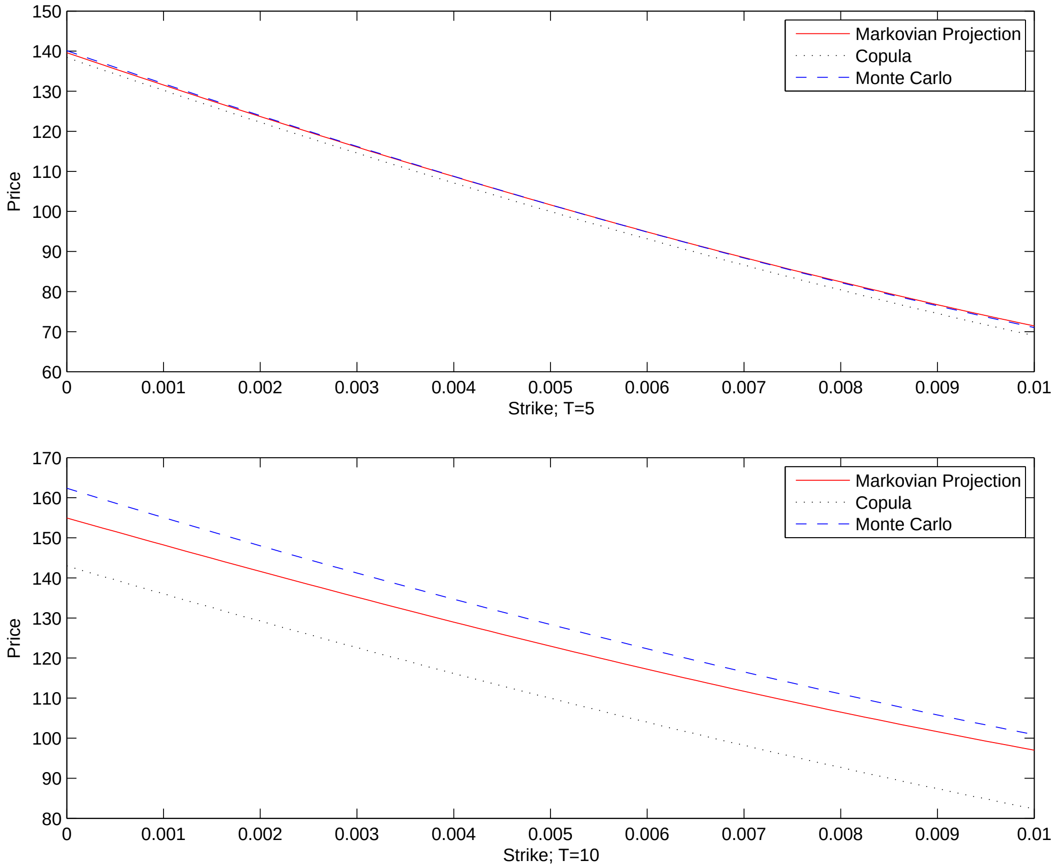

Figure 5: Strike dependent CMS spread call prices of a Markov Projection, a Copula and a Monte Carlo simulation with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$, and $\nu_2 = 0.4$. The first Figure is plotted with $T = 5$ and the second with $T = 10$.

values. The decorrelation parameter ξ shifts the prices parallel with a negative decorrelation leading to the lowest prices. This is due to the dependency of the spread distribution to the decorrelation. A lower decorrelation parameter shifts mass into the tails of the distribution. This comes clear by considering Figure (7) where two histograms of a SABR spread density are plotted for different values of ξ. If we change both cross skews $\gamma = \gamma_1 = \gamma_2$ simultaneously, the divergence in prices is smaller than by changing the decorrelation ξ with a slightly twist. As a result, if a multivariate SABR model is used to price baskets the decorrelation and cross skew parameters have a significant influence on the price.

Since the Markovian Projection is an approximation which is less accurate for long times to maturity, a proper valuation of a basket should in this case be done by a Monte Carlo simulation using the Markovian Projection for calibration. But the calibration is numerically very fast, since the Markovian Projection is an analytical approximation, while the Monte

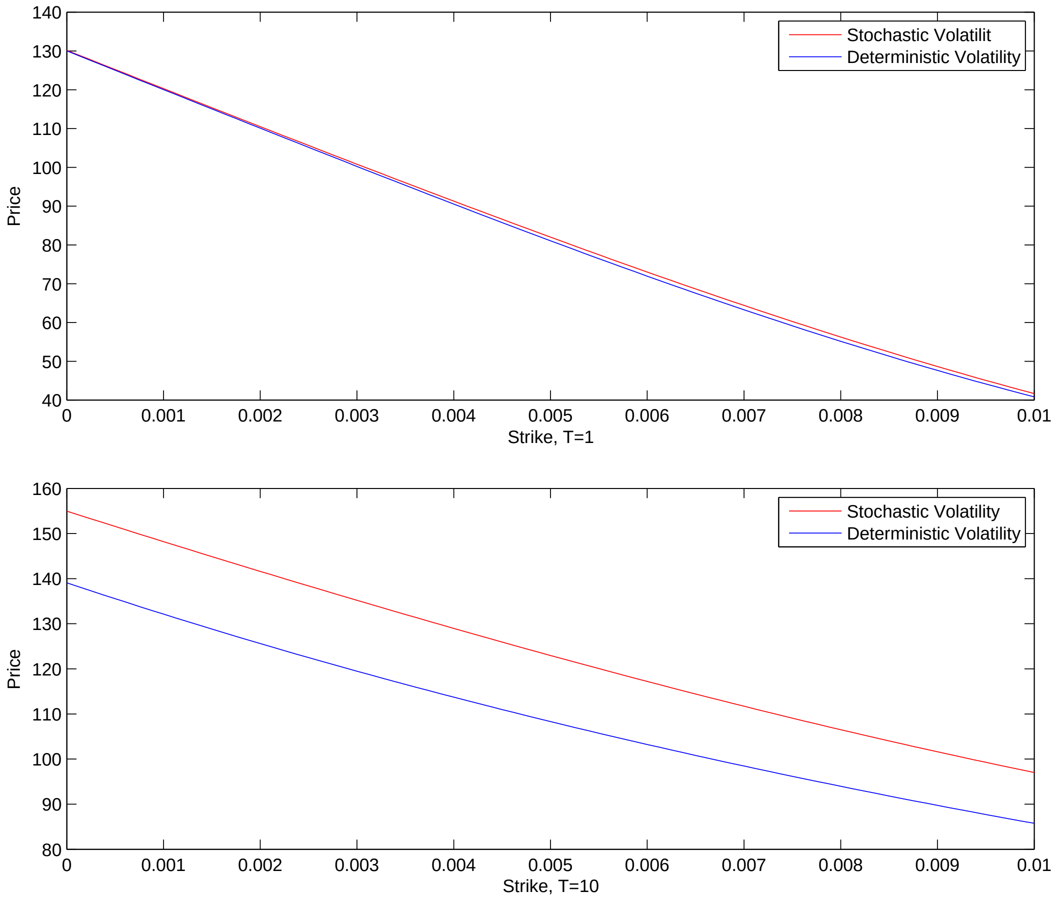

Figure 6: Strike dependent CMS spread call prices using GBM (deterministic volatility) and a SARR model (stochastic volatility) with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$ and $\nu_2 = 0.4$.

Carlo simulation and the Copula approach are plain numerical methods.

5 Conclusion

We have presented the application of the Markovian Projection technique to the SABR stochastic volatility model in multiple dimensions. As an example we have applied it to a popular interest rate derivative, the CMS spread option that significantly depends on the swaption volatility cube. The proposed technique takes into account all parameters modeling the dependence structure such as the correlation of the underlying forward CMS rates, the correlation between the rates and the volatility processes and the correlation between the volatility processes.

We find a good match with results obtained using Monte Carlo simulation. However, there are parameter sets where the fit is not reasonable. In particular changing the time to ma-

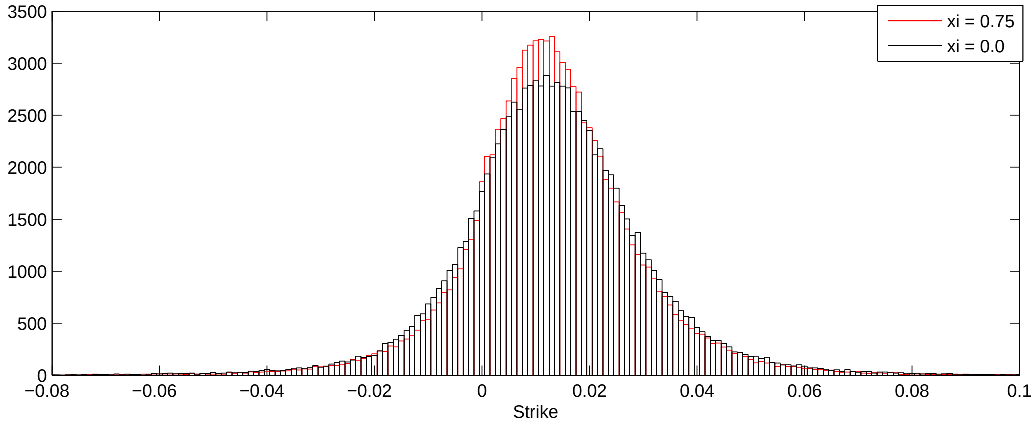

Figure 7: Histograms of CMS spread densities using a two-dimensional SABR model with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\beta = 0.7$, $\nu_1 = 0.4$, $\nu_2 = 0.4$ and $T = 10$.

turity makes the fir worse. We found that for short time to maturities the approximation is good whereas for large values the approximation gets weak. But even for long maturities the Markov Projection can still be used for calibration of the volatility and correlation parameters.

References

A. Antonov and M. Arneguy. Analytical formulas for pricing cms products in the libor market model with the stochastic volatility. *available at SSRN, http://ssrn.com/abstract=1352606*, 2009.

A. Antonov and T. Misirpashaev. Markovian projection onto a displaced diffusion. *International Journal of Theoretical and Applied Finance*, 12(4):507–522, 2009.

D. Belomestny, A. Kolodko, and J. Schoenmakers. Pricing cms spreads in the libor market model. *Weierstrass Institute for Applied Analysis and Stochastics working paper*, 1386, 2008.

E. Benhamou and O. Croissant. Local time for the sabr model: Connection with the complex black scholes and application to cms and spread options. *SSRN: http//ssrn.com/abstract=1064461*, 2007.

M. Berrahoui. Pricing cms spread options and digital cms spread options with smile. *Wilmott magazine*, pages 63–69, May 2004.

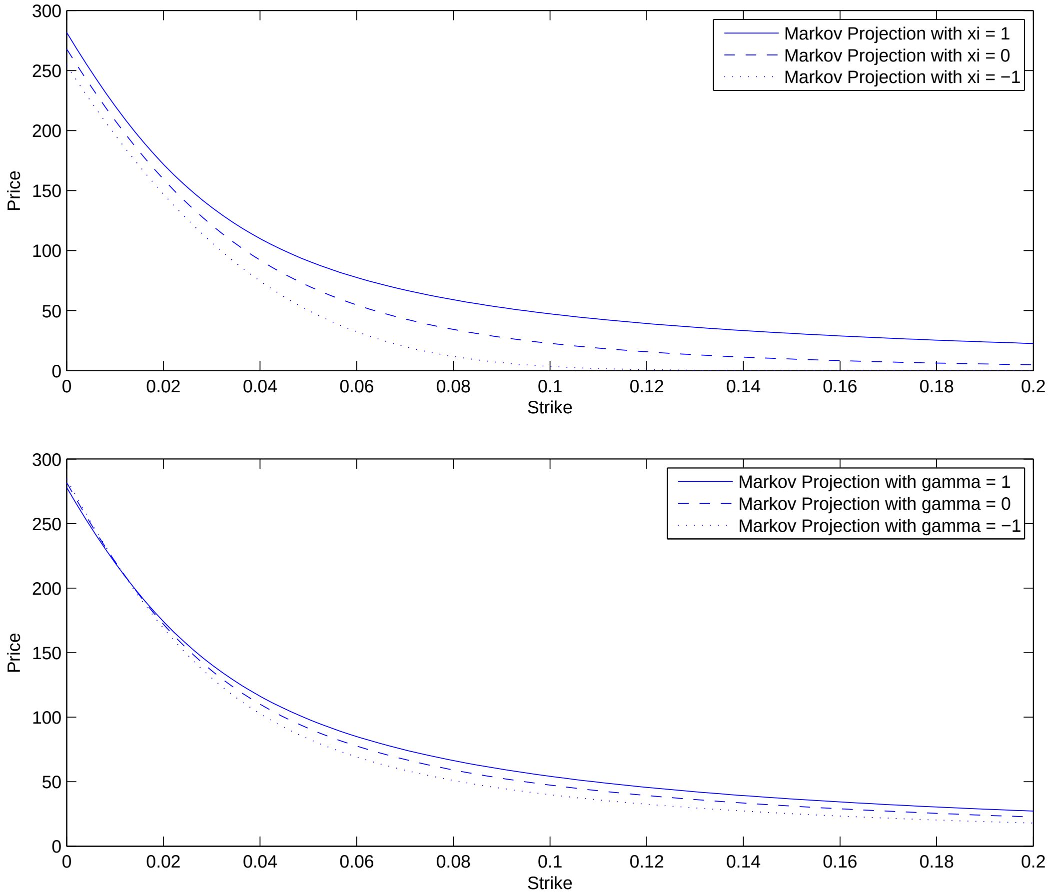

Figure 8: Strike dependent CMS spread call prices of a Markov Projection with $F_1 = 0.045$, $F_2 = 0.032$, $\alpha_1 = 0.2$, $\alpha_2 = 0.25$, $\rho_{1,2} = 0.8$, $\gamma_{1,1} = -0.2$, $\gamma_{2,2} = -0.3$, $\gamma_{1,2} = \gamma_{2,1} = -0.3$, $\xi_{1,2} = 0.75$, $\beta = 0.7$, $\nu_1 = 0.4$, $\nu_2 = 0.4$ and $T = 10$ for different cross skew and decorrelation parameters.

F. Black. The pricing of commodity contratcs. *Journal of Financial Economics*, 3:167–179, 1976.

M.A.H. Dempster and S.S.G. Hong. Spread option valuation and the fast fourier transform. *University of Cambridge - Judge Institute Working Paper Series*, 26, 2000.

B. Dupire. Pricing with a smile. *Risk*, pages 18–20, Jan. 1994.

I. Gyoengy. Mimicking the one-dimensional marginal distributions of processes having an ito differential. *Probability Theory and Related Fields*, 71:501–516, 1986.

P. S. Hagan. Convexity conundrums: Pricing cms swaps, caps and floors. *Wilmott Magazine*, pages 38–44, March 2003.

P. S. Hagan, D. Kumar, A. Lesniewski, and D. E. Woodward. Managing smile risk. *Wilmott Magazine*, 2003.

M. Joshi and C. Yang. Fast and accurate pricing and hedging of long dated cms spread options. *available at SSRN, http://ssrn.com/abstract=1520775*, 2009.

P.H. Labordere. A general asymptotic implied volatility for stochastic volatility models. *SSRN: http//ssrn.com/abstract=698601*, 2005.

R. Leon. Approximating cms spread options using the swap market model. *Honours Thesis, University of the Witwatersrand*, 2006.

Y. Lu and S. Neftci. Convexity adjustment and forward libor model: Case of constant maturity swaps. *FINRISK - Working Paper Series*, 2003.

M. Lutz and R. Kiesel. Efficient pricing of cms spread options in a stochastic volatility lmm. *available at SSRN: http://ssrn.com/abstract=1418571*, 2010.

W. Margrabe. The value of an option to exchange one asset for another. *Journal of Finance*, 33(1):177–186, 1978.

V. Piterbarg. Markovian projection method for volatility calibration. *Available at SSRN*, 2006.

M. Rubinstein. Displaced diffusion option pricing. *The Journal of Finance*, 38(1):213–317, 1983.