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Research Paper 244 January 2009

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ISSN 1441-8010 www.qfrc.uts.edu.au

PORTFOLIO ANALYSIS AND ZERO-BETA CAPM WITH HETEROGENEOUS BELIEFS

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ABSTRACT. With the standard mean variance framework, by assuming heterogeneity and bounded rationality of investors, this paper examines their impact on the market equilibrium and implications to the portfolio analysis. By constructing a market consensus belief, we establish market equilibrium prices of risky assets and show that the standard Black's zero-beta CAPM under homogeneous beliefs holds under the heterogeneous belief. We demonstrate that the biased belief (from the market consensus belief) of investors makes their optimal portfolio not necessarily locate on the market mean-variance frontier. We show that the traditional geometric relation of the mean variance frontiers with and without the riskless asset under the homogeneous beliefs does not hold under the heterogeneous beliefs. The results shed light on the risk premium puzzle, Miller's hypothesis, the lower market performance when the access to the riskfree asset is impossible, and the empirical finding that managed funds under-perform comparing to the market indices on average.

JEL Classification: G12, D84.
Keywords: Asset prices; heterogeneous beliefs; portfolio analysis; zero-beta CAPM.

Date: Current version: January 20, 2009.
We are grateful to seminar participants at Peking University and the National Chengchi University, and to participants at the 14th International Conference on Computing in Economics and Finance (Paris, June 2008) for helpful comments. Financial supports from the Australian Research Council (ARC) under Discovery Grant (DP0773776) and the Faculty Research Grant at UTS are gratefully acknowledged.
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1. INTRODUCTION

The Capital Asset Pricing Model (CAPM) developed simultaneously and independently by Sharpe (1964), Lintner (1965) and Mossin (1966) is perhaps the most influential equilibrium model in modern finance. It provides a theoretical foundation for relating risks linearly with expected return of assets. However, the CAPM is not without criticism, it assumes that (i) all investors make portfolio selection decisions according to the criteria set out in Markowitz (1959) within the mean-variance framework, (ii) investors have homogeneous beliefs about the future state of the market, (iii) unlimited borrowing and lending at a single risk-free rate, and (iv) a frictionless market. Among them, assumptions (ii) and (iii) might be the most restrictive and unrealistic ones. To provide some theoretical explanations on the early empirical tests of CAPM, Black (1972) has extended the CAPM by removing assumption (iii) and developed the well-known zero-beta CAPM. By relaxing assumption (ii), Chiarella, Dieci and He (2006*a*) derive a CAPM-like relationship with heterogeneous beliefs. This paper seeks to generalize the standard CAPM by removing assumptions (ii) and (iii) and examine the impacts of such generalization on the market equilibrium prices, the market mean-variance frontier, and the implications to the portfolio analysis.

Literatures have made a significant contribution to the understanding of the impact of heterogeneous beliefs amongst investors on market equilibrium. Some have considered the problem in discrete time (for example, see Lintner (1969), Rubinstein (1976), Fan (2003), Sun and Yang (2003), Chiarella et al (2006*a*) and Sharpe (2007)) and others in continuous time (for example, see Williams (1977), Detemple and Murthy (1994) and Zapatero (1998)). Equilibrium models have been developed to study the impact of heterogeneity either in the mean-variance framework (see, Lintner (1969), William (1977) and Sun and Yang (2003)) or in the Arrow-Debreu contingent claims economy (see, for example Rubinstein (1976) and Abel (1989, 2002)). Given the bounded rationality of investors, heterogeneity may be caused by difference in information or difference in opinion. In the first case, investors may update their beliefs as new information become available, Bayesian updating rule is often used (see, for example, Williams (1977) and Zapatero (1998)). In the second case, investor may revise their portfolio strategies as their views of the market change over time (see, for example Lintner (1969) and Rubinstein (1975)).

In the most of this literature, the impact of heterogeneous beliefs is studied for the case of a portfolio of one risky asset and one risk-free asset. Amongst all the literatures mentioned above, Lintner (1969) and Sun and Yang (2003) are the only ones considering market equilibrium and asset prices without assumptions (ii) and (iii) of the traditional CAPM. Lintner (1969) is the first paper considering the problem of market equilibrium without assumptions (ii) and (iii). He has shown that removing assumption (ii) does not change the structure of capital asset prices in any significant way, and removing assumptions (iii) further is just a mere extension of the case with a risk-free asset. Surprisingly, this significant contribution from Lintner has not been paid much attention until recent years. The main obstacle in dealing with heterogeneity is the complexity and heavy notation involved when the number of assets and the dimension of the heterogeneity increase. It might be due to this notational obstacle that makes the paper of Lintner hard to follow, and renders rather complicated analysis of the impact of heterogeneity on the market equilibrium prices. The question of whether the zero-beta CAPM still exists without assumptions (ii) and (iii) has been studied in Sun and Yang (2003) recently. They have provided conditions for the existence of the market equilibrium and shown that the zero-beta CAPM still holds under heterogeneous beliefs within the mean-variance framework. However, because of the same obstacle, they have not provided the market equilibrium price and examined the impact of heterogeneity on the market equilibrium price, mean-variance efficiency of the optimal portfolios of heterogeneous investors. This paper is devoted to overcome this obstacle, to present an explicit equilibrium price formula, and to examine the impacts of the heterogeneous beliefs on the market equilibrium.

Recently Chiarella et al (2006*a*, 2006*b*) introduce the concept of *consensus belief* and show that, when there is a riskless asset, the market consensus belief can be constructed explicitly as a weighted average of the heterogeneous beliefs. They show that the expected payoffs/returns of the risky assets under the consensus belief is a weighted average of that under investors' subjective beliefs, while the market equilibrium prices is a weighted average of the equilibrium prices perceived by each investor. Consequently, they establish an equilibrium relation between the market consensus expected payoff/returns of the risky assets and the market portfolio's expected payoff/returns, leading to a CAPM-like relation under heterogeneous beliefs. However, when there is no riskless asset, can we obtain a zero-beta CAPM-like relation under heterogeneous beliefs? The well known geometric tangency relation of portfolios with and without riskless asset in the standard mean variance portfolio analysis under homogeneous belief plays a very important role to the establishment of the CAPM. The question is that, does this geometric relationship still hold under heterogeneous beliefs? This paper is largely motivated by answering these two questions. We extend the analysis in Chiarella, Dieci and He (2006*a*, 2006*b*) by removing the assumption of a risk-free asset and examine the impact of the heterogeneity on the market equilibrium and the implication to the portfolio analysis.

The paper is structured as follows. In Section 2, we introduce and construct the market consensus belief linking the heterogeneous market with an equivalent homogeneous market, and present an explicit market equilibrium price formula. Consequently, a zero-beta CAPM under the heterogeneous beliefs is derived. In Section 3, we examines the impacts of different aspect of heterogenous beliefs on the market equilibrium. Through some numerical examples, Section 4 examines the implications of heterogeneity on the efficiency of the optimal portfolios of heterogeneous investors and shows that the optimal portfolios of investors is not locate on the equilibrium market frontier in general. We show that the geometric tangency relation of the portfolios with and without riskless asset under the standard homogeneous beliefs breaks down under the heterogeneous beliefs. Section 5 provides an alternative heterogeneous belief setup in returns and examines the robustness of the results for two different belief setups. Section 6 extends numerical analysis to a market with many investors and examines the impact of heterogeneity on the market when the belief dispersions are characterized by certain distribution. Section 7 summaries and concludes the paper. The proofs and details of some examples are provided in the appendices.

2. Equilibrium Asset Prices Under Heterogeneous Beliefs

When a financial market consists of investors with different views on the future movement of the market, it is important to understand how market equilibrium is obtained and the roles played by different investors. Within the standard mean-variance framework, in this section, we first introduce heterogeneous beliefs among investors and a concept of market consensus belief to reflect the market belief when market is in equilibrium. By constructing the consensus belief explicitly, we characterize the equilibrium asset prices. Consequently we show that a zero-beta CAPM-like relation still holds under heterogeneous beliefs.

2.1. **Heterogeneous Beliefs.** Following Lintner (1969) and Black (1972), we extend the static mean-variance model with homogeneous belief and consider a market in which there are many risky assets but there is no risk-free asset and investors have heterogeneous beliefs of the future returns of risky assets. Similar to Chiarella *et al.* (2006*a*), asset returns are measured in either the payoff in capital, which is considered in this section, or the rate of return, which will be examined in Section 5.

Consider a market with N risky assets, indexed by $j, k = 1, 2, \cdots, N$ and I investors indexed by $i = 1, 2, \cdots, I$. Let $\tilde{\mathbf{x}} = (\tilde{x_1}, \cdots, \tilde{x_N})^T$ be the random payoff vector of the risky assets. Assume that each investor has his/her own set of beliefs about the market in terms of means, variances and covariances of the payoffs of the assets, denoted by

$$y_{i,j} = E_i[\tilde{x}_j], \quad \sigma_{i,jk} = Cov_i(\tilde{x}_j, \tilde{x}_k) \qquad \text{for } 1 \le i \le I, \quad 1 \le j, k \le N. \tag{1}$$

For investor i, we define the mean vector and variance/covariance matrix of the payoffs of N assets as follows, $\mathbf{y}_i = \mathbb{E}_i(\tilde{\mathbf{x}}) = (y_{i,1}, y_{i,2}, \cdots, y_{i,N})^T$ and $\Omega_i = (\sigma_{i,jk})_{N\times N}$. Denote $\mathcal{B}_i = (\mathbb{E}_i(\tilde{\mathbf{x}}), \Omega_i)$ the set of subjective beliefs of investor i. Let $\mathbf{z}_i = (z_{i,1}, z_{i,2}, \cdots, z_{i,N})^T$ be the portfolio in the risky assets (in quantity) and $W_{i,o}$ be the initial wealth of investor i. Then the end-of-period portfolio wealth of investor i is given by $\tilde{W}_i = \tilde{\mathbf{x}}^T\mathbf{z}_i$. Under the belief $\mathcal{B}_i$ of investor i, the mean and variance of $\tilde{W}_i$ are given, respectively, by

$$\mathbb{E}_i(\tilde{W}_i) = \mathbf{y}_i^T\mathbf{z}_i, \qquad \sigma_i^2(\tilde{W}_i) = \mathbf{z}_i^T\Omega_i\mathbf{z}_i. \tag{2}$$

As in the standard mean-variance framework, we assume that investor i has a constant absolute risk aversion (CARA) utility function $U_i(w) = -e^{-\theta_i w}$, where θ_i is the CARA coefficient, and the end-of-period wealth $\tilde{W}_i$ of investor i is normally distributed. Under these assumptions, maximizing investor i's expected utility of wealth is equivalent to maximizing his/her certainty equivalent end-of-period wealth $\max_{\mathbf{z}_i} Q_i(\mathbf{z}_i)$ subject to the wealth constraint

$$\mathbf{p}_0^T\mathbf{z}_i = W_{i,o}, \tag{3}$$

where

$$Q_i(\mathbf{z}_i) := \mathbb{E}_i(\tilde{W}_i) - \frac{\theta_i}{2}\sigma_i^2(\tilde{W}_i) = \mathbf{y}_i^T\mathbf{z}_i - \frac{\theta_i}{2}\mathbf{z}_i\Omega_i\mathbf{z}_i$$

and $\mathbf{p}_0$ is the market price vector of the risky assets. Applying the first order conditions, we obtain the following Lemma on the optimal portfolio of the investor.

Lemma 2.1. *For given market price vector $\mathbf{p}_0$ of risky assets, the optimal risky portfolio $\mathbf{z}_i^*$ of investor i is uniquely determined by*

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}[\mathbf{y}_i - \lambda_i^*\mathbf{p}_0], \tag{4}$$

where

$$\lambda_i^* = \frac{\mathbf{p}_0^T\Omega_i^{-1}\mathbf{y}_i - \theta_i W_{i,o}}{\mathbf{p}_0^T\Omega_i^{-1}\mathbf{p}_0}. \tag{5}$$

Lemma 2.1 implies that the optimal demand of investor i depends on his/her absolute risk aversion (ARA) coefficient (θ_i), the expected payoffs and variance/covariance matrix of the risky asset payoffs, the Lagrange multiplier (λ_i^*), as well as the market price of the risky assets. Following Lintner (1969), λ_i^* is a shadow price, measuring the *marginal real (riskless) certainty-equivalent of investor i's end-of-period wealth*. In fact, applying the first order condition, we obtain $\frac{\partial Q_i(\mathbf{z}_i^*)}{\partial \mathbf{z}_i} = \lambda_i^* \mathbf{p}_o$, which leads to

$$\lambda_i^* = \frac{1}{p_{oj}} \frac{\partial Q_i(\mathbf{z}_i^*)}{\partial z_{ij}} \qquad \text{for all} \quad j = 1, 2, \cdots, N. \tag{6}$$

More precisely, equation (6) indicates that λ_i^* actually measures *investor i's optimal marginal certainty equivalent end-of-period wealth per unit of asset j relative to its market price* and it is a constant across all assets. In general, the shadow price is not necessary the same for all investors, however, it is the same when there exist a risk-free asset in the market. In fact, let the current price of the risk-free asset f be 1 and its payoff be $R_f = 1 + r_f$. Applying (6) to the risk-free asset leads to $\lambda_i^* = R_f$ for all investors, that is, the shadow price is equal to the payoff of the risk-free asset.

2.2. **Consensus Belief and Equilibrium Asset Prices.** We define the market equilibrium asset price vector $\mathbf{p}_o$ of the risky assets as the price vector under which individual's optimal demands (4) satisfy the market aggregation condition

$$\sum_{i=1}^{I} \mathbf{z}_i^* = \sum_{i=1}^{I} \bar{\mathbf{z}}_i := \mathbf{z}_m, \tag{7}$$

where $\bar{\mathbf{z}}_i$ is the endowment portfolio of investor i. Correspondingly, $\mathbf{z}_m$ is the market portfolio of the risky assets. It then follows from (7) and (4) that the market equilibrium price $\mathbf{p}_o$ is given, in terms of the heterogeneous beliefs of the investors, by

$$\mathbf{p}_0 = \Big(\sum_{i=1}^{I} \theta_i^{-1} \lambda_i^* \Omega_i^{-1} \Big)^{-1} \Big[\Big(\sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} \mathbf{y}_i \Big) - \mathbf{z}_m \Big]. \tag{8}$$

This expression defines the market equilibrium price $\mathbf{p}_o$ implicitly since λ_i^* depends on $\mathbf{p}_o$ as well. For the existence of the market equilibrium price, we refer to Sun and Yang (2003) and the references cited there. The concept of consensus belief has been used to characterize the market when investors are heterogeneous in different context and it is closely related to, but significantly different from, the concept of representative investor in classical finance literature. We now introduce the concept of consensus belief for the market with the heterogeneous beliefs.

Definition 2.2. *A belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$, defined by the expected payoff of the risky assets $\mathbb{E}_a(\tilde{\mathbf{x}})$ and the covariance matrix of the risky asset payoffs Ω_a, is called a market*

consensus belief *if the market equilibrium price under the heterogeneous beliefs is also the market equilibrium price under the homogeneous belief* $\mathcal{B}_a$.

When a consensus belief exists, the market with heterogeneous beliefs can be treated as a market with homogeneous consensus belief and then the classical Markowtiz portfolio analysis can be applied. Due to the complexity of heterogeneity, the existence and finding of such consensus belief is a difficult task in the literature. This obstacle makes the examination of the impact of the heterogeneity difficult. In the following, we construct the consensus belief explicitly, from which market equilibrium prices $\mathbf{p}_0$ can be determined explicitly in terms of the consensus belief. It is the explicit construction of the consensus belief that makes it easy to examine the role of heterogeneous beliefs played in determining the market equilibrium price and to derive the zero-beta CAPM relation.

Proposition 2.3. *Let*

$$\theta_a := \left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\right)^{-1}, \tag{9}$$

$$\lambda_a^* := \frac{1}{I}\theta_a\sum_{i=1}^{I}\theta_i^{-1}\lambda_i^*. \tag{10}$$

Then

(i) *the consensus belief* $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$ *is given by*

$$\Omega_a = \theta_a^{-1}\lambda_a^*\left(\frac{1}{I}\sum_{i=1}^{I}\lambda_i^*\theta_i^{-1}\Omega_i^{-1}\right)^{-1}, \tag{11}$$

$$\mathbf{y}_a := \mathbb{E}_a(\tilde{\mathbf{x}}) = \theta_a\Omega_a\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}\mathbb{E}_i(\tilde{\mathbf{x}})\right); \tag{12}$$

(ii) *the market equilibrium price* $\mathbf{p}_o$ *is determined by*

$$\mathbf{p}_0 = \frac{1}{\lambda_a^*}\left[\mathbf{y}_a - \frac{1}{I}\theta_a\Omega_a\mathbf{z}_m\right]; \tag{13}$$

(iii) *the equilibrium optimal portfolio of investor* i *is given by*

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}\left[(\mathbf{y}_i - \frac{\lambda_i^*}{\lambda_a^*}\mathbf{y}_a) + \frac{\lambda_i^*}{I\lambda_a^*}\theta_a\Omega_a\mathbf{z}_m\right]. \tag{14}$$

Proposition 2.3 shows how the consensus belief can be constructed explicitly from the heterogeneous beliefs. Under the consensus belief, the market equilibrium prices of the risky assets are determined in the standard way no risk-free assets. Intuitively

Proposition 2.3 indicates that the market consensus belief is a weighted average of the heterogeneous beliefs. More precisely, the market risk tolerance $(1/\theta_a)$ is simply an average of the risk tolerance of the heterogeneous investors, according to Huang and Lizenberger (1988), $\theta_a/I = (\sum_i^I \theta_i^{-1})^{-1}$ is called the *aggregate absolute risk aversion* and consequently $\theta_a W_{m0}/I$ is referred to as the aggregate relative risk aversion. The weighted average behaviour can also be viewed in the following way. Let $\tau_i = 1/\theta_i$ be the risk tolerance of investor i and $\tau_a = \sum_{i=1}^I \tau_i$ be the market aggregate risk tolerance. Then

$$\lambda_a^* = \sum_{i=1}^I \frac{\tau_i}{\tau_a}\lambda_i^*, \qquad \Omega_a^{-1} = \sum_{i=1}^I \frac{\tau_i \lambda_i}{\tau_a \lambda_a}\Omega_i^{-1}, \qquad \mathbb{E}_a(\tilde{\mathbf{x}}) = \Omega_a \sum_{i=1}^I \frac{\tau_i}{\tau_a}\Omega_i^{-1}\mathbb{E}(\tilde{\mathbf{x}})).$$

Hence the inverse of the covariance matrix (Ω_a^{-1}) for the market reflects an weighted average of the inverse covariance matrices of all investors and the market expected payoff is a weighted average of the covariances of the investors. The market equilibrium prices are determined such that each investor can choose their optimal portfolio subjectively and the market is cleared. It follows from (4) in Lemma 2.1 that $\mathbf{p}_0 = \frac{1}{\lambda_i^*}(\mathbf{y}_i - \frac{1}{\tau_i}\Omega_i \mathbf{z}_i^*)$ for $i = 1, \cdots, I$. However, if the entire market acts as an aggregate investor, then for the market to clear, the prices must be determined by the consensus belief as in (13) or equivalently as $\mathbf{p}_0 = \frac{1}{\lambda_a^*}(\mathbf{y}_a - \frac{1}{\tau_a}\Omega_a \mathbf{z}_m)$. This suggests that the consensus belief $\mathcal{B}_a$ must correspond to the belief of the aggregate market such that the market portfolio is an optimal portfolio. The expressions in Proposition 2.3 provide explicit relationships between the heterogeneous belief and the market consensus belief under the market aggregation. Their usefulness will be revealed when we derive a zero-beta CAPM-like relation and examine the impacts of the heterogeneity on the market equilibrium in the following subsection.

2.3. **The Zero-Beta CAPM under Heterogeneous Beliefs.** As a corollary of Proposition 2.3, we show now that a zero-beta CAPM-like relation holds under the constructed consensus belief with no risk-free asset.

Let the future payoff of the market portfolio $\mathbf{z}_m$ be given by $\tilde{W}_m = \tilde{\mathbf{x}}^T \mathbf{z}_m$ and its current market value is $W_{m,o} = \mathbf{z}_m^T \mathbf{p}_0 = \sum_{i=1}^I W_{i,o}$. Hence under the consensus belief $\mathcal{B}_a$, $\mathbb{E}_a(\tilde{W}_m) = \mathbf{y}_a^T \mathbf{z}_m$ and $\sigma_a^2(\tilde{W}_m) = \mathbf{z}_m^T \Omega_a \mathbf{z}_m$. Define the returns $\tilde{r}_j = \tilde{x}_j/p_{j,o} - 1$ and $\tilde{r}_m = \tilde{W}_m/W_{m,o} - 1$. Under the market consensus belief $\mathcal{B}_a$, we set

$$\mathbb{E}_a(\tilde{r}_j) = \frac{\mathbb{E}_a(\tilde{x}_j)}{p_{j,o}} - 1, \qquad \mathbb{E}_a(\tilde{r}_m) = \frac{\mathbb{E}_a(\tilde{W}_m)}{W_{m,o}} - 1 \qquad \sigma_a^2(\tilde{r}_m) = \frac{\sigma_a^2(\tilde{W}_m)}{W_{m,o}^2}$$

and

$$Cov_a(\tilde{r}_j, \tilde{r}_m) = \frac{1}{p_{j,o}W_{m,o}} Cov_a(\tilde{x}_j, \tilde{W}_m), \quad Cov_a(\tilde{r}_j, \tilde{r}_k) = \frac{1}{p_{j,o}p_{k,o}} Cov_a(\tilde{x}_j, \tilde{x}_j).$$

Then we have the following result.

Corollary 2.4. *In market equilibrium, the relation between expected return and risk under the heterogeneous beliefs can be expressed as*

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - (\lambda_a^* - 1)\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1)], \tag{15}$$

where

$$\lambda_a^* = \frac{\mathbf{z}_m^T \mathbf{y}_a - \theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m,o}}, \tag{16}$$

$$\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1) = \frac{\theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m0}} = \frac{1}{\tau_a} W_{m,o} \sigma_a^2(\tilde{r}_m) > 0 \tag{17}$$

and $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_N)^T$ *with*

$$\beta_j = \frac{Cov_a(\tilde{r}_m, \tilde{r}_j)}{\sigma_a^2(\tilde{r}_m)} = \frac{W_{mo}}{p_{oj}} \frac{Cov_a(\tilde{x}_j, \tilde{W}_m)}{\sigma_{a,m}^2}, \qquad j = 1, \cdots, N.$$

The equilibrium relation (15) is the standard Zero-Beta CAPM except that the mean and variance/covariance are calculated based on the consensus belief $\mathcal{B}_a$. We refer it as the *Zero-beta Heterogeneous Capital Asset Pricing Model* (ZHCAPM). For risky assets, relation (15) is equivalent to

$$\mathbb{E}_a[\tilde{r}_j] - (\lambda_a^* - 1) = \beta_j[\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1)], \qquad \text{for} \qquad j = 1, \cdots, N. \tag{18}$$

The zero-beta rate, $\lambda_a^* - 1$, corresponds to the expected return of the zero-beta portfolio of the market portfolio, where λ_a^* is the market shadow price. As in the standard case, the market risk premium, given by equation (17) is positively proportional to the aggregate relative risk aversion $W_{m,o}/\tau_a$ and the variance of the market portfolio returns $\sigma_a^2(\tilde{r}_m)$. The market price of risk under the consensus belief is given by $\phi = (\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1))/\sigma_a(\tilde{r}_m) = W_{m0}\sigma(\tilde{r}_m)/\tau_a$, which is proportional to the level of volatility of the market and the aggregate relative risk aversion.

As discussed earlier, investor i's shadow price becomes R_f across all investors when there exists a risk-free asset in the market. That is, $\lambda_i^* = \lambda_a^* = R_f$. Substituting this into Proposition 2.3 and Corollary 2.4 leads to the main results in Chiarella *et al.* (2006*a*).

3. The Impact of Heterogeneity

In this section, we use Proposition 2.3 and Corollary 2.4 to examine the impact of the heterogeneous beliefs on the market consensus belief and equilibrium price. To simplify the analysis, we focus on some special cases.

3.1. **The Shadow Prices and the Aggregation Property.** We first examine the relationship between individual shadow prices and the market consensus shadow price. Following (10), let $\lambda_a^* = f(\lambda_1^*, \lambda_2^*, \cdots, \lambda_I^*; \theta_1, \theta_2, \cdots, \theta_I)$. Then it is easy to see that $\frac{\partial f}{\partial \lambda_i^*} = \frac{\theta_a \theta_i^{-1}}{I} > 0$, showing that the market consensus shadow price increases as the shadow price of investor i increases, and the rate of increase depends on θ_i. It follows from $\frac{\partial^2 f}{\partial \lambda_i^* \partial \theta_i} = \frac{1}{I}\theta_i^{-3}\theta_a(\frac{1}{I}\theta_a - \theta_i)$ and $I\theta_i^{-1} > \theta_a^{-1}$ that $\frac{\partial^2 f}{\partial \lambda_i^* \partial \theta_i} < 0$. Therefore the market consensus shadow price is more sensitive to the change of the shadow price of investor who is less risk-averse.

According to Huang and Litzenberger (1988), the market satisfies the aggregation property when the market equilibrium prices are independent of the distribution of the initial wealth among investors. Huang and Litzenberger (1988) show that the aggregation property holds when investors have homogeneous belief, time-additive and state independent utility functions with linear risk tolerance and a common cautiousness coefficient. In a general two-period economy without specifying the type of utility function for any investors, Fan (2003) shows the *Second Welfare Theorem* holds. The theorem states that investors with large capital endowments would have lower marginal utilities of capital endowments and a stronger influence on the market equilibrium. In our case, the utility is measured by $Q_i(\mathbf{z})$. From (6), the marginal utility of investor i is represented by the shadow price (λ_i^*). It then follows from (5) that a large initial wealth or capital endowment leads to a lower marginal utility. Also, from the expression of the equilibrium price vector in (8), it can be seen that (λ_i^*) is inversely related to the price vector. This suggests that an investor with a lower shadow price or marginal utility has a stronger impact on the market equilibrium prices, and hence an investor with a larger capital is more influential in the market. This is consistent with the *Second Welfare Theorem*. In other words, the aggregation property does not hold in our case in general. However, if there is a risk-free asset in the market, then the shadow prices or marginal utilities is a constant across all investor. Correspondingly the market prices are independent of the initial wealth distribution. Summarizing the above analysis, we have the following Corollary.

Corollary 3.1. *With the heterogeneous beliefs and no risk-free asset, the aggregation property does not hold. Furthermore investors with lower shadow prices or marginal utilities have a stronger impact on the market equilibrium prices, and hence investors with larger capital are more influential in the market. However, if there is a risk-free asset, the aggregation property holds.*

3.2. **The Impact of Heterogeneous ARA Coefficients.** Proposition 2.3 indicates that the heterogeneous ARA coefficients or risk tolerance have complicated impact on the market consensus belief and equilibrium price. To illustrate such impact, we consider a special case when investors are homogeneous in the expected payoffs and covariance matrix but heterogeneous in ARA, that is, $\Omega_i = \Omega_a := \Omega_o, \mathbf{y}_i = \mathbf{y}_a := \mathbf{y}_o$ for all i. Accordingly the equilibrium price vector can be written as

$$\mathbf{p}_0 = \frac{1}{\lambda_a^*}\Big[\mathbf{y}_o - \frac{1}{I}\theta_a\Omega_o\mathbf{z}_m\Big], \qquad \lambda_a^* = \frac{\mathbf{z}_m^T\mathbf{y}_o - \theta_a\mathbf{z}_m^T\Omega_o\mathbf{z}_m/I}{W_{m0}}. \tag{19}$$

Equation (19) implies that, when the risk aversion coefficient is the only source of heterogeneity, the market equilibrium prices are independent of the initial wealth distribution amongst individuals and hence the aggregation property holds, a well-known result in the standard finance theory. For any risky asset j, (19) is equivalent to

$$p_{0,j} = \frac{1}{\lambda_a^*}\Big[y_{o,j} - \frac{1}{I}\theta_a Cov(\tilde{x}_j, \tilde{W}_m)\Big].$$

This, together with the market shadow price in equation (19), leads to $\frac{\partial p_{0,j}}{\partial \theta_a} = \frac{\sigma^2(\tilde{r}_m)(1-\beta_j)}{I\lambda_a^*}$. In the presence of a risk-free asset with payoff R_f, this becomes $\frac{\partial p_{0,j}}{\partial \theta_a} = -\frac{\sigma^2(\tilde{r}_m)\beta_j}{IR_f}$. Noting that, in this case, the equilibrium prices and expected returns are inversely related since the expected payoff is fixed. Together with the fact that $\frac{\partial \theta_a}{\partial \theta_i} = (\theta_a^{-1}\theta_i^{-1})^2/I > 0$ and $\frac{\partial^2\theta_a}{\partial \theta_i^2} = -2\frac{\partial \theta_a}{\partial \theta_i}(\frac{\partial \theta_a}{\partial \theta_i}\theta_a^{-1} + \theta_i^{-1}) < 0$, the above analysis leads to the following Corollary.

Corollary 3.2. *In a market with homogeneous beliefs and no risk-free assets,*

$$(\beta_j - 1)\frac{\partial p_{0,j}}{\partial \theta_i} < 0, \qquad (\beta_j - 1)\frac{\partial \mathbb{E}_o(\tilde{r}_j)}{\partial \theta_i} > 0$$

for $\beta_j \neq 1$ *and* $\frac{\partial p_{0,j}}{\partial \theta_i} = \frac{\partial \mathbb{E}_o(\tilde{r}_j)}{\partial \theta_i} = 0$ *for* $\beta_j = 1$. *If there exists a risk-free asset, then*

$$\beta_j\frac{\partial p_{0,j}}{\partial \theta_i} < 0, \qquad \beta_j\frac{\partial \mathbb{E}_o(\tilde{r}_j)}{\partial \theta_i} > 0$$

for $\beta_j \neq 0$ *and* $\frac{\partial p_{0,j}}{\partial \theta_i} = \frac{\partial \mathbb{E}_o(\tilde{r}_j)}{\partial \theta_i} = 0$ *for* $\beta_j = 0$. *The rate of change for both the equilibrium price and expected return is greater when investor is less risk averse.*

Corollary 3.2 indicates that the impact of ARA on the market equilibrium depends on the beta of the asset. When there is no risk-free asset, if an asset is more risky than the market ($\beta_j > 1$), an increase in ARA for any investor increases asset's price and decreases the expected future returns, and vice versa for a less risky asset. However, if there is a risk-free asset, the changes depend on the return correlation of the asset with the market. If the returns of the asset and market are positive correlated, an increase (decrease) in ARA of any investor leads to lower (higher) market equilibrium price and higher (lower) expected return for the asset. In addition, changing ARA of less risk averse investor has more significant impact on market equilibrium price and expected return. The market is dominated by less risk averse investors, because the market average risk aversion coefficient θ_a is a harmonic mean of θ_is, it aggravates the impact of the small θ_is. This suggests that when the risk aversion coefficients of the investors becomes more divergent with the average unchanged, the aggregate ARA would be reduced, resulting lower (higher) equilibrium price and higher (lower) expected return for assets with betas are below (above) the market level when there is no risk-free asset in the market. However, when there is a risk-free asset, the reduction of the market aggregate risk aversion leads to lower (higher) equilibrium price and higher (lower) expected return for assets that are negatively (positively) correlated with the market.

3.3. **The Impact of Heterogeneous Expected Payoffs.** We now assume that investors agree on the variances and covariances of asset payoffs, say $\Omega_i = \Omega_o$, but disagree on the expected future payoffs of the assets. Consequently $\Omega_a = \Omega_o$ and the equilibrium price for asset j becomes

$$p_{0,j} = \frac{1}{\lambda_a^*}\left[y_{a,j} - \frac{1}{\tau_a}Cov_o(\tilde{x}_j, \tilde{W}_m)\right], \tag{20}$$

where $\lambda_a^* = [\mathbf{z}_m^T\mathbf{y}_a - \mathbf{z}_m^T\Omega_o\mathbf{z}_m/\tau_a]/W_{m0}$ and $y_{a,j} = \sum_{i=1}^I(\tau_i/\tau_a)y_{i,j}$. This, together with (20), leads to

$$\frac{\partial p_{0,j}}{\partial y_{a,j}} = \frac{1-\alpha_j}{\lambda_a^*}, \tag{21}$$

where $\alpha_j = p_{0,j}z_{m,j}/W_{m0}$ is the market share of asset j in wealth. If there is a risk-free asset in the market with payoff R_f, then (21) simply becomes $\partial p_{0,j}/\partial y_{a,j} = 1/R_f$. Note that

$$\frac{\partial y_{a,j}}{\partial y_{i,j}} = \frac{1}{I}\frac{\theta_a}{\theta_i}y_{i,j} > 0, \qquad \frac{\partial^2 y_{a,j}}{\partial y_{i,j}\partial\theta_i} = \theta_a\theta_i^{-3}\frac{y_{i,j}}{I}(\frac{\theta_a}{I} - \theta_i) < 0. \tag{22}$$

Because of $\alpha \in [0,1]$, equations (21) and (22) indicates that investor i's subjective belief in the expected payoff of asset j is positively related to its equilibrium price. This is also true when there is a risk-free asset in the market. The positive correlation between the subjective beliefs in the expected payoff and the equilibrium price for asset j does not necessarily lead to a negative correlation between the subjective beliefs in the expected payoff and the market expected return for asset j. To see the exact relation, we have from $\mathbb{E}_a(\tilde{r}_j) = y_{a,j}/p_{o,j} - 1$ that

$$\frac{\partial \mathbb{E}_a(\tilde{r}_j)}{\partial y_{a,j}} = \frac{p_{o,j} - (1-\alpha_j)y_{a,j}/\lambda_a^*}{p_{o,j}^2}. \tag{23}$$

This expression is negative if and only if $(1+\mathbb{E}_a(\tilde{r}_j))(1-\alpha_j) > \lambda_a^*$. When this condition holds, the expected return decreases when the expected payoff increases for asset j. When there is a risk-free asset, $\lambda_a^* = R_f$ and equation (23) becomes $\frac{\partial \mathbb{E}_a(\tilde{r}_j)}{\partial y_{a,j}} = \frac{p_{o,j} - y_{a,j}/R_f}{p_{o,j}^2}$, which is negative if and only if $\mathbb{E}_a(\tilde{r}_j) > r_f$. When this condition holds, the expected return decreases when the heterogeneous belief in the expected payoff increases for asset j. Summarizing the above analysis, we obtain the following Corollary.

Corollary 3.3. *In a market with homogeneous beliefs in covariance matrix and no risk-free assets, if*

$$(1+\mathbb{E}_a(\tilde{r}_j))(1-\alpha_j) > \lambda_a^* \tag{24}$$

for asset j, then the market expected payoff increases and the expected return decreases when the heterogeneous belief in the expected payoff of any investor increases for asset j. When there is a risk-free asset, the condition (24) *becomes $\mathbb{E}_a(\tilde{r}_j) > r_f$.*

The following discussion is devoted to Miller's hypothesis (Miller (1977)) that asserts with high dispersion in beliefs have higher market price and lower expected future return than otherwise similar stocks. Empirical tests performed in Diether, Malloy and Scherbina (2002) support Miller's hypothesis. Intuitively, optimistic investors would increase the price of the asset and then reduce its expected future returns. However, there is a lack of theoretical modeling and explanation on this hypothesis. Let us consider a market in which investors have homogeneous beliefs in the covariance matrix but heterogeneous beliefs in the expected payoffs of two risky assets j and j'. Let the expected payoffs be $\mathbf{y}_j = (y_{1,j}, y_{2,j}, \cdots, y_{I,j})^T$ and $\mathbf{y}_{j'} = (y_{1,j'}, y_{2,j'}, \cdots, y_{I,j'})^T$ for asset j and j', respectively. Assume $y_{i,j'} = y_{i,j} + \epsilon_{i,j}$, where $\{\epsilon_{j,1}, \epsilon_{j,2}, \cdots, \epsilon_{j,I}\}$ is a set of real numbers such that $\sum_{i=1}^n \epsilon_{i,j} = 0$ and $\frac{1}{I}\sum_{i=1}^I (y_{i,j'} - \bar{y})^2 \geq \frac{1}{I}\sum_{i=1}^I (y_{i,j} - \bar{y})^2$, where $\bar{y} = (1/I)\sum_{i=1}^I y_{i,j}$. This condition means that investors have more divergence of opinions in the expected payoff for asset j' than asset j. According to Miller's hypothesis, asset j' would have higher market price and lower expected future return than asset j. To see if this is true, we consider the following simple example when $I = 2$.

Example 3.4. *Let $I = 2$. Given $\epsilon > 0$, consider two assets j and k with $y_{2,j} < y_{1,j}$, and $y_{1,k} = y_{1,j} + \epsilon$ and $y_{2,k} = y_{2,j} - \epsilon$. This specification indicates that the divergence of opinion about the asset's expected payoff is greater for asset k than for asset j. Then $y_{a,j} = \frac{\theta_1^{-1}}{\theta_1^{-1}+\theta_2^{-1}} y_{1,j} + \frac{\theta_1^{-1}}{\theta_1^{-1}+\theta_2^{-1}} y_{2,j}$ and $y_{a,k} = \frac{\theta_1^{-1}}{\theta_1^{-1}+\theta_2^{-1}}(y_{1,j} + \epsilon) + \frac{\theta_1^{-1}}{\theta_1^{-1}+\theta_2^{-1}}(y_{2,j} - \epsilon)$. Hence $y_{a,j} - y_{a,k} = \frac{\epsilon}{\theta_1^{-1}+\theta_2^{-1}}(\theta_2^{-1} - \theta_1^{-1})$. Accordingly, $y_{a,j} < y_{a,k}$ if and only if $\theta_1 < \theta_2$. This implies that if investor who is optimistic about the asset expected payoff is less risk averse, then a divergence of opinion among the two investors for the expected payoff for asset k leads to high expected payoff for the asset in equilibrium. This suggests that divergence of opinion on the asset expected payoffs generates higher market expected payoff if belief of assets' expected future payoffs is negatively correlated to risk aversion for any investor i. It then follows from Corollary 3.3 that, when both assets j and k satisfy the condition* (24)*, the divergence of opinion on the asset expected payoffs generates lower expected future return for the asset.*

To summarize, if our model is to be consistent with Miller's hypothesis that divergence of opinion causes asset price to increase and expected return to decrease, we need the investor with an optimistic view of the asset future payoff to be less risk-averse comparing with the relative pessimistic investor, also the asset return satisfies condition (24).

4. The Implications on the Portfolio Frontier and the Optimal Portfolios

In this section, we examine the mean-variance efficiency of the optimal portfolios of investors when the market is in equilibrium. Following the standard Markowitz method, we can construct the mean variance portfolio frontier based on the consensus belief. Because the consensus belief reflects the market belief when it is in equilibrium, we call this frontier the market equilibrium frontier. A portfolio is (in)efficient if it is (not) located on the market equilibrium frontier. When investors are homogeneous in their beliefs, it is well known that the investors will always choose an efficient portfolio. Furthermore, there exists a unique tangency portfolio between frontiers with and without a riskless security. The question is that whether these results still true when investors have heterogeneous beliefs? In this section, we provide a negative answer to the above question by considering first the market without risk-free asset and then comparing with the market with a risk-free asset.

4.1. **Efficiency of the Optimal Portfolios under Heterogeneous Beliefs without Risk-free Asset.** In the market we set up in Section 2, investors are bounded rational in the sense that they make their optimal decisions based on their beliefs. Based on investors' subjective beliefs, we can construct the mean-variance frontiers (in the standard deviation and expected return space) by using the standard Markwitz method. Of course, the optimal portfolios of the investors will be located on the efficient mean-variance frontiers under their own beliefs. Similarly, based on the consensus belief, the market equilibrium frontier can be constructed. By the market clearing condition and frontier construction, the market portfolio is always located on the market equilibrium frontier, hence always efficient. The question is whether the optimal portfolios of individual investors are located on the market equilibrium frontier. This is a very important question both theoretically and empirically. If the answer to the question is yes, then the optimal portfolio of the bounded rational heterogeneous investors are efficient under market aggregation. Otherwise, market fails to provide the mean-variance efficiency for the investors. If we refer to heterogeneous investors as fund managers and the market portfolio as the market index, the efficiency of the optimal portfolios will have important implication on whether fund managers can out perform the market index based on the mean-variance criteria.

To answer this question, we consider a consensus investor with the market consensus beliefs $\mathcal{B}_a$, risk aversion coefficient θ_i and initial wealth $W_{i,o}$. Then the optimal portfolio of the investor given by equation (14) becomes

$$\mathbf{z}_i^* = \left(1 - \frac{\lambda_i^*}{\lambda_a^*}\right)\theta_i^{-1}\Omega_a^{-1}\mathbf{y}_a + \frac{1}{I}\theta_i^{-1}\theta_a\frac{\lambda_i^*}{\lambda_a^*}\mathbf{z}_m. \tag{25}$$

Equation (25) shows that any consensus investor will divide his/her investment into two portfolios, namely, $\Omega_a^{-1}\mathbf{y}_a$ and the market portfolio $\mathbf{z}_m$, which is consistent with the *Two Fund Separation Theorem* (See Huang and Lizenberger (1988) Chapter 4, page 83) and such portfolios must be mean-variance efficient due to the construction, which means that the portfolios $\Omega_a^{-1}\mathbf{y}_a$ and $\mathbf{z}_m$ must be the frontier portfolios. It is easy to verify from (25) that the aggregate position of the portfolio $\Omega_a^{-1}\mathbf{y}_a$ of all investors is $\sum_i(1 - \frac{\lambda_i^*}{\lambda_a^*})\theta_i^{-1}\Omega_a^{-1}\mathbf{y}_a = \mathbf{0}$ when the market clearing condition (7) is satisfied.

However, when investor i's subjective belief ($\mathcal{B}_i$) differs from the market belief ($\mathcal{B}_a$), the optimal portfolio of investor i can be expressed as

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_a^{-1}(\mathbf{y}_i - \mathbf{y}_a) + \left(1 - \frac{\lambda_i^*}{\lambda_a^*}\right)\theta_i^{-1}\Omega_a^{-1}\mathbf{y}_a + \frac{1}{I}\theta_i^{-1}\theta_a\frac{\lambda_i^*}{\lambda_a^*}\mathbf{z}_m. \tag{26}$$

Then the composition of the portfolio depends also on the belief error $\mathbf{y}_i - \mathbf{y}_a$ of the investor i from the market. Analytically it is not easy to see if the optimal portfolio of investor i lies on the market equilibrium frontier. However, through Example F.1 in Appendix F, we can show that the optimal portfolios of investors are not located on the market equilibrium frontier in general. In this example, we consider a market with two investors and three risky assets. Given individuals' risk aversion coefficients, subjective beliefs and initial wealth, we first form the consensus belief and calculate the equilibrium price vector. Using the equilibrium price, we convert the consensus belief in asset payoffs to the consensus belief in asset returns and obtain the market expected returns and variances/covariances of asset returns. With the information provided in Table 3 in Appendix F, we can construct the portfolio frontiers for each investor and for the market equilibrium frontier in the mean-standard deviation space, and locate the optimal portfolios for individual investors as well as the market portfolio. Figure 1 exhibits the resulting graph.

Figure 1 shows two interesting and important features. Firstly, the market equilibrium portfolio frontier is located between two individual's frontiers. However, it is closer to that of investor 2. Intuitively, this is due to the fact that investor 2 is less risk averse and more optimistic about the market in the sense that he/she perceives higher expected payoffs and smaller standard deviations on the asset payoffs, hence he/she dominates the market. Secondly, it is verified that the optimal portfolios of the two investors are always located on their portfolio efficient frontiers based on their own beliefs and the market portfolio is located on the market efficient frontier under the consensus belief. However, in market equilibrium, the optimal portfolios of the two investors are strictly below the market equilibrium frontier. This may be hard to view in Figure 1. We provide a zoom-in version in Figure 2 to verify this observation.

Figure 2 clearly shows that the optimal portfolios of the two investors are not located on the frontier, though they are very close to it, and hence are inefficient. Intuitively, since both investors made "wrong guesses" about the market, investor 1 being pessimistic and investor 2 being optimistic, their optimal portfolios suffer from those "wrong guesses" in terms of mean-variance efficiency.

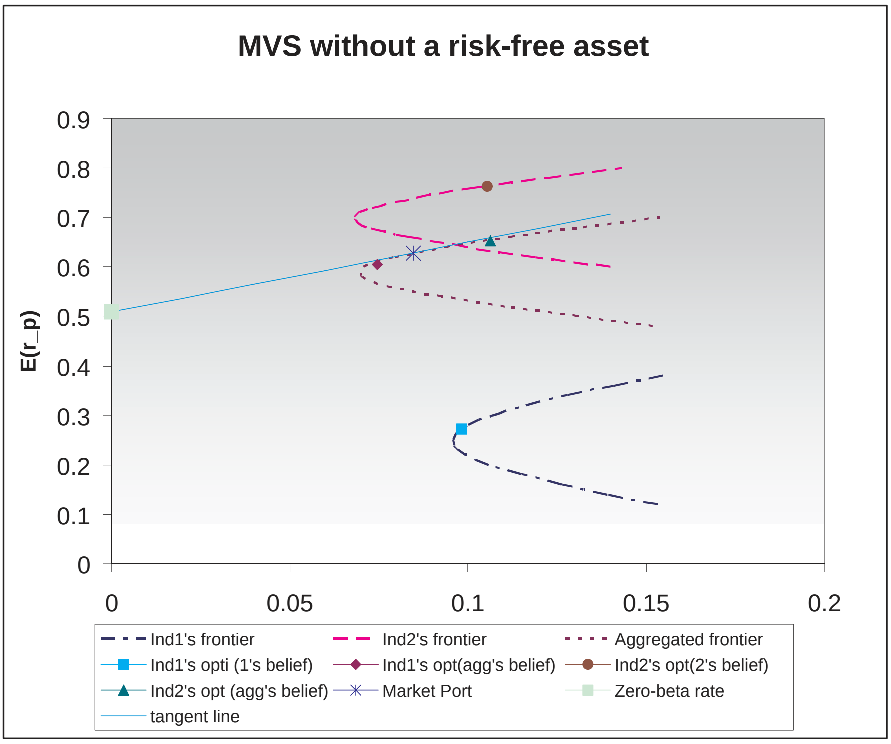


FIGURE 1. The mean variance frontiers under the heterogeneous beliefs and the market equilibrium consensus beliefs. The tangency line corresponding to the consensus belief has the market portfolio as the tangency portfolio and the expected return of the zero-beta portfolio of the market as the intercept with the expected return axis.

Sharpe (2007) simulates market trading using his latest program APSIM, the program assumes a risk-free asset and there is a true probability distribution of future states of the market. Although not directly compatible due to the different setups, Sharpe's findings[1] are consistent with ours. That is the market portfolio outperforms most of other portfolios in terms of mean-variance efficiency, superior fund managers can only at their best perform as well as the market portfolio. However, under our model, the "true" probability distribution of the future depends on the heterogeneous

[1]Sharpe shows in Chapter 6, case 18, that superior fund manager who makes the "correct guesses" about the future of the market (meaning their probability assessment of the future coincide with the hypothetical true probability assessment) has a Sharpe Ratio (of 0.367) slightly above the market's value (of 0.366), other investors who make the "wrong guess" (meaning that their probability assessment of the future differs the hypothetical true probability assessment) are mostly penalized in terms of efficiency (with the lowest Sharpe Ratio of 0.237). However a lucky investor still has the same Sharpe Ratio (of 0.367) as the superior fund managers.

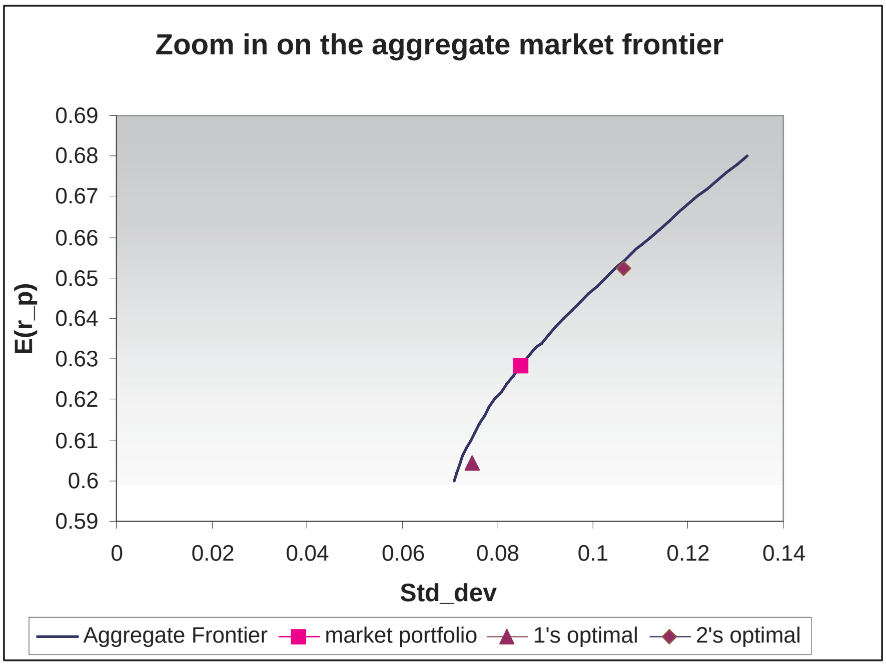


FIGURE 2. Close-up of the locations of individuals' optimal portfolios and the market portfolio relative to the market frontier when the market is in equilibrium.

beliefs of the investors. Sharpe (2007) explains the inferior performance of the active fund managers in the long run compare to the index funds by higher cost of active managers. To add to this, we suggest that it might be simply because it is very difficult for active mangers to consistently make correct predictions about the future, while index funds tracks the market portfolio, which is always mean-variance efficient. We conclude this numerical example by amending Sharpe's *Index Fund Premise*(IFP) to be as follows,

- IFPa Few of us are as smart as all of us.
- IFPb Few of us are as smart as all of us, and it is hard to identify such people in advance.
- IFPc Few of us are as smart as all of us, and it is hard to identify such people in advance, and they **definitely**[2] charge more than they are worth.

Because no optimal portfolio is mean-variance efficient unless the individual's belief coincides to the consensus belief, hence no one can beat the market portfolio when the market is in equilibrium.

[2]It reads "may" in Sharpe's book

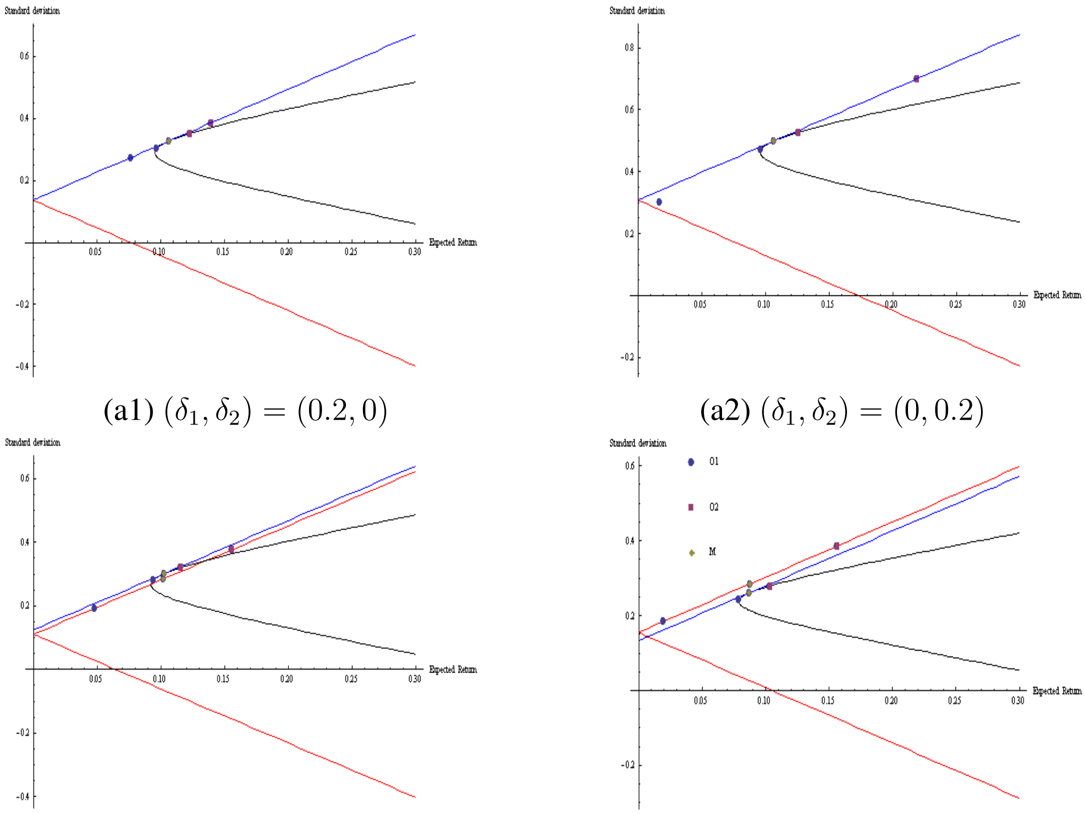
4.2. **The Geometric Relation of the Market Equilibrium Frontiers with and without Risk-free Asset.** To examine the *tangency relationship* of the traditional portfolio theory with heterogeneous beliefs, we consider the situation under which a riskless asset exists with future payoff R_f. Under the homogeneous belief, the classic portfolio theory tells us that the efficient portfolio frontier collapses to a straight line when a risk-free asset is added to the market. This straight line has one tangency point with the original frontier without a risk-free asset. This *tangency portfolio* is exactly the market portfolio when both the bond and equity markets clear. We now examine this equilibrium tangency relationship under heterogeneous beliefs through the following example.

Example 4.1. *Consider the case with $I = 2$ investors with beliefs $\mathcal{B}_i = (\Omega_i, \mathbf{y}_i)$ for $i = 1, 2$. There are $N = 3$ risky assets and a risk-free asset with payoff R_f. Let the absolute risk aversion coefficients $(\theta_1, \theta_2) = (5, 1)$, investors' initial wealth $W_{1,o} = W_{2,o} = \$10$, market endowment of risky assets $\mathbf{z}_m = (1, 1, 1)^T$, and $\mathbf{y}_o = (6.59, 9.34, 9.78)^T$, $\mathbf{1} = (1, 1, 1)^T$ and $\Omega_o = D_o C D_o$ where*

$$D_o = \begin{pmatrix} 0.7933 & 0 & 0 \\ 0 & 0.8770 & 0 \\ 0 & 0 & 1.4622 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0.2233 & 0.1950 \\ 0.2233 & 1 & 0.1163 \\ 0.1950 & 0.1163 & 1 \end{pmatrix},$$

in which D_o corresponds to the standard deviation matrix and C is the correlation matrix. Investors' beliefs are given by $\mathbf{y}_i = (1 + \delta_i)\mathbf{y}_o$ and $\Omega_i = D_i C D_i$, where $D_i = (1 + \epsilon_i) D_o$ for $i = 1, 2$. This implies that investors agrees on the correlation of asset payoffs, but disagree about the volatility and expected payoffs. Next we aggregate individuals' beliefs according to Proposition 2.3, first without a risk-free asset, then with a risk-free asset. The risk-free payoff R_f is determined such that the risk-free asset is in net-zero supply in equilibrium. To examine the tangency relationship, we plot the portfolio frontiers and optimal portfolios under the market consensus belief with and without risk-free asset for different values of δ_i and ϵ_i. Plots are shown in Figure 3.

When investors are homogeneous about the variances and covariances but heterogeneous about the expected payoffs of the risky asset, Figure 3 (a1) and (a2) show that the tangency relation still holds. This is not surprising because of the homogeneous belief of the variance-covariance matrix $\Omega_i = \Omega_o$, the consensus variance-covariance matrix is given by $\Omega_a = \Omega_o$. From the construction of the consensus belief, the expected payoff $\mathbf{y}_a$ is a risk tolerance weighted average of the heterogeneous beliefs in the expected



(a1) $(\delta_1, \delta_2) = (0.2, 0)$ (a2) $(\delta_1, \delta_2) = (0, 0.2)$

(a3) $(\epsilon_1, \epsilon_2) = (-0.2, 0)$ (a4) $(\epsilon_1, \epsilon_2) = (0, -0.2)$

FIGURE 3. Compare the geometric relationships between market portfolio frontiers with and without a risk-free asset, when the risk-free asset is in net-zero supply. In (a1) and (a2), $\mathbf{y}_1 \neq \mathbf{y}_2, \Omega_1 = \Omega_2$; in (a3) and (a4), $\mathbf{y}_1 = \mathbf{y}_2, \Omega_1 \neq \Omega_2$ with

payoffs. Therefore, the consensus belief $\mathcal{B}_a$ remains the same when a risk-free asset is added to the market. Furthermore, since the risk-free asset is in net-zero supply, it follows from equation (13) in Proposition 2.3 that

$$W_{m,o} = \mathbf{z}_m^T \mathbf{p}_0 = \frac{1}{\lambda_a^*}\left[\mathbf{y}_a^T \mathbf{z}_m - \frac{1}{I}\theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m\right] = \frac{1}{R_f}\left[\mathbf{y}_a^T \mathbf{z}_m - \frac{1}{I}\theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m\right].$$

Consequently, the riskless payoff R_f must equal to the zero-beta payoff λ_a^*. This implies that both the market's optimal marginal certainty equivalent wealth (CEW) and the equilibrium prices do not change when a risk-free asset is added to the market. Therefore the tangency relationship of the two market equilibrium frontiers with and without a risk-free asset holds with the market portfolio as the tangency portfolio. However, the efficiency of the optimal portfolios of the two investors depends on their expectations and risk aversion coefficients. On the one hand, when the more risk averse investor is optimistic and the less risk averse investor is pessimistic about the expected payoffs, intuitively this might not be the situation that one expects, Fig. 3 (a1) indicates that the optimal portfolios of both investors are located closer to the market portfolio and market frontiers. On the other hand, when the more risk averse investor is pessimistic and the less risk averse investor is optimistic about the expected payoffs, intuitively this might be the situation that one expects, Fig. 3 (a2) indicates that the optimal portfolios of both investors are located far away from the market portfolio and the equilibrium market frontiers, in particular, the optimal portfolio of the pessimistic investor may become inefficient, even more inefficient when the risk-free asset is available with lower expected return than the risk-free rate. This means that adding a risk-free asset in this situation may help investor 2 to achieve a higher expected return for his optimal portfolio by sacrificing the mean-variance efficiency of the optimal portfolio of investor 1.

When investors are heterogeneous in the variances of the asset payoffs but homogeneous in their expected payoffs, Figure 3 (a3) and (a4) illustrate that the tangency relation breaks down. The risk-free payoff is no longer guaranteed to equal to the zero-beta payoff, which results in a change in the market's optimal CEW and also the equilibrium prices. In particular, when the relative less risk averse investor, investor 2 in this case, is more confident (measured by the smaller variances), Figure 3 (a4) indicates that the existence of a risk-free asset actually pushes up the mean variance frontier, leading to higher expected return for the market portfolio. If one would believe that it is more likely that the less risk averse investor would be more confident in general, this implies that adding a risk-free asset would be more likely to push the portfolio frontier line above the tangency line of the frontier without the risk-free asset, leading to a higher market expected return. This observation would help us to explain the risk premium puzzle. However, when the relative more risk averse investor, investor 1 in this case, is more confident, Figure 3 (a3) implies that the existence of a risk-free asset actually pushes down the mean variance frontier, lowing the expected return of the market portfolio. This is an unexpected and surprising result. In the standard homogeneous case, the expected return of the market portfolio is independent of the existence of the risk-free asset which is in zero-net supply. The above analysis demonstrates that this is no longer the case when investors are heterogeneous. A restriction to the access of the risk-free security may lead to a lower market expected return, a phenomena we are experiencing now in the current financial crisis, and we leave the detailed discussion along this line to future research.

5. THE IMPACT OF THE HETEROGENEOUS BELIEFS IN ASSET RETURNS

In our discussion so far, the heterogeneous beliefs are formed in terms of asset payoffs, rather than asset returns as one may argue is the more accurate measure of return. One may also wonder if the different way in forming the heterogeneous beliefs would lead to different results. In this section, we examine the impact on the market and portfolio analysis when the heterogeneous beliefs are formed in returns. To be consistent with previous sections, we let $\mathbb{E}_i(\tilde{\mathbf{r}})$ be the expected asset returns vector and V_i be the variance/covariance matrix of asset returns for investor i. Denote $\mathcal{B}_i = (\mathbb{E}_i(\tilde{\mathbf{r}}), V_i)$ as the subjective beliefs of investor i, and let $\boldsymbol{\pi}_i = (\pi_{i,1}, \pi_{i,2} \cdots, \pi_{i,I})^T$ be the portfolio of investor i in dollar amount, and $W_{i,0}$ be the initial wealth of investor i.

5.1. **Individual's Portfolio Selection.** By assuming each investor's wealth is normal distributed, investor i's end-of-period expected utility maximization problem becomes $\max_{\boldsymbol{\pi}_i} Q_i(\boldsymbol{\pi}_i)$ subject to the wealth constraint $\boldsymbol{\pi}_i^T\mathbf{1} = W_{i,0}$, where $\boldsymbol{\mu}_i = \mathbb{E}_i(\tilde{\mathbf{r}}_i) + \mathbf{1}$ and $Q_i(\boldsymbol{\pi}) = \boldsymbol{\mu}_i^T\boldsymbol{\pi}_i - \frac{\theta_i}{2}\boldsymbol{\pi}_i V_i \boldsymbol{\pi}_i$ is the certainty equivalent end-of-period wealth of investor i in terms of his dollar investment in each asset. Similar to the payoff setup in the previous sections, we can solve this optimization problem and yield the following result, $\boldsymbol{\pi}_i^* = \theta_i^{-1}V_i^{-1}(\boldsymbol{\mu}_i - \lambda_i^*\mathbf{1})$, where $\lambda_i^* = \frac{\mathbf{1}^T V_i^{-1}\boldsymbol{\mu}_i - \theta_i W_{i,0}}{\mathbf{1}^T V_i^{-1}\mathbf{1}}$ (see Appendix D for the proof). Clearly, $\lambda_i^* = \partial Q_i(\boldsymbol{\pi}_i^*)/\partial \pi_{i,j}$ is still the shadow price of investor i, however it's meaning is slightly different in the sense that λ_i^* is the *optimal marginal certainty equivalent end-of-period wealth per dollar investment in asset j for investor i*. It is the same across all risky assets, but different across different investors. Obviously, if there exists a risk-free asset (f) with return r_f, then it must be true that $\lambda_i^* = 1 + r_f = R_f$ for all investors .

5.2. **The Consensus Belief, Market Equilibrium, and the Zero-Beta CAPM.** We now show that a consensus belief can be constructed in the newly defined market under new market aggregate condition $\boldsymbol{\pi}_m = \sum_{i=1}^I \boldsymbol{\pi}_i^*$ with $W_{m,o} = \boldsymbol{\pi}_m^T\mathbf{1}$. In this case the equilibrium price vector is determined by

$$\mathbf{p}_0 = Z^{-1}\sum_{i=1}^{I}\theta_i^{-1}V_i^{-1}(\boldsymbol{\mu}_i - \lambda_i^*\mathbf{1}), \tag{27}$$

where Z is an $N \times N$ diagonal matrix with diagonal elements z_j representing the market supply in absolute number of shares of asset j. It turns out that we can still construct a consensus belief such that, if held by all investors, it generates the same equilibrium price vector as in equation (27).

Proposition 5.1. *Let* $\theta_a := (\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1})^{-1}$ *and* $\lambda_a^* := \frac{1}{I}\theta_a\sum_{i=1}^{I}\theta_i^{-1}\lambda_i^*$. *Then*

(i) *the consensus belief* $\mathcal{B}_a$ *is given by*

$$V_a = \theta_a^{-1}\lambda_a^*\left(\frac{1}{I}\sum_{i=1}^{I}\lambda_i^*\theta_i^{-1}V_i^{-1}\right)^{-1}, \tag{28}$$

$$\boldsymbol{\mu}_a = \mathbb{E}_a(\mathbf{1}+\tilde{\mathbf{r}}) = \theta_a V_a\left(\frac{1}{I}\sum_{i=1}^{I}\theta_i^{-1}V_i^{-1}\boldsymbol{\mu}_i\right), \qquad \mathbb{E}_a(\tilde{\mathbf{r}}) = \boldsymbol{\mu}_a - \mathbf{1}; \tag{29}$$

(ii) *the market equilibrium price* $\mathbf{p}_o$ *is determined by*

$$\mathbf{p}_0 = I\theta_a^{-1}Z^{-1}V_a^{-1}(\boldsymbol{\mu}_a - \lambda_a^*\mathbf{1}); \tag{30}$$

(iii) *let* $\Omega_i = P_0V_iP_0$ *and* $\mathbf{y}_i = P_0\mathbb{E}_i(\tilde{\mathbf{r}})$ *where* $P_0 = Diag(\mathbf{p}_0)$, *then the equilibrium asset prices can also be written as*

$$\mathbf{p}_0 = \frac{1}{\lambda_a^*}\left[\mathbf{y}_a - \frac{1}{I}\theta_a\Omega_a\mathbf{z}_m\right]; \tag{31}$$

where $\mathbf{y}_a$ *and* $\boldsymbol{\Omega}_a$ *are calculated as in Proposition 2.3;*

(iv) *the Zero-beta CAPM relation*

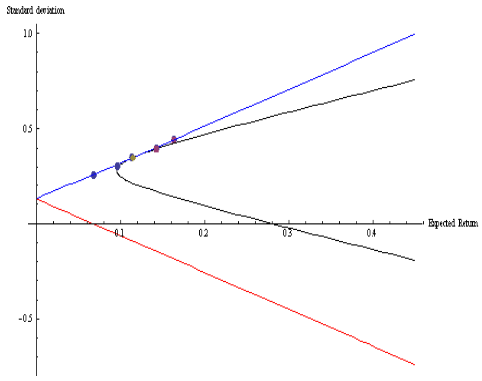
$$\mathbb{E}_a[\tilde{\mathbf{r}}] - (\lambda_a^* - 1)\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - (\lambda_a - 1)], \tag{32}$$

holds, where

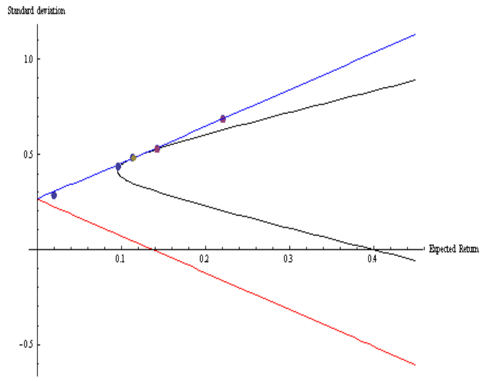
$$\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_N)^T, \qquad \beta_j = \frac{Cov_a(\tilde{r}_m, \tilde{r}_j)}{\sigma_a^2(\tilde{r}_m)}, \qquad j = 1, \cdots, N.$$

Therefore, we see from Proposition 5.1 that both equilibrium and a consensus belief also exist if we allow investors to form their beliefs in terms of asset return instead of asset payoffs. However, this set up in asset returns has some advantages; (i) since the shadow price λ_i^* no longer depends on the equilibrium prices, the equilibrium price vector can now be solved explicitly, which is a computational convenience; (ii) ZH-CAPM can be derived in a much simpler manner (see Appendix E). We now investigate the different impact this setup has on market equilibrium in comparison with the payoff setup.

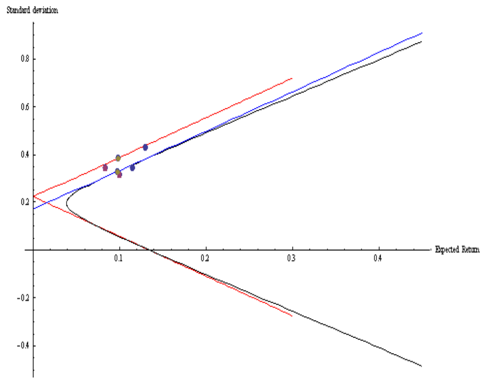
5.3. **The Impact on the Market and Comparison for Both Setups.** In this subsection, we aim to illustrate the different impact of heterogeneity on the market efficient portfolio frontiers and individual optimal portfolios when investors form their beliefs in terms of rate of returns instead of payoffs. Similarly, we examine the optimal portfolios and the tangency relation of the market equilibrium frontiers with and without risk-free asset through an numerical example detailed in Example F.2 in Appendix F.



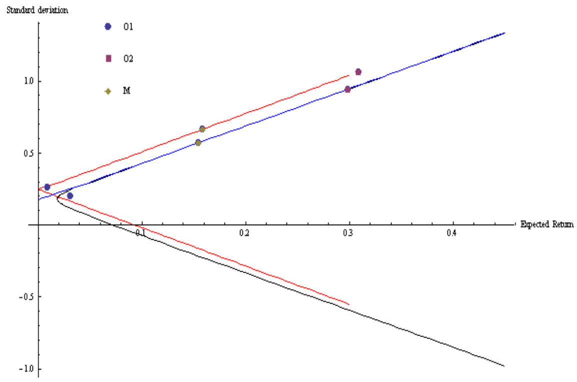
(b1) $(\delta_1, \delta_2) = (0.2, 0)$



(b2) $(\delta_1, \delta_2) = (0, 0.2)$



(b3) $(\epsilon_1, \epsilon_2) = (-0.1, 0)$



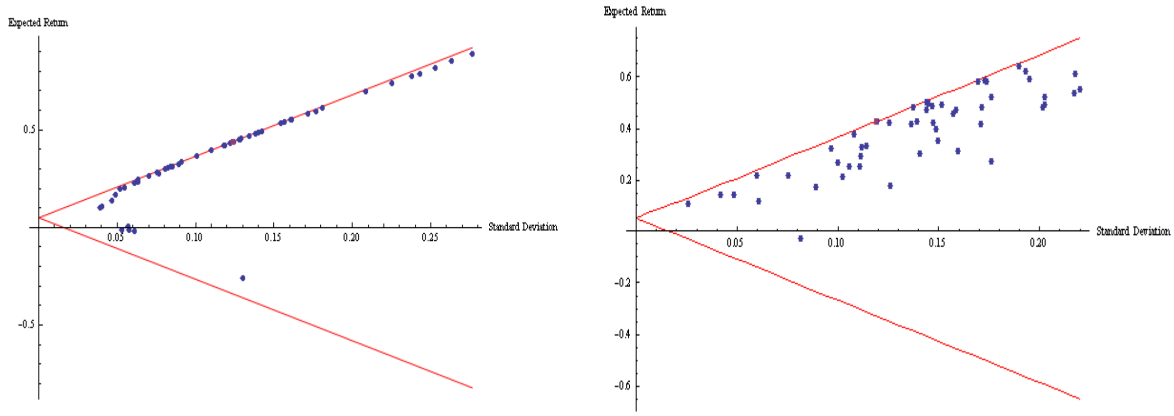
(b4) $(\epsilon_1, \epsilon_2) = (0, -0.1)$

FIGURE 4. Compare the optimal and market portfolios and geometric relationships of market equilibrium frontiers with and without a risk-free asset when the risk-free asset is in net-zero supply and investors form their beliefs in terms of asset returns.

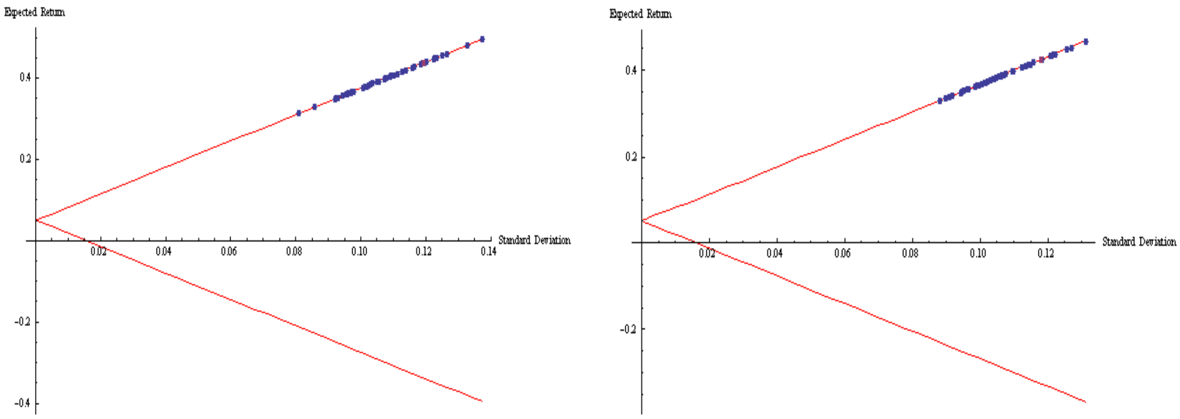
Comparing Figure 3 for payoff setup and Figure 4 for the return setup, we observe some interesting similarities and differences. (i) When investors are homogeneous in covariances but heterogeneous in the means, for both setups and with/without riskless asset, the optimal portfolios of the two investors are not located on the market frontiers, though they can be very close to the frontiers, as indicated by comparing Figure 4 (b1) and (b2) to Figure 3 (a1) and (a2). Also, the tangency relation of the equilibrium market frontiers with and without riskless asset still holds. In addition, when there is a risk-free asset, the optimal portfolio are located much further apart from the market portfolio, in particular, for the optimal portfolio of investor who is relatively more risk averse but pessimistic in his/her belief in expected returns (payoffs), see Figure 3 (a2) and Figure 4 (b2). In other words, both setups of the heterogeneity in mean have very similar impact. (ii) When the investors have heterogeneous beliefs in variances, there is a significant difference between two setups, comparing Figures 4 (b3) and (b4) to Figures 3 (a3) and (a4). In the return setup, heterogeneity asset volatility has a significantly greater impact on the portfolio frontiers and the mean-variance efficiency of optimal portfolios. Adding a risk-free asset can push the equilibrium market frontier line far above to the tangency line of the frontier without risk-free asset, leading to a much higher market expected return. The optimal portfolios of the two investors also benefit significantly by adding a risk-free asset. The gain of the market expected return by adding a riskless asset can be one of potential explanations on the risk premium puzzle and the reduction of the market expected return by restriction of access to the riskless security could be used to explain market returns of the current bear market and financial crisis.

6. The Impact of Heterogeneity on the Market with Many Investors

From the traditional portfolio theory under homogeneous beliefs, we know that all investors hold portfolios located on the market's efficient frontier, which is a hyperbola when there is no riskless asset and a straight line connecting the risk-free asset and the market portfolio, which is called the *Capital Market Line* (CML). When investors are heterogeneous, we have shown through numerical examples of three assets and two investors in the previous sections that investors no longer held portfolios on the equilibrium market frontier unless investors held the market consensus belief. Essentially, this is due to the fact that the consensus belief is determined endogenously by the heterogeneous beliefs and no individual knows the consensus belief in advance. In this section, we extend the analysis in the previous sections to a market consisting of many different investors and we want to see whether those features observed for the market with two investors also hold for the market with many investors. To characterize the beliefs of the investors, we assume that investors' beliefs follow a certain probability distribution, noise in the beliefs can be either additive or multiplicative, univariate or multivariate. For consistency, we consider multiplicative noise for the payoff setup and additive noise for the return setup. This provides us with four possible cases, we use numerical examples to examine the efficiency of the optimal portfolios of investors, their relative position to the CML, and the difference between two setups.



(c1) Heterogeneity in expected payoffs $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0.2, 0)$.



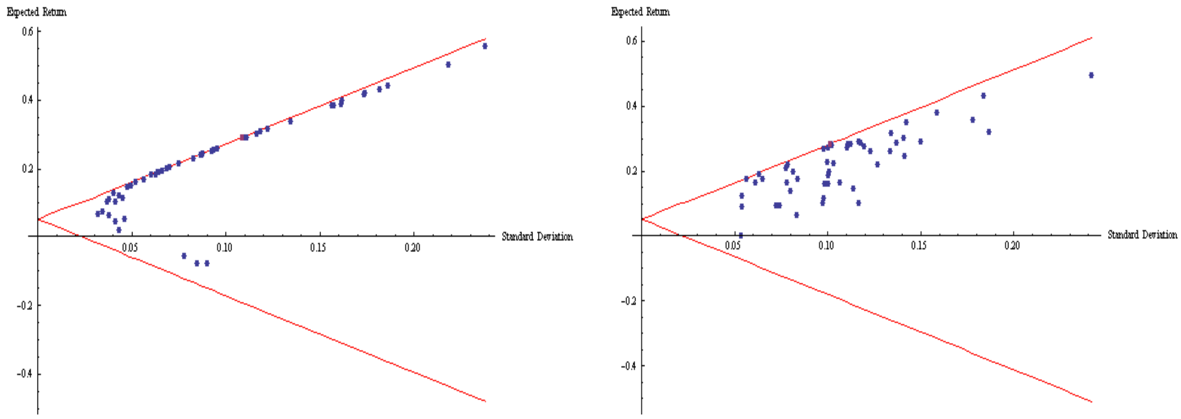
(c2) Heterogeneity in variances $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0, 0.03)$.

FIGURE 5. The optimal portfolios of all the 50 investors and their relative position to the CML when investors' beliefs in terms of asset payoffs is homogeneous in variances and heterogeneous in expected payoffs in (c1) or homogeneous in expected payoffs and heterogeneous in variances in (c2). The left (right) panels correspond to univariate (multivariate) distribution in beliefs.

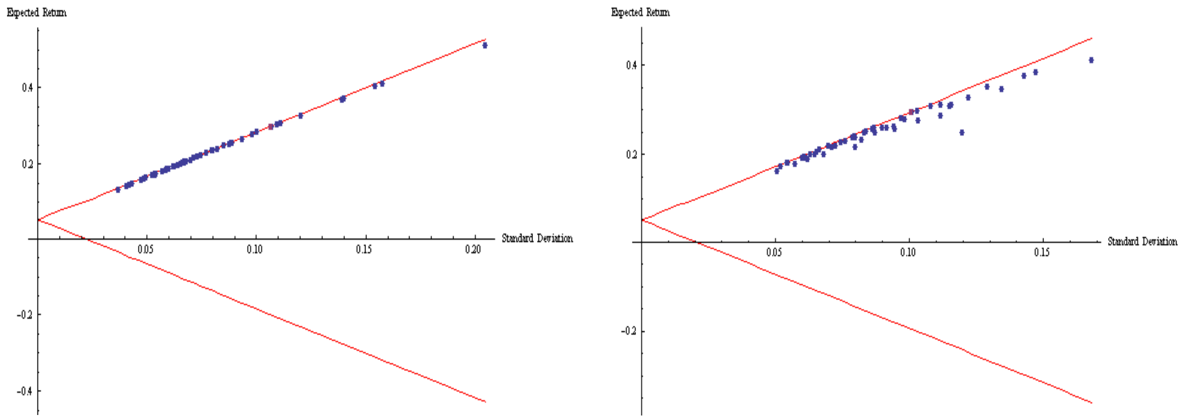
In our first example (see Example F.3 in Appendix F for the details), we consider the payoff setup and the market with three risky assets and a risk-free asset and 50 investors. We assume that all 50 investors who agree on the correlation matrix in both setups of the beliefs. Beliefs dispersions in expected payoffs are multiplicative. In the first case, each investor's belief dispersion is driven by a univariate, normally distributed noise. But in the second case, the dispersion is driven by multivariate, normally distributed noise for each investor. The results for the two cases are plotted in Figure 5 in which the optimal portfolios of all 50 investors and their relative position to the CML are plotted. Figure 5(c1) illustrates the case when investors are homogeneous in variances but heterogeneous in the expected payoffs, while Figure 5(c2) illustrates the case other way around. The left panels correspond to the case with univariate noise and the right panels correspond to the case with multivariate noise. Figure 5 leads to the following interesting observations. (i) The optimal portfolios of all investors are almost on the CML when investors are heterogeneous in variances but homogeneous in the expected payoffs, illustrated by Figure 5(c2). The same effect is observed with multivariate noise (right panel) as for univariate noise (left panel). This shows that heterogeneity in covariances plays insignificant role for the efficiency of the optimal portfolios of investors. (ii) The heterogeneity in expected payoff has significant impact on the location of the optimal portfolios of the investors, illustrated in Figure 5(c1). The optimal portfolios become less efficient, in particular, when the belief dispersions are multivariate normally distributed (the right panel). Some optimal portfolios are far below the CML, even have lower expected return than the risk free rate (the left panel). In addition, when the belief dispersions are univariate, the optimal portfolios seem to form a hyperbolic curve under the equilibrium market efficient frontier (the left panel). However, when the divergence of opinions are not the same for each asset, optimal portfolios are scattered under the frontier without any significant pattern. This example shows that heterogeneity in expected payoff has more significant impact on the efficiency of optimal portfolios of investors than the heterogeneity in variances.

In our second example (see Example F.4 in Appendix F for the details) we consider a similar situation but with the return setup. The belief dispersions are additive (which is normal for case (iii) and multi-variate normal for case (iv)). The results are reported in Figure 6. Comparing Figure 6 (d1) to Figure 5(c1), we see a similar impact of the heterogeneity in the expected returns on the optimal portfolios to the payoff setup. However, in comparing Figure 6 (d2) to Figure 5(c2), we notice that the impact of the heterogeneity of variances are very different from the payoff setup, especially when ϵ_is are multivariate normally distributed (the right panel). For the return setup, many of investors' optimal portfolios are far below the equilibrium frontier compared to the payoff set up, leading to lower Sharpe ratios.

Based on the above two examples with many investors and two setups in beliefs, we find that heterogeneity in expected payoff/returns has significant impact on the optimal portfolios of investors than the heterogeneity in variances. Also, different distribution of the beliefs has different impact. Between two setups in beliefs, the return setup leads the optimal portfolios of investors to be less efficient than the payoff setup. Overall, we can see that, due to the heterogeneous beliefs, the market fails to provide investors with



(d1) Heterogeneity in the expected returns $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0.2, 0)$.

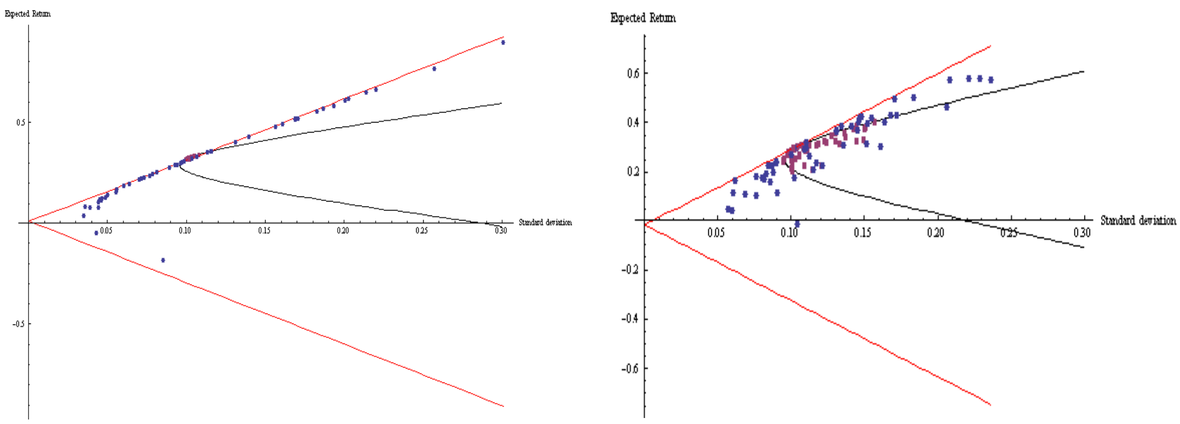


(d2) Heterogeneity in the variances $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0, 0.03)$.

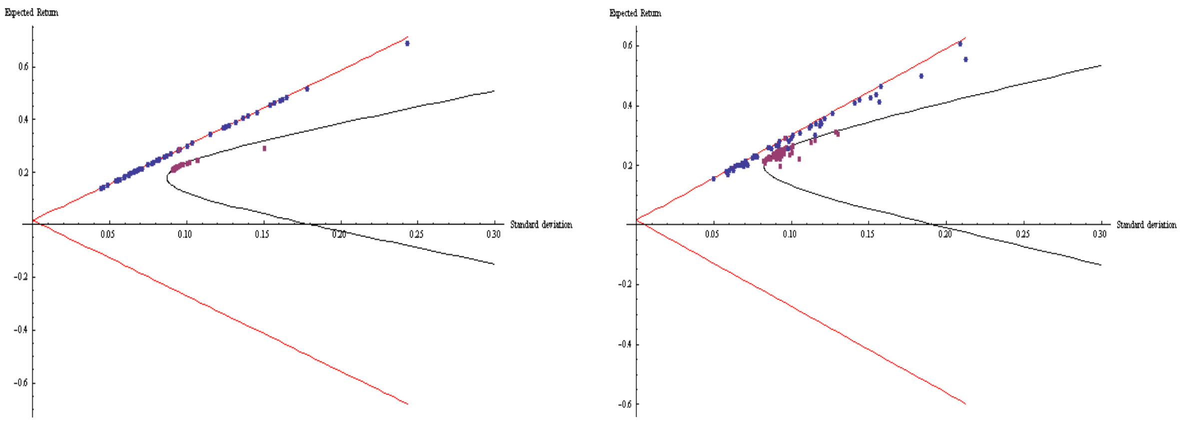
FIGURE 6. The optimal portfolios of all the 50 investors and their relative position to the CML when investors' beliefs in terms of asset returns is homogeneous in variances and heterogeneous in expected payoffs in (d1) and homogeneous in expected payoffs and heterogeneous in variances in (d2). The left panels are when (δ_i, ϵ_i) are univariate random variables, the right-panels are when (δ_i, ϵ_i) are multivariate random variables.

efficient portfolio, this generic feature is not what we would expect in a homogeneous market. It shows that heterogeneous investors can never beat the market when the performance is measured by the Sharpe ratio.

Finally, we examine the *tangency relation* once again for the market with many investors through an example under the return setup (see Example F.5 in Appendix F for details). The result is illustrated in Figure 7. One can see that without a riskless security the optimal portfolios of the investors are grouped much closer together, especially



(e1) Heterogeneity in the expected returns $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0.2, 0)$.



(e2) Heterogeneity in the variances $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0, 0.03)$.

FIGURE 7. The optimal portfolios of all the 50 investors and their relative position to the CML in a market with a risk-free asset and in a market without a risk-free asset. Investors' beliefs in terms of asset returns is homogeneous in variances and heterogeneous in expected payoffs in (d1) and homogeneous in expected payoffs and heterogeneous in variances in (d2). The left panels are when (δ_i, ϵ_i) are univariate random variables, the right-panels are when (δ_i, ϵ_i) are multivariate random variables.

when the investors' divergence of opinion about future returns are the same across all stocks. The tangency relation holds when investors have heterogeneous beliefs about the expected future returns but homogeneous beliefs on the covariance matrix. However it no longer holds when beliefs in the varainces/covariances of asset returns are heterogeneous, and adding a risk-free asset in zero-net supply to the market can improve the expected return of the market portfolio significantly.

7. Conclusion

Within the mean-variance framework, by assuming that investors are heterogeneous, this paper examines the impact of the heterogeneity on the market equilibrium prices and equilibrium mean-variance portfolio frontier in a market with many risky assets and no riskless asset. The heterogeneity is measured by the risk aversion coefficients, expected payoffs/returns, and variance/covariance matrices of risky assets of heterogeneous investors. Investors are bounded rational in the sense that, based on their beliefs, they make their optimal portfolio decisions. To characterize the market equilibrium prices of the risky assets, we introduce the concept of consensus belief of the market and show how the consensus or market belief can be constructed from the heterogeneous beliefs. Basically, under the market aggregation, the consensus belief is a weighted average of the heterogeneous beliefs. In both setups of beliefs, explicit formula for the market equilibrium prices of the risky assets are derived. As a by-product of the consensus belief and equilibrium price formula, we show that the standard Black's zero-beta CAPM with homogeneous beliefs still holds with heterogeneous beliefs. The impact of the heterogeneity on the market equilibrium, mean variance frontier and the efficiency of the optimal portfolios of the investors are analyzed. In particular, through some numerical examples, we show that, under market aggregation, the biased belief (from the market belief) of an investor can push his/her optimal portfolio below the equilibrium market frontier (although they are very close to the efficient frontier). This demonstrates that bounded rational investors may never achieve their mean-variance efficiency (measured by the Sharpe ratio) in marker equilibrium. If we refer the heterogeneous investors as fund managers and the market portfolio as a market index, then our result offers a theoretic explanation on the empirical finding that, according to the mean-variance criteria, managed funds under-perform the market indices on average. We also offer an explanation on Miller's proposition that "divergence of opinion corresponds to lower future asset returns" and the subsequent empirical findings on this. Furthermore, the well known *tangency relation* of the frontiers with and without risk-free asset under the homogeneous beliefs breaks down under the heterogeneous beliefs, in particular when investors are heterogeneous in variances. Adding a risk-free asset to the market with many risky assets can have very complicated effect on the market in general. In the homogeneous market, the expected return of the market portfolio is independent of the existence of the risk-free asset which is in zero net supply. However, in the heterogeneous market, adding a risk-free asset to the market with many risky assets leads to a higher expected return for the market portfolio in equilibrium. This result can be used to explain the risk premium puzzle and the current financial market crisis.

The implication of the heterogeneity on the market under different market conditions is far more complicated than it seems and it deserves further study. It would be interesting to extend the current static framework to a dynamic setting that so that the intertemporal effect can be examined. It is also interesting to allow investors to learn overtime from the market through various learning mechanisms, such as the Bayesian updating rule, and the expectation feedback mechanisms. These extension will give us a richer modelling environment and hopefully lead to a better understanding of the phenomenons in our financial market. We leave these issues to the future research.

REFERENCES

Abel, A. (1989), Asset prices under heterogeneous beliefs: implications for the equity premium, working paper 09-89, Rodney L. White Center for Financial Research.

Abel, A. (2002), 'An exploration of the effects of pessimism and doubt on asset returns', *Journal of Economic Dynamics and Control* **26**, 1075–1092.

Black, F. (1972), 'Capital market equilibrium with restricted borrowing', *Journal of Business* **45**, 444–454.

Chiarella, C., Dieci, R. and He, X. (2006*a*), Aggregation of heterogeneous beliefs and asset pricing theory: a mean-variance analysis, Technical Report 186, Quantitative Finance Research Center, University of Techonology, Sydney.

Chiarella, C., Dieci, R. and He, X. (2006*b*), A dynamic heterogeneous beliefs capm, working paper (draft), University of Technology, Sydney.

Detemple, J. and Murthy, S. (1994), 'Intertemporal asset pricing with heterogeneous beliefs', *Journal of Economic Theory* **62**, 294–320.

Diether, K., Malloy, C. and Scherbina, A. (2002), 'Differences of opinion and cross section of stock returns', *Journal of Finance* **57**, 2113–2141.

Fan, S. (2003), GCAPM(i): A microeconomic theory of investments, SSRN working paper series, Fan Asset Management LLC and Institutional Financial Analytics.

Huang, C.-F. and Litzenberger, R. (1988), *Foundations for Financial Economics*, Elsevier, North-Holland.

Lintner, J. (1965), 'The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets', *Review of Economics and Statistics* **47**, 13–37.

Lintner, J. (1969), 'The aggregation of investor's diverse judgements and preferences in purely competitive security markets', *Journal of Financial and Quantitative Analysis* **4**, 347–400.

Markowitz, H. (1959), *Portfolio Selection: Efficient Diversification of Investment*, J. Wiler and Sons, New York. Second Edition by Blackwell, Oxford, 1991.

Miller, E. (1977), 'Risk, uncertainity, and divergence of opinion', *Journal of Finance* **32**, 1151–1168.

Mossin, J. (1966), 'Equilibrium in a capital asset market', *Econometrica* **35**, 768–783.

Rubinstein, M. (1975), 'Security market efficiency in an arrow-debreu economy', *American Economic Review* **65**, 812–824.

Rubinstein, M. (1976), 'The strong case for the generalized logarithmic utility model as the premier model of financial markets', *Journal of Finance* **31**, 551–571.

Sharpe, W. (1964), 'Capital asset prices: A theory of market equilibrium under conditions of risk', *Journal of Finance* **19**, 425–442.

Sharpe, W. (2007), *Investors and Markets, Portfolio Choices, Asset Prices, and Investment Advice*, Princeton University Press.

Sun, N. and Yang, Z. (2003), 'Existence of equilibrium and zero-beta pricing forumla in the capitial asset pricing model with heterogeneous beliefs', *Annals of Economics and Finance* **4**, 51–71.

Williams, J. (1977), 'Capital asset prices with heterogeneous beliefs', *Journal of Financial Economics* **5**, 219–239.

Zapatero, F. (1998), 'Effects of financial innovations on market volatility when beliefs are heterogeneous', *Journal of Economic Dynamics and Control* **22**, 597–626.

APPENDIX A. PROOF OF LEMMA 2.1

Let λ_i be the Lagrange multiplier and set

$$L(\mathbf{z}_i, \lambda_i) := \mathbf{y}_i^T \mathbf{z}_i - \frac{\theta_i}{2} \mathbf{z}_i \Omega_i \mathbf{z}_i + \lambda_i [\mathbf{p}_0^T \mathbf{z}_i - W_0^i] \tag{33}$$

Since $U_i(.)$ is concave, the optimal portfolio of agent i is determined by the first order condition

$$\frac{\partial L}{\partial \mathbf{z}_i} = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{z}_i = \theta_i^{-1} \Omega_i^{-1} [\mathbf{y}_i - \lambda_i \mathbf{p}_0]. \tag{34}$$

Substituting (34) into (3) yields (5), this completes the proof.

APPENDIX B. PROOF OF PROPOSITION 2.3

On the one hand, from Definition 2.2, if the consensus belief $\mathcal{B}_a = (\mathbb{E}_a(\tilde{\mathbf{x}}), \Omega_a)$ exists, then it must be true that

$$\mathbf{z}_i^* = \theta_a^{-1} \Omega_a^{-1} [\mathbf{y}_a - \lambda_a^* \mathbf{p}_0]. \tag{35}$$

Applying the market equilibrium condition to (35), we must have

$$\mathbf{z}_m = \sum_{i=1}^{I} \mathbf{z}_i^* = I \left[\theta_a^{-1} \Omega_a^{-1} [\mathbf{y}_a - \lambda_a^* \mathbf{p}_0] \right]. \tag{36}$$

This leads to the equilibrium price (13).

On the other hand, it follows from the individuals demand (4) and the market clearing condition (7) that, under the heterogenous beliefs

$$\mathbf{z}_m = \sum_{i=1}^{I} \mathbf{z}_i^* = \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} [\mathbf{y}_i - \lambda_i^* \mathbf{p}_0]. \tag{37}$$

Under the definitions (11) and (12), we can re-write equation (37) as

$$\mathbf{z}_m = \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} \mathbf{y}_i - \left(\sum_{i=1}^{I} \theta_i^{-1} \lambda_i^* \Omega_i^{-1} \right) \mathbf{p}_0 = I \theta_a^{-1} \Omega_a^{-1} \mathbf{y}_a - I \theta_a^{-1} \lambda_a^* \Omega_a^{-1} \mathbf{p}_0, \tag{38}$$

which leads to the same market equilibrium price (13). This shows that $\mathcal{B}_a = \{\Omega_a, \mathbf{y}_a\}$ defined in (11) and (12) is the consensus belief. Inserting (13) into (4) will give the equilibrium optimal portfolio (14) of investor i.

APPENDIX C. PROOF OF COROLLARY 2.4

The equilibrium price vector in (13) can be re-written to express the price of each asset

$$p_{0,j} = \frac{1}{\lambda_a^*}(y_{a,j} - \theta_a/I \sum_{k=1}^{N} \sigma_{j,k} z_{m,k}) = \frac{1}{\lambda_a^*}[y_{a,j} - \frac{\theta_a}{I} Cov_a(\tilde{x}_j, \tilde{W}_m)]. \tag{39}$$

It follows from (39) that $y_{a,j} - \lambda_a^* p_{0,j} = \frac{\theta_a}{I} Cov_a(\tilde{x}_j, \tilde{W}_m)$ and hence

$$\frac{y_{a,j}}{p_{0,j}} - \lambda_a^* = \frac{1}{p_{0,j}} \frac{\theta_a}{I} Cov_a(\tilde{x}_j, \tilde{W}_m).$$

Therefore

$$\mathbb{E}_a(\tilde{r}_j) - (\lambda_a^* - 1) = \frac{1}{p_{0,j}} \frac{\theta_a}{I} Cov_a(\tilde{x}_j, \tilde{W}_m). \tag{40}$$

It follows from $W_{m0} = \frac{1}{\lambda_a^*} \mathbf{z}_m^T (\mathbf{y}_a - \theta_a \Omega_a \mathbf{z}_m / I)$ that

$$\lambda_a^* = \frac{\mathbf{z}_m^T \mathbf{y}_a - \theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m0}}. \tag{41}$$

Using the definition of λ_a^* in (41), we obtain

$$\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1) = \frac{\mathbf{y}_a^T \mathbf{z}_m}{\mathbf{z}_m^T \mathbf{p}_0} - \lambda_a^* = \frac{\mathbf{y}_a^T \mathbf{z}_m}{W_{m0}} - \frac{\mathbf{z}_m^T \mathbf{y}_a - \theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m0}}.$$

Thus

$$\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1) = \frac{\theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m0}} \neq 0. \tag{42}$$

Dividing (40) by (42) leads to

$$\begin{aligned}\frac{\mathbb{E}_a(\tilde{r}_j) - (\lambda_a^* - 1)}{\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1)} &= \frac{\left(\frac{1}{p_{0,j}} \frac{\theta_a}{I} Cov_a(\tilde{x}_j, \tilde{W}_m)\right)}{\left(\frac{\theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m0}}\right)} = \frac{\frac{1}{p_{0,j}} Cov_a(\tilde{x}_j, \tilde{W}_m)}{\frac{\sigma_{a,m}^2}{W_{m0}}} \\ &= \frac{Cov_a\left(\frac{\tilde{x}_j}{p_{0,j}}, \frac{\tilde{W}_m}{W_{m0}}\right)}{\frac{\sigma_{a,m}^2}{W_{m0}^2}} = \frac{Cov_a(\tilde{r}_j, \tilde{r}_m)}{\sigma_a^2(\tilde{r}_m)} = \beta_j \end{aligned} \tag{43}$$

leading to the CAPM-like relation in (15).

APPENDIX D. THE SOLUTION TO THE OPTIMAL PORTFOLIO SELECTION PROBLEM IN SUBSECTION 5.1

The optimization problem to solve is

$$\max_{\boldsymbol{\pi}_i} \left(\boldsymbol{\mu}_i^T \boldsymbol{\pi}_i - \frac{\theta_i}{2} \boldsymbol{\pi}_i V_i \boldsymbol{\pi}_i \right) \tag{44}$$

subject to the wealth constraint $\boldsymbol{\pi}_i^T\mathbf{1} = W_{i,0}$. Let λ_i be the Lagrange multiplier and set

$$L(\boldsymbol{\pi}_i, \lambda_i) := \boldsymbol{\mu}_i^T\boldsymbol{\pi}_i - \frac{\theta_i}{2}\boldsymbol{\pi}_i V_i \boldsymbol{\pi}_i + \lambda_i[\boldsymbol{\pi}_i^T\mathbf{1} - W_{i,0}]. \tag{45}$$

Then the optimal portfolio of agent i is determined by the first order conditions

$$\frac{\partial L}{\partial \boldsymbol{\pi}_i} = \mathbf{0} \qquad \Rightarrow \qquad \boldsymbol{\pi}_i^* = \theta_i^{-1}V_i^{-1}(\boldsymbol{\mu}_i - \lambda_i\mathbf{1}). \tag{46}$$

Substituting (46) into the wealth constraint yields $\lambda_i^* = \frac{\mathbf{1}^T V_i^{-1}\boldsymbol{\mu}_i - \theta_i W_{i,0}}{\mathbf{1}^T V_i^{-1}\mathbf{1}}$.

APPENDIX E. PROOF OF PROPOSITION 5.1

From equation (27) resulted from the aggregation condition, we know that the market equilibrium prices in terms of investors' subjective beliefs are given by

$$\mathbf{p}_0 = Z^{-1}\bigg(\sum_{i=1}^{I}\theta_i^{-1}V_i^{-1}\boldsymbol{\mu}_i - \sum_{i=1}^{I}\theta_i^{-1}\lambda_i^* V_i^{-1}\mathbf{1}\bigg). \tag{47}$$

Using equation (28) and (29), we can re-write the equilibrium prices as

$$\mathbf{p}_0 = Z^{-1}\bigg(I\theta_a^{-1}V_a^{-1}\boldsymbol{\mu}_a - I\theta_i^{-1}\lambda_i^* V_a^{-1}\mathbf{1}\bigg) = \bigg(\frac{Z}{I}\bigg)^{-1}\theta_a^{-1}V_a^{-1}(\boldsymbol{\mu}_a - \lambda_a^*\mathbf{1}), \tag{48}$$

which corresponds to the equilibrium price specified in equation (30). This means that if every investor in the market has the same belief $\mathcal{B}_a := (V_a, \boldsymbol{\mu}_a)$, then the equilibrium prices in the homogeneous market are identical to the ones in the heterogeneous market. Hence one can conclude that $\mathcal{B}_a$ is a consensus belief.

Now if we substitute $V_i = P_0^{-1}\Omega_i P_0^{-1}$ and $\boldsymbol{\mu}_i = P_o^{-1}\mathbf{y}_i$ into equation (27), then we can recover the expression for equilibrium prices in equation (8). This expression for equilibrium prices corresponds to the consensus belief $\mathcal{B}_a := (\Omega_a, \mathbf{y}_a)$ in terms of asset payoffs, which are defined in Proposition 2.3. Then the equilibrium prices in terms of the consensus belief is stated in equation (13) which is identical to (30).

Finally, we prove the ZHCAPM under the setup where investors form their beliefs about the future asset returns. Define the expected market return under the consensus belief as $\mathbb{E}_a(\tilde{r}_m) = \mathbb{E}_a(\frac{\tilde{W}_m}{W_{m0}} - 1)$, where $W_{m0} = \boldsymbol{\pi}_m^T\mathbf{1}$ is the total initial market wealth. Let

$$\mu_{am} = \mathbb{E}_a(\frac{\tilde{W}_m}{W_{m0}}) = \mathbb{E}_a(\tilde{r}_m) + 1 \qquad \text{and} \qquad \boldsymbol{\omega}_m = \frac{\boldsymbol{\pi}_m}{W_{m0}}.$$

From the fact that $\tilde{W}_m = \boldsymbol{\pi}_m^T(\tilde{\mathbf{r}} + \mathbf{1})$ and $\boldsymbol{\omega}_m^T\mathbf{1} = 1$, it follows that

$$\mu_{am} - \lambda_a^* = \boldsymbol{\omega}_m^T(\boldsymbol{\mu}_a - \lambda_a^*\mathbf{1}) = \boldsymbol{\omega}_m^T\bigg(\frac{1}{I}\theta_a^{-1}V_a\boldsymbol{\pi}_m\bigg) = \frac{W_{m0}}{I}\theta_a^{-1}\sigma_{am}^2. \tag{49}$$

Also we can rearrange equation (30) to get

$$\boldsymbol{\mu}_a - \lambda_a^*\mathbf{1} = \frac{1}{I}\theta_a^{-1}V_a\boldsymbol{\pi}_m, \tag{50}$$

which can be written for each asset j as follows

$$\mu_{a,j} - \lambda_a^* = \frac{W_{m0}}{I}\theta_a^{-1}\sum_{k=1}^{N}\sigma_{a,jk}\omega_{mk} = \frac{W_{m0}}{I}\theta_a^{-1}Cov_a(\tilde{r}_j, \tilde{r}_m). \tag{51}$$

Therefore

$$\frac{\mu_{a,j} - \lambda_a^*}{\mu_{am} - \lambda_a^*} = \frac{Cov_a(\tilde{r}_j, \tilde{r}_m)}{\sigma_{am}^2} = \beta_j. \quad (52)$$

Equation (52) leads to

$$\boldsymbol{\mu}_a - \lambda_a^* \mathbf{1} = \boldsymbol{\beta}(\mu_{am} - \lambda_a^*), \quad (53)$$

leading to the relation in (32), which completes the proof.

APPENDIX F. NUMERICAL EXAMPLES

Example F.1. *Let $I = 2$ and $N = 3$. Consider the set up in Table 1*

Initial Wealth	*Risk Aversion*	*Expected payoffs*	*Variance/Covariance of payoffs*
$W_0^1 = 10$	$\theta_1 = 5$	$\mathbf{y}_1 = \begin{pmatrix} 6.60 \\ 9.35 \\ 9.78 \end{pmatrix}$	$\Omega_1 = \begin{pmatrix} 0.6292 & 0.1553 & 0.2262 \\ & 0.7692 & 0.1492 \\ & & 2.1381 \end{pmatrix}$
$W_0^2 = 10$	$\theta_2 = 1$	$\mathbf{y}_2 = \begin{pmatrix} 9.60 \\ 12.35 \\ 12.78 \end{pmatrix}$	$\Omega_2 = \begin{pmatrix} 0.4292 & -0.0447 & 0.0262 \\ & 0.5692 & -0.0508 \\ & & 1.7381 \end{pmatrix}$

TABLE 1. Market specifications and heterogeneous beliefs.

Assuming there is one share available for each asset, that is, $\mathbf{z}_m = (1, 1, 1)^T$. Based on the information in Table 1, we use equation (8) and Excel Solver to solve for the equilibrium price vector and obtain the market equilibrium price $\mathbf{p}_0 = (5.6436, 7.4328, 6.9236)^T$. The optimal portfolios and shadow prices of the investors are given by $\mathbf{z}_1^* = (0.380, 0.768, 0.310)^T$, $\lambda_1^* = 0.7894$ for investor 1 and $\mathbf{z}_2^* = (0.620, 0.232, 0.690)^T$ and $\lambda_2^* = 1.6520$ for investor 2. Using Proposition 2.3, we construct the consensus belief $\mathcal{B}_a$, the aggregate risk aversion coefficient θ_a, and the aggregate shadow price λ_a^*, and obtain the result in Table 2. We then use

Total Market Initial Wealth	Shadow Price	Risk Aversion	Expected payoffs	Variance/Covariance of payoffs
$W_{m0} = 20$	$\lambda_a^* = 1.5083$	$\theta_a = 1.6667$	$\mathbf{y}_a = \begin{pmatrix} 8.88 \\ 11.63 \\ 12.06 \end{pmatrix}$	$\Omega_a = \begin{pmatrix} 0.4383 & -0.0356 & 0.0352 \\ & 0.5783 & -0.0417 \\ & & 1.9472 \end{pmatrix}$

TABLE 2. The market consensus beliefs, shadow price and ARA.

the market equilibrium price to convert the consensus belief from payoffs to returns as follows. Let $P_0 = diag[\mathbf{p}_0] = diag(5.6436, 7.4328, 6.9236)$ and

$$\mathbb{E}_i(\tilde{\mathbf{r}}) := P_0^{-1}\mathbf{y}_i - \mathbf{1}, \qquad V_i(\tilde{\mathbf{r}}) := P_0^{-1}\Omega_i P_0^{-1}, \qquad i = 1, 2, a;$$

$$\mathbf{w}_i^* := \frac{1}{W_{i,o}} P_0 \mathbf{z}_i^*, \qquad \mathbb{E}_i(\tilde{r}_{ip}^*) := \mathbb{E}_i(\tilde{\mathbf{r}})^T \mathbf{w}_i^*, \qquad \sigma_{ip}^* = (\mathbf{w}_i^{*T} V_i(\tilde{\mathbf{r}}) \mathbf{w}_i^*)^{1/2}, \quad i = 1, 2;$$

$$\mathbb{E}_a(\tilde{r}_{ip}^*) := \mathbb{E}_a(\tilde{\mathbf{r}})^T \mathbf{w}_i^*, \qquad \sigma_{ip}^a = (\mathbf{w}_i^{*T} V_a(\tilde{\mathbf{r}}) \mathbf{w}_i^*)^{1/2}, \qquad i = 1, 2;$$

$$\mathbf{w}_m := \frac{1}{W_{m,o}} P_0 \mathbf{z}_m, \qquad \mathbb{E}_a(\tilde{r}_m) := \mathbb{E}_a(\tilde{\mathbf{r}})^T \mathbf{w}_m,$$

$$\sigma_{a,m} = (\mathbf{w}_m^T V_a(\tilde{\mathbf{r}}) \mathbf{w}_m)^{1/2}, \qquad \boldsymbol{\beta} := \frac{V_a(\tilde{\mathbf{r}}) \mathbf{w}_m}{\sigma_{a,m}^2}.$$

Expected returns	Variance/Covariance of returns	optimal portfolio weights	Portfolio Return/SD
$\mathbb{E}_1(\tilde{\mathbf{r}}) = \begin{pmatrix} .1690 \\ .2577 \\ .4126 \end{pmatrix}$	$V_1 = \begin{pmatrix} .0198 & .0037 & .0058 \\ & .0139 & .0029 \\ & & .0446 \end{pmatrix}$	$\mathbf{w}_1^* = \begin{pmatrix} .2144 \\ .5711 \\ .2145 \end{pmatrix}$	$\mathbb{E}_1(r_{1p}^*) = .2719$ $\sigma_{1p}^* = .09824$ $\mathbb{E}_a(r_{1p}^*) = .6043$ $\sigma_{1p}^a = .0748$
$\mathbb{E}_2(\tilde{\mathbf{r}}) = \begin{pmatrix} .7006 \\ .6613 \\ .8459 \end{pmatrix}$	$V_2 = \begin{pmatrix} .0135 & -.0011 & .0007 \\ & .0103 & -.0010 \\ & & .0404 \end{pmatrix}$	$\mathbf{w}_2^* = \begin{pmatrix} .3499 \\ .1722 \\ .4778 \end{pmatrix}$	$\mathbb{E}_2(r_{2p}^*) = .7633$ $\sigma_{2p}^* = .1054$ $\mathbb{E}_a(r_{2p}^*) = .6522$ $\sigma_{2p}^a = .1065$
$\mathbb{E}_a(\tilde{\mathbf{r}}) = \begin{pmatrix} .5729 \\ .5644 \\ .7418 \end{pmatrix}$	$V_a = \begin{pmatrix} .0138 & -.0008 & .0009 \\ & .0105 & -.0008 \\ & & .0406 \end{pmatrix}$	$\mathbf{w}_m = \begin{pmatrix} .2822 \\ .3716 \\ .3462 \end{pmatrix}$	$\mathbb{E}_a(r_m) = .6283$ $\sigma_{a,m} = .0848$

$$\boldsymbol{\beta} = \begin{pmatrix} 0.5390 & 0.4681 & 1.9468 \end{pmatrix}^T$$

TABLE 3. Heterogeneous beliefs and the consensus belief, the individual optimal and market portfolios in equilibrium, and the means and standard deviations of these portfolios under heterogeneous and consensus belies, respectively.

We then obtain the following results.

In the above definitions, $\mathbb{E}_i(\mathbf{r})$ and $V_i(\tilde{\mathbf{r}})$ are the expected returns vectors and covariance matrices in terms of asset returns for each investor.Subsequently, $\mathbf{w}_i^*$ are the individuals' optimal portfolio weights, $\mathbb{E}_i(\tilde{r}_{ip}^*)$ and σ_{ip}^* are the expected return and standard deviations of the optimal portfolios of investors, respectively, under their subjective beliefs, $\mathcal{B}_i = (\mathbb{E}_a(\tilde{r}_{ip}^*), \sigma_{ip}^a)$. Similarly, under the market belief $\mathcal{B}_a$, $\mathbf{w}_m$ is the market portfolio weight vector, $\mathbb{E}_a(\tilde{r}_m)$ and $\sigma_{a,m}$ are the market return and volatility under the market belief respectively. Finally $\boldsymbol{\beta}$ is the vector of beta coefficients. According to these definitions, we obtain results in Table 3.

Example F.2. *In this example, we re-examine the tangency relation and the mean-variance efficiency of both individual optimal portfolios and the market portfolio with investors' beliefs formed in terms of asset returns, $\mathcal{B}_i = (V_i, \mathbb{E}_i(\tilde{\mathbf{r}}))$. Assume $I = 2$ and $N = 3$, the risk-free payoff is R_f. The initial wealth of investors, ARA coefficients and market portfolio of risky assets are assumed to be the same as in Example 4.1. Let $\mathbb{E}_o(\tilde{\mathbf{r}}) = (0.169, 0.258, 0.413)^T$ and $V_o = d_o C d_o$, where C is defined in Example 4.1 and $d_o = Diag[(0.1406, 0.1180, 0.2112)^T]$*[3]*. To introduce heterogeneity into investors' beliefs, we let $\mathbb{E}_i(\tilde{\mathbf{r}}) = \mathbb{E}_o(\tilde{\mathbf{r}}) + \delta_i \mathbf{1}$ and $V_i = d_i C d_i$, where $d_i = d_o + \epsilon_i(Diag(\mathbf{1}))$. Then we can study the geometric relationship between portfolio frontiers with and without a risk-free asset by aggregating individual belief using Proposition 5.1. Risk-free asset is determined in equilibrium by assuming it is in net-zero supply. The numerical results are plotted in Figure 4.*

Example F.3. *Let the number of investors $I = 50$, number of risky assets $N = 3$, and market portfolio of risky assets is given by $\mathbf{z}_m = (25, 25, 25)^T$ (so that the average number of each stock per investor stays at 0.5 as in the previous examples). Assume that there is a risk-free asset with payoff $R_f = 1.05$. Investors' initial wealth $W_{0,i} = \$10$, the ARA coefficients*

[3] $(\mathbb{E}_o(\tilde{\mathbf{r}}), V_o)$ is the equivalent homogeneous beliefs to $(\mathbf{y}_o, \Omega_o)$ in Example 4.1 in the sense that they both generate identical equilibrium prices with all other variable being equal.

$\theta_i \sim \mathcal{N}(\theta_o, \sigma^2_{\theta_i})$ *with* $\theta_o = 3$ *and* $\sigma_{\theta_i} = 0.3$ *for* $i = 1, 2, \cdots, I$. *Let the beliefs be formed in payoffs and consider two types of probability distributions for investors' beliefs;*

(i). $\mathbf{y}_i = (1 + \delta_i)\mathbf{y}_o$ *and* $\Omega_i = D_i C D_i$, $D_i = (1 + \epsilon_i) D_o$ *for* $i = 1, \cdots, 50$, *where* C, $\mathbf{y}_o$ *and* D_o *are defined in Example 4.1 and* $\delta_i \sim \mathcal{N}(0, \sigma^2_{\delta_i})$ *and* $\epsilon_i \sim \mathcal{N}(0, \sigma^2_{\epsilon_i})$*;*

(ii). $\mathbf{y}_i = \delta_i + \mathbf{y}_o$ *and* $\Omega_i = D_i C D_i$, $D_i = Diag[\epsilon_i + (0.7933, 0.8770, 1.4622)^T]$ *for* $i = 1, \cdots, 50$, *where* $\delta_i \sim \mathcal{MN}(\mathbf{0}, \Sigma_{\delta_i})$ *and* $\epsilon_i \sim \mathcal{MN}(\mathbf{0}, \Sigma_{\epsilon_i})$ *and* $\Sigma_{\delta_i} = \sigma_{\delta_i} Diag[\mathbf{1}]$ *and* $\Sigma_{\epsilon_i} = \sigma_{\epsilon_i} Diag[\mathbf{1}]$.

Symbols $\mathcal{N}$ *and* $\mathcal{MN}$ *stand for normal distribution and multivariate normal distribution respectively.*

Example F.4. *In this example, we repeat the analysis in Example F.3 with investors' beliefs formed in terms of asset returns. We again consider two types of probability distributions for investors' beliefs;*

(iii). $\mathbb{E}_i(\tilde{\mathbf{r}}) = \delta_i + \mathbb{E}_o(\tilde{\mathbf{r}})$ *and* $V_i = d_i C d_i$, $d_i = \epsilon_i Diag[\mathbf{1}] + d_o$ *for* $i = 1, \cdots, 50$, *where* $\mathbb{E}_o(\tilde{\mathbf{r}})$ *and* d_o *are defined in Example F.2,* $\delta_i \sim \mathcal{N}(0, \sigma^2_{\delta_i})$ *and* $\epsilon_i \sim \mathcal{N}(0, \sigma^2_{\epsilon_i})$*;*

(iv). $\mathbb{E}_i(\tilde{\mathbf{r}}) = \delta_i + \mathbb{E}_o(\tilde{\mathbf{r}})$ *and* $V_i = d_i C d_i$, $d_i = Diag[\epsilon_i + (0.1406, 0.1180, 0.2112)^T]$ *for* $i = 1, \cdots, 50$, *where* $\delta_i \sim \mathcal{MN}(\mathbf{0}, \Sigma_{\delta_i})$ *and* $\epsilon_i \sim \mathcal{MN}(\mathbf{0}, \Sigma_{\epsilon_i})$, $\Sigma_{\delta_i} = \sigma_{\delta_i} Diag[\mathbf{1}]$ *and* $\Sigma_{\epsilon_i} = \sigma_{\epsilon_i} Diag[\mathbf{1}]$.

Example F.5. *In this example, we repeat the analysis in Example F.4, however we now consider first a market without a risk-free asset and then a market with a risk-free asset. In the latter case, we assume that the risk-free asset is in net-zero supply and the risk-free rate is determined in equilibrium by the following equation,*

$$1 + r_f = \frac{\mathbf{1}^T V_a^{-1}(\mathbf{1} + \mathbb{E}_a(\tilde{r})) - W_{m0}\theta_a/I}{\mathbf{1}^T V_a^{-1}\mathbf{1}}. \tag{54}$$

Equation (54) comes from that the fact that total equity available in the market must equal to the sum of investors' wealth, that is $\boldsymbol{\pi}_m^T \mathbf{1} = W_{m0}$, *where* $\boldsymbol{\pi}_m = Z\mathbf{p}_0$