2008/69

■

Discrete-continuous analysis of optimal equipment replacement

Yuri Yatsenko and Natali Hritonenko

CORE
Voie du Roman Pays 34
B-1348 Louvain-la-Neuve, Belgium.
Tel (32 10) 47 43 04
Fax (32 10) 47 43 01
E-mail: corestat-library@uclouvain.be
http://www.uclouvain.be/en-44508.html

CORE DISCUSSION PAPER
2008/69

Discrete-continuous analysis of optimal equipment replacement

Yuri YATSENKO [1] and Natali HRITONENKO[2]

December 2008

Abstract

In Operations Research, the equipment replacement process is usually modeled in discrete time. The optimal replacement strategies are found from discrete (or integer) programming problems, well known for their analytic and computational complexity. An alternative approach is represented by continuous-time vintage capital models that explicitly involve the equipment lifetime and are described by nonlinear integral equations. Then the optimal replacement is determined via the optimal control of such equations. These two alternative techniques describe essentially the same controlled dynamic process. We introduce and analyze a model that unites both approaches. The obtained results allow us to explore such important effects in optimal asset replacement as the transition and long-term dynamics, clustering and splitting of replaced assets, and the impact of improving technology and discounting. In particular, we demonstrate that the cluster splitting is possible in our replacement model with given demand in the case of an increasing asset lifetime. Theoretical findings are illustrated with numeric examples.

Keywords: vintage capital models, optimization, equipment lifetime, discrete-continuous models.

JEL Classification: E20, O40, C60

[1] Houston Baptist University, USA; Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium. E-mail: yyatsenko@hbu.edu

[2] Prairie View A&M University, USA. E-mail: nahritonenko@pvamu.edu

Yuri Yatsenko expresses his gratitude to CORE for the financial support of his extended stay in the Fall 2008.

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1 Introduction

In Operations Research (OR), the equipment replacement models are usually represented as sequential decision problems (see, e.g., [5-9, 13, 21-24, 26-30]). In economic growth theory, similar processes are described by *vintage capital models* (VCMs) [2, 4, 12, 15-20, 25, 31, 35, 36], which explicitly involve the lifetime of capital equipment. These two alternative techniques describe the same controlled dynamic process and possess well developed theories. The majority of OR replacement models are discrete while the most of VCMs are continuous. A critical comparison of these models would be beneficial for both management science and economics. This paper attempts to combine these two modeling approaches to obtain new insight into the rational equipment replacement under improving technology. It provided a rigorous mathematical analysis of continuous and discrete replacement models and compares the outcomes with existing results. A similar discrete-to-continuous analysis was recently provided by Bambi [1] for optimization models with time-to-build structure of capital.

Mathematically, the VCMs are represented by non-linear Volterra integral equations with unknowns in the integration limits. Following R.Solow [31], the *replacement decision* in VCMs is to efficiently replace the old vintages of capital with new capital *under technological change*. Malcomson [25] developed an optimization VCM for the capital equipment replacement decisions of an individual firm. In the 90's, the VCMs were intensively used in economic growth theory (see Benhabib and Rustichini [2], Boucekkine, Germain, and Licandro [4], Cooley, Greenwood, and Yorukoglu [10], Greenwood, Herkowitz, and Krusell [12], van Hilten [14], and the references therein). A consistent optimization technique for the VCMs with endogenous capital lifetime was developed by Yatsenko in [32-34] and applied to various models by Hritonenko and Yatsenko in [15-20].

The OR equipment replacement problems fall into two fundamental categories: *serial* and *parallel replacement.* Following Hartman [13], the parallel replacement considers assets[3] that are economically interdependent and operate in parallel. The *economic interdependence* can be caused by various economic factors, which include:

- the requirement of keeping a prescribed number of assets in service at all times (capacity demand constraints) as in Sethi and Chand [30], Bylka, Sethi, and Sorger [5], Chand, McClurg, and Ward [7];

[3] In this paper, we use the terms *machines*, *equipment*, and *assets* interchangeably.

- prespecified output demand constraints as in Rajagopalan [27];
- restrictions on capital expenditures (rationing budgeting constraints) as in Hartman [13] or Karabakal, Lohmann, and Bean [23];
- additional fixed replacement costs that occur when one or more assets are replaced as in Jones, Zydiak, and Hopp [22].

The first three factors exhibit *constant return-to-scales* with respect to the number of machines. The last forth assumption (fixed charge) introduces the economy of scale in purchase prices that makes the replacement problem essentially more difficult. The *parallel replacement problem* with a fixed charge *in stationary environment* was solved by Jones, Zydiak, and Hopp in [22] and expanded to the case of capital budgeting constraints and a finite horizon by Hartman in [13]. Another possible case of non-constant return-to-scales is the dis-economy of scale in maintenance costs considered by Jones and Zydiak in [21].

Technological change (TC) is the key external factor that impacts the replacement process. TC is commonly described as the improved efficiency and reduced operating and maintenance costs of possible replacement assets. Rogers and Hartman [29] distinguish continuous TC and discontinuous TC. The mainstream of studies on the equipment replacement under TC (e.g., Bethuyne [3], Regnier, Sharp, and Tovey [28], Rogers and Hartman [29], and Sethi and Chand [30]) considers the serial replacement. On the other hand, the continuous VCMs involve the amount of assets as a continuous variable and consider the parallel replacement under TC.

The present paper considers a deterministic *parallel replacement* model with the economic interdependence caused by demand and budgeting constraints under general *continuous TC*. The paper contribution into the OR replacement literature includes new analytic and qualitative techniques and rules for the optimal equipment lifetime under continuous TC.

The majority of equipment replacement models are *discrete* (or *integer*) *programming* problems based the following assumptions: the time is discrete (integer), the service lives of machines are discrete (integer), and the numbers of machines are discrete (integer). This paper investigates what happens if we relax these assumptions and consider the corresponding continuous optimal control problems.

The paper is organized as follows. In Section 2, a discrete model of parallel equipment replacement (similar to [7]) is constructed in the terms of unknown equipment lifetime. Section 3 introduces the modified discrete replacement model with the elementary period length η, $0<\eta\leq 1$. Section 4 proves that the parameterized model leads to the discrete model of Section 2 at $\eta=1$ and

a continuous VCM at $\eta \to 0$. Section 5 explores the corresponding continuous replacement problem that appears to be well known in the VCM theory. Section 6 returns to the discrete model and analyzes its dynamics, particularly the clustering and splitting effects first discovered by Jones, Zydiak, and Hopp [22] in the absence of TC. Section 7 summarizes the major paper results, their possible generalizations, and some open problems.

2 Basic discrete model of equipment replacement

We start with a parallel replacement model under the same assumptions as in [7, 8, 26] but expressed in the terms of unknown machine lifetime. Following [5-8, 26, 30], a production shop keeps P, $P \geq 1$, machines of a particular type at all times ($P=1$ in [5, 30]). The industry operates under *conditions of improving technology*, which means that newer vintages of machines are better (require less maintenance). In economics, these conditions are known as *technological change* (TC) embodied in new capital equipment (new *vintages* of machines). The performance of operating machines (measured by maintenance costs) deteriorates as the machines become older. So, the shop should consider selling each machine at a certain point of time and buying a new machine. The same situation repeats with the new machine so the shop will make a *chain of replacement decisions*.

Let us consider a (finite or infinite) planning horizon $\{1,\ldots,T\}$ in the integer *discrete time* $t=\ldots,-1, 0,1,2,\ldots$. To simplify the model, we assume that only one technology (type of machines, vintage) is available at each time t (see Section 7 about the case of many available technologies). As in [5, 7, 30], we define *the technology* as $(\Pi_t, Q_{t,k}, S_{t,k})$, where Π_t is the purchased price and installation cost of a machine bought at time t (the machine of vintage t), $Q_{t,k}$ is the maintenance cost for the vintage t machine during the time period $k \geq t$, and $S_{t,k}$ is the salvage value of the vintage t machine at the end of period $k \geq t$. Since the purchased price Π_t involves a certain installation (switching) cost, we assume that $\Pi_t > S_{t,k}$ for all $k \geq t$. Because of deterioration, the sequence $Q_{t,k}$ non-decreases and $S_{t,k}$ non-increases when the machine age $a=t-k$ increases. At this point, we make a general assumption of the continuous TC that the sequence $Q_{t,k}$ decreases in t for any fixed machine age $a=t-k$. More specific cases of this dependence will be considered below.

Jones, Zydiak, and Hopp [22] analyzed a parallel replacement problem in the stationary environment and discovered two important replacement principles: the "*older cluster*

replacement rule" (an optimal replacement policy always replaces older machines first) and the "*no-splitting rule"* (machines of the same age are either kept or replaced at the same time period). As shown in [22], the first rule holds *in stationary environment* under *continuous deterioration* (when the maintenance cost is non-decreasing and the salvage value is non-increasing in the asset age).

In this paper, we presume that the "older cluster replacement rule" is valid in the *non-stationary environment* under continuous deterioration and TC. Under this rule, we can choose the machine life as a control. Namely, instead of tracking the replacement chain for each machine as in [7, 30], we introduce the following decision variables:

- the lifetime (service life) L_t of the youngest machine *replaced* in period t and

- the number m_t of new machines purchased during the period t, $1 \le t \le T$.

In terms of [26, 30], the machine lifetime L_t is equal to the difference $t - A_t$ between the regeneration point t and the purchase point A_t of the machine. For the clarity, we assume that machines are replaced at the end of a period. Then, a machine purchased in period $t - L_t$ will be used during the periods $t - L_t + 1$ through t and be replaced in period t. The requirement that the total number of operating machines is equal to P is expressed by the following *demand constraint*:

$$\sum_{k=t-L_t+1}^{t} m_k = P, \quad t = 1,...,T. \tag{1}$$

Let the shop be in business for a while and have P machines at $t=0$. Then, the initial condition of the replacement problem at $t=0$ is

$$\sum_{k=-L_0+1}^{0} m_k^0 = P, \tag{2}$$

where m_k^0 is the known number of machines purchased in each period k, $-L_0 \le k \le 0$, during the known prehistory. The constraints (1)-(2) completely determine vector $\{L_j,\ 1 \le j \le t\}$ under a given $\{m_j,\ 1 \le j \le t\}$, and vice versa.

The discrete replacement models use the *regeneration monotonicity property*, which means that scrapped and replaced machines should not be used again. In the terms of L_j, this requirement can be expressed as

$$L_{j+1} \le L_j + 1, \quad 1 \le j \le T. \tag{3}$$

Under the no-splitting rule, the machines of the same age are always replaced during the same period. Then, the number of machines replaced in period $[j, j+1]$ is obtained by subtracting the demand constraint (1) at $t=j$ from itself at $t=j-1$:

$$m_j = \sum_{k=j-L_{j-1}}^{j-L_j} m_k \ . \tag{4}$$

Depending on the L_j dynamics, the following situations are possible: $m_j=0$ if $L_j=L_{j-1}+1$ (no machine is replaced), $m_j = m_{j-L_j}$ if $L_j = L_{j-1}$, or $m_j = m_{j-L_{j-1}} + \ldots + m_{j-L_j}$ if $L_j<L_{j-1}$ (the machines of several vintages $j-L_{j-1},\ldots,j-L_j$ are replaced in the same period j).

As shown below in Section 6, the "no-splitting rule" is common in the optimal replacement dynamics under TC. However, we will not assume it in the general case. If this rule is violated, then the capacity demand constraint (1) holds but the number of replaced machines is found differently. Namely, if the cluster of currently replaced assets m_{j-L_j} purchased at $j-L_j$ is split over the period $[j, j+l]$, $l>0$, then instead of (4) we have

$$m_{j-L_j} = \sum_{k=j}^{j+l} m_k \ , \tag{5}$$

where m_k, $j \le k \le j+l$, are determined by additional requirements (see Section 6.2).

Following [1,3,7,8,13,22,28], the discounted total cost of replacement policy over the T-period horizon $[0,T]$ can be written as

$$J(T) = \sum_{j=1}^{T} \rho^j \Pi_j m_j + \sum_{j=1}^{T} \rho^j \sum_{k=j-L_j}^{j-1} Q_{j,k} m_k - \sum_{j=1}^{T} \rho^j \sum_{k=L_{j-1}}^{L_j} S_{j,j-k} m_{j-k}, \tag{6}$$

where the first term represents the total price of purchased machines, the second term is the total maintenance cost, and the last term stands for the total salvage value of the replaced machines. The parameter ρ, $0<\rho \le 1$, denotes the discount factor during the unit time interval.

Now we can formulate the machine replacement problem as the nonlinear integer-valued *optimization problem* (*OP*)

$$\min_{m_j, L_j,\, j=1,\ldots,T} J(T) \tag{7}$$

with the unknown controls $L_j \in \mathbf{I}$ and $m_j \in \mathbf{I}$, $L_j \geq 0$, $m_j \geq 0$, $1 \leq j \leq T$, subjected to constraints (1), (3) and the initial condition (2) ($\mathbf{I}$ is the set of integer numbers).

In the case of one machine, we have $P=1$ and $m_j=0$ or $m_j=1$, $1 \leq j \leq T$. Then, the purchase points are the instants when $m_j=1$ and the machine lifetime can be determined as $L_k = k-j+1$ if $m_j=1$, $m_s=0$ at $j<s<k$, and $m_k=1$.

3 Parameterized discrete model of equipment replacement

In this section, we construct a modified version of discrete model (1)-(7) with the step length η, $0<\eta \leq 1$. The goal is to obtain a *parameterized* model, which will produce a continuous-time model of equipment replacement when the parameter η goes to zero. A challenge is that some model characteristics (such as productivity rates and expenses) depend on the time scale.

To define an appropriate time scale parameterization for model (1)-(7), we split each unit period $[j, j+1]$ of the original time $j=1,\ldots,T$ into N smaller *equidistant time intervals* $[j, j+1/N, j+2/N, \ldots, j+(N-1)/N, j+1]$, where $N>1$ is an integer. We will need case $N=1$ as well. Then the *time scale* parameter η is the length of the *elementary* time period: $\eta = 1/N$, $0<\eta \leq 1$.

As result, we have the *parameterized discrete time scale* $t_k = k/N$, $k=\ldots,-1,0,1,2,\ldots$, in addition to the original integer time scale $j=\ldots,-1, 0,1,2,\ldots$. The planning horizon $[0,T]$ is now split into NT *of elementary intervals* $[t_{k-1}, t_k]$ of the length η, $k=1,\ldots, NT$, $N=1/\eta$. The parameter η, $0<\eta \leq 1$, can be interpreted as a *characteristic time of the replacement decision*. Roughly speaking, η reflects how often the replacement decision is implemented (every day, week, month, etc.). Smaller values of η mean that the replacement decision is made more often and uses $N>1$ values of technology parameters Π_t, $Q_{t,k}$, $S_{t,k}$ during each unit interval $[j, j+1]$.

The lifetime L_j of the machines, their purchased price Π_j and salvage value $S_{j,k}$ do not depend on the time scale and remain the same as in model (1)-(6) but are supposed to be measured more often at times $t_k = k/N$.

However, the model also includes some scale-independent characteristics with the meaning of rates per time unit. Indeed, if the original time unit $[0,1]$ is one year and $[0, \eta]$ is one day, then the replacement of one machine *per year* does not mean replacing one machine *every day* of the year. As we divide each unit-length period $[j, j+1]$ into N sub-periods $[t_k, t_k+\eta]$, $k = jN, jN+1,\ldots,$

$j(N+1)$-1, our decision variable m_j obviously depends on N: the number of machines *replaced during the elementary period* $[t_{k-1}, t_{k-1}+\eta]$ is equal to $m_j/N = \eta m_j$. If N=10 and we replace 2 machines per unit period, then we replace 2 * η = 0.2 machines per each "sub-period" of the length η=0.1. To handle this situation, we use the new scale-independent variable: the *machine replacement intensity* (replacement speed) $\hat{m}_k$, related to the number m_k of replacements during the elementary period as $m_k=\eta \hat{m}_k$.

Similarly, we introduce the scale-independent *maintenance expense intensity* $\hat{Q}_{j,k}$, such that the maintenance cost $Q_{j,k}$ of vintage t_j *during* the *elementary period* $[t_{k-1}, t_k]$ is $Q_{j,k}=\eta \hat{Q}_{j,k}$.

Finally, the discount factor over the *elementary period* of length η is $\rho^{1/N}=\rho^{\eta}$.

Now we can construct the modified replacement model in parameterized time $t_k = k\eta$, $k\in \mathbf{I}$. The *decision variables* in the modified model will be the replacement intensity $\hat{m}_k$ and the machine lifetime L_k, $1\le k\le NT$. Then, the total number of operating machines is equal to

$$\eta \sum_{j=k-L_k/\eta+1}^{k} \hat{m}_j = P, \quad k = 1,...,T/\eta. \tag{8}$$

The discounted total cost of replacement policy over the horizon $[0,T]$ can be written as

$$J_\eta(T) = \eta \sum_{j=1}^{T/\eta} \rho^{\eta j} \Pi_j \hat{m}_j + \eta \sum_{j=1}^{T/\eta} \rho^{\eta j} \eta \sum_{k=j-L_j/\eta}^{j-1} \hat{Q}_{j,k} \hat{m}_k - \eta \sum_{j=1}^{T/\eta} \rho^{\eta j} \sum_{k=L_{j-1}/\eta}^{L_j/\eta} S_{j,j-k} \hat{m}_{j-k}, \tag{9}$$

and we formulate the machine replacement problem in *the parameterized discrete time* as

$$\min_{\hat{m}_j, L_j,\, j=1,...,NT} J_\eta(T) \tag{10}$$

with $2TN$ discrete-valued unknown variables L_k and $\hat{m}_k$, $1\le k\le NT$, subjected to constraint (8),

$$L_k N\in \mathbf{I}, \quad \hat{m}_k N\in \mathbf{I}, \quad L_k\ge 0, \quad \hat{m}_k\ge 0, \quad L_{k+1}\le L_k+\eta,$$

and the initial condition

$$\eta \sum_{j=-NL_0+1}^{0} \hat{m}_j^0 = P, \tag{11}$$

where the given numbers $\hat{m}_j^{\,0}$, $-L_0N \le j \le 0$, represent the replacement intensity during the known prehistory $[-L_0, 0]$.

If η=1, then $\hat{m}_j = m_j$, $\hat{Q}_{j,k} = Q_{j,k}$, and model (8)-(11) coincides with the basic model (1)-(7).

4 Continuous time replacement model

The model (8)-(11) produces a continuous optimization problem when $\eta \to 0$.

Theorem 1. The discrete-time discrete-valued OP (8)-(11) corresponds to the continuous time problem $\min_{m,L} I$,

$$I = \int_0^T e^{-rt}[\Pi(t)m(t) + \int_{t-L(t)}^{t} Q(t,\tau)m(\tau)d\tau - S(t-L(t),t)m(t-L(t))(1-L'(t))]dt, \quad (12)$$

$r = -\ln\rho$, with respect to the unknown functions $m(t)$ and $L(t)$, $t \in [0,T)$, $T \le \infty$, that satisfy constraints

$$P = \int_{t-L(t)}^{t} m(\tau)d\tau, \quad (13)$$

$$L(t) \ge 0, \quad L'(t) \le 1, \quad (14)$$

$$m(t) \ge 0, \qquad t \in [0, T), \quad (15)$$

and the initial conditions

$$L(0) = L_0 > 0, \ m(\tau) = m_0(\tau), \ \tau \in [-L_0, 0]. \quad (16)$$

Proof. Expressions (12)-(13) follow from applying the standard definition of Riemann integral to formulas (8),(9), and (11). The conversion of (8) to (13) is trivial and (11) also leads to (13) at t=0. To prove the transformation of (9) into (12), let us consider two interior sums in (9) first. These sums are correct because the numbers L_k/η in their limits are integer by the restriction $L_kN \in \mathbf{I}$. Then,

$$\eta \sum_{k=j-L_j/\eta}^{j-1} \hat{Q}_{j,k}\hat{m}_k \underset{\eta\to 0}{\to} \int_{t-L(t)}^{t} \hat{Q}(\tau,t)\hat{m}(\tau)d\tau \quad \text{where} \quad t = j\eta \quad \text{and}$$

$$\sum_{k=L_{j-1}/\eta}^{L_j/\eta} S_{j,j-k}\hat{m}_{j-k} = \sum_{k=j-L_{j-1}/\eta}^{j-L_j/\eta} S_{j,k}\hat{m}_k \approx \frac{1}{\eta}\int_{t-\eta-L(t-\eta)}^{t-L(t)} S(\tau,t)\hat{m}(\tau)d\tau$$

$$\approx S(t-L(t),t)\hat{m}(t-L(t))\frac{[t-L(t)]-[t-\eta-L(t-\eta)]}{\eta} \underset{\eta\to 0}{\to} S(t-L(t),t)\hat{m}(t-L(t))[1-L'(t)]$$

Finally, at $\rho = e^{-r}$, (9) leads to

$$J_\eta(T) = \eta\sum_{j=1}^{T/\eta} e^{-\eta r j}\left[\Pi_j\hat{m}_j + \sum_{k=j-L_j/\eta}^{j-1}\eta\hat{Q}_{j,k}\hat{m}_k - \sum_{k=L_{j-1}/\eta}^{L_j/\eta} S_{j,j-k}\hat{m}_{j-k}\right]$$

$$\underset{\eta\to 0}{\to}\int_0^T e^{-rt}[\Pi(t)\hat{m}(t) + \int_{t-L(t)}^{t}\hat{Q}(\tau,t)\hat{m}(\tau)d\tau - S(t-L(t),t)\hat{m}(t-L(t))(1-L'(t))]dt$$

For brevity, we omit the "hat" symbol in the obtained functions $\hat{m}(t)$ and $\hat{Q}(\tau,t)$.

The theorem is proved. ■

The *unknown variables* of the OP (12)-(16) are the lifetime $L(t)$ of the machines replaced at time t and the replacement intensity (the *instantaneous* speed of the replacement) $m(t)$, $t\in[0,T)$. Inequality (14) is equivalent to the regeneration monotonicity property (3) of the discrete model (1)-(6). The functions $\Pi(t)$, $Q(\tau,t)$, and $S(\tau,t)$ are assumed to be given at $t\in[0,T)$, $\tau\in[-L_0,T)$. The functions $\Pi(\tau)$ and $Q(\tau,t)$ represent the embodied TC and decrease in τ (newer equipment is more efficient).

The salvage cost component in the objective functional (12) includes the derivative of the unknown $L(t)$ and causes certain difficulties during the OP analysis. Let us assume that the salvage value is negligible $S(\tau,t)=0$ (it is true, at least, for high-tech products such as computer, networks, electronics, see also [28]). Then, the OP (12)-(16) is

$$\min_{m,L} I = \min_{m,L}\int_0^T e^{-rt}[\Pi(t)m(t) + \int_{t-L(t)}^{t} Q(\tau,t)m(\tau)d\tau]dt \qquad (17)$$

with respect to the unknown functions m and L under restrictions (13)-(15) and initial conditions (16).

The problem (13)-(17) is, in fact, a well-known problem of the cost minimization for a firm using a continuum of vintage capital equipment. Similar optimization problems were first introduced by Malcomson [25] and investigated by van Hilten in [14], Boucekkine, Germain, and Licandro in [4], Hritonenko and Yatsenko in [17-20, 35]. Mathematically, a major new feature of such

problems lies in the *new type of control functions* that appear in the integration limits of integral equations. A systematic treatment of the OP (13)-(17) was provided in [35] for the case of variable $P(t)$. Possible dependence of P on t includes the *non-stationary demand* considered in the discrete settings in [7, 30]. Also, an additional restriction $m(t) \le m_{\max}(t)$ in [35] reflects *budgeting constraints* common in the discrete replacement models.

5 Optimal equipment lifetime in continuous model

The papers [20, 35] describe the complete dynamics of the OP (13)-(17) such as the solution structure and turnpike properties for both infinite ($T=\infty$) and finite ($T<\infty$) horizons. Here we omit technical details and refer an interested reader to [15-20, 32-35].

Lemma *(necessary and sufficient condition for an extremum)* [35]. A measurable function $m^*(t)$, $t \in [0,T)$, and the corresponding $L^*(t)$, $t \in [0,T)$, $T \le \infty$, are a solution to the OP (13)-(17) *if and only if*

$$I'(t) \ge 0 \qquad \text{at} \qquad m^*(t) = 0,$$

$$I'(t) = 0 \qquad \text{at} \qquad m^*(t) > 0, \quad t \in [0, T),$$

where

$$I'(t) = \int_t^{\min\{t+\hat{L}^*(t),T\}} e^{-r(u-t)}[Q(u - L^*(u), u) - Q(t,u)]d\tau - \Pi(t), \quad t \in [0,T), \quad (18)$$

is the OP gradient $I'(t)$ with respect to m, $\hat{L}(t)$ is the future lifetime of the machine bought at t, $t + \hat{L}(t)$ is the instant when this machine should be scrapped,

$$t + \hat{L}(t) = [t - L(t)]^{-1}, \quad (19)$$

and $[x(t)]^{-1}$ denotes the inverse of $x(t)$.

Gradient (18) relates the equipment lifetime $L(t)$ of the machine *replaced* at t to the future lifetime $\hat{L}(t)$ of the machine *bought* at t. It defines the future marginal profit from the purchase of a new machine at time t and is equal to the difference of the marginal revenue (the future rental value) of the new machine and its price. The future rental value naturally depends on the future lifetime of machines. The independence of gradient (18) on m reflects a *constant return-to-scales*

economy and has essential implications. Namely, the structure of OP solutions appears to be defined by the nonlinear integral-functional equation $I'(t)=0$ or

$$\int_t^{[t-L(t)]^{-1}} e^{-r(u-t)}[Q(u-L(u),u)-Q(t,u)]du=\Pi(t), \quad t\in[0,\infty), \tag{20}$$

with respect to the unknown function $\tilde{L}(t)$, $t\in[0,\infty)$. Equation (20) has been first derived in [32] and investigated in [15-17, 33]. It states that, in the rational strategy of equipment replacement under the embodied TC, the profit of putting a new machine into service and scrapping an older obsolete machine is equal to the price of the new machine. A solution $\tilde{L}$ of (20) (if it exists), is called *the turnpike trajectory* of the OP (13)-(17).

5.1 The structure of optimal replacement

The qualitative analysis of the OP (13)-(17) reveals interesting patterns of the rational equipment replacement strategies under TC. In this paper, we focus on the infinite-horizon OP ($T=\infty$). Then the structure of the OP solutions is pretty simple and is described by the following statement.

Theorem 2 (Yatsenko and Hritonenko [35]). If a unique solution $\tilde{L}$ of equation (20) exists and $P(t)$=const, then the OP (13)-(17) with restriction $m(t)\le m_{\max}(t)$ has a unique solution (m^*, L^*) of the following structure:

A. *Transition dynamics*:

$$m^*(t)=\begin{cases} 0 & \text{if } \tilde{L}(0)>L_0, \\ m_{\max}(t) & \text{if } \tilde{L}(0)<L_0, \end{cases} \quad t\in[0,\mu), \tag{21}$$

where the corresponding $L^*(t)$ is found on $[0,\mu)$ from (13) at the given m^* and is increasing at $\tilde{L}(0)>L_0$ and decreasing at $\tilde{L}(0)<L_0$. The length $\mu\ge 0$ of the interval $[0,\mu)$ is determined from the condition $L^*(\mu)=\tilde{L}(\mu)$ and depends on $|\tilde{L}(0)-L_0|$; $\mu=0$ at $\tilde{L}(0)=L_0$.

B. *Long-time dynamics*:

$$L^*(t)=\tilde{L}(t), \tag{22}$$

$$m^*(t)=m^*(t-\tilde{L}(t))[1-d\tilde{L}(t)/dt], \quad t\in[\mu,\infty). \tag{23}$$

The proof is based on the necessary and sufficient condition for an extremum (Lemma) and is provided in [35]. Theorem 2 classifies three possible types of the initial machine distribution:

(1) $\tilde{L}(0)<L_0$, all active machines are too young and will be replaced later;

(2) $\tilde{L}(0)>L_0$, some active machines are too old and should be replaced immediately;

(3) $\tilde{L}(0)=L_0$, no transition period.

Theorem 2 may be also interpreted as a *turnpike theorem in the strongest form.* It states that the optimal $L^*(t)$ coincides with the turnpike trajectory $\tilde{L}(t)$ except for some initial interval $[0,\mu)$. The turnpike properties are well known for other (non-integral) models in growth theory, and their presence often serves as an indicator of the quality of an optimization model. The turnpike properties deliver some important patterns for strategic replacement decisions. For the finite-horizon OP (13)-(17), a *turnpike theorem in normal form* was proved in [17, 35].

By Theorem 2, the optimal investment control m^* possesses *replacement echoes* [4, 17, 20, 35] caused by initial condition (16). Indeed, if the initial condition $L(0)=L_0$, $m(t)=m_0(t)$, $t\in[-L_0, 0]$, at the left end $t=0$ of the horizon $[0, \infty)$ is such that $L_0 \neq \tilde{L}(0)$, then it causes the appearance of the boundary-valued section $m(t)\equiv 0$ or $m_{\max}(t)$, $t\in[0,\mu]$, in the optimal investment trajectory $m^*(t)$ during the first replacement cycle and the dissemination of the corresponding *replacement echoes* through the whole horizon $[0, \infty)$. Such echoes are absent when $L_0 = \tilde{L}(0)$. If $\tilde{L}(t)$ decreases, then the replacement interval shortens and the replacement echoes increase from one interval to other (and converse).

The established structure of OP solutions shows that the optimal equipment lifetime L^* possesses turnpike properties, whereas the optimal investment m^* does not strive to any limit. The jumping behavior of the investment m^* is common for the replacement problems.

5.2 The dynamics of optimal equipment lifetime

By Theorem 2, the dynamics of the optimal equipment lifetime $\tilde{L}$ is determined by the nonlinear integral equation (20). This equation is a key for the optimal replacement decision. The equation can be solved numerically for any given smooth functions Π and Q. There is no general theory for such equations, so one needs to consider meaningful special cases. Hritonenko and Yatsenko

in [16] analyze this equation for exponent, power, and logarithmic functions $\Pi(\tau)$ and $Q(\tau)$. Here, we consider a more general case:

$$Q(\tau,t)=Q_d(t-\tau)\,e^{-c_q\tau}, \qquad \Pi(\tau)=\Pi_0\,e^{-c_p\tau}, \qquad c_q,\ \Pi_0>0,\ c_p\geq 0, \tag{24}$$

where the function $Q_d(t-\tau)>0$ reflects the *equipment deterioration* and depends on the equipment age $t-\tau$ (not necessarily monotonically). The exponential TC is reflected by the dependence of formulas (24) on τ and can be different for the machine price $\Pi(\tau)$ and operating expense $Q(\tau,t)$. Then, the following statements are valid

Theorem 3. Under conditions (21) and the exponential deterioration

$$Q_d(t-\tau) = Q_0\,e^{c_d(t-\tau)}, \qquad c_d \geq 0, \tag{25}$$

equation (20) has a unique solution $\tilde{L}(t)>0$, $t\in[0,\infty)$, such that :

- if $c_q < c_p$, then $\tilde{L}(t)$ monotonically decreases and approaches 0 as $t\to\infty$;
- if $c_q > c_p$, then $\tilde{L}(t)$ monotonically increases and approaches ∞ as $t\to\infty$;
- if $c_q = c_p$, then $\tilde{L}(t)\equiv L$, $t\in[0,\infty)$, where the constant L is determined from the non-linear equation

$$r\,e^{(c_q+c_d)L} + (c_q+c_d)\,e^{-rL} = (r+c_q+c_d)[1+r\Pi_0/Q_0] \quad \text{at} \quad r>0 \tag{26}$$

or $$e^{(c_q+c_d)L} - (c_q+c_d)L = 1 + (c_q+c_d)\Pi_0/Q_0 \qquad \text{at} \quad r = 0. \tag{27}$$

In particular, $L \approx [2\Pi_0/(Q_0(c_q+c_d))]^{1/2}$ for $0 < c_q+c_d < r << 1$.

Proof is based on a similar result obtained in Yatsenko and Hritonenko [35] for the integral equation (20) without deterioration at $Q_d(t-\tau)=Q_0$. Using conditions (24) and (25), equation (20) is rewritten in the following form:

$$Q_0\int_t^{[t-L(t)]^{-1}} e^{-r(u-t)}[e^{-c_q(u-L(u))}e^{c_dL(u)} - e^{-c_qt}e^{c_d(u-t)}]du = \Pi_0 e^{-c_pt}, \quad t\in[0,\infty). \tag{28}$$

Multiplying (28) by e^{-c_dt} and separating the factor $e^{c_d(u-t)}$ in the integrand, we obtain

$$Q_0\int_t^{[t-L(t)]^{-1}} e^{-r(u-t)}e^{c_d(u-t)}[e^{-c_q(u-L(u))-c_d(u-L(u))} - e^{-(c_q+c_d)t}]du = \Pi_0 e^{-(c_p+c_d)t}, \quad t\in[0,\infty).$$

Now, setting $c_1=c_q+c_d$, $c_2=c_p$, $c_3=r+c_q$ converts the last equation into equation (33) of [35]. Applying Theorem 3 from [35] to this equation proves the theorem. ■

The theorem does not cover all possible combinations of the parameters. In particular, equation (28) has a finite solution $\tilde{L}(t)$, $t\in[0,\infty)$, while $c_q+c_d>0$ and $c_p+c_d>0$. In the critical case $c_q+c_d=0$, equation (28) may have an indefinitely increasing solution $\tilde{L}(t)$ but such that the function $t-\tilde{L}(t)$ is bounded on the interval $[0,\infty)$. Finally, in the case $c_q+c_d<0$, equation (28) has no solution on the infinite interval $[0,\infty)$.

The obtained results produce some qualitative rules of the dynamics of the optimal equipment lifetime:

Property 1. In the case (24)-(25), if the operating cost $Q(\tau,t)$ decreases slower in τ than the machine price $\Pi(\tau)$, then the long-term optimal machine lifetime $\tilde{L}(t)$ decreases (and converse).

Property 2. In the case (24)-(25), if the TC rates c_q and c_p of the operating cost $Q(\tau,t)$ and machine price $\Pi(\tau)$ are the same, then the long-term optimal machine lifetime $\tilde{L}(t)$ is constant. This constant depends only on the TC and deterioration rates, the discount factor, and the proportion Π_0/Q_0 between the initial machine price and operating cost.

Moreover, if the TC rates of the operating cost and equipment price are equal, $c_q=c_p$, then the equipment lifetime is constant for *any deterioration law* (not necessarily exponential). Namely, the following result holds.

Theorem 4 (Yatsenko and Hritonenko [35]). If condition (24) holds and $c_q=c_p$, then for any function $Q_d(t-\tau)$ such that $Q_d(x)e^{c_q x}$ increases, equation (20) has the constant solution $\tilde{L}(t)\equiv L$, $t\in[0,\infty)$, determined from the non-linear equation:

$$\int_0^L e^{-ru}[e^{c_q(u-L)}Q_d(u-L)-Q_d(0)]du=\Pi_0. \qquad (29)$$

Equation (29) has a solution $L>0$ even when $c_q=c_p=0$ but $Q_d(x)$ strictly decreases.

As stressed in [7], theoretic research should provide useful heuristics for practical decision making. Theorems 3 and 4 lead to certain approximate rules about the long-term optimal lifetime of equipment. These rules do not depend on the production scale and are defined only by the technology ($Q(\tau,t)$, $\Pi(\tau)$) and the discount rate r. Hence, they can be used by any business. The

theorems allow us to analyze how the dynamics of the optimal lifetime depends on the intensity of TC and discounting. Namely, the following properties hold.

Property 3 (*TC impact on the optimal assets lifetime*). If condition (24) holds, $c_q=c_p$, and

$$c_q + \frac{Q_d'(x)}{Q_d(x)} > 0, \tag{30}$$

then the solution $\tilde{L}$ of equation (29) is smaller in the case of more intense TC. Specifically, if $c_q^1 > c_q$, then the corresponding solution $\tilde{L}^1 < \tilde{L}$.

Proof. Let us denote the left-hand part of equation (29) by the implicit function $F(L, c_q)$ of two variables L and c_q. Using the implicit function theorem, we obtain that $dL/dc_q = -\partial F/\partial c_q / \partial F/\partial L$. By (29),

$$\begin{aligned} \partial F/\partial c_q &= \int_0^L e^{-ru} e^{c_q(u-L)} Q_d(u-L)du > 0, \\ \partial F/\partial L &= -\int_0^L e^{-ru}[-c_q e^{c_q(u-L)} Q_d(u-L) - e^{c_q(u-L)} Q_d'(u-L)]du = \\ &= \int_0^L e^{-ru} e^{c_q(u-L)}[-c_q Q_d(u-L) - Q_d'(u-L)]du < 0 \end{aligned} \tag{31}$$

at condition (30). Hence, $dL/dc_q>0$ and the optimal L decreases when c_q increases, i.e., when the TC intensity increases. The property is proved. ■

Property 3 states that the TC acceleration decreases the optimal lifetime of equipment, hence, the paradox [9] does not appear in our continuous model. Another interesting issue raised by Chand, Hsu, and Sethi [6] is the impact of discounting.

Property 4 (*The impact of discounting on the optimal lifetime*). Under conditions (24) and $c_q=c_p$, the solution $\tilde{L}$ of equation (29) is larger in the case of more intensive discounting. Specifically, if $r^1 > r$, then the corresponding $\tilde{L}^1 > \tilde{L}$.

Proof is analogous to the previous one. Considering the left-hand part of (29) as the implicit function $F(L, r)$ of L and r, we obtain that $dL/dr = -\partial F/\partial r / \partial F/\partial L$ and

$$\partial F/\partial r = -r\int_0^L e^{-ru} e^{c_q(u-L)}[Q_d(u-L) - Q_d(0)]du < 0,$$

hence, $dL/dc_q>0$ and the optimal L increases when r increases. The property is proved. ■

The finite-horizon case is technically more complicated because of the "zero-investment period" $(\theta,T]$, $\theta<T$ (see Hilten [14], Hritonenko and Yatsenko [17, 20, 35]) and is not considered here. Finite-horizon optimization is important for the management science that explores real situations when a decision maker truncates the horizon to a finite value. The well-known techniques for reducing the end effects in the equipment replacement models [5, 6, 8, 30], such as *rolling horizon* and *minimum forecast horizon*, are also promising for VCMs.

6 Examples: clustering and splitting

Let us return to the discrete replacement model. Despite the growing number of papers on the assets replacement under TC (Bethuyne [3], Cheevaprawatdomrong and Smith [9], Rogers and Hartman [29], Regnier, Sharp, and Tovey [28], Sethi and Chand [30]), its qualitative properties in discrete case are still unclear. Even the question whether TC delays or speeds up the optimal replacement is still debated (compare Bethuyne [3] and Rogers and Hartman [29]). Bethuyne [3] applies a necessary extremum condition to his continuous model and derives an equation for the optimal assets lifetime. However, his model assumptions are different from ours and lead to the conclusion that the optimal assets lifetime increases under TC (that contradicts our Property 3). A similar paradox is obtained in a discrete replacement model by Cheevaprawatdomrong and Smith [9] and analyzed by Hritonenko and Yatsenko in [18]. Rogers and Hartman [29] show that the optimal lifetime of assets decreases under more intensive TC for both exponential continuous TC and the discontinuous TC in the form of technological breakthroughs (the same is stated in Property 3).

A discrete model of serial replacement closely related to the present paper is constructed by Regnier, Sharp, and Tovey [28]. They consider the case of different TC rates in the operating cost and machine price and show that it leads to the *variable optimal* lifetime. In particular, it is proved that the optimal lifetime decreases when the machine price decreases faster than the operating cost, and inverse (see our Property 1).

One can expect the optimal lifetime $L^*=\{L_k^*,\ k=1,\dots,NT\}$ in the discrete model to be close to the solution $L^*(t)$, $t\in[0,T]$, of the continuous OP (13)-(17) for small η. The following result states that if the OP (13)-(17) solution is integer-valued, then both solutions coincide.

Theorem 5. Let the continuous OP (13)-(17) has a unique solution (m^*, L^*). If the functions m^* and L^* are piecewise–constant:

$$m^*(t) = m_k^*, \quad L^*(t) = L_k^*, \quad t \in [t_{k-1}, t_k] = [\eta(k\text{-}1), \eta k], \quad k=1,\ldots,NT, \quad \eta = 1/N, \tag{32}$$

and such that $m_k^* N \in \mathbf{I}$ and $L_k^* N \in \mathbf{I}$, then $\{m_k^*\}$, $\{L_k^*\}$, $k=1,\ldots,NT$, is a solution of the equivalent discrete OP (8)-(11) obtained from the OP (13)-(17) at $\rho = e^{-r}$, $S_{jk} = 0$, and

$$\Pi_k = \int_{\eta(j-1)}^{\eta j} \Pi(t)dt, \qquad Q_{kj} = \int_{\eta(j-1)}^{\eta j} \int_{\eta(k-1)}^{\eta k} Q(\tau,t)d\tau dt, \\ j = -NL_0 + 1, \ldots, NT, \quad k = 1, \ldots, NT. \tag{33}$$

Proof. The discrete OP (8)-(11) is obtained from the continuous OP (13)-(17) when the unknown functions $L(t)$, $m(t)$, $t \in [0,T]$, are piecewise-constant on the elementary intervals $[t_{k-1}, t_k]$, $k=1,\ldots,NT$. Then the substitution of (32) into expressions (13) and (17) leads to formulas (33) for the given machine maintenance cost $Q(\tau,t)$ and price $\Pi(t)$. The set $\mathbf{\Omega}_I$ of the piecewise–constant admissible OP solutions (32) is a subset of the domain $\mathbf{\Omega}$ of *all* admissible OP solutions: $\mathbf{\Omega}_I \subset \mathbf{\Omega}$. Therefore, under condition (33) the optimal (minimal) value J^* of functional (9) cannot be smaller than the optimal value I^* of functional (17), $I^* \le J^*$. In the case (32), we have $I^* = J^*$, hence, the values (32) deliver a solution of the discrete OP (8)-(11),(33). The theorem is proved. ■

Now, we can apply the results of Section 4 about the VCM (13)-(17) to the discrete models (1)-(7) and (8)-(11). Next subsection considers a special case of the discrete model (1)-(7), when an exact solution of the equivalent continuous OP can be constructed and is integer-valued.

Unlike the majority of previous works that describe the serial replacement under TC, our discrete model considers the parallel replacement. It allows us to observe the clustering effect and cluster splitting under TC.

6.1 Geometric TC and geometric deterioration

The *geometric TC* means that the maintenance cost Q_{kj} at a fixed age k-j and the equipment acquisition cost Π_j drop by constant factors C_q and C_p after each time period:

$$\Pi_j = C_p \Pi_{j\text{-}1}, \qquad Q_{kj} = C_q Q_{k\text{-}1,j\text{-}1}, \qquad 0 < C_q < 1, \quad 0 < C_p \le 1 \tag{34}$$

(see, e.g., Cheevaprawatdomrong and Smith [9], Regnier, Sharp, and Tovey [28]). Routine calculations show that (34) corresponds to the *exponential TC* (24) at $c_p = -\ln(C_p)$ and $c_q = -\ln(C_q)$ in the continuous model. Indeed, comparing (6) and (17) at $t=j$, we obtain that

$$\int_0^T e^{-rt}\Pi(t)m(t)dt = \sum_{k=1}^{T}\rho^k m_k \int_{k-1}^{k}\Pi(t)dt = \sum_{k=1}^{T}\rho^k m_k \Pi_k \ ,$$

where $\Pi_k = \Pi_0 \int_{k-1}^{k} e^{-c_p t}dt = \Pi_0 e^{-c_p k}\,\frac{e^{c_p}-1}{c_p}$, hence $\Pi_k = e^{-c_p}\,\Pi_{k-1}$ and $C_p = e^{-c_p}$. Analogously, $C_q = e^{-c_q}$. Regnier, Sharp, and Tovey [28] also confirm that their discrete geometric TC model is similar to the continuous negative exponential TC.

The *geometric deterioration* means that the maintenance cost Q_{kj} increases by a constant factor C_d when the age *j*-*k* of a machine in service increases by 1 :

$$Q_{k,j+1} = C_d Q_{k,j}\,, \qquad Q_{k,j} = C_d Q_{k-1,j}\,, \qquad C_d \geq 1. \tag{35}$$

It corresponds to the *exponential deterioration* (25) at $c_d = \ln(C_d)$ in the continuous model.

When the TC rates in the operation cost and the purchase price are the same, $c_p = c_q$, then by Theorem 2 the (long-term) optimal lifetime is constant and determined by equation (26). We consider this case to illustrate the presence and nature of clustering.

Let r=0.1, $c_q = c_p$=0.03, c_d =0.05, Q_0=1, and Π_0 =2.55 in the continuous model (13)-(17). It corresponds to $\rho \approx 0.9048$, $C_q = C_p \approx 0.9704$, $C_d \approx 1.051$ in the discrete model (1)-(7). Then, by (26), the long-term optimal lifetime is $\tilde{L} \equiv 8$, $t \in [0,\infty)$, in the continuous model. By Theorem 5, the lifetime $\tilde{L} \equiv 8$ is also optimal in the discrete case if the corresponding replacement amounts are integer. The complete dynamics of the optimal replacement is analyzed below in several examples. Let the machine number be P=8.

Example 1 (no transition dynamics). If there is no machine older than 8 years at time t=0, then we can choose the matching initial condition L_0=8, and μ=0 by Theorem 1. Then:

- If the initial machine distribution $\{m_k^0\}$ is flat, m_k^0=1, k=-7,…,0, then the optimal replacement is determined by (5) as $m_k = m_{k-8}$ =1, k=1,…,∞.
- If the initial distribution $\{m_k^0\}$ is uneven, then the optimal replacement reproduces it through the infinite horizon (*replacement echoes*). For example, if

  m_k^0=0, k=-7,-6,-5,-4 and m_k^0=2, k=-3,-2,-1,0,

  then by (5) $m_j = m_{j-8} = \begin{cases} 0 \text{ if } j = 1+8k, 2+8k, 3+8k, 4+8k \\ 2 \text{ if } j = 5+8k, 6+8k, 7+8k, 8+8k \end{cases}$, k=1,…,∞.

No clustering appears in this case.

Example 2 (clustering during transition dynamics). In this case, some machines are older than the long-time optimal age at time t=0. Let L_0=10, m_k^0=2, k=-9,…,-5, and m_k^0=0, k=-4,…,0. Then the machines purchased at k=-9,-8, are older than the optimal age $\tilde{L} \equiv 8$ and should be replaced as soon as possible. Formally, we can choose $m_{\max}$=6 in the OP (13)-(17), then $L^*(t)$ decreases from $L^*(0)$=10 to $L^*(1)$=8, the transition dynamics period $[0,\mu]$ is [0,1], and

$$m^*(t) = \begin{cases} 6, & t \in [8k, 8k+1] \\ 2, & t \in [8k+1, 8k+3] \\ 0, & t \in [8k+3, 8(k+1)] \end{cases}, \qquad k=1,\ldots,\infty,$$

in the continuous OP. By Theorem 5, (L^*, m^*) also delivers a solution to the discrete OP (1)-(7). The optimal replacement during the first replacement period is determined by formula (5) as

$$m_1 = \sum_{k=j-10}^{j-8} m_k = m_{-9} + m_{-8} + m_{-7} = 6, \quad m_2 = m^0_{2-8} = m^0_{-6} = 2, \text{ and } m_k = m^0_{k-8} = 0, \quad k=3,4,5,6,7,8.$$

So, the machines m_{-9}, m_{-8}, and m_{-7} are combined into a *cluster* of 6 machines during the first replacement. Later, this cluster is repeated indefinitely as :

$$m_j = m_{j-8} = \begin{cases} 6 & \text{if} \quad j = 1+8k, \\ 2 & \text{if} \quad j = 2+8k, \\ 0 \text{ if} & j = 3+8k, \ldots, 8+8k, \end{cases} \qquad k=1,\ldots,\infty.$$

The optimal replacement policy is

{2,2,2,2,2,0,0,0,0,0};6,2,0,0,0,0,0,0,0,0;6,2,0,0,0,0,0,0,0,0;6,2,0,0,0,0,0,0,0,0,…

So, the clusters naturally appear even in the case of constant asset lifetime because of the "non-optimal" initial distribution of the equipment. No cluster splitting occurs, so, the *no-splitting rule* is valid in the case of the constant lifetime.

If the TC rates in the operation cost and the purchase price are different, $c_p \neq c_q$, then the optimal assets lifetime is not constant and decreases or increases depending on the sign of $c_p - c_q$. A similar result has also been proved for a discrete serial replacement model by Regnier, Sharp, and Tovey in [28]. Such a case is analyzed below.

6.2 Increasing optimal lifetime of assets

Now we assume that the deterioration and the TC in operating cost are exponential but the TC in purchase price is not. Namely, let r=0.1, c_q=0.04, c_d=0.04, Q_0=1, and

$$\Pi(t) = 13.77 + 4.12\, e^{-0.1t} - 16.67\, e^{-0.04t} \qquad (36)$$

in the continuous model (13)-(17) ($\Pi(t)$ monotonically increases from $\Pi(0)$=1.22 to $\Pi(\infty)$=13.77). Then the (long-term) optimal lifetime of machines replaced at t is

$$\tilde{L}(t)=t/2+4, \qquad t\in[0,\infty), \qquad (37)$$

which can be verified by the direct substitution of r, c_q, c_d, Q_0, (25), and (36) into equation (20). Correspondingly, $[t-\tilde{L}(t)]^{-1} = 2t+8$ in (20) and the future lifetime of machines bought at t is $\hat{L}(t) = [t-\tilde{L}(t)]^{-1}-t = t+8$. The optimal lifetime $\hat{L}(t)$ doubles every replacement period: $\hat{L}(0)=8$, $\hat{L}(8)=16$, $\hat{L}(24)=32$, $\hat{L}(56)=64$, $\hat{L}(120)=128$, and so on.

Let P=4. Then L_0=4, hence, by Theorem 2, there is no transition period, μ=0, and the *exact solution* of the continuous OP is

$$L^*(t) = t/2+4, \qquad m^*(t) = m^*(t/2-4)/2, \qquad t\in[0,\infty). \qquad (38)$$

While (38) is not integer–valued, it shows the trend of optimal replacement. If we apply the rule (38) to the discrete model, we can choose an *ε-approximate* optimal lifetime as $L_j = j/2+4$, j=1,…,∞, and determine the corresponding replacement amounts. Let the initial machine distribution $\{m_k^0\}$ be concentrated around the point k=-3:

$$m^0(t) = \begin{cases} 4, & t\in[-4,-3), \\ 0, & t\in[-3,0] \end{cases}$$

or, in the terms of discrete model, at the point k=-3: $m_{-3}^0=4$ and $m_{-2}^0=m_{-1}^0=m_0^0=0$. Then, by (38), the optimal replacement will be

$$m_j = m_{j-L_j}/2 = \begin{cases} 2 & \text{if} \quad j=1,2 \\ 0 & \text{if} \quad j=3,4,5,6,7,8 \end{cases}$$

during the first replacement interval,

$$m_{j+8} = \begin{cases} 1 & \text{if} \quad j = 1,2,3,4 \\ 0 & \text{if} \quad j = 5,...,16 \end{cases}$$

during the second replacement interval, and fractional machine replacements after.
Therefore, the optimal replacement policy is

{4,0,0,0}; 2,2,0,0,0,0,0,0; 1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0;…

So, the initial cluster of 4 machines is split into two “two-machine clusters” during the first replacement and into four “one-machine clusters” during the second replacement. Next, the splitting pattern continues in the continuous model.

Thus, the cluster splitting is possible in the continuous model in the case of the increasing asset lifetime. Hence, the *no-splitting rule* is not valid in the general case of replacement under TC with the given demand. To obtain a similar result in the discrete model, we need to relax only one assumption about integer-valued lifetime of the equipment.

7 Conclusion

In this paper, we have established new relationships between the discrete-time equipment replacement models and continuous-time vintage capital models (VCMs). These models describe the same production process but use different mathematical tools. The comparative analysis of these two approaches provides a new insight into some open issues of equipment renovation.

1. We have analyzed the *parallel machine replacement* under the general *continuous TC* when the economic interdependence is caused by the capacity demand constraints and, possibly, capital budgeting constraints. This case is well explored in the corresponding continuous VCMs. Then, the optimal machine lifetime is separated from the optimal investment amount, is determined by a nonlinear integral equation, and does not depend on the economy scale. The corresponding optimal investment amount depends on the initial distribution of the machines. In the special case of the exponential TC, new analytic formulas and qualitative rules for the optimal machine lifetime are provided.

2. The no-splitting rule, earlier established by Jones, Zydiak, and Hopp [22] for the equipment replacement in stationary environment, is valid when the optimal equipment lifetime is constant. It is shown that the machine clusters naturally appear because of the non-optimal initial distribution of machines. In the case of an increasing optimal equipment lifetime, the possibility of cluster splitting is demonstrated in the continuous model.

3. A more challenging and interesting case arises when several technological alternatives exist for possible replacement of a machine [5, 7]. This case has been investigated in discrete settings in [5] and in continuous vintage settings in [19, 20, 36]. The corresponding discrete-continuous analysis similar to the one provided in this paper would be helpful in choosing the optimal technological parameters of the replaced machines.

4. Management applications discussed in this paper can lead to new open problems in the VCM theory. The models can incorporate a stochastic behavior of external prices and technology. An interesting issue is to assume random equipment failures in the discrete equipment replacement models, derive and analyze the corresponding integral equations for optimal equipment lifetime. It will lead to other types of technology dynamics $Q(\tau,t)$, $\Pi(\tau)$, different from the exponential TC. In particular, the shape of the deterioration factor $Q_d(t-\tau)$ in (24) can be non-monotonic and reflect various failure distributions (Weibull, "bathtub", and so on). Another important case occurs when every replacement involves an additional fixed cost that introduces the economy of scale. The dynamics of the optimal equipment lifetime in such cases is an open problem. Finally, while the optimization with a linear utility function is a common choice in the equipment replacement management problems, it is interesting to consider the case of concave utility that describes a risk adverse decision-making behavior under uncertainty.

References

[1] Bambi, M., Unifying time-to-built theory, CER-ETH Economics Working Paper 08/98, Center of Economic Research at ETH, Zurich, 2008. http://www.cer.ethz.ch/ research/wp_08_98.pdf.

[2] Benhabib, J., and A. Rustichini, "A vintage capital model of investment and growth", in *General Equilibrium, Growth, and Trade. II The Legacy of Lionel W. McKenzie*, R. Becker (Editor), Academic Press, New York, 1993, pp.248-301.

[3] Bethuyne, G., Optimal replacement under variable intensity of utilization and technological progress, *The Engng Econ*.. 43(1998), 85-106.

[4] Boucekkine, R., M. Germain, and O. Licandro, Replacement echoes in the vintage capital growth model, *J Econ Theory*, 74(1997), 333-348.

[5] Bylka, S., S.P. Sethi, and G. Sorger, Minimal forecast horizons in equipment replacement models with multiple technologies and general switching costs, *Naval Res Logist*, 39 (1992), 487-507.

[6] Chand, S., V.Hsu, and S.P. Sethi, Forecast, solution, and rolling horizons in operations management problems: a classified bibliography, *Manufact Service Oper Manage*, 4 (2002), 25-43.

[7] Chand, S., T. McClurg, and J. Ward, A model for parallel machine replacement with capacity expansion, *Eur J Oper Res*, 121(2000), 519-531.

[8] Chand, S., S.P. Sethi, and G. Sorger, Forecast horizons in the discounted dynamic lot size model, *Manage Sci*, 38 (1992), 1034-1048.

[9] Cheevaprawatdomrong, T., and R.L. Smith, A paradox in equipment replacement under technological improvement, *Oper Res Letters*, 31(2003), 77-82.

[10] Cooley, T., J. Greenwood, and M. Yorukoglu, The replacement problem, *J Monetary Econ*, 40 (1997), 457-499.

[11] Fattorini, H., *Infinite Dimensional Analysis Optimization and Control Theory*, Cambridge University Press, Cambridge, 1999.

[12] Greenwood, J., Z. Herkowitz, and P. Krusell, Long-run implications of investment-specific technological change, *Amer Econ Rev*, 87 (1997), 342-362.

[13] Hartman, J., The parallel replacement problem with demand and capital budgeting constraints, *Naval Res Logist*, 47(2000), 40-56.

[14] Hilten, O. van, The optimal lifetime of capital equipment, *J Econ Theory*, 55 (1991), 449-454.

[15] Hritonenko, N. and Yu. Yatsenko, *Applied mathematical modeling of engineering problems*, Kluwer Academic Publishers, Dordrecht, 2003.

[16] Hritonenko, N. and Yu. Yatsenko, Integral-functional equations for optimal renovation problems, *Optimization*, 36 (1996), 249-261.

[17] Hritonenko, N. and Yu. Yatsenko, *Modeling and optimization of the lifetime of technologies*, Kluwer Academic Publishers, Dordrecht, 1996.

[18] Hritonenko, N. and Yu. Yatsenko, Optimal equipment replacement without paradoxes: a continuous analysis, *Oper Res Letters*, 35(2007), 245-250.

[19] Hritonenko, N. and Yu. Yatsenko, Optimization of financial and energy structure of productive capital, *IMA Journal of Management Mathematics*, 17(2006), 245-255.

[20] Hritonenko, N. and Yu. Yatsenko, Turnpike properties of optimal delay in integral dynamic models, *J Opt Theory Appl*, 127(2005), 109-127.

[21] Jones, P., and J. Zydiak, The fleet design problem, *The Engng Econ*, 38(1993), 83-98.

[22] Jones, P., J. Zydiak, and W. Hopp, Parallel machine replacement, *Naval Res Logistics*, 38(1991), 351-365.

[23] Karabakal, N., J. Lohmann, and J. Bean, A Lagrangian algorithm for the parallel replacement problem with capital rationing constraints, *Mgmt Sci*, 40(1994), 305-319.

[24] Love, C.E. and R. Guo, Utilizing Weibull failure rates in repair limit analysis for equipment replacement/preventive maintenance decisions, *J Oper Res Society*, 47 (1996), 1366-1376.

[25] Malcomson, J.M., Replacement and the rental value of capital equipment subject to obsolescence, *J Econ Theory*, 10 (1975), 24-41.

[26] McClurg, T. and S. Chand, A parallel machine replacement model, *Naval Res Logist*, 49(2002), 275-287.

[27] Rajagopalan, S., Capacity expansion and equipment replacement: a unified approach, *Operations Research*, 46(1998), 846-857.

[28] Regnier, E., G. Sharp, and C. Tovey, Replacement under ongoing technological progress, *IIE Trans*, 36(2004), 497-508.

[29] Rogers, J. and J. Hartman, Equipment replacement under continuous and discontinuous technological change, *IMA J Manag Math*, 16(2005), 23-36.

[30] Sethi, S.P. and S. Chand, Planning horizon procedures in machine replacement models, *Manage Sci*, 25 (1979), 140-151.

[31] Solow, R., J. Tobin, C., von Weizsacker, and M. Yaari, Neoclassical growth with fixed factor proportions, *Rev Econ Studies*, 33 (1966), 79-115.

[32] Yatsenko, Yu., *Integral models of the systems with controllable memory*, Naukova Dumka Press, Kiev, 1991 (in Russian).

[33] Yatsenko, Yu., Volterra integral equations with unknown delay time, *Methods Applic Anal*, 2 (1995), 408-419.

[34] Yatsenko, Yu., Maximum principle for Volterra integral equations with controlled delay time, *Optimization*, 53 (2004), 177-187.

[35] Yatsenko, Yu. and N. Hritonenko, Optimization of the lifetime of capital equipment using integral models, *J Ind Manag Optimization*, 1(2005), 415-432.

[36] Yatsenko, Yu. and N. Hritonenko, Network economics and optimal replacement of age-structured IT capital, *Mathematical Methods of Operations Research*, 65 (2007), 483-497.

Recent titles

CORE Discussion Papers

2008/31. Helmuth CREMER, Philippe DE DONDER, Dario MALDONADO and Pierre PESTIEAU. Taxing sin goods and subsidizing health care.

2008/32. Jean J. GABSZEWICZ, Didier LAUSSEL and Nathalie SONNAC. The TV news scheduling game when the newscaster's face matters.

2008/33. Didier LAUSSEL and Joana RESENDE. Does the absence of competition *in* the market foster competition *for* the market? A dynamic approach to aftermarkets.

2008/34. Vincent D. BLONDEL and Yurii NESTEROV. Polynomial-time computation of the joint spectral radius for some sets of nonnegative matrices.

2008/35. David DE LA CROIX and Clara DELAVALLADE. Democracy, rule of law, corruption incentives and growth.

2008/36. Jean J. GABSZEWICZ and Joana RESENDE. Uncertain quality, product variety and price competition. 2008/37. Gregor ZOETTL. On investment decisions in liberalized electricity markets: the impact of price caps at the spot market.

2008/38. Helmuth CREMER, Philippe DE DONDER, Dario MALDONADO and Pierre PESTIEAU. Habit formation and labor supply.

2008/39. Marie-Louise LEROUX and Grégory PONTHIERE. Optimal tax policy and expected longevity: a mean and variance approach.

2008/40. Kristian BEHRENS and Pierre M. PICARD. Transportation, freight rates, and economic geography.

2008/41. Gregor ZOETTL. Investment decisions in liberalized electricity markets: A framework of peak load pricing with strategic firms.

2008/42. Raouf BOUCEKKINE, Rodolphe DESBORDES and Hélène LATZER. How do epidemics induce behavioral changes?

2008/43. David DE LA CROIX and Marie VANDER DONCKT. Would empowering women initiate the demographic transition in least-developed countries?

2008/44. Geoffrey CARUSO, Dominique PEETERS, Jean CAVAILHES and Mark ROUNSEVELL. Space-time patterns of urban sprawl, a 1D cellular automata and microeconomic approach.

2008/45. Taoufik BOUEZMARNI, Jeroen V.K. ROMBOUTS and Abderrahim TAAMOUTI. Asymptotic properties of the Bernstein density copula for dependent data.

2008/46. Joe THARAKAN and Jean-Philippe TROPEANO. On the impact of labor market matching on regional disparities.

2008/47. Shin-Huei WANG and Cheng HSIAO. An easy test for two stationary long processes being uncorrelated via AR approximations.

2008/48. David DE LA CROIX. Adult longevity and economic take-off: from Malthus to Ben-Porath.

2008/49. David DE LA CROIX and Gregory PONTHIERE. On the Golden Rule of capital accumulation under endogenous longevity.

2008/50. Jean J. GABSZEWICZ and Skerdilajda ZANAJ. Successive oligopolies and decreasing returns.

2008/51. Marie-Louise LEROUX, Pierre PESTIEAU and Grégory PONTHIERE. Optimal linear taxation under endogenous longevity.

2008/52. Yuri YATSENKO, Raouf BOUCEKKINE and Natali HRITONENKO. Estimating the dynamics of R&D-based growth models.

2008/53. Roland Iwan LUTTENS and Marie-Anne VALFORT. Voting for redistribution under desert-sensitive altruism.

2008/54. Sergei PEKARSKI. Budget deficits and inflation feedback. 2008/55. Raouf BOUCEKKINE, Jacek B. KRAWCZYK and Thomas VALLEE. Towards an understanding of tradeoffs between regional wealth, tightness of a common environmental constraint and the sharing rules.

2008/56. Santanu S. DEY. A note on the split rank of intersection cuts.

2008/57. Yu. NESTEROV. Primal-dual interior-point methods with asymmetric barriers.

2008/58. Marie-Louise LEROUX, Pierre PESTIEAU and Gregory PONTHIERE. Should we subsidize longevity?

Recent titles

CORE Discussion Papers - continued

2008/59. J. Roderick McCRORIE. The role of Skorokhod space in the development of the econometric analysis of time series.
2008/60. Yu. NESTEROV. Barrier subgradient method.
2008/61. Thierry BRECHET, Johan EYCKMANS, François GERARD, Philippe MARBAIX, Henry TULKENS and Jean-Pascal VAN YPERSELE. The impact of the unilateral EU commitment on the stability of international climate agreements.
2008/62. Giorgia OGGIONI and Yves SMEERS. Average power contracts can mitigate carbon leakage.
2008/63. Jean-Sébastien TANCREZ, Philippe CHEVALIER and Pierre SEMAL. A tight bound on the throughput of queueing networks with blocking.
2008/64. Nicolas GILLIS and François GLINEUR. Nonnegative factorization and the maximum edge biclique problem.
2008/65. Geir B. ASHEIM, Claude D'ASPREMONT and Kuntal BANERJEE. Generalized time-invariant overtaking.
2008/66. Jean-François CAULIER, Ana MAULEON and Vincent VANNETELBOSCH. Contractually stable networks.
2008/67. Jean J. GABSZEWICZ, Filomena GARCIA, Joana PAIS and Joana RESENDE. On Gale and Shapley '*College admissions and stability of marriage*'.
2008/68. Axel GAUTIER and Anne YVRANDE-BILLON. Contract renewal as an incentive device. An application to the French urban public transport sector.
2008/69. Yuri YATSENKO and Natali HRITONENKO. Discrete-continuous analysis of optimal equipment replacement.

Books

Y. POCHET and L. WOLSEY (eds.) (2006), *Production planning by mixed integer programming.* New York, Springer-Verlag.
P. PESTIEAU (ed.) (2006), *The welfare state in the European Union: economic and social perspectives.* Oxford, Oxford University Press.
H. TULKENS (ed.) (2006), *Public goods, environmental externalities and fiscal competition.* New York, Springer-Verlag.
V. GINSBURGH and D. THROSBY (eds.) (2006), *Handbook of the economics of art and culture.* Amsterdam, Elsevier.
J. GABSZEWICZ (ed.) (2006), La différenciation des produits. Paris, La découverte.
L. BAUWENS, W. POHLMEIER and D. VEREDAS (eds.) (2008), *High frequency financial econometrics: recent developments*. Heidelberg, Physica-Verlag.
P. VAN HENTENRYCKE and L. WOLSEY (eds.) (2007), *Integration of AI and OR techniques in constraint programming for combinatorial optimization problems*. Berlin, Springer.

CORE Lecture Series

C. GOURIÉROUX and A. MONFORT (1995), Simulation Based Econometric Methods.
A. RUBINSTEIN (1996), Lectures on Modeling Bounded Rationality.
J. RENEGAR (1999), A Mathematical View of Interior-Point Methods in Convex Optimization.
B.D. BERNHEIM and M.D. WHINSTON (1999), Anticompetitive Exclusion and Foreclosure Through Vertical Agreements.
D. BIENSTOCK (2001), Potential function methods for approximately solving linear programming problems: theory and practice.
R. AMIR (2002), Supermodularity and complementarity in economics.
R. WEISMANTEL (2006), Lectures on mixed nonlinear programming.