Allocation of fixed costs: characterization of the (dual) weighted Shapley value

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Abstract

The weighted value was introduced by Shapley in 1953 as an asymmetric version of his value. Since then several axiomatizations have been proposed including one by Shapley in 1981 specifically addressed to cost allocation, a context in which weights appear naturally. It was at the occasion of a comment in which he only stated the axioms. The present paper offers a proof of Shapley's statement as well as an alternative set of axioms. It is shown that the value is the unique rule that allocates *additional* fixed costs fairly: only the players who are concerned contribute to the fixed cost and they contribute in proportion to their weights. A particular attention is given to the case where some players are assigned a zero weight.

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1. Introduction

There is a large literature on cost allocation based on solutions to cooperative games with transferable utility.[1] Among the solutions that have been used in actual cost allocation problems, the Shapley value is definitely predominant. The value was introduced and axiomatized by Shapley (1953a and b), and its use in cost allocation was suggested by Shubik (1962). Since then a number of alternative axiomatizations have been proposed among which Young (1985a), Chun (1989) and van den Brink (2001). Shapley (1953a) also considered an *asymmetric* version of the value obtained by introducing exogenous weights to cover asymmetries that are not included in the underlying game. This "weighted" value has been studied by Owen (1968, 1972) and its computation has been considered by Dragan (2008). It has been axiomatized by Kalai and Samet (1987), Hart and Mas-Colell (1987), and Weber (1988).[2] In the meantime, at the occasion of a comment at an accounting conference, Shapley (1981) had suggested an axiomatization of his weighted value within the specific framework of cost allocation.

The aim of the present paper is to complete and extend Shapley's 1981 note by addressing explicitly the question of the allocation of fixed costs. We offer in this way a set of axioms that further supports the use of the weighted Shapley value in cost allocation.

In the symmetric case, weights are equal and fairness suggests that any *additional* fixed cost (e.g. the cost of a common facility) should be allocated uniformly among the players who are concerned. This is what the Shapley value does while examples indicate that other solution concepts like the nucleolus generally do not. Actually, we show that the Shapley value is the *only* allocation rule that allocates additional fixed cost uniformly.

In the asymmetric case, weights are assigned to players and fixed costs are divided according to these weights. The resulting cost allocation corresponds to the allocation derived from the standard formula applied to the *dual* game and not to the associated surplus sharing game.

Weights come up naturally in the context of cost allocation. Shapley illustrates his 1981 comment with the well known problem of dividing the travel costs of a scientist who visits a series of institutions. There the weights are the number of days spent in each institution. Weights are also included in contracts signed by the owners of a condominium and used to divide the cost of building or maintaining common facilities. Another example is data or patent pooling among firms where the size of the firms, measured for instance by their market

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[1] See for instance Roth and Verrecchia (1979), Moulin (1988 and 2003) or Young (1985b).

[2] Kalai and Samet allow for zero weights. Hart and Mas-Colell base their characterization on their potential function. Weber defines the weighted value as a particular random order value.

shares, are natural weights.[3] The case where a player is assigned a zero weight is of particular interest. For instance, in a data sharing framework, a player may hold data and be willing to share them while not being otherwise part of the cooperative project.

The paper is organized as follows. Section 2 is devoted to the definition of weighted cost games and related concepts, in particular the marginal cost vectors and the probability distributions over players' permutations associated to given (positive) weights. Following Weber (1988), the weighted Shapley value is defined in Section 3 as a *random order value* where the relative weight of a player is the probability of being *first* in a players' permutation.[4] We give a proof of Shapley's 1981 proposition and provide an alternative set of axioms that are then compared to the axioms proposed by van den Brink (2001) in the symmetric case. The case where some players are assigned a zero weight is considered apart. We show that to compute the weighted value, nonzero-weight players and zero-weight players can be treated separately. Concluding remarks are offered in Section 4.

2. Cost games

2.1 Weighted cost games

A set $N = \{1,...,n\}$ of players, $n \geq 2$, face the problem of dividing the cost of some facility. The cost of realizing it to the benefit of any coalition $S \subset N$ is also known. This defines a real-valued set function C on the subsets of N. Any set function C such that $C(\varnothing) = 0$ defines a *cost function*. The *dual* of a set function C on N is defined by $C^*(S) = C(N) - C(N \backslash S)$. For any given player set N, the restriction C_T of a set function C on a subset $T \subset N$ is simply defined by $C_T(S) = C(S \cap T)$.

Linear combinations of cost functions on a common set N are cost functions on N. Consequently the space of all cost functions on a set N is a vector space that can be identified to $\mathbb{R}^{2^n-1}$ where $2^n–1$ is the number of nonempty coalitions. The collection of $2^n–1$ *unanimity games*

$$u_T(S) = 1 \quad \text{if } T \subset S$$
$$= 0 \quad \text{if not}$$

defined for all $T \subset N$, $T \neq \varnothing$, forms a basis of $\mathbb{R}^{2^n-1}$.

[3] See Dehez and Tellone (2011) for an analysis of the resulting "*data games*".

[4] This is in contrast with the standard weighted Shapley value where the relative weight of a player is the probability of being *last* in a players' permutation.

Here we shall use the alternative basis formed by the duals of the unanimity games:

$$e_T(S) = 1 \quad \text{if } S \cap T \neq \varnothing$$
$$\qquad = 0 \quad \text{if not}$$

defined for all $T \subset N, T \neq \varnothing$. Unanimity games were introduced by Shapley in 1953 to prove existence and uniqueness of the value. The duals of unanimity games are the *representation games* introduced by Kalai and Samet (1987).[5]

Any cost function C on a set N can be uniquely decomposed into a linear combination of representation games i.e. there exists a *unique* vector $\alpha = (\alpha_T)$ that defines C in the basis (e_T):

$$C(S) = \sum_{\substack{T \subset N \\ T \neq \varnothing}} \alpha_T e_T(S) \;=\; \sum_{T : T \cap S \neq \varnothing} \alpha_T \tag{1}$$

The coefficients α_T are given by:

$$\alpha_T = \sum_{S : S \cup T = N} (-1)^{t+s+1-n} C(S) \tag{2}$$

A cost function C on a set N is *subadditive* if:

$$S \cap T = \varnothing \;\Rightarrow\; C(S \cup T) \geq C(S) + C(T)$$

From now on, we assume that cost functions are subadditive. Consequently, the set $G(N)$ of cost functions on a given set N is a *convex cone* in $\mathbb{R}^{2^n - 1}$. A cost function is *concave* if

$$C(S \cup T) + C(S \cap T) \geq C(S) + C(T) \;\text{ for all } S, T \subset N$$

Concave cost function are subadditive and the set of concave cost function is itself a convex cone. Alternatively a cost function is concave if *marginal costs* are nonincreasing with respect to set inclusion:

$$i \in S \subset T \;\Rightarrow\; C(S) - C(S \setminus i) \leq C(T) - C(T \setminus i)$$

Example 1 Consider the 3-player cost function defined by:

$$\begin{array}{lll} C(1) = 6 & C(12) = 9 & \\ C(2) = 7 & C(13) = 13 & \\ C(3) = 10 & C(23) = 15 & C(123) = 16 \end{array}$$

[5] Representation games can be seen as *normalized fixed cost games*: coalitions containing members of T entail a fixed cost equal to 1.

It is concave and thereby subadditive. The coefficients α_T's are given by:

$$\begin{array}{lll} \alpha_1 = 1 & \alpha_{12} = 2 & \\ \alpha_2 = 3 & \alpha_{13} = 1 & \\ \alpha_3 = 7 & \alpha_{23} = 0 & \alpha_{123} = 2 \end{array}$$

For a given a player set *N*, we denote by $w = (w_1, ..., w_n)$ the vector of *weights* that are assigned to players, with $w_i > 0$ for all *i*. The extension to situations where some players are assigned a zero weight will be considered in Section 35. The symmetric case is defined by equal weights.

A triple (N, C, w) defines a *weighted cost game.* A *rule* is a mapping φ that associates to any weighted cost game (N, C, w) a *n*-dimensional vector $y = \varphi(N, C, w)$.

Notations: Lower case letters *n, s, t,…* denote the sizes of the sets *N, S, T,…* For any vector *y*, $y_S = (y_i \mid i \in S)$ is the subvector corresponding to a subset *S* and $y(S) = \sum_{i \in S} y_i$. Specific coalitions may be identified as *ijk…* instead of $\{i, j, k, ...\}$. For a coalition *S*, $S \backslash i$ denotes the subset from which player *i* has been removed.

2.2 Fixed costs

Starting from a cost function $C \in G(N)$ and a nonempty coalition $T \subset N$, a *fixed cost* is an additional amount $f \in \mathbb{R}$ that only affects coalitions including members of T:[6]

$$\begin{aligned} C'(S) &= C(S) + f \text{ for all } S \subset N \text{ such that } S \cap T \neq \varnothing \\ &= C(S) \qquad \text{otherwise} \end{aligned} \tag{3}$$

Put differently, the cost function C' is defined by $C' = C + f\ e_T$.

Example 2 *Airport games* are defined by nonnegative vectors $c = (c_1, ..., c_n)$ such that $c_1 \leq c_2 \leq ... \leq c_n$. The associated cost function is given by $C(S) = \mathrm{Max}_{i \in S}\, c_i$. Airport games are concave. Setting $c_0 = 0$, we observe that the additional cost $c_i - c_{i\text{-}1}$ is a fixed cost for the players in $T_i = \{i, …, n\}$.[7]

We denote by (N, f) the *fixed cost game* corresponding to a facility whose cost is fixed and equal to *f*. The corresponding cost function *C* is defined by $C = f\ e_N$ i.e. $C(S) = f$ for all $S \subset N, S \neq \varnothing$. The dual of the fixed cost game (N, f) is a *pure bargaining game*:

$$C(S) = f \text{ for all } S \subset N \ \Rightarrow\ C^*(S) = 0 \text{ for all } S \neq N \text{ and } C^*(N) = f$$

[6] Costs as well as fixed costs can be negative. A fixed cost can alternatively be interpreted as a tax when positive and a subsidy (or bonus) when negative.

[7] See Littlechild and Owen (1973) or, more recently, Thomson (2007).

2.3 Marginal cost vectors

For a given a set of players *N*, we denote by Π_N the set of the $n!$ players' permutations. The *vector* of marginal costs $t^{\pi}(N,C)$ associated to a cost function $C \in G(N)$ and a permutation $\pi = (i_1,...,i_n)$ is defined by:

$$t^{\pi}_{i_1}(N,C) = C(i_1) - C(\varnothing) = C(i_1)$$

$$t^{\pi}_{i_k}(N,C) = C(i_1,...,i_k) - C(i_1,...,i_{k-1}) \quad (k = 2,...,n)$$

Marginal cost vectors are *imputations* i.e. individually rational allocations:

$$\text{for all } \pi \in \Pi_N, \ \sum_{i \in N} t^{\pi}_i(N,C) = C(N) \ \text{ and } \ t^{\pi}_i(N,C) \le C(i) \text{ for all } i \in N$$

Furthermore

$$t^{\pi}(N,C^*) = t^{\pi^{-1}}(N,C) \qquad (4)$$

where $\pi^{-1} = (i_n,...,i_1)$ denotes the *reversed* permutation. The marginal cost vectors associated to the fixed cost game (N,f) are given by:

$$t^{\pi}_i(N,f) \ = f \quad \text{if } i \text{ is first in } \pi$$

$$= 0 \quad \text{otherwise}$$

For a given a set of players *N*, we denote by $\Delta(\Pi_N)$ the set of probability distributions over the set permutations Π_N. Let us assume for a moment that the w_i's are natural numbers. If there were w_i players of type *i*, the probability $P_w(\pi)$ that the permutation $\pi = (i_1,...,i_n)$ comes out through a sequence of independent drawings is given by:

$$P_w(\pi) = \frac{w_{i_1}}{w_{i_1} + ... + w_{i_n}} \frac{w_{i_2}}{w_{i_2} + ... + w_{i_n}} ... \frac{w_{i_{n-1}}}{w_{i_{n-1}} + w_{i_n}} = \prod_{k=1}^{n} \frac{w_{i_k}}{\sum_{j=k}^{n} w_{i_j}} \qquad (5)$$

knowing that, each time a player is drawn, all players of the same type are removed. In the symmetric case, the w_i's are all equal and $P_w(\pi) = 1/n!$ for all $\pi \in \Pi_N$. For *normalized* weights, $w(N) = 1$ and w_i is then the probability that player *i* comes *first* in an arbitrary permutation:

$$\sum_{\pi \in \Pi_{N\backslash i}} P_w(i,\pi) = w_i \sum_{\pi \in \Pi_{N\backslash i}} P_{w\backslash w_i}(\pi) = w_i$$

Players *i* and *j* are *substitutes* in a cost game (N,C,w) if they have identical weights and identical marginal costs: $w_i = w_j$ and $C(S) - C(S\backslash i) = C(S) - C(S\backslash j)$ for all *S* containing *i* and *j*. A player *i* is a *dummy* in a cost game (N,C,w) if his or her marginal costs are all zero: $C(S) - C(S\backslash i) = 0$ for all $S \subset N$.

3. The Shapley value

3.1 The weighted value as a random order value

A *random order value* is the mean marginal cost vector corresponding to some probability distribution $\lambda \in \Delta(\Pi_N)$. Hence, the set of allocations corresponding to random order values is the convex hull of the marginal cost vectors, a set known as the *Weber set*. It contains the core.[8] The (dual) weighted value of a cost game (N, C, w) is the random order value associated to the probability distribution induced by *w*:

$$\varphi(N, C, w) = \sum_{\pi \in \Pi_N} P_w(\pi)\, t^{\pi}(N, C) \qquad (6)$$

3.2 Shapley's characterization

Shapley (1981) proposed to characterize the weighted value in a cost allocation context by the following set of axioms:

A1 Full cost allocation (efficiency)

For all N, $C \in G(N)$ and $w \in \mathbb{R}^n_{++}$: $\sum_{i \in N} \varphi_i(N, C, w) = C(N)$

A2 Symmetry

If, for some N, $C \in G(N)$ and $w \in \mathbb{R}^n_{++}$ i and j are substitute players:

$$\varphi_i(N, C, w) = \varphi_j(N, C, w)$$

A3 Dummy elimination

If for some N, $C \in G(N)$ and $w \in \mathbb{R}^n_{++}$, i is a null player:

$$\varphi_i(N, C, w) = 0 \ \text{ and } \ \varphi_j(N, C, w) = \varphi_j(N \backslash i, C_{N \backslash i}, w_{N \backslash i}) \ \text{ for all } j \neq i$$

where $C_{N \backslash i} \in G(N \backslash i)$ denotes the restriction of C to $N \backslash i$.

A4 Homogeneity

φ is homogeneous of degree 0 in w:

$$\text{for all } N, C \in G(N), w \in \mathbb{R}^n_{++} \text{ and } \beta > 0,\ \varphi(N, C, \beta w) = \varphi(N, C, w)$$

A5 Additivity

For all N, $C_1, C_2 \in G(N)$ and $w \in \mathbb{R}^n_{++}$:

$$\varphi(N, C_1 + C_2, w) = \varphi(N, C_1, w) + \varphi(N, C_2, w)$$

[8] See Weber (1988).

A6 Shared facility

For all $N, w \in \mathbb{R}^n_{++}$ and $f \in \mathbb{R}$:

$$w(N) = 1 \Rightarrow \varphi_i(N, f, w) = w_i f \quad \text{for all } i \in N$$

These are standard axioms except for the dummy player axiom that is stronger than usual, and for the fixed cost axiom that applies specifically to cost allocation problems. The dummy player axiom says that a dummy player does not contribute *and* that removing a dummy player does not affect what the other players pay.[9]

The fixed cost axiom provides a natural interpretation of the (normalized) weights: they are the proportions into which a fixed cost is to be divided and they coincide with the probability of being first in a players' permutation.[10]

Shapley takes the opportunity of his comment to insist on the implicit requirement that an allocation method should only depend on *N*, *C* (and *w*), a "hidden axiom" that makes additivity a natural requirement in a transferable utility framework: "*It is the hidden "domain" axiom, not the additivity axiom, that demands that we shut our eyes to the structural detail that stands behind the characteristic function.*" (1981, p.132) [11]

Lemma The weighted value as defined by (6) satisfies the axioms A1 to A6.

Proof Consider a cost game (N, C, w). Since marginal cost vectors are imputations, A1 immediately follows.

Assume *i* and *j* are substitute players and consider an arbitrary permutation $\pi \in \Pi_N$. Then the permutation π' obtained from π by exchanging *i* and *j* has the same probability of occurrence than π. A2 then follows since the amount player *i* is asked to pay if π occurs is equal to the amount player *j* is asked to pay if π' occurs.

Homogeneity of the probability distributions P_w implies A4. A5 and the first part of A3 are immediate consequences of (6).

The second part of A3 requires some attention. Using (5) we observe that removing a player, say *j*, results in the probability distribution over $\Pi_{N \setminus j}$ given by:

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[9] This part of the axiom is the "*null player out*" axiom introduced by Derks and Haller (1999).

[10] In Shapley's words: "*The weights represent the proportions into which we want to divide the total cost in a situation where there is a single jointly-used facility that costs a fixed amount K, regardless of the number of users.*" (1981, p.135)

[11] Ignoring this hidden axiom is a common source of misunderstanding. Shapley refers in particular to the critique of the additivity axiom made by Luce and Raiffa (1957, p.248).

$$P_{w_{N\backslash j}}(\pi') = \sum_{\substack{\pi\in\Pi_N \\ \pi_{N\backslash j}=\pi'}} P_w(\pi) \tag{7}$$

If j is a dummy, we have:

$$C(S) - C(S\backslash i) = C(S\backslash j) - C(S\backslash ij) \text{ for all } S \subset N \tag{8}$$

Combining (7) and (8), we then get:

$$\sum_{\pi\in\Pi_N} P_w(\pi)\, t_i^{\pi}(N,C) = \sum_{\pi'\in\Pi_{N\backslash j}} P_{w_{-j}}(\pi')\, t_i^{\pi'}(N\backslash j, C) \quad \text{for all } i \in N\backslash j$$

It remains to verify A6. We have already observed that, in the marginal cost vectors associated to a fixed cost game (N, f), only players who are first pay and they pay exactly f. Since the probability that player i comes first in an arbitrary permutation is given by $w_i \Big/ \sum_j w_j$ A6 follows. •

Shapley suggests using the duality relation but he provides no proof that his axioms define uniquely (6). It happens that efficiency and symmetry are actually redundant.

Theorem 1 The weighted Shapley value is the unique allocation rule that satisfies *dummy elimination* (A3), *homogeneity* (A4), *additivity* (A5) and *shared facility* (A6).

Proof We start from a game (N,C,w) and the unique representation of C in the basis (e_T) given by (1). Consider some coalition $T \subset N$. Players outside T are dummies in the game $(N,\alpha_T e_T, w)$. Hence, by A3:

$$\varphi_i(N,\alpha_T e_T, w) = 0 \quad \text{for all } i \in N\backslash T$$

and we may restrict ourselves to the game defined on T. A4 allows using normalized weights $\bar{w}_i = w_i / w(T)$. Applying A6, we then have:

$$\varphi_i(N,\alpha_T e_T, w) = \varphi_i(T,\alpha_T e_T, w_T) = \bar{w}_i\ \alpha_T \quad \text{for all } i \in T$$

Using A5, we may then extend φ to the entire game (N,C,w): [12]

$$\varphi_i(N,C,w) = \sum_{\substack{T\subset N \\ T\neq\varnothing}} \varphi_i(N,\alpha_T e_T, w) = \sum_{\substack{T\subset N \\ T\ni i}} \frac{w_i}{w(T)}\alpha_T \quad (i = 1...n) \tag{9}$$

Hence A3, A4, A5 and A6 define a unique allocation rule. By the Lemma, it is the rule defined by (6). •

From the proof, it is easily seen that keeping efficiency, only the null player out part of A3 is needed.

[12] Any function v can be written as $v = (v - u) + u$. If v, u and $v - u$ are all subadditive functions, additivity can be applied to deal with negative coefficients.

Applying (9) to Example 1, the weighted value associated to the weights w = (1, 2, 3) is given by (2.25, 5, 8.75) to be compared to the symmetric value (3.17, 4.67, 8.17). This is confirmed by the following table that, according to (4), lists the marginal cost vectors and probability distributions associated to the game (N, C, w) and its dual (N, C^*, w).

	1	**2**	**3**	$\boldsymbol{P_w}$	**1**	**2**	**3**	$\boldsymbol{P_w}$*
123	6	3	7	1/15	1	5	10	1/3
132	6	3	7	1/10	1	7	8	1/4
213	2	7	7	1/12	3	3	10	1/6
231	1	7	8	1/4	6	3	7	1/10
312	3	3	10	1/6	2	7	7	1/12
321	1	5	10	1/3	6	3	7	1/15

3.3 Alternative characterization

Consider the game $(N, C + f\ e_T, w)$ defined by (3) where a fixed cost f concerns some subset $T \subset N$ of players. Fairness suggests that only players concerned should contribute to the fixed cost:

A7 Fairness [13]

For all N, $C \in G(N)$, $w \in \mathbb{R}^n_{++}$, $f \in \mathbb{R}$ and $T \subset N$:

$$\varphi_i(N, C + f\ e_T, w) = \varphi_i(N, C, w) \quad \text{for all } i \in N \backslash T$$

Furthermore, they should contribute in proportion to their weights:

A8 Shared additional facility

For all N, $C \in G(N)$, $w \in \mathbb{R}^n_{++}$, $f \in \mathbb{R}$ and $T \subset N$, there exists $b \in \mathbb{R}$ such that

$$\varphi_i(N, C + f\ e_T, w) = \varphi_i(N, C, w) + w_i b \quad \text{for all } i \in T$$

Together the axioms A7 and A8 replace additivity. It is easily verified that (6) satisfies A7 and A8. The following proposition establishes that, together with A1, A7, and A8 define uniquely the weighted value.

Theorem 2 The weighted value is the unique allocation rule that satisfies *efficiency* (A1), *fairness* (A7) and *shared additional facility* (A8).

[13] This axiom is implied by Young's monotonicity axiom: if a game changes so that a player's contribution to all coalitions does not decrease then that player's allocation does not decrease. See Young (1985a).

Proof Consider a game (N,C,w), a fixed cost f and an arbitrary subset $T \subset N$. Using A7 and A8, we get:

$$\varphi_i(N,0+f\ e_T,w)=\varphi_i(N,0,w) \qquad \text{for all } i \in N\backslash T$$

$$\varphi_i(N,0+f\ e_T,w)=\varphi_i(N,0,w)+w_i b \quad \text{for all } i \in T$$

Summing over N and using A1, we obtain $b=f/w(T)$. As a consequence,

$$\sum_{i\in N\backslash T}\varphi_i(N,f\ e_T,w)=0 \ \Rightarrow \sum_{i\in N\backslash T}\varphi_i(N,0,w)=0$$

Applying this argument to $T=N\backslash i$ for an arbitrary $i\in N$, we conclude that $\varphi_i(N,0,w)=0$ for all $i\in N$. Hence, applied to the unique representation of C in the basis (e_T) given by (1), we have:

$$\varphi_i(N,\alpha_T e_T,w)=0 \qquad \text{for all } i\in N\backslash T$$

$$\varphi_i(N,\alpha_T e_T,w)=\frac{w_i}{w(T)}\alpha_T \quad \text{for all } i\in T$$

Applying A7 repeatedly to every subset $T\subset N$, $S\neq\varnothing$, φ is then extended to the entire game (N,C,w) as in (9). •

The following observations confirm the logical independence of the axioms A1, A7 and A8. The proportional division

$$\varphi_i(N,C,w)=\frac{w_i}{w(N)}C(N)$$

satisfies A1 and A8 but not A7. The symmetric Shapley value satisfies A1, A7 but not A8. Multiplying the sum of marginal costs of any given player by his or her weight defines a rule

$$\varphi_i(N,C,w)=w_i\sum_{\substack{S\subset N\\ S\ni i}}\big(C(S)-C(S\backslash i)\big)$$

that satisfies A7 and A8 but not A1. Indeed

$$\varphi_i(N,C+fe_T,w)=w_i\sum_{\substack{S\subset N\\ S\ni i}}\big(C(S)-C(S\backslash i)\big)+w_i\ f\sum_{\substack{S\subset N\\ S\ni i}}\big(e_T(S)-e_T(S\backslash i)\big)$$

where

$$\begin{aligned} i\notin T:\quad e_T(S)-e_T(S\backslash i)&=0-0=0 \quad \text{if } S\cap T=\varnothing\\ &=1-1=0 \quad \text{if } S\cap T\neq\varnothing \end{aligned}$$

$$\begin{aligned} i\in T:\quad e_T(S)-e_T(S\backslash i)&=1-0=1 \quad \text{if } S\cap T=\{i\}\\ &=1-1=0 \quad \text{if } S\cap T\neq\{i\} \end{aligned}$$

Hence A7 is satisfied. So is A8 because the number of coalition S including a given player i and whose intersection with a given T reduces to $\{i\}$ is a positive constant independent of i.

To illustrate how the axioms A1, A7 and A8 determine a unique solution, consider the airport game defined in Example 2. Knowing that $c_i - c_{i-1}$ is a fixed cost for the players in $T_i = \{i,\dots,n\}$, we obtain the following allocation:

$$
\begin{aligned}
y_1 &= \frac{w_1}{w_1+\dots+w_n} c_1 \\
y_2 &= \frac{w_2}{w_1+\dots+w_n} c_1 + \frac{w_2}{w_2+\dots+w_n}(c_2-c_1) \\
&\dots \\
y_k &= \frac{w_k}{w_1+\dots+w_n} c_1 + \frac{w_k}{w_2+\dots+w_n}(c_2-c_1) + \dots + \frac{w_k}{w_k+\dots+w_n}(c_k-c_{k-1}) \quad k=3,\dots,n-1 \\
&\dots \\
y_n &= \frac{w_n}{w_1+\dots+w_n} c_1 + \frac{w_n}{w_2+\dots+w_n}(c_2-c_1) + \dots + \frac{w_n}{w_{n-1}+w_n}(c_{n-1}-c_{n-2}) + (c_n-c_{n-1})
\end{aligned}
$$

This is exactly the cost allocation derived from (6).

3.4 Equal weights

In the symmetric case, weights are equal and the axioms A1, A7 and A8 become:

A1' Full cost allocation (efficiency)

For all N, $C \in G(N)$: $\sum_{i\in N} \varphi_i(N,C) = C(N)$

A7' Fairness

For all N, $C \in G(N)$, $f \in \mathbb{R}$ and $T \subset N$: $\varphi_i(N, C + f\, e_T) = \varphi_i(N,C)$ for all $i \in N \backslash T$

A8' Shared additional facility

For all N, $C \in G(N)$, $f \in \mathbb{R}$ and $T \subset N$, there exists $b \in \mathbb{R}$ such that

$$\varphi_i(N, C + f\, e_T) = \varphi_i(N,C) + b \quad \text{for all } i \in T$$

By Theorem 2, the (symmetric) Shapley value is uniquely defined by A1', A7' and A8'. Actually, Shared additional facility can be replaced by Symmetry.[14]

A2' Symmetry:

If, for some N, $C \in G(N)$, i and j are substitute players: $\varphi_i(N,C) = \varphi_j(N,C)$

[14] Thanks are due to René van den Brink for suggesting this proposition.

Theorem 3 The symmetric value is the unique allocation rule that satisfies *efficiency* (A1'), *symmetry* (A2'), and *fairness* (A7').

Proof Consider a symmetric cost game (N,C). By A1' and A2', $\varphi_i(N,0)=0$ for all $i\in N$. Given the unique representation of C in the basis (e_T) given by (1), A2' implies that there exist a and b in $\mathbb{R}$ such that:

$$\varphi_i(N,\alpha_T e_T)=a \quad \text{for all } i\in N\backslash T$$

$$=b \quad \text{for all } i\in T$$

where $a=0$ by A7' and $b=\alpha_T/t$ by A1'. Applying fairness repeatedly to all subsets $T\subset N$, φ is then extended to the entire game (N,C) as in (9). •

The axioms A7 and A8 must be compared to the fairness axiom introduced by van den Brink (2001) to characterize the Shapley value:

> Consider two games (N,v) and (N,w), and a subset $T\subset N$ of players who are substitute in (N,w). There exists $a\in\mathbb{R}$ such that $\varphi_i(N,v+w)-\varphi_i(N,C,v)=a$ for all $i\in T$ i.e. the change in the allocation is the same for all substitute players.

Applied to the allocation of an additional fixed cost, van den Brink's axiom implies the following property:

For all N, $C\in G(N)$, $f\in\mathbb{R}$ and $T\subset N$, there exists $a,b\in\mathbb{R}$ such that :

$$\varphi_i(N,C+f\,e_T)-\varphi_i(N,C)=a \quad \text{for all } i\in N\backslash T$$

$$\varphi_i(N,C+f\,e_T)-\varphi_i(N,C)=b \quad \text{for all } i\in T$$

The dummy player axiom is however needed to characterize the Shapley value. Indeed, a = 0 and the above combination of A7 and A8 holds.

The nucleolus is another game theoretic solution concept that has been used in cost allocation problems.[15] To illustrate how the nucleolus allocates fixed costs in a symmetric situation, as compared to the Shapley value, consider Example 1. The cost allocations derived from the Shapley value and the nucleolus are given by $(3.17, 4.67, 8.17)$ and $(2.75, 4.75, 8.5)$ respectively. Adding a fixed cost $f=3$ affecting all players, the cost allocations become $(4.17, 5.67, 9.17)$ and $(3.67, 5.67, 9.17)$ respectively: the Shapley value imposes to each player the same additional contribution while the nucleolus imposes different additional contributions.

[15] The nucleolus has been introduced by Schmeidler (1967). It selects a core allocation when the core is nonempty and defines an allocation rule that satisfies efficiency, symmetry, dummy player but not additivity.

Example 3 Consider the 4-player cost function defined by:

$C(1) = 7$, $C(2) = 9$, $C(3) = 13$, $C(4) = 15$

$C(12) = 15$, $C(13) = 19$, $C(14) = 21$, $C(23) = 20$, $C(24) = 22$, $C(34) = 25$

$C(123) = 27$, $C(124) = 26$, $C(134) = 24$, $C(234) = 26$, $C(1234) = 30$

The cost allocations derived from the Shapley value and the nucleolus are respectively given by (5.08, 6.75, 8.58, 9.58) and (5.5, 7.5, 8.25, 8.75). Adding a fixed cost $f = 2$ that affects players 1 and 2, the cost allocations then become respectively (6.08, 7.75, 8.58, 9.58) and (63.3, 83.3, 85, 88.3): the fixed cost is divided equally between player 1 and 2 by the Shapley value while the nucleolus imposes additional contributions to all four players.

3.5 Zero weights

So far we have considered the case where weights were positive. In some applications it may be justified to assign a zero-weight to some players. For instance, in a data sharing process some players may hold data and be willing to share them while not being otherwise part of the cooperative project. Furthermore, to generate the set of *all* weighted values, it is necessary to extend the definition of the value to the case where some weights are zero. This has been done by Kalai and Samet (1987).

The set of weighted Shapley values is a subset of the Weber set[16] and, surprisingly, the core happens to be a subset of the set of weighted values. This was shown by Monderer, Samet and Shapley (1992) who also prove that a cost game is concave *if and only if* its core coincides with the set of weighted values. Consequently, the three sets coincide on the class of concave cost games: the core, the set of weighted values and the Weber set.

Here we are interested in further characterizing weighted values when some players are assigned a zero weight. We shall see that what nonzero-weight players pay coincides with the weighted value of the cost game restricted to the set of nonzero-weight player. The question then concerns the allocation of what remains among zero-weight players.

For a vector of weights $w \in \mathbb{R}^n_{++}$, the probability that a given permutation $\pi = (i_1, \ldots, i_n)$ occurs is given by (5). It can alternatively be written as:

$$P_w(\pi) = \prod_{k=1}^{n-1} \frac{1}{1 + \sum_{j=k+1}^{n} w_{i_j} / w_{i_k}} \tag{10}$$

[16] See Weber (1988) and Vasiliev (2007).

For a given set N of players with weights $w \in \mathbb{R}_{+}^{n}$, the set of all random order values is obtained from probability distributions in the set

$$F_N(w) = \left\{ p \in \Delta(\Pi_N) \mid p(\pi) = \lim_{w^m \to w} P_{w^m}(\pi) \text{ for some converging sequence } (w^m) \subset \mathbb{R}_{++}^{n} \right\}$$

If there is a *single* zero-weight player, the limit distribution is uniquely defined: he or she is *last* with probability 1 and pays his or her marginal cost with respect to the grand coalition $C(N) - C(N\backslash i)$. In Example 1, the value corresponding to the weights (0, 1, 2) is given by (1, 5.67, 9.33). Only the permutations (2,3,1) and (3,2,1) enter into account, with probabilities 1/3 and 2/3, respectively.

If instead there are several zero-weight players, the limit distribution depends on the speed of convergence to zero of the different weights. We shall see that to compute the weighted value nonzero-weight players and zero-weight players can be treated separately.

Theorem 4 **T**he set of weighted values of the cost game (N,C,w) where $w \in \mathbb{R}_{+}^{n}$ consists of the allocations y defined by:

$$i \in N\backslash Z: \; y_i = \varphi_i(N\backslash Z, C_{N\backslash Z}, w_{N\backslash Z}) \text{ where } Z = \{i \in N \mid w_i = 0\}$$

$$i \in Z: \quad y_i = \sum_{\pi'' \in \Pi_Z} \lambda(\pi'')\, t_i^{(\pi',\pi'')}(N,C) \text{ for some } \lambda \in F_Z(0)$$

independently of $\pi' \in \Pi_{N\backslash Z}$.

Proof From (5) or (10) it is clear that distributions in $F(w)$ assign a *zero* probability to permutations in which a zero-weight player *precedes* a nonzero-weight player. Hence, only permutations of the form $\pi = (\pi', \pi'')$ where $\pi' \in \Pi_{N\backslash Z}$ and $\pi'' \in \Pi_Z$ may be given a positive probability. By continuity of (10), for any sequence (w^m) in $\mathbb{R}_{++}^{n}$ converging to w, the sequence of probability distributions (P_{w^m}) converges to a distribution $p \in F(w)$ of the form

$$p(\pi', \pi'') = P_{w_{N\backslash Z}}(\pi')\, \lambda(\pi'') \quad \text{for all } (\pi', \pi'') \in \Pi_{N\backslash Z} \times \Pi_Z$$

$$= 0 \qquad \text{otherwise}$$

for some probability distribution $\lambda \in F_Z(0)$. The corresponding allocation is then given by:

$$y_i = \sum_{\substack{\pi' \in \Pi_{N\backslash Z} \\ \pi'' \in \Pi_Z}} P_{w_{N\backslash Z}}(\pi')\, \lambda(\pi'')\, t_i^{(\pi',\pi'')}(N,C)$$

where $t_i^{(\pi',\pi'')}$ is independent of π'' for all $i \in N\backslash Z$ and independent of π' for all $i \in Z$. Hence, for a player $i \in N\backslash Z$, we have:

$$y_i = \sum_{\pi' \in \Pi_{N\backslash Z}} P_{w_{N\backslash Z}}(\pi')\, t_i^{(\pi',\pi'')}(N,C) \sum_{\pi'' \in \Pi_Z} \lambda(\pi'') = \sum_{\pi' \in \Pi_{N\backslash Z}} P_{w_{N\backslash Z}}(\pi')\, t_i^{\pi'}(N\backslash Z, C_{N\backslash Z})$$

i.e. $y_i = \varphi_i(N\backslash Z, C_{N\backslash Z}, w_{N\backslash Z})$ for all $i \in N\backslash Z.$ For a player $i \in Z,$ we have:

$$y_i = \sum_{\pi''\in\Pi_Z} \lambda(\pi'')\, t_i^{(\pi',\pi'')}(N,C) \sum_{\pi'\in\Pi_{N\backslash Z}} P_{w_{N\backslash Z}}(\pi') \;=\; \sum_{\pi''\in\Pi_Z} \lambda(\pi'')\, t_i^{(\pi',\pi'')}(N,C)$$

This concludes the proof. •

As a consequence, computing the weighted value involves only the coalitions that are either subsets of $N\backslash Z$ or of the form $N\backslash Z \cup T$ for some $T \subset Z:$ only coalitions of nonzero weight players or coalition containing all nonzero weight players matter. The contributions of the zero-weight players can then be written as

$$y_i = \sum_{\pi\in\Pi_Z} \lambda(\pi)\, t_i^{\pi}(Z,C') \tag{11}$$

where the marginal contributions vectors $t^{\pi}(Z,C')$ are based on the game (Z,C') defined by

$$C'(S) = C(S \cup N\backslash Z) - C(N\backslash Z)$$

Actually *any* probability distribution on Π_Z is possible: $F_Z(0) = \Delta(\Pi_Z).$ In line with the principle "*equal treatment of equals*", the natural selection consists to treat zero-weight players symmetrically by using the uniform distribution $\lambda(\pi) = 1/z!$ for all $\pi \in \Pi_Z.$ It corresponds to common sequences $w_i^n = v^n \to 0$ for all $i \in Z.$ The contributions of zero-weight players are then given by the (symmetric) value associated to the game (Z,C'):

$$\varphi_i(N,C,w) \;= \frac{1}{z!} \sum_{\pi\in\Pi_Z} t_i^{\pi}(Z,C')$$

Using the data of Example 3 with $w = (1,2,0,0),$ we get $\varphi(N,C,w) \simeq (6.33, 8.66, 8, 7).$

4. Concluding remarks

In the framework of surplus sharing games, weights define the shares in the cake to be divided in a pure bargaining game and, when normalized, they are the probabilities of being *last* in an arbitrary permutation. Owen (1968, 1972) has shown that the relation between what the weighted value assigns to a player and his or her weight may not be monotonic.[17] It is on that basis that he questioned the interpretation of weights as a measure of bargaining ability, an interpretation originally suggested by Shapley. In the present cost allocation context, the weights define the shares in a fixed cost. Given that the cost function and the weights are independent elements, it is no surprise that what a player is asked to pay may not be increasing in his or her weight. Actually Monderer, Samet and Shapley (1992) have shown

[17] See also Haeringer (2006).

that monotonicity is a necessary *and* sufficient condition for concavity. This is illustrated by airport games.

As a last remark, it must be stressed that the axiomatizations that are formulated here apply to the case where weights are positive: they do not cover Theorem 4 and its symmetric extension (11). It remains an open question.

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