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Strategic complementarities and nested potential games

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Abstract

This note shows that every finite game of strategic complementarities is a nested pseudo-potential game defined in Uno (2007, *Economics Bulletin* **3** (17)) if the action set of each player is one-dimensional, except possibly for one player.

Keywords: Strategic complementarities, potential games, existence of a pure strategy Nash equilibrium.

JEL Classification: C72

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1 Introduction

Whenever we analyze a strategic situation as a game, we face the issue of the existence of a Nash equilibrium, especially a pure strategy Nash equilibrium for economic applications. In the literature, several sufficient conditions for existence of a pure strategy Nash equilibrium have been provided.[3] Among such conditions, those in terms of "strategic complementarities" by Topkis (1973) and Vives (1990) and those in terms of "potential functions" by Rosenthal (1973) and Monderer and Shapley (1996), can be applied even if action sets are finite.[4] This note investigates the relationship between these two conditions.

A game of strategic complementarities is a game in which if the competitors turn more aggressive, the agent's optimal reaction is to become more aggressive as well. Many economic models belong to this class of games.[5] The weakest version of such strategic complementarities is the weak strategic complementarities discussed in Dubey *et al.* (2006). Games of weak strategic complementarities are those in which, for each player i, there exists a selection within i's best response correspondence which is non-decreasing in the other players' action.

On the other hand, several versions of potential functions also have been proposed since Monderer and Shapley (1996).[6] These potential functions have a common feature: a potential function is a real-valued function over the set of action profiles of a game, and every maximizer of a potential function is a pure strategy Nash equilibrium of the games. That is, in games with a potential function, known as a potential game, the problem of finding a Nash equilibrium is a simple maximization problem rather than a fixed point problem. This implies that every potential game possesses a pure strategy Nash equilibrium if action sets are finite. The weakest version of such potential functions is the nested pseudo-potential function introduced in Uno (2007a).

[3]For example, see Debreu (1952), Glicksberg (1952), Nikaido and Isoda (1955), Dasgupta and Maskin (1986), Topkis (1979), Vives (1990), Rosenthal (1973), Monderer and Shapley (1996), Milchtaich (1996) and so on.

[4]In fact, Topkis (1973) used the term of supermodular instead of strategic complementarities. Strategic complementarities were originally used by Bulow, Geanakoplos, and Klemperer (1985). Rosenthal (1973) did not use the term of potential function, but essentially used the same concept as exact potential function defined by Monderer and Shapley (1996). Milchtaich (1996) provided a sufficient condition in congestion games with player specific utilities, where action sets are finite.

[5]See Topkis (1998), Vives (1999), and Amir (2005).

[6]For example, exact potentials, weighted potentials, ordinal potentials, generalized ordinal potentials are introduced in Monderer and Shapley (1996); (ordinal) best response potentials in Voorneveld (2000); pseudo-potentials in Dubey *et al.* (2006); best response potentials and better response potentials in Morris and Ui (2004); generalized potentials, monotone potentials, and local potentials in Morris and Ui (2005); iterated potentials in Oyama and Tercieux (2004); nested best response potentials in Uno (2007b) and so on.

The nested pseudo-potential functions generalize the pseudo-potential functions defined by Dubey *et al.* (2006). A pseudo-potential function of a game is a real-valued function f over its set of action profiles such that any best-response of each player i if endowed with f as payoff function is a best-response as well in the original game. As for the other versions of potential functions, every maximizer of a pseudo-potential function of a game is a Nash equilibrium of the game. It is as if the pseudo-potential functions are payoff functions of one representative agent who chooses strategies for all players.

In considering a nested pseudo-potential function, we think of a representative agent for a subset T of players instead of all of them: for each player i in T, given any strategy profile for other players, maximizing this representative agent's payoff f_T yields a best-response for player i. Suppose that there is a partition $\mathcal{T}$ of players such that, for each member T of $\mathcal{T}$, there is such a representative agent whose payoff function is f_T.[7] Then the collection of f_T's can be seen as a new complete information game, where each member T in $\mathcal{T}$ is regarded as a single player. That is, the original game is reduced to a game with a smaller number of players.

Notice that such reduction can be nested: the new game among step 1 representative agents may be reduced to a game with an even smaller number of players, by considering a step 2 representative agent for step 1 representative agents, and then a representative agent of these, and so on. We say that a game has a nested pseudo-potential if a game is reduced to a game with one representative agent through this process, where the payoff functions of representative agents are pseudo-potential functions.

In earlier literature, Dubey *et al.* (2006) showed that the set of pseudo-potential games strictly includes the set of games of weak strategic complementarities if the action sets are one-dimensional and each payoff function depends on her own action and the aggregator of the other players' actions, i.e., in the case of a game with an aggregator. Otherwise, a game of weak strategic complementarities may not be a pseudo-potential game, as shown in Example 6.1 below.

This note shows that the set of nested pseudo-potential games strictly includes the set of games of weak strategic complementarities if the action sets are one-dimensional, except possibly for one player, and finite (Theorem 5.1). The above relationships are illustrated in Figure 1. This result establish that the existence of nested pseudo-potential function rather than weak strategic complementarities suffices to guarantee the existence of a pure strategy Nash equilibrium as long as we consider games where each player's action set is one-dimensional and finite.[8]

[7] This idea also has appeared as q-potential in Monderer (2007).

[8] Unfortunately, the proof of our result depends on Taraski's fixed point theorem.

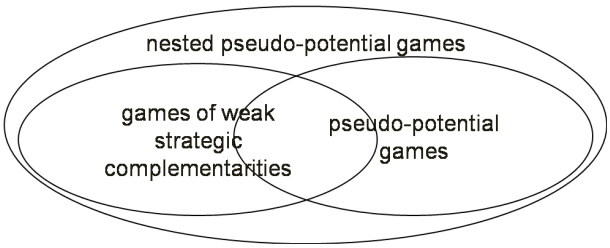

Figure 1: Strategic complementarities and nested potential games when the action set of each player is one-dimensional, except possibly for one player

Our result provides an answer to the question of which games are nested pseudo-potential games that are not be pseudo-potential games. The answer is that, for example, a game of weak strategic complementarities may not be a pseudo-potential game but still be a nested pseudo-potential game by the result (Theorem 5.1).

2 Preliminaries

Let X be a finite subset of the m-dimensional Euclidean space $\mathbb{R}^m$. The inequality $x \geq y$ means $x_i \geq y_i$ for each i, while $x > y$ means $x \geq y$ and there exists i such that $x_i > y_i$.

For $x, y \in X$, let $\inf_X\{x, y\}$ denote the greatest lower bound for x and y in X, and let $\sup_X\{x, y\}$ denote the least upper bound for x and y in X.

A set X in $\mathbb{R}^m$ is a *lattice* if X contains the least upper bound and the greatest lower bound of each pair of its elements, i.e., for each $x, y \in X$, $\inf_X\{x, y\} \in X$ and $\sup_X\{x, y\} \in X$.

Tarski (1955) showed that the collection of fixed points of a non-decreasing function from a nonempty finite lattice into itself is a nonempty lattice, and he gave the form of the greatest fixed point and the least fixed point:[9]

Theorem 2.1 (Tarski, 1955) *Suppose that f is a non-decreasing function from a nonempty finite lattice X to X. Then the set of fixed points of f in X is a nonempty lattice,* $\sup\{x \in X | x \leq f(x)\}$ *is the greatest fixed point, and* $\inf\{x \in X | x \geq f(x)\}$ *is the least fixed point.*

[9]In fact, Tarski (1955) provides the fixed point theorem for an non-decreasing function on a complete lattice instead of a finite lattice.

3 Strategic Complementarities

A strategic form game consists of a finite player set $N = \{1, \ldots, n\}$, an action set A_i for $i \in N$, and the payoff function $g_i : A \to \mathbb{R}$ for $i \in N$, where $A := \prod_{i \in N} A_i$. Since we fix the set A of action profiles, we denote a strategic form game $(N, (A_i)_{i \in N}, (g_i)_{i \in N})$ simply by $\mathbf{g}^N := (g_i)_{i \in N}$. For notational convenience, we write $a = (a_i)_{i \in N} \in A$; for $i \in N$, $A_{-i} = \prod_{j \neq i} A_j$ and $a_{-i} = (a_j)_{j \neq i} \in A_{-i}$; and for $T \subseteq N$, $A_T = \prod_{i \in T} A_i$, $a_T = (a_i)_{i \in T} \in A_T$, $A_{-T} = \prod_{i \in N \setminus T} A_i$, and $a_{-T} = (a_i)_{i \in N \setminus T} \in A_{-T}$. For each $T \subseteq N$, for any $a_{-T} \in A_{-T}$, let $\mathbf{g}^N|_{a_{-T}}$ denote the game where the action profile of players outside T is fixed to a_{-T}.

Since Topkis (1979), various notions of strategic complementarities have been introduced.[10] Among them, the weakest notion is the game of weak strategic complementarities. A game has weak strategic complementarities if, for each player, there exists a non-decreasing selection in the player's best-response correspondence:

Definition 3.1 (Dubey *et al.*, 2006) A game $\mathbf{g}^N$ is a *finite game of weak strategic complementarities* if, for each $i \in N$, $A_i \subset \mathbb{R}^{m_i}$ is a finite lattice,[11] where $m_i \in \mathbb{N}$, and there exists a function $b_i : A_{-i} \to A_i$ such that

1. b_i is i's best-response selection: $b_i(a_{-i}) \in \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, and

2. b_i is non-decreasing with a_{-i}: $b_i(a_{-i}) \leq b_i(a'_{-i})$ whenever $a_{-i} < a'_{-i}$.

4 Nested Potential Games

Let $\mathbf{g}^N$ be a strategic form game. Beginning with Monderer and Shapley (1996), various notions of potential games have been proposed. Among them, one of the weakest notions is the nested pseudo-potential games introduced in Uno (2007a). To introduce the nested pseudo-potential games, we introduce the pseudo-potential games proposed by Dubey *et al.* (2006). A pseudo-potential of a game $\mathbf{g}^N$ is a real valued function f on the set A of action profiles such that, for each player i, i's best-response against the other players' actions a_{-i} in the alternative game where i's payoff function is given by f is a best-response to a_{-i} in the original game $\mathbf{g}^N$ as well:

[10] For example, the supermodular games introduced by Topkis (1979), the games of strategic complementarities introduced by Bulow *et al.* (1985), the quasi-supermodular game introduced by Milgrom and Shannon, and so on.

[11] We can also consider a version of games with compact action sets. In the version, it is difficult to show our main result hold, as we will discuss in Remark 6.4 later.

Definition 4.1 (Dubey *et al.*, 2006) A function $f : A \to \mathbb{R}$ is a *pseudo-potential* of $\mathbf{g}^N$ if, for each $i \in N$,

$$\arg\max_{a_i \in A_i} f(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i}) \tag{1}$$

for all $a_{-i} \in A_{-i}$. If $\mathbf{g}^N$ has a pseudo-potential, $\mathbf{g}^N$ is called a *pseudo-potential game.*[12]

We say that an action profile a^* is a *pseudo-potential maximizer* of $\mathbf{g}^N$ if $f(a^*) \geq f(a)$ for all $a \in A$. Dubey *et al.* (2006) showed that a pseudo-potential maximizer, if it exists, is a Nash equilibrium of the underlying game:

Proposition 4.2 (Dubey *et al.*, 2006) *If $\mathbf{g}^N$ is a pseudo-potential game with a pseudo-potential maximizer a^*, then a^* is a Nash equilibrium of $\mathbf{g}^N$.*

As a consequence, whenever action sets are finite, every pseudo-potential game has a pure strategy Nash equilibrium, since there always exists a maximizer for a function whose domain is finite.

Corollary 4.3 (Dubey *et al.*, 2006) *Every pseudo-potential game with finite action sets has a pure strategy Nash equilibrium.*

We shall extend Proposition 4.2 by introducing a weaker notion of potential where a 'pseudo-potential' is considered for each subset of players instead of the entire set. For a partition $\mathcal{T}$ of N, we define the partition $\mathcal{T}$ pseudo-potentials as follows:

Definition 4.4 (Uno, 2007a) Let $\mathcal{T}$ be a partition of N. A *partition $\mathcal{T}$ pseudo-potential* of $\mathbf{g}^N$ is a tuple $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$, where, for each $T \in \mathcal{T}$, $f_T : A \to \mathbb{R}$ satisfies that, for each $i \in T$,

$$\arg\max_{a_i \in A_i} f_T(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$$

for all $a_{-i} \in A_{-i}$.

We denote such a partition $\mathcal{T}$ pseudo-potential $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$ by $\mathbf{f}^{\mathcal{T}} := (f_T)_{T\in\mathcal{T}}$ since action sets $(A_T)_{T\in\mathcal{T}}$ can be derived from the partition $\mathcal{T}$ of N and the set A of action

[12] If the inclusion of (1) can be replaced by the equality, f is called an (ordinal) *best-response potential*, which is introduced in Voorneveld (2000). The pseudo-potentials generalize thus the (ordinal) best-response potentials. Morris and Ui (2004, 2005) also introduced alternative best-response potentials, which are special classes of (ordinal) best-response potentials of Voorneveld (2000) and the pseudo-potentials in Dubey *et al.* (2006). See Morris and Ui (2004) for more discussion of this notion. We can apply the analogous arguments in this section to these best-response potentials of Morris and Ui (2004).

profiles in the original game $\mathbf{g}^N$.[13]

Note that we can regard each $\mathcal{T}$-pseudo-potential $\mathbf{f}^{\mathcal{T}}$ as a strategic form game, where $\mathcal{T}$ is the player set; for each $T \in \mathcal{T}$, A_T is the action set of T; and for each $T \in \mathcal{T}$, f_T is the payoff function of T. The idea of the nested pseudo-potential games is to construct such games iteratively for a nested sequence of partitions:

Definition 4.5 (Uno, 2007a) A function $f : A \to \mathbb{R}$ is a *nested pseudo-potential* of $\mathbf{g}^N$ if there exist a positive integer K and a sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=1}^K = ((f_T^k)_{T\in\mathcal{T}^k})_{k=1}^K$ such that

- $\{\mathcal{T}^k\}_{k=1}^K$ is a nested sequence of N: $\{\mathcal{T}^k\}_{k=1}^K$ is an increasingly coarser sequence of partitions of N with $\mathcal{T}^1 = \{\{i\}|i \in N\}$ and $\mathcal{T}^K = \{N\}$;
- $\mathbf{f}^{\mathcal{T}^1} = (f_T^1)_{T\in\mathcal{T}^1}$ is the original game $\mathbf{g}^N$: for each $i \in N$, $f_{\{i\}}^1(a) = g_i(a)$ for all $a \in A$;
- for each $k = 2, 3, \ldots, K$, $\mathbf{f}^{\mathcal{T}^k} = (f_T^k)_{T\in\mathcal{T}^k}$ is a $\mathcal{T}^k$-pseudo-potential of $\mathbf{f}^{\mathcal{T}^{k-1}} = (f_T^{k-1})_{T\in\mathcal{T}^{k-1}}$, where $\mathbf{f}^{\mathcal{T}^{k-1}}$ is regarded as a strategic form game as above: for each $T^k \in \mathcal{T}^k$ and each $T^{k-1} \in \mathcal{T}^{k-1}$ with $T^{k-1} \subseteq T^k$,

$$\arg \max_{a_{T^{k-1}} \in A_{T^{k-1}}} f_{T^k}^k(a_{T^{k-1}}, a_{-T^{k-1}}) \subseteq \arg \max_{a_{T^{k-1}} \in A_{T^{k-1}}} f_{T^{k-1}}^{k-1}(a_{T^{k-1}}, a_{-T^{k-1}})$$

  for all $a_{-T^{k-1}} \in A_{-T^{k-1}}$; and
- $\mathbf{f}^{\mathcal{T}^K} = (f_N^K)$ is such that $f_N^K(a) = f(a)$ for all $a \in A$.

A game that admits a nested pseudo-potential is called a *nested pseudo-potential game.*

We say that an action profile a^* is a *nested pseudo-potential maximizer* of $\mathbf{g}^N$ if $f(a^*) \geq f(a)$ for all $a \in A$.

The essential property shared by all existing versions of potential games is that maximizers of a potential function are Nash equilibria as in Proposition 4.2. The nested pseudo-potential proposed here inherits this property. Indeed, Uno (2007a) showed that a nested pseudo-potential maximizer, if it exists, is a Nash equilibrium of the underlying game:

Proposition 4.6 (Uno, 2007a) *Let $\mathbf{g}^N$ be a nested pseudo-potential games with a nested pseudo-potential maximizer a^*. Then, a^* is a Nash equilibrium of $\mathbf{g}^N$.*

[13] The partition $\mathcal{T}$ pseudo-potential generalizes Monderer (2007)'s q-potential: a strategic form game $\mathbf{g}^N$ has a q-potential if and only if $\mathbf{g}^N$ has a partition $\mathcal{T}$-potential, where q refers to the number of elements in $\mathcal{T}$ and the potentials in $(f_T)_{T\in\mathcal{T}}$ are meant to be the exact potentials in Monderer and Shapley (1996). If $\mathbf{g}^N$ is a q-potential game, then it has a partition $\mathcal{T}$ pseudo-potential such that the number of elements of $\mathcal{T}$ is q. The converse is not true, since there is a pseudo-potential game without an exact potential.

Proposition 4.6 implies the following corollary.

Corollary 4.7 (Uno, 2007a) *Every nested potential game with finite action sets possesses a pure strategy Nash equilibrium.*

5 Nested Potentials in Games of Strategic Complementarities

This section shows that games of weak strategic complementarities are nested pseudo-potential games if the action set of each player is one-dimensional, except possibly for one player.

Theorem 5.1 *Let $\mathbf{g}^N$ be a finite game of weak strategic complementarities, where $A_i \subset \mathbb{R}^m$, $m \geq 2$, for at most one unique player $i \in N$, and $A_j \subset \mathbb{R}$ for any $j \neq i$. Then $\mathbf{g}^N$ is a nested pseudo-potential game.*

To prove the above theorem, we will use the following four facts.

Firstly, a game of weak strategic complementarities has a property that, for each subset T of N, the Nash equilibria of the restriction $\mathbf{g}^N|_{a_{-T}}$ of $\mathbf{g}^N$ for any action $a_{-T} \in A_{-T}$ of all players outside T has a selection that is non-decreasing with respect to a_{-T}:

Lemma 5.2 *Let $\mathbf{g}^N$ be a game of weak strategic complementarities. Let T be a subset of N. For any $a_{-T} \in A_{-T}$, let $\mathbf{g}^N|_{a_{-T}}$ be the restricted game by a_{-T}. Then, there exists a function $e_T : A_{-T} \to A_T$ such that*

1. *e_T is an equilibrium selection: $e_T(a_{-T})$ is a pure strategy Nash equilibrium of $\mathbf{g}^N|_{a_{-T}}$ for any $a_{-T} \in A_{-T}$; and*
2. *$e_T(a_{-T})$ is non-decreasing with a_{-T}: $e_T(a_{-T}) \leq e_T(a'_{-T})$ whenever $a_{-T} < a'_{-T}$.*

This lemma resembles the result from monotone comparative statics establishing that a function from a nonempty lattice into itself has a fixed point that is non-decreasing with the parameter. The proof is also similar to that of the monotone comparative statics in Milgrom and Roberts (1994) or Topkis (1998, p.41, Theorem 2.5.2).

Proof. See Appendix. ■

Secondly, in a pseudo-potential game, for each pure strategy Nash equilibrium, we can find a pseudo-potential such that the Nash equilibrium is the unique maximizer of the pseudo-potential:

Lemma 5.3 *Let $\mathbf{g}^N$ be a pseudo-potential game. If a^* is a pure strategy Nash equilibrium of $\mathbf{g}^N$, then there exists a pseudo-potential $f : A \to \mathbb{R}$ such that $\{a^*\} = \arg\max_{a \in A} f(a)$.*

Proof. See Appendix. ■

Thirdly, we have the following characterization of partition pseudo-potentials:

Lemma 5.4 *$(f_T)_{T \in \mathcal{T}}$ is a partition $\mathcal{T}$ pseudo-potential of $\mathbf{g}^N$ if and only if, for each member T of $\mathcal{T}$, for any action $a_{-T} \in A_{-T}$ of all players outside T, $f_T(\cdot, a_{-T})$ is a pseudo-potential of the restricted game $\mathbf{g}^N|_{a_{-T}}$ by a_{-T}.*

Finally, a finite two-person game of weak strategic complementarities has a pseudo-potential. Indeed, Dubey *et al.* (2006) showed that a two-person finite game of weak strategic complementarities, where each action set is one-dimensional, has a pseudo-potential:[14]

Proposition 5.5 (Dubey *et al*, 2006) *Let $\mathbf{g}^{\{1,2\}}$ be a two-person finite game with $A_1, A_2 \subset \mathbb{R}$. If $\mathbf{g}^{\{1,2\}}$ has weak strategic complementarities, then it is a pseudo-potential game.*

We extend Proposition 5.5 to the case where the action set of one player is multi-dimensional.

Proposition 5.6 *Let $\mathbf{g}^{\{1,2\}}$ be a finite two-person game with $A_1 \subset \mathbb{R}^m$, where $m \in \mathbb{N}$, and $A_2 \subset \mathbb{R}$. If $\mathbf{g}^{\{1,2\}}$ has weak strategic complementarities, then it is a pseudo-potential game.*

Proof. See Appendix. ■

We prove Theorem 5.1 by applying Lemmas 5.2, 5.3, 5.4 and Proposition 5.6 iteratively. The outline of the proof is the following: let $\mathbf{g}^N$ be a finite game of weak strategic complementarities. Firstly, by Lemma 5.4 and Proposition 5.6, we know there exists a partition $\{\{1,2\}, 3, \ldots, n\}$ pseudo-potential of $\mathbf{g}^N$. Next, by Lemmas 5.2 and 5.3, in particular, we can find a partition $\{\{1,2\}, 3, \ldots, n\}$ pseudo-potential $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ such that a best-response of representative agent $\{1,2\}$ is non-decreasing with $a_{-\{1,2\}}$. Then, we can regard $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ as a game of weak strategic complementarities. Moreover, by applying Lemmas 5.2, 5.3, 5.4 and Proposition 5.6, we have a partition $\{\{1,2,3\}, \ldots, n\}$ pseudo-potential $\mathbf{f}^{\{\{1,2,3\},\ldots,n\}}$ of $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ such that $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ is a game of weak strategic complementarities, and so on. Finally, we can find a partition $\{\{1,2,\ldots,n\}\}$ pseudo-potential $(f^{\{\{1,2,\ldots,n\}\}})$ of $\mathbf{f}^{\{\{1,2,\ldots,n-1\},n\}}$. Thus, we have a nested pseudo-potential $f = f^{\{\{1,2,\ldots,n\}\}}$.

Proof of Theorem 5.1. See Appendix. ■

[14] In fact, Dubey *et al.* (2006) showed that games with an aggregator of weak strategic complementarities or weak strategic substitutes are pseudo-potential games.

6 Examples

In what follows, we show by way of examples that, when the action set of a single player is allowed multi-dimensional, the relationship among strategic complementarities, a pseudo potential and a nested pseudo potential is given as in Figure 1 of Introduction.

As mentioned in Proposition 5.5, Dubey *et al.* (2006) showed that two-person games of weak strategic complementarities are pseudo-potential games. However, games with more than two players of weak strategic complementarities may not be a pseudo-potential game as the following example shown.

Example 6.1 Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ in Table 1, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.

0	0	1	1	0	1
0	$4,4,4$	$0,0,1$	0	$1,0,0$	$0,1,0$
1	$0,1,0$	$1,0,0$	1	$0,0,1$	$4,4,4$

Table 1: (g_1, g_2, g_3)

We can show that $\mathbf{g}^{\{1,2,3\}}$ has weak strategic complementarities.

However, this game is not a pseudo-potential game. Indeed, note that $\mathbf{g}^{\{1,2,3\}}$ has a strict best-response cycle $(1,0,0) \to (1,0,1) \to (0,0,1) \to (0,1,1) \to (0,1,0) \to (1,1,0) \to (1,0,0)$. Since pseudo-potential games cannot have strict best-response cycles as shown by Kukushkin (2004), this game is not a pseudo-potential game. On the other hand, $\mathbf{g}^{\{1,2,3\}}$ is a nested pseudo-potential game. Indeed, $(f^2_{\{1,2\}}, f^1_{\{3\}})$ given in Table 2 is a $\{\{1,2\},\{3\}\}$-pseudo-potential of $\mathbf{g}^{\{1,2,3\}}$, where $f^1_{\{3\}}(\cdot) = g_3(\cdot)$.

	0	1
$(0,0)$	$3,4$	$1,0$
$(0,1)$	$0,1$	$2,0$
$(1,0)$	$2,0$	$0,1$
$(1,1)$	$1,0$	$3,4$

Table 2: $(f^1_{\{1,2\}}, f^1_{\{3\}})$

	0	1
$(0,0)$	2	0
$(0,1)$	1	0
$(1,0)$	0	1
$(1,1)$	0	2

Table 3: $f^2_{\{1,2,3\}}$ or f

Regarding the $\{\{1,2\},\{3\}\}$-pseudo-potential $(f^2_{\{1,2\}}, f^1_{\{3\}})$ as a strategic form game, we can show that $(f^2_{\{1,2,3\}})$ defined in Table 3 is a $\{\{1,2,3\}\}$-pseudo-potential of $(f^2_{\{1,2\}}, f^1_{\{3\}})$. Thus $\mathbf{g}^{\{1,2,3\}}$ is a nested pseudo-potential game.

A pseudo-potential game may not have strategic complementarities as in the following example.

Example 6.2 Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ in Table 5, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.

0	0	1	2
0	0, 0, 0	1, 1, 0	0, 0, 1
1	0, 0, 0	0, 0, 1	1, 1, 0

1	0	1	2
0	1, 1, 1	1, 0, 1	0, 0, 0
1	1, 0, 1	0, 0, 0	1, 0, 1

Table 4: $\mathbf{g}^{\{1,2,3\}}$ is a pseudo-potential without weak strategic complementarities

We can show that $\mathbf{g}^{\{1,2,3\}}$ has a pseudo-potential f in Table 5. We can also show that $\mathbf{g}^{\{1,2,3\}}$ does not have weak strategic complementarities.

0	0	1	2
0	0	2	1
1	0	1	2

1	0	1	2
0	4	3	0
1	1	0	3

Table 5: a pseudo-potential f of $\mathbf{g}^{\{1,2,3\}}$

The following game, which appeared in Uno (2007a), strategic complementarities or a pseudo-potential game but it is a nested pseudo-potential game.

Example 6.3 (Uno, 2007a) Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ in Table 6, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix; players 1 and 2 have identical interests, player 3's payoff is the same as others when player 1 chooses a_1, but is reversed otherwise as in the matching pennies game.

0	0	1
0	3, 3, 3	0, 0, 0
1	−1, −1, 1	1, 1, −1

1	0	1
0	0, 0, 0	2, 2, 2
1	1, 1, −1	−1, −1, 1

Table 6: (g_1, g_2, g_3)

Note that $\mathbf{g}^{\{1,2,3\}}$ is not a game of strategic complementarities. Note also that $\mathbf{g}^{\{1,2,3\}}$ is not a pseudo-potential game. Indeed, $\mathbf{g}^{\{1,2,3\}}$ has a strict best-response cycle $(1,0,0) \to (1,1,0) \to (1,1,1) \to (1,0,1) \to (1,0,0)$. Since pseudo-potential games cannot have strict best response cycles as shown by Kukushkin (2004), the game is not a pseudo-potential game. However, we can show that $\mathbf{g}^{\{1,2,3\}}$ is a nested pseudo-potential game.

Remark 6.4 Dubey *et al.* (2006) presented a more general version of Proposition 5.5 where action sets are compact subsets of $\mathbb{R}$, provided that, for each player i, i's best-response selection b_i is continuous in the set A_{-i} of the other players' action profiles as well. But, we cannot immediately extend Theorem 5.1 to games with compact action sets. This is because it is difficult to guarantee that there exists a partition potential $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ of a game $\mathbf{g}^N$ such that a best-response selection $b_{\{1,2\}}$ of representative agent $\{1,2\}$ is continuous in the set $A_{-\{1,2\}}$ of action profiles of players outside $\{1,2\}$, since a game of weak strategic complementarities does not always have a continuous non-decreasing equilibrium selection.

Appendix: Proofs

Proof of Lemma 5.2. Suppose that $\mathbf{g}^N$ is a game of weak strategic complementarities. For each $i \in N$, let $b_i : A_{-i} \to A_i$ be i's best-response selection such that $b_i(a_{-i}) \leq b_i(a'_{-i})$ whenever $a_{-i} < a'_{-i}$. Fix any $T \subseteq N$. For any $a_{-T} \in A_{-T}$, let $b_T(\cdot, a_{-T}) : A_T \to A_T$ be the function defined by $b_T(a_T, a_{-T}) := (b_i(a_{T\setminus\{i\}}, a_{-T}))_{i\in T}$ for any $a_T \in A_T$. For any $a_{-T} \in A_{-T}$, since $b_T(\cdot, a_{-T})$ is a non-decreasing function, by Tarski's fixed point theorem (Theorem 2.1), there exists the greatest (least) fixed point of $b_T(\cdot, a_{-T})$, i.e., the greatest (least) pure strategy Nash equilibrium of $\mathbf{g}^N|_{a_{-T}}$.

Pick any $a_{-T}, a'_{-T} \in A_{-T}$ with $a_{-T} < a'_{-T}$. Let $e_T(a_{-T})$ and $e_T(a'_{-T})$ be the greatest pure strategy Nash equilibria of $\mathbf{g}^N|_{a_{-T}}$ and $\mathbf{g}^N|_{a'_{-T}}$, respectively. Because $e_T(a_{-T}) = b_T(e_T(a_{-T}), a_{-T})$ and $b_T(e_T(a_{-T}), a_{-T}) \leq b_T(e_T(a_{-T}), a'_{-T})$, we have $e_T(a_{-T}) \leq b_T(e_T(a_{-T}), a'_{-T})$. By Theorem 2.1, $\sup\{a_T \in A_T | a_T \leq b_T(a_T, a'_{-T})\}$ is the greatest pure strategy Nash equilibrium of $\mathbf{g}^N|_{a'_{-T}}$. Thus, we have $e_T(a_{-T}) \leq e_T(a'_{-T})$. ■

Proof of Lemma 5.3. Suppose that $\mathbf{g}^N$ is a game with pseudo-potential f. Let a^* be a pure strategy Nash equilibrium of $\mathbf{g}^N$. Let $c \in \mathbb{R}$ be a sufficiently large number such that $c > \max_{a\in A} f(a)$. Define a function $\hat{f} : A \to \mathbb{R}$ such that, for each $a \in A$,

$$\hat{f}(a) = \begin{cases} c & \text{if } a = a^* \\ f(a) & \text{otherwise} \end{cases}$$

Then, we have $\{a^*\} = \arg\max_{a\in A} \hat{f}(a)$. And, we can show that $\hat{f}$ is also a pseudo-potential of $\mathbf{g}^N$. Indeed, fix any $i \in N$ and any $a_{-i} \in A_{-i}$. If $a_{-i} \neq a^*_{-i}$, we have $\arg\max_{a_i\in A_i} \hat{f}(a_i, a_{-i}) = \arg\max_{a_i\in A_i} f(a_i, a_{-i})$. Since f is a pseudo-potential of $\mathbf{g}^N$,

$\arg\max_{a_i \in A_i} \hat{f}(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$. If $a_{-i} = a^*_{-i}$, we have $\{a^*_i\} = \arg\max_{a_i \in A_i} \hat{f}(a_i, a_{-i})$. Since a^* is a Nash equilibrium, we have $a^*_i \in \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$. Thus we have $\arg\max_{a_i \in A_i} \hat{f}(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$. Hence $\hat{f}$ is a pseudo-potential of $\mathbf{g}^N$. ■

Proof of Proposition 5.6. Suppose that $\mathbf{g}^{\{1,2\}}$ has weak strategic complementarities. Then, for $i, j = 1, 2$ with $i \neq j$, there exists a function $b_i : A_j \to A_i$ such that $b_i(a_j) \in \arg\max_{a_i \in A_i} g_i(a_i, a_j)$ for all $a_j \in A_j$, and $b_i(a_j) \leq b_i(a'_j)$ whenever $a_j < a'_j$. Let A'_1 be the range of b_1, i.e., $A'_1 := \{a_1 \in A_1 |$ there exists $a_2 \in A_2$ such that $a_1 = b_1(a_2)\}$. Since A'_1 is linearly ordered and finite, there exist a subset A_1 of $\mathbb{R}$ and a bijection h from A'_1 to A_1 such that for each $a_1, a'_1 \in A'_1$, $a_1 < a'_1$ if and only if $h(a_1) < h(a'_1)$. Such $\hat{A}_1$ exists by the property of b_1. Let $\hat{g}_1 : \hat{A}_1 \times A_2 \to \mathbb{R}$ be the function defined by $\hat{g}_1(\hat{a}_1, a_2) = g_1(h^{-1}(\hat{a}_1), a_2)$ for all $\hat{a}_1 \in \hat{A}_1$ and all $a_2 \in A_2$.

Consider a two-person game $(\hat{g}_1, g_2)$ given by $(\hat{g}_1, g_2) := (\{1, 2\}, (\hat{A}_1, A_2), (\hat{g}_1, g_2))$. It then follows that there exists player 1's best-response selection $\hat{b}_1 : A_2 \to \hat{A}_1$ such that $\hat{b}_1(a_2) \leq \hat{b}_1(a'_2)$ whenever $a_2 < a'_2$, since A'_1 and $\hat{A}_1$ are order isomorphic, $b_1(a_2) \in A'_1$ for any $a_2 \in A_2$, and $b_1(a_2) \leq b_1(a'_2)$ whenever $a_2 < a'_2$. Since $\mathbf{g}^{\{1,2\}}$ has strategic complementarities, there exists also player 2's best-response selection $\hat{b}_2 : \hat{A}_1 \to A_2$ such that $\hat{b}_2(a_1) \leq \hat{b}_2(a'_1)$ whenever $a_1 < a'_1$. Thus, $(\hat{g}_1, g_2)$ has weak strategic complementarities. By proposition 5.5, $(\hat{g}_1, g_2)$ has a pseudo-potential $\hat{f} : \hat{A}_1 \times A_2 \to \mathbb{R}$.

Let $c \in \mathbb{R}$ be sufficiently small so that $c < \min_{(\hat{a}_1, a_2) \in \hat{A}_1 \times A_2} \hat{f}(\hat{a}_1, a_2)$, which exists since $\hat{A}_1 \times A_2$ is finite.

Let $f : A \to \mathbb{R}$ be a function such that, for all $a_1 \in A_1$ and all $a_2 \in A_2$,

$$f(a_1, a_2) = \begin{cases} \hat{f}(h(a_1), a_2) & \text{if } a_1 \in A'_1 \\ c & \text{if } a_1 \in A_1 \backslash A'_1 \text{ and } a_2 \in b_2(a_1) \\ c - 1 & \text{otherwise} \end{cases} \tag{2}$$

We will show that f is a pseudo-potential of $\mathbf{g}^{\{1,2\}}$. Fix any $a_2 \in A_2$. Pick any $a^{**}_1 \in \arg\max_{a_1 \in A_1} f(a_1, a_2)$. Then, $a^{**}_1 \in A'_1$ must hold by the choice of constant c in the construction of f. Since $a^{**}_1 \in \arg\max_{a_1 \in A'_1} f(a_1, a_2)$, we have $h(a^{**}_1) \in \arg\max_{\hat{a}_1 \in \hat{A}_1} \hat{f}(\hat{a}_1, a_2)$. Since $\hat{f}$ is a pseudo-potential of $(\hat{g}_1, g_2)$, we have $h(a^{**}_1) \in \arg\max_{\hat{a}_1 \in \hat{A}_1} \hat{g}_1(\hat{a}_1, a_2)$. Since A'_1 and $\hat{A}_1$ are order isomorphic, we have $a^{**}_1 \in \arg\max_{a_1 \in A'_1} g_1(a_1, a_2)$. And, since $a^{**}_1 \in A'_1$, we have $g_1(a^{**}_1, a_2) \geq g_1(a_1, a_2)$ for all $a_1 \in A_1 \backslash A'_1$. Thus, we have $a^{**}_1 \in \arg\max_{a_1 \in A_1} g_1(a_1, a_2)$.

Fix any $a_1 \in A_1$. Pick any $a^{**}_2 \in \arg\max_{a_2 \in A_2} f(a_1, a_2)$. If $a_1 \in A'_1$, we have $a^{**}_2 \in$

$\arg\max_{a_2 \in A_2} g_2(a_1, a_2)$, since $\hat{f}$ is a pseudo-potential of $(\hat{g}_1, g_2)$. If $a_1 \in A_1 \backslash A'_1$, we must have $a_2^{**} \in b_2(a_1)$ by the construction of f. Thus, we have $a_2^{**} \in \arg\max_{a_2 \in A_2} g_2(a_1, a_2)$. Hence, f is a pseudo-potential of $\mathbf{g}^{\{1,2\}}$. ■

Proof of Theorem 5.1. Without loss of generality, we will assume that $m_1 \in \mathbb{N}$ and $m_i = 1$ for each $i \neq 1$. Suppose that $\mathbf{g}^N$ is a game of weak strategic complementarities. We shall show that, for each $l = 1, 2, \ldots, n$, there exists a function $f^l_{\{1,\ldots,l\}} : A \to \mathbb{R}$ such that

1. $(f^l_{\{1,\ldots,l\}}, g_{l+1}, \ldots, g_n)$ is a $\{\{1, \ldots, l\}, \{l+1\}, \ldots, \{n\}\}$-pseudo-potential of $(f^{l-1}_{\{1,\ldots,l-1\}}, g_l, \ldots, g_n)$, where $(f^0_{\{0\}}, g_1, \ldots, g_n) := (g_1, \ldots, g_n)$;

2. there exists a function $b_{\{1,\ldots,l\}} : A_{-\{1,\ldots,l\}} \to A_{\{1,\ldots,l\}}$ with

   - $b_{\{1,\ldots,l\}}(a_{-\{1,\ldots,l\}}) \in \arg\max_{a_{\{1,\ldots,l\}} \in A_{\{1,\ldots,l\}}} f^l_{\{1,\ldots,l\}}(a_{\{1,\ldots,l\}}, a_{-\{1,\ldots,l\}})$ for all $a_{-\{1,\ldots,l\}} \in A_{-\{1,\ldots,l\}}$, and
   - $b_{\{1,\ldots,l\}}(a_{-\{1,\ldots,l\}}) \geq b_{\{1,\ldots,l\}}(a'_{-\{1,\ldots,l\}})$ whenever $a_{-\{1,\ldots,l\}} > a'_{-\{1,\ldots,l\}}$.

The proof proceeds by induction on l. First, when $l = 1$, let $f^1_{\{1\}}(\cdot) := g_1(\cdot)$. Then, $(f^1_{\{1\}}, \ldots, g_n)$ is a $\{\{1\}, \ldots, \{n\}\}$-potential of $(g_1, \ldots, g_n)$. Moreover, since $\mathbf{g}^N$ is a game of weak strategic complementarities, there exists a function $b_{\{1\}} : A_{-\{1\}} \to A_{\{1\}}$ with $b_{\{1\}}(a_{-\{1\}}) \in \arg\max_{a_{\{1\}} \in A_{\{1\}}} f^1_{\{1\}}(a_{\{1\}}, a_{-\{1\}})$ for all $a_{-\{1\}} \in A_{-\{1\}}$, and $b_{\{1\}}(a_{-\{1\}}) \geq b_{\{1\}}(a'_{-\{1\}})$ whenever $a_{-\{1\}} > a'_{-\{1\}}$.

Suppose that, for each $l \leq k-1 \leq n-1$, there exist functions $f^l_{\{1,\ldots,l\}} : A \to \mathbb{R}$ and $b_{\{1,\ldots,l\}} : A_{-\{1,\ldots,l\}} \to A_{\{1,\ldots,l\}}$ such that the conditions 1 and 2 hold. We will show that there exists such functions $f^k_{\{1,\ldots,k\}} : A \to \mathbb{R}$ and $b_{\{1,\ldots,k\}} : A_{-\{1,\ldots,k\}} \to A_{\{1,\ldots,k\}}$.

Fix any $a_{-\{1,\ldots,k\}} \in A_{-\{1,\ldots,k\}}$. Consider a restricted game $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, \ldots, g_n)|_{a_{-\{1,\ldots,k\}}}$ by $a_{-\{1,\ldots,k\}}$. By the assumption of induction, there exists a function $b_{\{1,\ldots,k-1\}}(\cdot, a_{-\{1,\ldots,k\}}) : A_k \to A_{\{1,\ldots,k-1\}}$ with $b_{\{1,\ldots,k-1\}}(a_k, a_{-\{1,\ldots,k\}}) \in \arg\max_{a_{\{1,\ldots,k-1\}} \in A_{\{1,\ldots,k-1\}}} f^{k-1}_{\{1,\ldots,k-1\}}(a_{\{1,\ldots,k-1\}}, a_k, a_{-\{1,\ldots,k\}})$ for all $a_k \in A_k$, and $b_{\{1,\ldots,k-1\}}(a_k, a_{-\{1,\ldots,k\}}) \geq b_{\{1,\ldots,k-1\}}(a'_k, a_{-\{1,\ldots,k\}})$ whenever $a_k > a'_k$. And, since b_k is player k's best-response selection such that $b_k(a_{\{1,\ldots,k-1\}}, a_{-\{1,\ldots,k\}}) \geq b_k(a'_{\{1,\ldots,k-1\}}, a_{-\{1,\ldots,k\}})$ whenever $a_{\{1,\ldots,k-1\}} > a'_{1,\ldots,k-1}$, we can regard the restricted game $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, \ldots, g_n)|_{a_{-\{1,\ldots,k\}}}$ by $a_{-\{1,\ldots,k\}}$ as a two-person game of weak strategic complementarities, where $N = \{\{1, \ldots, k-1\}, \{k\}\}$, $A_{\{1,\ldots,k\}} \subset \mathbb{R}^{m+k-1}$, and $A_k \subset \mathbb{R}$. By Proposition 5.6, $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, g_{k+1}, \ldots, g_n)|_{a_{-\{1,\ldots,k\}}}$ has a pseudo-potential.

Now, consider a restricted game $\mathbf{g}^N|_{a_{-\{1,\ldots,k\}}}$ for any $a_{-\{1,\ldots,k\}} \in A_{-\{1,\ldots,k\}}$. Since $\mathbf{g}^N$ has weak strategic complementarities, by Lemma 5.2, there exists an equilibrium selection

$e_{\{1,\ldots,k\}} : A_{-\{1,\ldots,k\}} \to A_{\{1,\ldots,k\}}$ of $\mathbf{g}^N|_{a_{-\{1,\ldots,k\}}}$ such that $e_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}}) \leq e_{\{1,\ldots,k\}}(a'_{-\{1,\ldots,k\}})$ whenever $a_{-\{1,\ldots,k\}} < a'_{-\{1,\ldots,k\}}$.

For any $a_{-\{1,\ldots,k\}} \in A_{-\{1,\ldots,k\}}$, since $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, \ldots, g_n)|_{a_{-\{1,\ldots,k\}}}$ has a pseudo-potential, by Lemma 5.3, there exists a pseudo-potential $f^k_{\{1,\ldots,k\}}(\cdot, a_{-\{1,\ldots,k\}}) : A_{\{1,\ldots,k\}} \to \mathbb{R}$ such that $e_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}})$ is a unique maximizer of $f^k_{\{1,\ldots,k\}}$:

$$\{e_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}})\} = \arg \max_{a_{\{1,\ldots,k\}} \in A_{\{1,\ldots,k\}}} f^k_{\{1,\ldots,k\}}(a_{\{1,\ldots,k\}}, a_{-\{1,\ldots,k\}}). \tag{3}$$

Recall that, for any partition $\mathcal{T}$ of N, $\mathbf{g}^N$ has a partition $\mathcal{T}$ pseudo-potential if and only if, for each member T of $\mathcal{T}$, for any $a_{-T} \in A_{-T}$, the restricted game $\mathbf{g}^N|_{a_{-T}}$ by a_{-T} is a pseudo-potential game (Definition 4.1). For any partition $\mathcal{T}$ of N, recall that $(f_T)_{T\in\mathcal{T}}$ is a partition $\mathcal{T}$ pseudo-potential of $\mathbf{g}^N$ if and only if, for each member T of $\mathcal{T}$, for any $a_{-T} \in A_{-T}$, $f_T(\cdot, a_{-T})$ is a pseudo-potential of the restricted game $\mathbf{g}^N|_{a_{-T}}$ by a_{-T}. Thus, $(f^k_{\{1,\ldots,k\}}, g_{k+1}, \ldots, g_n)$ is a $\{\{1,\ldots,k\},\{k+1\},\ldots,\{n\}\}$-pseudo-potential of $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, g_{k+1}, \ldots, g_n)$. Hence, $f^k_{\{1,\ldots,k\}}$ satisfies Condition 1.

Let $b_{\{1,\ldots,k\}} : A_{-\{1,\ldots,k\}} \to A_{\{1,\ldots,k\}}$ be the function defined by $b_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}}) := e_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}})$ for any $a_{-\{1,\ldots,k\}} \in A_{-\{1,\ldots,k\}}$. Then, $b_{\{1,\ldots,k\}}$ satisfies Condition 2, since (3) holds and $e_T(a_{-\{1,\ldots,k\}})$ is non-decreasing with $a_{-\{1,\ldots,k\}}$. Thus, $\mathbf{g}^N$ is a nested pseudo-potential game. ■

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