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Abstract

In this paper we propose benchmark values for the coefficients of relative risk aversion and relative prudence on the basis of a binary choice model where the decision maker chooses between aggregating or disaggregating multiplicative risks. We relate our results to the decision maker's willingness to trade-off the second with the first and the third (central) moment of his wealth distribution.

Keywords: relative risk aversion, relative prudence.

JEL Classification: D81

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1 Introduction

Since the early 70's, it has been known that unity is a benchmark value for the coefficient of relative risk aversion (RRA for short; defined as $-\frac{u''(X)}{u'(X)}X$ where u is the vNM utility function). Indeed, in papers dealing with portfolio and/or saving decisions such as Hahn (1970) and Rothschild - Stiglitz (1971), results are shown to depend upon a comparison between unity and the RRA coefficient. Later papers (e.g., Fishburn - Porter (1976), Cheng - Magill - Shäfer (1987) and Hadar - Seo (1990) to name just a few) have confirmed the early results and the reader can refer to Meyer - Meyer (2005) for a recent survey.

Since the concept of prudence is more recent than that of risk aversion, the notion of relative prudence (RP for short; defined as $-\frac{u'''(X)}{u''(X)}X$) is much less discussed than that of relative risk aversion. However, the scant literature that exists suggests that the benchmark value for RP is 2 (see, e.g., Hadar - Seo (1990) and Choi - Kim - Snow (2001)). In these papers, it appears indeed that a second order dominant shift in the return of a risky asset increases its demand if RP is lower than 2. These results are summarized in Gollier (2001).

Quite interestingly, in all these models the benchmark values for RRA and RP are the consequences of an optimizing behavior and they do not result from an analysis of individual preferences[1].

The purpose of this note is precisely to justify the two benchmark values without reference to any specific choice problem. We show indeed that the two benchmark values can be obtained from a preference for disaggregating harms in a model of "risk apportionment". This idea was recently used by Eeckhoudt - Schlesinger (2006) to give an interpretation of the signs of successive derivatives of a utility function in a context of additive risks. By considering instead multiplicative risks, we obtain the benchmark values for RRA and RP as a consequence of a preference for "harm disaggregation".

Our note is organized as follows. In the second section, we illustrate the concept of risk apportionment applied to multiplicative risks. The next section contains the main result related to the benchmark values for RRA and RP. In the fourth section, we give an interpretation of the results in terms of attitudes toward moments of a distribution. We then briefly conclude.

2 Multiplicative risks

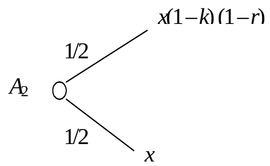
In 2006, Eeckhoudt and Schlesinger have used the principle of "preference for pain disaggregation" and they have applied it to additive risks to justify the alternating sign of successive derivatives of a utility function. In this way they

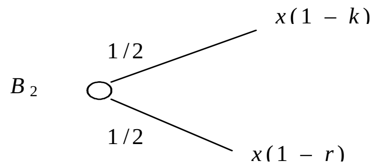
[1] The only exception that we are aware of is a paper by Choi - Menezes (1983) who discuss the value of RRA from the concept of a "probability premium".

could reinterpret the concepts of prudence and temperance outside the decision models, in which these concepts had been defined, e.g., by Kimball (1990, 1992).

Loosely speaking, pain disaggregation means that faced with two equally likely states of nature, a decision maker always prefers to receive one of two harms for certain (i.e., one in each state) as opposed to either facing the two harms jointly or facing none of them.

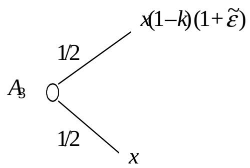
To apply the idea to multiplicative lotteries, consider an individual with initial wealth x of which a share k $(0 < k < 1)$ may be lost with probability equal to $\frac{1}{2}$. If there is also a probability of $\frac{1}{2}$ to lose a share r $(0 < r < 1)$, the principle of pain disaggregation means that lottery B_2 is preferred to lottery A_2 with

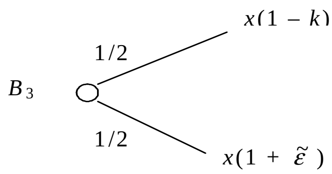




Lottery B_2 is preferred to lottery A_2 since indeed the two harms are disaggregated in B_2: they do not occur jointly as in A_2.

Instead of two sure losses, consider now a situation with one sure multiplicative loss $(-k)$ and a zero-mean random (multiplicative) return, $\widetilde{\varepsilon} \in [-1, \infty)$, which is a harm for a risk averse decision maker. Preference for pain disaggregation states that B_3 should be preferred to A_3 with





In the next section, we show that in the expected utility model (EU), such preferences induce the benchmark values of respectively RRA and RP.

3 Benchmark values

We now consider a decision maker with utility function $u(\cdot)$ who obeys the EU axioms and is risk averse $(u'' < 0)$ and prudent $(u''' > 0)$. We propose to characterize an individual who prefers lottery B_2 to lottery A_2 and lottery B_3 to lottery A_3 as defined in the previous section.

Consider first a decision maker faced with the two lotteries A_2 and B_2. Then we have that for any pair $(k, r) \in (0,1)^2$,

$$\begin{array}{rcl} B_2 & \succ & A_2 \\ & \Updownarrow & \\ \frac{1}{2}u\left[x\left(1-k\right)\right]+\frac{1}{2}u\left[x\left(1-r\right)\right] & > & \frac{1}{2}u\left[x\left(1-k\right)\left(1-r\right)\right]+\frac{1}{2}u\left[x\right] \\ & \Updownarrow & \\ u\left[x\left(1-k\right)\right]-u\left[x\right] & > & u\left[x\left(1-k\right)\left(1-r\right)\right]-u\left[x\left(1-r\right)\right]. \end{array}$$

To examine the RRA coefficient, we proceed in two steps. First, we define a function v such that $v\left(r,k;x\right) \overset{\text{def}}{=} u\left[x\left(1-k\right)\left(1-r\right)\right]-u\left[x\left(1-r\right)\right]$, and obtain that $B_2 \succ A_2$ iff $v\left(0,k;x\right) > v\left(r,k;x\right)$ for all $(k, r) \in (0,1)^2$. A necessary and sufficient condition for $B_2 \succ A_2$ is thus that

$$\begin{array}{rcl} v_r\left(r,k,x\right) & < & 0 \\ & \Updownarrow & \\ -x\left(1-k\right)u'\left[x\left(1-k\right)\left(1-r\right)\right]+xu'\left[x\left(1-r\right)\right] & < & 0 \\ & \Updownarrow & \\ u'\left[x\left(1-r\right)\right] & < & \left(1-k\right)u'\left[x\left(1-k\right)\left(1-r\right)\right]. \end{array}$$

Now define function w such that $w\left(r,k,x\right) \overset{\text{def}}{=} \left(1-k\right)u'\left[x\left(1-k\right)\left(1-r\right)\right]$. A necessary and sufficient condition for $B_2 \succ A_2$ (all $(k, r) \in (0,1)^2$), is then that $w\left(r,k,x\right)$ is an increasing function in k, that is

$$\begin{array}{c} w_k\left(r,k,x\right) > 0 \text{ (all } (k,r) \in (0,1)^2\text{)} \\ \Updownarrow \\ -u'\left[x\left(1-k\right)\left(1-r\right)\right] \\ -x\left(1-r\right)\left(1-k\right)u''\left[x\left(1-k\right)\left(1-r\right)\right] > 0 \text{ (all } (k,r) \in (0,1)^2\text{)} \\ \Updownarrow \\ 1+X\frac{u''\left[X\right]}{u'\left[X\right]} < 0 \text{ (all } X > 0\text{)}. \end{array}$$

We therefore obtain that lottery B_2 is preferred to lottery A_2 iff RRA exceeds 1. Notice at this stage the difference with additive risks. For additive risks, as shown in Eeckhoudt - Schlesinger (2006), concavity of the utility function (i.e., risk aversion) is sufficient to justify a preference for disaggregating additive sure pains. Our analysis shows that for multiplicative pains, matters are less simple: for a decision maker to accept the disaggregation of sure multiplicative pains, his degree of risk aversion must be "high" enough ($RRA > 1$). On the other hand, Chiu and Madden (2007) obtain that some criminal activities are less desirable when risk increases if individual admits a $RRA < 1$.

Consider now the possibility of a zero-mean risk of return loss. Lottery B_3 is preferred to lottery A_3 iff for all zero-mean random variables, $\widetilde{\varepsilon} \in [-1, \infty)$, and all k in $(0,1)$,

$$\begin{array}{rcl}\frac{1}{2}u\left[x\left(1-k\right)\right]+\frac{1}{2}Eu\left[x\left(1+\widetilde{\varepsilon}\right)\right] & > & \frac{1}{2}Eu\left[x\left(1-k\right)\left(1+\widetilde{\varepsilon}\right)\right]+\frac{1}{2}u\left[x\right] \\ & \Updownarrow & \\ Eu\left[x\left(1+\widetilde{\varepsilon}\right)\right]-u\left[x\right] & > & Eu\left[x\left(1-k\right)\left(1+\widetilde{\varepsilon}\right)\right]-u\left[x\left(1-k\right)\right].\end{array}$$

In a similar way as previously, we define function v such that $v\left(k,x\right)\overset{\text{def}}{=}Eu\left[x\left(1-k\right)\left(1+\widetilde{\varepsilon}\right)\right]-u\left[x\left(1-k\right)\right]$ and obtain that $B_3\succ A_3$ (for all $k\in(0,1)$) iff $v\left(0,x\right)>v\left(k,x\right)$ (all $k\in(0,1)$). A necessary and sufficient condition for $B_3\succ A_3$ is then that

$$\begin{gathered}v_k\left(k\right)<0\\ \Updownarrow\\ -xE\left(1+\widetilde{\varepsilon}\right)u'\left[x\left(1-k\right)\left(1+\widetilde{\varepsilon}\right)\right]+xu'\left[x\left(1-k\right)\right]<0\\ \Updownarrow\\ u'\left[x\left(1-k\right)\right]<E\left(1+\widetilde{\varepsilon}\right)u'\left[x\left(1-k\right)\left(1+\widetilde{\varepsilon}\right)\right].\end{gathered}$$

In a second step, we define function w such that $w\left(k,\varepsilon,x\right)\overset{\text{def}}{=}\left(1+\varepsilon\right)u'\left[x\left(1-k\right)\left(1+\varepsilon\right)\right].$ Remembering that $E\widetilde{\varepsilon}=0$ and $var\widetilde{\varepsilon}>0$, we can write that $B_3\succ A_3$ (for all $k\in(0,1)$) iff $Ew\left(k,\widetilde{\varepsilon},x\right)>w\left(k,E\widetilde{\varepsilon},x\right)$. This condition is satisfied if function w is strictly convex for all $\varepsilon\in[-1,\infty)$, that is

$$\begin{gathered}w_{\varepsilon\varepsilon}(k,\varepsilon,x)>0\text{ (for all }k\in(0,1),\varepsilon\in[-1,\infty))\\ \Updownarrow\\ 2x\left(1-k\right)u''\left[x\left(1-k\right)\left(1+\varepsilon\right)\right]\\ +x^2\left(1-k\right)^2\left(1+\varepsilon\right)u'''\left[x\left(1-k\right)\left(1+\varepsilon\right)\right]>0\text{ (for all }k\in(0,1),\varepsilon\in[-1,\infty))\\ \Updownarrow\\ 2+X\frac{u'''(X)}{u''(X)}<0\text{ (for all }X>0).\end{gathered}$$

A comment similar to the one made for risk aversion applies here. When the sure loss and the zero mean risk are additive, positive prudence ($u'''>0$) implies a preference for pain disaggregation. However, in the multiplicative case the condition is more demanding: the preference for pain disaggregation requires that prudence be strong enough ($RP>2$).

4 Moments

In this section, we provide an interpretation for RRA and RP as measures for the decision maker's willingness to trade-off different moments of her wealth distribution, and we show that the benchmark values obtained in the previous section are limiting trade-offs.

Consider the first two moments of lotteries B_2 and A_2. Easy computations reveal that

$$\begin{aligned} EX_{B_2} &= x[1-\frac{1}{2}k-\frac{1}{2}r] && (1)\\ EX_{B_2}-EX_{A_2} &= -\frac{1}{2}krx<0, \text{ and} && (2)\\ varX_{B_2}-varX_{A_2} &= -kr[1-\frac{1}{2}(k+r)+\frac{1}{4}kr]x^2<0. && (3)\end{aligned}$$

Taking a second order Taylor expansion of u around the mean outcome under lottery B_2, and taking expectations, we get

$$\begin{aligned} Eu(X_{B_2}) &\simeq u(EX_{B_2})+\frac{1}{2}u''(EX_{B_2})varX_{B_2}, \text{ and}\\ Eu(X_{A_2}) &\simeq u(EX_{B_2})+u'(EX_{B_2})(EX_{A_2}-EX_{B_2})\\ &\quad +\frac{1}{2}u''(EX_{B_2})[varX_{A_2}+(EX_{B_2}-EX_{A_2})^2].\end{aligned}$$

Therefore,

$$\begin{aligned} Eu(X_{B_2}) &\gtrless Eu(X_{A_2})\\ &\Updownarrow\\ u''(EX_{B_2})\left[varX_{B_2}-varX_{A_2}-(EX_{B_2}-EX_{A_2})^2\right] &\gtrless u'(EX_{B_2})(EX_{A_2}-EX_{B_2}).\end{aligned}$$

Using expressions (2) and (3), this comparison results in

$$\begin{aligned} Eu(X_{B_2}) &\gtrless Eu(X_{A_2})\\ &\Updownarrow\\ -\frac{u''(EX_{B_2})}{u'(EX_{B_2})}\left[EX_{B_2}+\frac{1}{2}krx\right] &\gtrless 1,\end{aligned}$$

so that for small values of k and r the critical value for RRA approaches 1.

When RRA equals 1, the lower variance of lottery B_2 exactly compensates for its lower expectation and the decision maker is indifferent between B_2 and A_2. Should RRA exceed 1, lottery B_2 would be preferred. Indeed when RRA exceeds 1, risk aversion is pretty strong so that the lower variance of B_2 becomes relatively more attractive.

Turning now to lotteries B_3 and A_3 we notice they have the same mean, but they differ in their variance and third central moment (denoted μ_3):

$$\begin{aligned} EX_{B_3} &= EX_{A_3}=x[1-\frac{1}{2}k] && (4)\\ varX_{A_3}-varX_{B_3} &= -\frac{1}{2}x^2[1-(1-k)^2]E\varepsilon^2<0 && (5)\\ \mu_3X_{A_3}-\mu_3X_{B_3} &= -\frac{1}{2}x^3[1-(1-k)^3]E\widetilde{\varepsilon}^3-\frac{3}{4}x^3k[1+(1-k)^2]E\varepsilon^2. && (6)\end{aligned}$$

Because lotteries B_3 and A_3 have the same mean, the decision maker now faces a trade-off between the second and third moments. Taking expectations of the third order Taylor expansions of $u(X)$ around the (common) mean gives

$$\begin{aligned} Eu(X_{A_3}) &\simeq u(EX) + \frac{1}{2}u''(EX)varX_{A_3} + \frac{1}{6}u'''(EX)\mu_3 X_{A_3} \\ Eu(X_{B_3}) &\simeq u(EX) + \frac{1}{2}u''(EX)varX_{B_3} + \frac{1}{6}u'''(EX)\mu_3 X_{B_3}, \end{aligned}$$

so that

$$\begin{aligned} Eu(X_{B_3}) &\gtreqless Eu(X_{A_3}) \\ &\Updownarrow \\ -\frac{u'''(EX)EX}{u''(EX)} &\gtreqless 3\frac{varX_{B_3} - varX_{A_3}}{\mu_3 X_{B_3} - \mu_3 X_{A_3}} EX. \end{aligned} \tag{7}$$

Absolute prudence is thus trice the willingness to substitute the second for the third central moment. Under the assumption that $E\widetilde{\varepsilon} = 0$, making use of (4)-(6), the *rhs* of (7) reduces to

$$\frac{[1-(1-k)^2](2-k)}{k[1+(1-k)^2]},$$

with the limiting property

$$\lim_{k \to 0} \frac{[1-(1-k)^2](2-k)}{k[1+(1-k)^2]} = 2.$$

Thus a decision maker becomes exactly indifferent between small gambles of type B_3 and A_3 when his RP equals 2. Then the increase in variance in B_3 is exactly compensated for by the increase in the third central moment[2].

5 Conclusion

The existing literature on savings and portfolio choices under risk has revealed that quite often decisions taken by individuals depend upon the values of their coefficients of relative risk aversion and relative prudence. More specifically the benchmark values of RRA and RP, taken into consideration inside these models of choice, are respectively 1 and 2.

In this note, we have given a more fundamental interpretation of these benchmark values which is related to the individual's preferences and not to a specific problem of choice. This result has been obtained by applying to multiplicative risks the notion of risk apportionment that was used for additive risks in order to justify the sign of successive derivatives of a utility function.

[2] For a related discussion around the measure of absolute prudence, see Chiu (2005).

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