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Abstract

This paper develops an endogenous growth model based on the idea of new combinations of input factors as a growth mechanism. The model integrates the idea of several technologies used simultaneously in producing final output. Innovations are of the horizontal and vertical type and in addition of the type of new technologies which can be combined with existing ones. All types of innovations are endogenous and the occurrence of a new technology has stochastic elements as well. This leads to endogenous dynamics in the growth rates of final output production.

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1 Introduction

In an empirical contribution Levin, Klevorick, Nelson and Winter (1995) made the observation that R&D expenditures across manufacturing sectors in the U.S. are very heterogenous. The argument put forward explaining these empirical facts relies on the heterogeneity in technological opportunity across these sectors. These opportunities might be bounded from above so that investments in R&D might not pay off equally in all sectors. However new opportunities can show up if new technologies are developed and can be combined with existing technologies used by existing sectors. This is seen as one engine of growth by Levin, Klevorick, Nelson and Winter (1995).

The role of integration of new production factors into the production technology for goods was also stressed by Schumpeter (1912). From his point of view economic development through new combinations was defined by the following five characteristics:

- Production of new goods or new qualities of existing goods,
- introduction of new production technologies,
- opening of a new market,
- introduction of new factors into the production process,
- change in the organization of markets.

Clearly, the first two characteristics are able to produce exponential growth in theoretical models. These arguments have been heavily used by e.g. Romer (1990), Grossman and Helpman (1991) or Aghion and Howitt (1992). However the forth argument for economic development has not been used extensively in modern growth models. This paper will stress the role of new production factors in the process of generating growth dynamics. The mechanism is based on the idea that new technologies, providing new input factors for final good production, can be combined with existing production factors in order to yield technological progress.

The idea of utilitizing new combinations in growth models has also been used by Weitzman (1998). In his "Recombinant Growth Model", parvise new combinations of existing knowledge leads to growth that can be exponential. The aim of Weitzman's (1998) article was to open up the black box of the generation process for new technologies. However, the model presented there is very much based on combinatory mathematics and is not so much nested in the usual models of endogenous economic growth. This might make it difficult to use the model in a mainstream economic analysis as a building stone to work on economic problems beyond economic growth.

The first aim of this paper is therefore to use the idea of new production factors as a mechanism for economic development and a source of growth in an endogenous growth model which is designed in accordance with modern growth models. By this, it is attempted to open up the black box of how new technologies can lead to ongoing gains in total factor productivity and hence economic growth. The model will use standard assumptions about horizontal and vertical innovations and will add the introduction of new technologies.

The second aim is to elaborate on an issue that is partially discussed in Levin, Klevorick, Nelson and Winter (1995) and is not so much recognized by the new growth theory. It is the possibility that innovative behavior of firms becomes harder and harder the more developed an existing technology is. This might be true for the horizontal and vertical dimension of innovation. It might be the case for one sector of the economy which is already well developed, with many different varieties of goods existing next to each other, it is harder to come up with yet a new variant. But a similar argument might be true also within a particular variant of one sector. The more technologically or qualitatively advanced this variant is, the harder it becomes to design a new generation with an increased quality of that variant. Thus new technologies might have a higher growth potential than older ones, regarding this dimension of innovation.

(Endogenous) growth models of the first and second generation generally do not take account of these "fishing out" effects. In general, long run exponential growth is produced in these models by making use of an ad hoc "standing on shoulders of giants" argument. This is true for the horizontal as well as the vertical dimension of innovation. Past progress in the degree of horizontal differentiation of input factors or past progress in the quality level of goods improve the current opportunities of R&D. This is the fundamental assumption guaranteeing economic growth. One exception is, of course, the model in Jones (1995), where growth is generated by an increasing degree of horizontal differentiation and the "standing on shoulders of giants" effect might also be negative, i.e. a "fishing out" effect exists and it becomes more difficult to invent new horizontal differentiated variants of goods as more already exist. For long run growth it is then however necessary, that the resources devoted to R&D must grow exponentially; this works as a replacement for the "standing on shoulders of giants" assumption.

There is thus some lack of a justification of the "standing on shoulders of giants" argument in the existing growth literature regarding models in the new growth theory. This paper aims to fill this gap by explicitly taking account of a "fishing out" effect in the horizontal dimension of innovation and by modelling the way the "shoulders of giants" are created for the vertical dimension of innovation. At the same time, the paper offers an analytically answer of how new technologies affect growth. The mechanism behind this is partially the possibility of forming new combinations of new technologies with existing ones. And second, if there are many technologies present in the production technology of final goods, "fishing out" effects in some technologies are not that problematic as new technologies with a high growth potential can compensate for this. In the Schumpetrian sense it is the introduction of new production factors in the process of producing final goods that opens up new channels of growth.

The model developed below in the paper has some aspects in common with the literature on general purpose technologies (see e.g. Helpman 1998). In this literature, growth is generated as a consequence of major technology breakthroughs which are followed up by less drastic technology changes yielding progress in productivity. A model somewhat related to the present model is that of Aghion and Howitt (1998).

This model extents the work in Aghion and Howitt (1992) by a second innovation stage capturing the idea of invention of new general purpose technologies. This model as well as others in Helpman (1998) are developed to study the effects of the introduction of new general purpose technologies on e.g. productivity in the short and in the long run. The present paper has something to add to this literature as it captures endogenous introduction of new technologies, i.e. general purpose technologies, as a rare event as well as endogenous horizontal and vertical innovations as a common event. Therefore the dynamics of the growth rate of the economy can be studied.

The paper is organized as follows. Section two gives the basic idea behind the model by presenting an illustrative easy example of the modelling strategy to be developed fully in the later sections. Section three presents the fully formulated model and formalizes implications for the development of total factor productivity and economic growth. Finally section four discusses the results and concludes.

2 The Basic Idea

Romer (1987) introduced the following production function which was subsequently used in many (semi-)endogenous growth models. In a discrete formalization this production technology is given by

$$Y = L^{\alpha} \sum_{i=1}^{N} x_i^{1-\alpha}. \tag{1}$$

One direct interpretation of this production technology is that a set of N horizontally differentiated intermediate input factors of quantity x_i can be used in combination with labor, L, to yield final output Y. Since all workers in this setup can simultaneously make use of the N available variants of input factors, growth can be generated by growth in N. However, for growth in N to be exponentially, a "standing on shoulders of giants" argument has to be employed and no fishing out effects can be present.

Now imagine a switch from production with technology (1) with one set of horizontally differentiated input factors to a technology with two sets of differentiated input factors. These two sets of input factors can be thought of being two different technology sectors or two different basic or general purpose technologies. One production technique using these two technology sectors could be of the following form

$$Y = L^{\alpha}\left(\sum_{i=1}^{N_1} x_i^{\alpha_1}\right)\left(\sum_{i=1}^{N_2} z_i^{\alpha_2}\right) \quad \text{with} \quad \alpha_1 + \alpha_2 = 1 - \alpha. \tag{2}$$

Simplifying the argument even further, suppose there were at the beginning $N = 2$ differentiated intermediate input factors in the production technology (1) and production switched to the technology (2) with sets $N_1 = N_2 = 2$ input factors. First, there were two input factors that could be combined with labor, but then after the switch there are four combinations of input factors that can be combined with labor to yield final output. This is the basic mechanism behind the fully developed model in the sections to come.

What this exercise clarify, is that, first, the degree of horizontal differentiation in one technology sector can be fixed and growth can still take place as long as new technology sectors are developed. Second, vertical innovations are also not necessary to create growth. In fact the productivity of intermediate input factors in final goods production was normalized to one in the above example. Thus, this is an extreme example of fishing out effects in both the horizontal as well as the vertical dimension of innovation. Both types of innovation were bounded from above in the example and still growth is possible by forming new combinations of new technology sector inputs with existing ones.

In the more elaborate model below, the degree of horizontal differentiation and vertical dimension of innovation are allowed to be determined endogenously by profit seeking technology firms.

3 The Model

This section deals with the fully developed model and the basic assumptions will be stated one by one in the following.

3.1 Consumers

It is assumed that the economy allows for a two period overlapping generations model where agents work only in their first period of life where they inelastically supply one unit of labor. Savings in period τ build the capital stock of the economy in period $\tau + 1$. As will be shown later on, the economic environment is governed by uncertainty. Agents are assumed to maximize expected life time utility given by

$$U_\tau = E_\tau \left[\ln c_\tau + \frac{1}{1+\rho} \ln c_{\tau+1} \right]. \tag{3}$$

where c_τ is real consumption expenditure in period τ and ρ is the rate of time preference. $E_\tau[\cdot]$ is the expectation operator conditioned on information available in period τ. The timing of events is as follows. At the beginning of the time period the capital stock is given by past savings. Total factor productivity which will be defined later is then revealed and with this knowledge firms decide on employment and households on savings. Time is assumed to be discrete and all figures correspond to the current time period τ if not stated differently.

Allowing for only two living periods can be justified in this context because the model below deals with the influence of major technology changes, which in general do not occur frequently. Therefore the time periods are interpreted in terms of generations.

Maximization takes place subject to the intertemporal budget constraint for the representative agent

$$c_{\tau+1} = (1 + r_{\tau+1})(w_\tau - c_\tau),$$

where $r_{\tau+1}$ is the net interest rate and w_τ is the real wage rate.

This leads directly to the optimality condition that individual savings s_τ are given by

$$s_\tau = \frac{1}{2+\rho} w_\tau \tag{4}$$

Due to the log preferences uncertainty seems to has disappeared in this optimality condition. This is of course due to the independence of the propensity to save from the interest rate. Further, it is assumed that each representative agent supplies inelastically one unit of labor in each period of time. The total population which forms the labor force is assumed to be stationary although nothing in the model would change if population grows over time. Aggregate individual assets form the capital stock of the economy in the next period, i.e. $K_{\tau+1} = \frac{1}{2+\rho} w_\tau L$. L denotes the total working age population in the economy.

3.2 The Production Technology

The production technology of the model is assumed to be of a more general form than in the introductory example in the preceding section. It takes the form

$$Y = L^\alpha \prod_{j=1}^{M} \sum_{i=1}^{N_j} \left(\frac{\lambda_{j,i} x_{j,i}}{\alpha_j} \right)^{\alpha_j} \quad \text{with} \quad \sum_{j=1}^{M} \alpha_j = 1 - \alpha, \tag{5}$$

where M is the number of technology sectors or general purpose technologies which can be used in combination with labor L. N_j is the degree of horizontal differentiation of the jth technology sector. $\lambda_{j,i}$ denotes the quality level of the ith intermediate input factor in technology j and is thus capturing the vertical innovation dimension. In addition to the very simplified model in the preceding section, vertical innovations are explicitly modelled here by changes in $\lambda_{j,i}$. Finally, $x_{j,i}$ is the quantity of the ith intermediate input factor used in sector j. Due to the assumption that $\sum \alpha_j = 1 - \alpha$ it is clear that this production function has constant returns to scale in labor and intermediate input factors. The occurrence of the output elasticities in the denominators of the above expression simplifies the following analysis without affecting the general result of the model. Where it is appropriate, comments on the implications of this simplifying assumption will be given.

3.3 Horizontal and Vertical Innovations

This section deals with the innovating behavior in established well known technology sectors. This innovating behavior is driven by horizontal and vertical innovations. It will be shown under what conditions these may lead to long run sustainable growth.

The growth mechanism clearly is driven by the intermediate input factors. It is assumed that each variant of these factors, regardless in which technology sector it is used, is produced from one particular firm. As it will be assumed that fixed costs in every period of time are involved in producing these factors, there is only room for one producer which supplies the whole market with its particular variant. Competitors will rather develop a new variant than compete within a market for an existing variant in order to maximize profits. There are two technologies available to produce the differentiated input factors. One is a monopolistic technology with fixed costs and one is a competitive with constant marginal costs. It is assumed that the monopolistic technology involves an up-front fixed cost in order to develop the particular variant and constant marginal costs thereafter. The competitive technology can be used to copy one particular variant. Fixed costs need not be incurred with this technology, but marginal costs are higher by a factor $\gamma > 1$ than in the case of the monopolistic technology.

The original developer of one particular variant of intermediate input factors has to invest a fixed cost in every period of time in order to be able to produce this variant. This is an application of the idea in Young (1998), but here the problem is simplified to a more static environment where investment and production takes place within the same period of time[1]. The fixed cost are in terms of the final good of the economy, Y, and are given by the following real cost function

$$F_{j,i} = \begin{cases} \eta e^{\mu \lambda_{j,i}/\bar{\lambda}_{\tau-1}} \alpha_j Y & \text{if } \lambda_{j,i} \geq \bar{\lambda}_{\tau-1}, \\ \eta e^{\mu} \alpha_j Y_\tau & \text{otherwise,} \end{cases} \tag{6}$$

There are two cases for the fixed costs, one in which it is optimal for the producer to increase the quality level of its particular variant, and one where increasing quality does not pay and hence quality is not further improved.

Looking first at the case where quality improvement is desired. As can be seen from the function (6) the fixed costs depend first on the desired quality level of the intermediate input factor and, second, on the size of the market for which the

[1]In Young (1998) the production process is divided into a two period problem. In period one the producer has to incur the fixed cost, production then takes place in period two.

particular variant is directed. The higher the desired quality level, the higher are the fixed costs exactly as in Young (1998). There is a "standing on shoulders of giants" effect through the dependence on a quality index $\bar{\lambda}_{\tau-1}$ determined in the preceding period. The higher this quality index in the preceding period is, the lower are the fixed costs for a given new quality level. This quality index represents the state of the art in intermediate input factor quality and is determined by intermediate input factor producers active in R&D for enhancing quality. It is defined by a geometric mean according to[2]

$$\ln \bar{\lambda}_\tau = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \frac{h(\lambda_{j,i} - \bar{\lambda}_{\tau-1})}{\sum_{j=1}^{M} \sum_{i=1}^{N_j} h(\lambda_{j,i} - \bar{\lambda}_{\tau-1})} \ln \lambda_{j,i}, \tag{7}$$

where $h(z) = 0$ for $z = 0$ and $h(z) = 1$ for $z > 0$. Hence, only qualities enter this geometric mean for firms which set their quality level above the index of the preceding period. The economic intuition behind this assumption is the idea that quality improvements in period τ become common knowledge in period $\tau+1$. Firms who did not improve quality above the index in period τ do not contribute to this common knowledge. This way of modelling the "standing on shoulders of giants" effect has the advantage that it can be traced back where in the economy it originates. Therefore, this a way of partly giving a meaningful foundation to this effect.

It is assumed that the number of firms and sectors is large enough so that no single producer of intermediate input factors has a significant influence on the quality index. Therefore each producer neglects its influence on $\bar{\lambda}$ when choosing its $\lambda_{j,i}$. This has the effect that all intertemporal aspects are removed from the optimization problem of the individual firm.

The second argument of the function (6) involves the size of the technology sector for which the particular variant is designed for. This size is given by $\alpha_j Y$ which is the part of sector j in final output Y of the economy. The larger one sector, the higher are the fixed costs for setting up a variant of the intermediate input factors for that sector. This assumption should reflect a "fishing out" effect implying that

[2] The assumption of that particular mean is not critical for the major results to be developed below. If this assumption is changed it will only affect the result (22) below.

it is harder to develop a product for a market that is already highly developed.

If the market environment is such that it is not beneficiary to improve the quality level above the average level, there is still a need for fixed costs. These fixed costs are justified by the plausible assumption that even if product quality is kept fixed relative to the average, there is still a necessity to keep the product technologically up to date or compatible with other developing input factors on the market. The fixed costs are then only proportionate to the size of the technology sector for which the input factor is developed for.

After the fixed costs for research and development have been incurred in order to raise product quality, the firm can produce its particular variant at constant marginal costs. The production technology is assumed to be linear with unit productivity in the single input factor which is capital. The capital has to be rented from the household sector at the gross interest rate $r + \delta$, where δ is the rate of depreciation. The competitive production technology does not require fixed costs but has a capital productivity of γ^{-1}.

What still needs to be determined is when and to what extent quality improvements above the index are desirable for a firm producing a particular variant of the intermediate input factors. As is indicated by the assumptions above, the firm is assumed to set a limit price by the factor γ over marginal costs. The price χ_{ji} for variant i in sector j is thus given by $\gamma(r + \delta)$. Hence prices for all variants are equal, $\chi_{ji} = \chi$. The demand function for variant i in sector j is determined by its marginal product and is given by

$$x_{j,i} = \alpha_j \chi^{-\frac{1}{1-\alpha_j}} \lambda_{j,i}^{\frac{\alpha_j}{1-\alpha_j}} L^{\frac{\alpha}{1-\alpha_j}} \prod_{k \neq j} \left[\sum_{h=1}^{N_k} \left(\frac{\lambda_{k,h} x_{k,h}}{\alpha_k} \right)^{\alpha_k} \right]^{\frac{1}{1-\alpha_j}} . \tag{8}$$

It is important to see the dependence of demand on the quality level through the term $\lambda_{j,i}^{\frac{\alpha_j}{1-\alpha_j}}$. The demand increases with product quality in a way that is determined by the importance of the particular sector j in final goods production measured by the output elasticity α_j. The greater this elasticity, the higher is the effect of product quality on demand. This is what will determine the results below.

Since prices χ are already determined by the limit pricing rule, the only variable which is left for maximizing profits is the quality level $\lambda_{j,i}$. Intermediate input factor producers set this quality level in such a way that it maximizes net profits given by

$$\pi_{j,i} = (\gamma - 1)(r + \delta)x_{j,i} - F_{j,i}, \tag{9}$$

where $x_{j,i}$ and $F_{j,i}$ are given by equations (7) and (6) above.
Maximization of (8) with respect to $\lambda_{j,i}$ gives the optimality conditions

$$\lambda_{j,i\tau} = \begin{cases} \frac{1}{\mu}\frac{\alpha_j}{1-\alpha_j}\bar{\lambda}_{\tau-1} & \text{if} \quad \frac{1}{\mu}\frac{\alpha_j}{1-\alpha_j} > 1, \\ \bar{\lambda}_{\tau-1} & \text{else} \end{cases} \tag{10}$$

which are quite analogous to Young (1998). This result already incorporates the assumption that input factor producers in each technology sector enter the market as long net profits given by (8) are positive. In equilibrium these profits are zero. This result directly shows the influence of the output elasticity α_j on the development of the quality level. If one sector j is important, quality grows rapidly, but if the other case prevails, quality behaves like the average past quality level. A high output elasticity works as an incentive to invest in quality of the variants in a particular technology sector j. Therefore, important sectors are contributing much to the development of quality and thereby creating a positive spillover effect onto less important technologies.
With the result for quality growth in intermediate input factors in hands, the fixed costs for each input factor producer are determined. This in turn determines the number of input factor producers in each technology sector who compete monopolistically. This number drives net profits down to zero when no incentive for further market entry exists and is given by[3]

$$N_j = \frac{\gamma - 1}{\gamma}\frac{1}{\eta e^{\mu\frac{\lambda_{j,i,\tau}}{\bar{\lambda}_{\tau-1}}}}, \tag{11}$$

[3]Strictly speaking, N_j is given by the integer part of the right hand side of equation (10) due to the discrete structure of the model. Then, there would be some net profits left for intermediate input factor producers. Assuming that these firms are equally owned by consumers and that these profits are equally distributed among them, restores equilibrium. Additionally, if N_j is large, there is not a significant difference between the right hand side of (10) and its integer part. To avoid heavy notation in the following, all calculations are based on the exact result in (10).

where the ratio $\frac{\lambda_{j,i,\tau}}{\lambda_{\tau-1}}$ is given by (9). Two forces work in order to limit the number of input factor producers in each sector. First, if the output elasticity is large there are large incentives to invest in the quality level which drives up the fixed costs. Second, if a technology sector is large, i.e. a high α_j, the fixed costs component which adjusts for sector size is large as well, also driving up fixed costs. In the decentralized optimum the number of firms in the market is given by the above figure.

As can be seen from the results so far, the horizontal dimension of innovation, i.e. the number of variants of intermediate input factors per technology sector, can not cause long run growth. For a given growth rate of the quality level of intermediate input factors the number of horizontally differentiated input factors is stationary.

If the number of technology sectors M is constant over time, the only mechanism that creates long run sustainable exponential growth is an increasing level of quality of intermediate input factors. As can be seen from equation (9), which determines the optimal development of the quality level over time, this is possible as long as there is at least one technology sector with an output elasticity which satisfies the condition $\frac{1}{\mu}\frac{\alpha_j}{1-\alpha_j} > 1$. This sector is then responsible for creating growth in the quality levels and creating a positive externality for the remaining technologies. If this is not the case, growth in quality will not take place at all.

On the other hand, if M increases over time, the output elasticities α_j must become smaller on average. This means that, ceteris paribus, the incentive to invest in the quality level of intermediate input factors decreases. At some point the incentive is so small, that the quality level is given by the average level in the preceding period of time, as can be seen from the optimality conditions (9). This might happen for all sectors so that in general the vertical dimension of innovation, i.e. growth in the quality level, does not guarantee long run exponential growth. But the source of long run economic growth in this case can be already seen if the production function

(5) is written in reduced form as[4]

$$Y = (1-\alpha)^{-(1-\alpha)} L^{\alpha} K^{1-\alpha} \prod_{j=1}^{M} \left(N_j^{1-\alpha_j} \bar{\lambda}_j^{\alpha_j} \right). \tag{12}$$

This reduced form can be obtained by using the marginal product condition for one variant of the intermediate input factor in one sector and aggregating over all sectors. Additionally one has to use the capital market resource constrain that $x_{j,i} = \frac{K_j}{N_j}$, with K_j the capital stock used in technology sector j. Equilibrium in the capital market requires that the interest rate is equal for all technology sectors which implies $\frac{K_j}{K_l} = \frac{\alpha_j}{\alpha_l}$ and that $K = \sum K_j$.

Total factor productivity will play an important role in what follows in the next subsections. It is defined as

$$TFP = \prod_{j=1}^{M} \left(N_j^{1-\alpha_j} \bar{\lambda}_j^{\alpha_j} \right). \tag{13}$$

From the reduced from (11) one can easily see that despite a stationary N_j and possibly also a stationary[5] $\bar{\lambda}_j = \frac{1}{N_j} \sum_{i=1}^{N_j} \lambda_{j,i}$ growth is still possible if M, the number of technology sectors, grows linearly. The economic intuition behind this is the formalization of the idea of forming new combinations of input factors of new technologies with existing ones. In the production function for final output this is realized through the multiplicative influence of the degrees of horizontal differentiation N_j. The issue addressed in the next subsection is how new technology sectors are created and get integrated into the production technology for final output.

3.4 The Introduction of New Technologies

The creation of new technology sectors is in general governed by more uncertainty than the introduction of new good in a well known and established industry. Something fundamentally new must created which might not be that controllable as the creation of a new variant of an intermediate input factor in an existing technology sector.

[4]Using the expression $\lambda_{j,i} x_{j,i}$ instead of $\frac{\lambda_{j,i} x_{j,i}}{\alpha_j}$ in the production function (5) would result in an additional multiplicative term $\prod \alpha_j^{\alpha_j}$. Implications of this term are discussed further below.

[5]Due to the symmetry in the model $\lambda_{j,i} = \lambda_{j,k}$ for all i and $k \in \{1, ..., N_j\}$.

The assumptions in this process are as follows. There is an endogenous number of research firms engaged in producing new technology sectors embodied in a single variant of an intermediate input factor. Once one of these firms has discovered this variant of the new technology sector it has for one period the exclusive knowledge of how this new technology is produced. After this period this knowledge diffuses and potential other firms can set up their own variants in that sector. Thus, the case of horizontally differentiated sectors which was analyzed in the preceding section applies only to technology sectors which are old in the sense that the necessary knowledge has already been diffused.

With these assumptions, it is clear that the degree of horizontal differentiation of any new technology is $N_n = 1$[6]. The output elasticity of such a new technology, if integrated into the production technology for final output, is denoted by α_n.

Unfortunately, the process of integrating new technologies can not be analyzed for the general case of unrestricted output elasticities α_j. Since it must be true that $\sum \alpha_j = 1 - \alpha$ before and after the integration of new technology sectors, the unrestricted case would imply unspecified changes in the output elasticities of already established sectors which prohibits clear cut statements about the behavior of the economy. This can be avoided by assuming a clear process for the development of the output elasticities.

Assume that once new general purpose technologies or new technology sectors are discovered, they partially crowd out the existing technology sectors by taking away some of their importance in producing the final good if they are integrated. This is formalized by a depreciation of the existing output elasticities with rate δ_α. This results in a behavior given by $\alpha_{j,\tau+1} = (1-\delta_\alpha)\alpha_{j,\tau}$ when new technology sectors are discovered in period $\tau + 1$ and are integrated. The economic intuition behind this depreciation scheme is that the importance of technologies declines with their age, i.e. a technology sector is most important when it is just discovered and becomes more and more "replaced" by newer technologies created as time passes on. Applying

[6] If there are fixed costs involved in creating different variants, the creator of the new technology will naturally set up only one variant.

this rule to all elasticities, it gives a particular new technology sector an output elasticity of $\delta_\alpha(1-\alpha)/m_+$ to start with, where m_+ is the number of symmetric new technology sectors. Therefore α_n equals $\delta_\alpha(1-\alpha)/m_+$.

As said before, the process of creating a new technology is partially governed by uncertainty. The firm conducting research in this area is nevertheless able to influence the probability of success through investing a quasi-fixed cost which does only depend on the probability of success and the sector size of the new technology if it were to be integrated, but not on the quantity of units produced. As before this front up investment also determines the quality level of the intermediate good in the new technology. After being successful, it depends on the producers of final goods whether they integrate this new technology in the production process of final goods. Furthermore the R&D activities of different firms are treated as stochastically independent. If the owners of these firms are well diversified, it is reasonable to assume that firms engaged in creating new technologies maximize expected profits and hence are risk neutral.

The fixed costs, in terms of the final good of the economy, determining the probability of success and the quality level are given by

$$F_n = \eta e^{\nu \frac{p_n}{1-p_n} + \mu \frac{\lambda_n}{\lambda_\tau - 1}} \alpha_n Y_I, \tag{14}$$

and have to be incurred whether or not the innovation process is successful. As can be seen, these costs increase in the probability of success, p_n, the desired quality level λ_n and the size of the new sector, $\alpha_n Y_I$. $\nu > 0$ is an exogenously given productivity parameter. As before, the argument for including the sector size is the idea that R&D costs increase with the size of the sector. This part of the R&D costs is determined conditional on the integration of the new technology sector, i.e. the sector size that would emerge if the new technology is integrated into the production technology of final goods. Y_I denotes final output in the economy if the new technology sector is integrated. Therefore this term is deterministic for the developer of the new technology.

If the firm succeeds in creating a new technology it can capture the whole market for the differentiated input factor for this new technology. The demand for this variant is then given by

$$x_n = [\gamma(r+\delta)]^{-1}\alpha_n Y_I,$$

since the number of variants in this new sector is restricted to one by assumption. The demand function has this particular simple representation because a constant share of final goods production falls on each sector. The share is given by the output elasticity. The assumptions about the price mark-up over marginal costs are as in the preceding sections.

Profits for the firm are then given by

$$\pi_n = \tilde{I}_n(\gamma-1)(r+\delta)x_n - F_n,$$

where $\tilde{I}_n$ is a random variable which takes on the value 1 with probability $\tilde{p}_n$ and 0 with probability $1-\tilde{p}_n$. $\tilde{p}_n$ is the probability with which the firm succeeds in creating a new technology and is selected by final goods producers to be included in the technology of producing final goods.

The term $\tilde{I}_n$ can be split into two separate independent random variables, $\tilde{I}_n = I_n I_s$. I_n takes on the value 1 with probability p_n if the firm is successful in creating a new technology and 0 with probability $1-p_n$ if not. I_s takes on the value 1 if this new technology is selected to be included into the production technology of final goods. Whether this is the case, clearly depends on the decision of final goods producers and on the number of competitive new technologies. Since these are the realizations of independent random variables which are beyond the control of the firm, the optimization problem of choosing the probability of success and the optimal quality level can already be solved.

Following these arguments, expected profits can be written as

$$E[\pi_n] = p_n E[I_s]\frac{\gamma-1}{\gamma}\alpha_n Y - \eta e^{\nu\frac{p_n}{1-p_n}+\mu\lambda_n}\alpha_n Y.$$

Setting the first derivative of the expected profits equal to zero and assuming that entry into the market for new technologies takes place until the expected profits are driven down to zero, gives the optimal probability as

$$p_n = 1 + \frac{\nu}{2} - \sqrt{\left(1+\frac{\nu}{2}\right)^2 - 1}. \tag{15}$$

This expression for p_n is always larger than zero and smaller than one and characterizes a maximum. Furthermore since the parameter ν is identical for all firms engaged in creating new technologies, the chosen probability is identical across firms. This makes the overall probability of all firms successful in creating a new technology for being integrated into the final goods production technology symmetric.
Solving the maximization problem with respect to the quality level of the new technology leads to the result

$$\lambda_n = \frac{1}{\mu}\frac{\alpha_n}{1-\alpha_n}\bar{\lambda}_{\tau-1}, \tag{16}$$

which quite analogous to the result in (9).
So far, nothing is said about the number of new technology sectors to be integrated in a particular period of time, $m_+ \in \{0, 1, 2, ...\}$. The decision about integration has to be made by the producers of final goods. In general, this is a complex dynamic program, because the decision of how many sectors to integrate in one period has an influence on all forthcoming periods. A less complex problem can be established if one is willing to allow for some additional assumptions about the market environment for final goods. Assume that entry and exit into the market for final goods is costless and any combination of technology sectors used in the past by at least one final good producer can be copied at no costs by a competitor. Due to the first assumption there is perfect competition in the market for final goods. Because of the second, the only strategy that causes no losses is to maximize the current total factor productivity by choosing the myopic m_+.
The problem is then easiest solved if one first looks on the problem of deciding on the number of new technologies, given that this number is at least 1. Looking at the reduced form of the production function in (11) and interpreting M as the number of established technology sectors, already available in the previous period, the relevant term of the new total factor productivity is given by

$$TFP_{m_+>0} = \prod_{j=1}^{M}\left(N_{j,+}^{1-(1-\delta_\alpha)\alpha_j}\bar{\lambda}_{j,+}^{(1-\delta_\alpha)\alpha_j}\right)\prod_{n=1}^{m_+}\left(N_n^{1-\delta_\alpha(1-\alpha)/m_+}\lambda_n^{\delta_\alpha(1-\alpha)/m_+}\right),$$

where $N_{j,+}$ is the new degree of horizontal differentiation in established sectors and $\bar{\lambda}_{j,+}$ the new average level of quality in sector j. Both numbers do not depend on m_+

because the development of the quality level depends only on the already existing sectors. What is depending on m_+ is the very right product term which can be written as

$$\prod_{n=1}^{m_+}\left(N_n^{1-\delta_\alpha(1-\alpha)/m_+}\lambda_n^{\delta_\alpha(1-\alpha)/m_+}\right)=\left(\frac{1}{\mu}\frac{\delta_\alpha(1-\alpha)/m_+}{1-\delta_\alpha(1-\alpha)/m_+}\bar{\lambda}_{\tau-1}\right)^{\delta_\alpha(1-\alpha)},$$

since $N_n = 1$. As can clearly be seen, this expression declines with m_+. The reason for the total factor productivity to decline with the number of new technologies is the split of $\delta_\alpha(1-\alpha)$ between the m_+ new technologies as the output elasticity. Due to this, the incentive to invest in quality in the new sector decreases as more sectors get integrated. For the producers of final goods it is therefore optimal, if they decide to integrate new technologies, to integrate only one. What still needs to be determined is whether m_+ is zero or one.

The decision between zero or one new technology sectors can be seen as a decision between growth through quality improvements of existing technologies or growth through integration of new technologies and combinations between them. To see the determining factors for this decision, it is helpful to look at the sequence of output elasticities with and without the new technology sector, A_+ and A

$$\begin{aligned}
A_+ &= \{\delta_\alpha(1-\alpha), \delta_\alpha(1-\delta_\alpha)(1-\alpha), \delta_\alpha(1-\delta_\alpha)^2(1-\alpha), ..., \\
&\quad \delta_\alpha(1-\delta_\alpha)^{M+m_+-2}(1-\alpha), (1-\delta_\alpha)^{M+m_+-1}(1-\alpha)\}, \qquad (17)\\
A &= \{\delta_\alpha(1-\alpha), \delta_\alpha(1-\delta_\alpha)(1-\alpha), \delta_\alpha(1-\delta_\alpha)^2(1-\alpha), ..., \\
&\quad (1-\delta_\alpha)^{M-1}(1-\alpha)\}. \qquad (18)
\end{aligned}$$

The first $M-1$ elements of both sequences are identical, the last element in A is split into two elements in A_+.

One term in the total factor productivity consists of the quality levels in all technology sectors, $\prod \bar{\lambda}_j^{\alpha_j}$. Denote this term as Q_+ in the case of $m_+ = 1$ and Q in the case of $m_+ = 0$. Then Q_+ and Q can be written according to the sequences (16) and (17) together with the optimality conditions (9) and (15) as

$$
\begin{aligned}
Q_{+} &= \left\{ \prod_{k=1}^{M+m_{+}-1} \left[\max\left(\frac{1}{\mu} \frac{(1-\delta_{\alpha})^{k-1}\delta_{\alpha}(1-\alpha)}{1-(1-\delta_{\alpha})^{k-1}\delta_{\alpha}(1-\alpha)}, 1 \right) \bar{\lambda}_{\tau-1} \right]^{(1-\delta_{\alpha})^{k-1}\delta_{\alpha}(1-\alpha)} \right\} \times \\
&\times \left[\max\left(\frac{1}{\mu} \frac{(1-\delta_{\alpha})^{M+m_{+}-1}(1-\alpha)}{1-(1-\delta_{\alpha})^{M+m_{+}-1}(1-\alpha)}, 1 \right) \bar{\lambda}_{\tau-1} \right]^{(1-\delta_{\alpha})^{M+m_{+}-1}(1-\alpha)}, \\
Q &= \left\{ \prod_{k=1}^{M-1} \left[\max\left(\frac{1}{\mu} \frac{(1-\delta_{\alpha})^{k-1}\delta_{\alpha}(1-\alpha)}{1-(1-\delta_{\alpha})^{k-1}\delta_{\alpha}(1-\alpha)}, 1 \right) \bar{\lambda}_{\tau-1} \right]^{(1-\delta_{\alpha})^{k-1}\delta_{\alpha}(1-\alpha)} \right\} \times \\
&\times \left[\max\left(\frac{1}{\mu} \frac{(1-\delta_{\alpha})(1-\alpha)^{M-1}}{1-(1-\delta_{\alpha})(1-\alpha)}, 1 \right) \bar{\lambda}_{\tau-1} \right]^{(1-\delta_{\alpha})^{M-1}(1-\alpha)}.
\end{aligned}
$$

If there is a significant number of existing technology sectors M, it is reasonable to look at the case where, through ongoing depreciation of the output elasticities α_j, the growth factors in the last two terms in Q_+ and in the last term in Q, representing the oldest technologies, equal one. In this case, however, it holds that $Q_+ = Q$ and hence there is no change in total factor productivity if m_+ is zero or one. With respect to this part of the total factor productivity final good producers are indifferent between integrating a new technology sector or not. Quality growth does not influence the decision regarding the integration of new technologies.

But there is another part of total factor productivity with is to be considered, i.e. the term reflecting the horizontal degree of differentiation of the technology sectors, $\prod N_j^{1-\alpha_j}$. Remember from the result (10) that in established technology sectors this degree is given by $\frac{\gamma-1}{\gamma}\frac{1}{\eta e^{\mu\lambda_j/\bar{\lambda}_{\tau-1}}}$. As technology sectors get older, the incentive to invest in quality decreases and from some point $\lambda_j = \bar{\lambda}_{\tau-1}$ holds so that these sectors reach their maximum degree of differentiation of magnitude $\frac{\gamma-1}{\gamma}\frac{1}{\eta e^{\mu}}$. If a new technology sector is integrated, i.e. $m_+ = 1$, its initial degree of differentiation is $N_n = 1$ as pointed out above. Therefore the term representing the degree of differentiation in total factor productivity, denoted by N_+ in the case $m_+ = 1$ and

N in the case of $m_+ = 0$, can be written as

$$\begin{aligned}
N_+ &= 1^{1-\delta_\alpha(1-\alpha)} \times \\
&\times \prod_{k=1}^{M-1} \left[\min \left(\frac{\gamma-1}{\gamma} \frac{1}{\eta e^{\frac{\delta_\alpha(1-\delta_\alpha)^k(1-\alpha)}{1-\delta_\alpha(1-\delta_\alpha)^k(1-\alpha)}}}, \frac{\gamma-1}{\gamma}\frac{1}{\eta e^\mu} \right)^{1-\delta_\alpha(1-\delta_\alpha)^k(1-\alpha)} \right] \times \\
&\times \min \left(\frac{\gamma-1}{\gamma} \frac{1}{\eta e^{\frac{(1-\delta_\alpha)^{M+m_+-1}(1-\alpha)}{1-(1-\delta_\alpha)^{M+m_+-1}(1-\alpha)}}}, \frac{\gamma-1}{\gamma}\frac{1}{\eta e^\mu} \right)^{1-(1-\delta_\alpha)^{M+m_+-1}(1-\alpha)}, \\
N &= \left(\frac{\gamma-1}{\gamma} \frac{1}{\eta e^{\frac{\delta_\alpha(1-\alpha)}{1-\delta_\alpha(1-\alpha)}}} \right)^{1-\delta_\alpha(1-\alpha)} \times \\
&\times \prod_{k=1}^{M-2} \left[\min \left(\frac{\gamma-1}{\gamma} \frac{1}{\eta e^{\frac{\delta_\alpha(1-\delta_\alpha)^k(1-\alpha)}{1-\delta_\alpha(1-\delta_\alpha)^k(1-\alpha)}}}, \frac{\gamma-1}{\gamma}\frac{1}{\eta e^\mu} \right)^{1-\delta_\alpha(1-\delta_\alpha)^k(1-\alpha)} \right] \times \\
&\times \min \left(\frac{\gamma-1}{\gamma} \frac{1}{\eta e^{\frac{(1-\delta_\alpha)^{M-1}(1-\alpha)}{1-(1-\delta_\alpha)^{M-1}(1-\alpha)}}}, \frac{\gamma-1}{\gamma}\frac{1}{\eta e^\mu} \right)^{1-(1-\delta_\alpha)^{M-1}(1-\alpha)}
\end{aligned}$$

As noted above the sequences of output elasticities are identical from the beginning on with the difference in the last terms, the last to elements in A_+ sum up to the last element in A. Denote the sector which is just below its maximum degree of differentiation as sector with number $\bar{m}$, where sectors are ordered with increasing age. $\bar{m}$ is then also the $\bar{m}$th element in A_+ and A. The sum of the remaining elasticities in A_+ and A is identical although the number of the remaining elasticities is different. Therefore $\prod_{k=\bar{m}+1}^{M+m_+} N_k^{1-\alpha_k}$ with $\alpha_k \in A_+$ is equal to $\left(\frac{1}{\mu}\frac{\gamma-1}{\gamma}\frac{1}{\eta e^\mu}\right)\prod_{k=\bar{m}+1}^{M} N_k^{1-\alpha_k}$ with $\alpha_k \in A$. Therefore, dividing N_+ through N gives the result

$$\frac{N_+}{N} = \left(\frac{\gamma-1}{\gamma}\frac{1}{\eta}\right)^{\delta_\alpha(1-\alpha)} e^{\delta_\alpha(1-\alpha)-\mu}. \tag{19}$$

With the condition $\frac{1}{\mu}\frac{\delta_\alpha(1-\alpha)}{1-\delta_\alpha(1-\alpha)} > 1$, i.e. at least the newest technology sector finds it profitable to rise its quality level above the quality index of the preceding period, it can easily be shown that the term above is larger than 1. The reason for this result is that as a new technology sector is integrated, an old one reaches is full degree of horizontal differentiation due to lower fixed costs through lower investments in quality. At the the same time the new technology has a degree of differentiation of

1. Therefore it is, from the myopic point of view of the producers of final goods, necessary to integrate one new technology sector, if it is available[7].

Next, turn to the issue of availability of a new technology sector for integration into the production technology for final goods. This issue is closely related to the properties of the indicator variable $\tilde{I}_n = I_n I_s$, where I_n is one if the representative firm engaging in creation of a new technology is successful and I_s is one if this particular technology is selected by final good producers. The properties of I_n have already been determined, i.e. it takes on the value 1 with probability p_n, given by equation (14), and 0 with probability $1 - p_n$. I_s depends on the competitive environment of the representative firm. It is certainly 1 if no other firm was successful in creating a competitive new technology. It is assumed that it takes on the value $\frac{1}{2}$ if an additional firm was successful, $\frac{1}{3}$ if two additional firms were successful and so forth. Thus it is assumed that if more than one new technology is available, all new technologies have the same probability of being selected by final good producers. If a new technology is not selected in one period it has no second chance, i.e. it can not be integrated in the following periods. Economically this can be justified by the argument that a newly created technology is based on the state of the art of technology. If the technology changes, e.g. through integration of another new technology, the not selected technologies created in the past are not longer compatible. Another interpretation that would lead to the same result could be that to achieve compatibility again, additional R&D costs have to be incurred which are again given by (13).

Since all firms engaged in creating new technologies set the same probability of success given by (14) and research projects are assumed to be stochastically inde-

[7] Using the expression $\lambda_{j,i} x_{j,i}$ instead of $\frac{\lambda_{j,i} x_{j,i}}{\alpha_j}$ in the production function (5) would imply an additional term in $\frac{N_+}{N}$ which would be smaller than and asymptotical equal to 1. This is because $\prod \alpha_j^{\alpha_j}$ then appears as an additional multiplicative term in TFP given by (12). However, this term converges under the assumption of constant depreciation from above to $[\delta_\alpha(1-\alpha)]^{1-\alpha}(1-\delta_\alpha)^{(1-\delta_\alpha)(1-\alpha)/\delta_\alpha}$ as M increases. Therefore asymptotically nothing changes, however, for small M the right hand side of (18) has to be sufficiently larger than 1 in order to compensate for the loss in $\prod \alpha_j^{\alpha_j}$.

pendent, the probabilities of being selected have a binomial distribution. Denote be $\tilde{N}$ the number of firms engaging in creation of new technologies, the expected value of I_s is given by

$$E[I_s] = \sum_{h=0}^{\tilde{N}-1} \frac{1}{1+h} P(x=h), \tag{20}$$

where $P(x = h)$ is the probability that h of the remaining $\tilde{N} - 1$ developers of technology sectors are successful in creating a new one. Since the probabilities have a binomial distribution, $P(x = h)$ is given by

$$P(x=h) = \left(\begin{array}{c} \tilde{N}-1 \\ h \end{array} \right) p_n^h (1-p_n)^{\tilde{N}-1-h} = \frac{(\tilde{N}-1)!}{h!(\tilde{N}-1-h)!} p_n^h (1-p_n)^{\tilde{N}-1-h}.$$

The expected value in equation (19) can alternatively be written as

$$\begin{aligned} E(I_s) &= \sum_{h=0}^{\tilde{N}-1} \frac{1}{p_n} \frac{1}{\tilde{N}} \frac{\tilde{N}!}{(h+1)!(\tilde{N}-1-h)!} p_n^{h+1} (1-p_n)^{\tilde{N}-1-h} = \\ &= \frac{1}{p_n} \frac{1}{\tilde{N}} (1-p_0), \end{aligned}$$

where p_0 is the probability that no new technology sector is created, i.e. $p_0 = (1-p_n)^{\tilde{N}}$. With this result, the expected value of the net profits for a representative firm engaged in creating a new technology are given by

$$E(\pi_n) = \frac{1}{\tilde{N}} \left[1-(1-p_n)^{\tilde{N}} \right] \frac{\gamma-1}{\gamma} \alpha_n Y - \eta e^{\nu \frac{p_n}{1-p_n} + \mu \lambda_n} \alpha_n Y, \tag{21}$$

where p_n is given by equation (14) and λ_n by (15). Assuming entry into the market for new technologies until net profits are driven down to zero results in the following condition

$$\frac{\tilde{N}}{1-(1-p_n)^{\tilde{N}}} = \frac{\gamma-1}{\gamma} \frac{1}{\eta e^{\nu \frac{p_n}{1-p_n} + \mu \lambda_n}}, \tag{22}$$

which determines the number of competitive research firms $\tilde{N}$ engaged in creating new technologies. As before p_n and λ_n are determined by equations (14) and (15). The left hand side of equation (21) is strictly increasing in $\tilde{N}$ which can be seen as $\frac{1-(1-p_n)^{\tilde{N}}}{\tilde{N}} - \frac{1-(1-p_n)^{\tilde{N}-1}}{\tilde{N}-1} = \frac{p_0+p_1-1}{\tilde{N}(\tilde{N}-1)} < 0$, where p_0 and p_1 are the probabilities of a successful creation of exactly 0 and 1 new technologies economy wide. Thus, expected net profits in equation (20) are decreasing in $\tilde{N}$, as long as $\tilde{N}$ is larger than one. If $\tilde{N}$ is growing large, the term $(1-p_n)^{\tilde{N}}$ converges to zero and $\tilde{N}$ is approximately given by

$$\tilde{N} \approx \frac{\gamma - 1}{\gamma} \frac{1}{\eta e^{\nu \frac{p_n}{1-p_n} + \mu \lambda_n}},$$

which is quite analogous to the degree of horizontal differentiation in older technologies, but here it applies to competitors. This approximation is just for illustration purposes and will not be used in the following.

3.5 Growth of Total Factor Productivity

So far it has been shown that, if at least one new technology sector has been created, one new technology will be integrated into the production technology for final goods. However, the creation of new technologies is governed by uncertainty and it is possible that in a particular period of time no new technology is ready for integration. This happens with probability $p_0 = (1-p_n)^{\tilde{N}}$. The probability that a new technology is integrated is thus $1 - p_0$.

First, turn to the growth rate in production of final goods if no new technology sector is integrated. If this happens to be the case, then growth still takes place in this special case of constant depreciation of the output elasticities through at most two channels. First, growth in the level of quality of intermediate input factors takes place. This applies only to the sectors which qualify for a growth factor larger than one according to the condition in (9). Second, if in the preceding period of time one new technology sector has been integrated, its degree of horizontal differentiation jumps from 1 to the value given in (10), which gives the diversification for an established technology sector. Thus, there are two cases to be considered. In the pure quality growth case the growth factor for total factor productivity is determined by the growth factor $\gamma_{\bar{\lambda}}$ of the quality index $\bar{\lambda}$

$$\gamma_{TFP,1} = \gamma_{\bar{\lambda}}^{1-\alpha} = \prod_{s=1}^{\bar{m}} \left(\frac{1}{\mu} \frac{(1-\delta_\alpha)^{s-1}\delta_\alpha(1-\alpha)}{1-(1-\delta_\alpha)^{s-1}\delta_\alpha(1-\alpha)} \right)^{\frac{(1-\alpha)}{\bar{m}}}, \tag{23}$$

where $\bar{m}$ again denotes the $\bar{m}$th technology sector for which it still pays to invest in quality to rise it above the index of the preceding period. The expression for $\gamma_{\bar{\lambda}}$ follows from the definition (**??**) and the optimality condition (9). The growth factor $\gamma_{TFP,1}$ is realized with an unconditional probability of p_0^2, i.e. two consecutive periods with no integration of a new technology.

In the case where in the preceding period a new technology sector has been integrated, there is an additional source of growth through horizontal differentiation of this sector. This degree of diversification is given by equation (10) and leads to a growth factor of

$$\gamma_{TFP,2} = \left(\frac{\gamma - 1}{\gamma} \frac{1}{\eta e^{\frac{\delta_\alpha (1-\alpha)}{1-\delta_\alpha (1-\alpha)}}} \right)^{1-\delta_\alpha(1-\alpha)} \gamma_{TFP,1}. \tag{24}$$

Therefore the growth factor of total factor productivity in this case is strictly larger than in the case where no new technology has been integrated in the previous period. This growth factor turns up with unconditional probability $(1 - p_0)p_0$.

Next, turn to the second general case where a new technology sector is integrated into the production technology for final goods. Again, two sub cases have to be considered, the first considers no integration of a new technology in the preceding period of time and the second where also in the preceding period a new technology has been integrated. In the first sub case, growth takes places through quality growth in the intermediate input factors of the newest $\bar{m}$ technologies. Also contributing to growth is the fact of an additional technology sector reaching its maximum degree of horizontal differentiation. Growth is negatively affected by the differentiation of 1 of the new technology sector. With the results of the previous section the growth factor for this case can be written as

$$\gamma_{TFP,3} = \left(\frac{\gamma - 1}{\gamma} \frac{1}{\eta} \right)^{\delta_\alpha(1-\alpha)} e^{\delta_\alpha(1-\alpha)-\mu} \gamma_{TFP,1}. \tag{25}$$

Again, the quality component of growth is identical by its construction. The unconditional probability of this case is $p_0(1 - p_0)$.

The second sub-case is the scenario of integration of new technologies in two subsequent periods of time. As before the behavior of quality growth is not affected and is given by $\gamma_{TFP,1}$. The second source of growth is the full horizontal differentiation of an additional technology sector as in the first sub case. But the negative effect of the first sub case is now missing, since a new technology sector with horizontal degree of differentiation of 1 is present in both time periods. The growth factor is therefore given by

$$\gamma_{TFP,4} = \left(\frac{\gamma - 1}{\gamma} \frac{1}{\eta e^{\mu}} \right) \gamma_{TFP,1}, \tag{26}$$

which is realized with an unconditional probability $(1 - p_0)^2$. Following the arguments put forward so far, the growth factors can be ordered as followed

$$\begin{aligned} \gamma_{TFP,4} &> \gamma_{TFP,3}, \\ \gamma_{TFP,2} &> \gamma_{TFP,1}, \\ \gamma_{TFP,3} &> \gamma_{TFP,1}. \end{aligned}$$

Whether $\gamma_{TFP,3}$ is larger or smaller than $\gamma_{TFP,2}$ essentially depends on the values of the parameters of the model. This can be seen as $\frac{\gamma_{TFP,3}}{\gamma_{TFP,2}} = e^{-\mu} \left(\frac{\gamma-1}{\gamma} \frac{1}{\mu} \right)^{2\delta_\alpha(1-\alpha)-1}$. Note that the decision between integrating a new technology sector or not, as discussed in the previous section, is not based on these figures. The reason for this is that the comparative scenario is not the previous period but the relevant scenario of the current period, i.e. what would happen if the new technology sector were not to be integrated.

Summarizing the above cases, which correspond to the subscripts of the growth factors and will be used in the following, gives the four possibilities

1. No integration of a new technology sector in period τ and $\tau + 1$,
2. Integration of a new technology sector in period τ, but not in $\tau + 1$,
3. No integration of a new technology sector in period τ, but in $\tau + 1$,
4. Integration of a new technology sector in period τ and $\tau + 1$.

Since the event of integration of a new technology sector is a random event, the sequence of TFP growth factors is a random variable. However, with the results so far some light on the expected behavior of this growth factor can be shaded. The probability of non availability of new technology sector in one period of time is given by $p_0 = (1-p_i)^{\tilde{N}}$. The time interval t_i until the creation of a new technology sector and its integration into the production technology for final goods has therefore an expected value of

$$E[t_i] = 1(1-p_0) + 2p_0(1-p_0) + 3p_0^2(1-p_0) + 4p_0^3(1-p_0) + ... = \frac{1}{1-p_0}.$$

Therefore, on average, every $\frac{1}{1-p_0}$ periods one new technology sector is integrated into the production technology for final goods. The growth factor of TFP follows on average for $\frac{1}{1-p_0} - 1$ periods $\gamma_{TFP,1}$, then for one period $\gamma_{TFP,3}$ and for an additional period $\gamma_{TFP,2}$ before returning to $\gamma_{TFP,1}$.

Indeed, the random variable which gives the state of the world in period τ and defines which of the four growth rates of TFP realizes follows a homogenous first order Markov chain with transition matrix

$$P = \begin{bmatrix} p_0 & 0 & 1-p_0 & 0 \\ p_0 & 0 & 1-p_0 & 0 \\ 0 & p_0 & 0 & 1-p_0 \\ 0 & p_0 & 0 & 1-p_0 \end{bmatrix}, \tag{27}$$

where the element $p_{n,m}$ in row n and column m denotes the probability to move from state n to state m. The states are defined in correspondence with the numbering of the growth factors of total factor productivity above. The eigenvalues of this transition matrix are 1,0,0,0, hence the process is also ergodic (see e.g. Hamilton (1994) chap. 22).

Due to its structure, this Markov chain converges after two periods of time to the stationary solution $[p_0^2, p_0(1-p_0), p_0(1-p_0), (1-p_0)^2]'$ which gives the unconditional probabilities of the four states of the economy.

3.6 Growth in Production

The preceding subsection dealt with the behavior of total factor productivity over time. To draw conclusions about the behavior of final good production, the behavior of the capital stock of the economy has to be considered as well. Due to the log preferences of consumers, the capital stock in $\tau+1$ does not depend on $r_{\tau+1}$ but only on wages in τ, i.e.

$$K_{\tau+1} = \frac{1}{2+\rho}\alpha Y_\tau = \frac{1}{2+\rho}\alpha TFP_\tau L^\alpha K_\tau^{1-\alpha}.$$

Using this equation iteratively gives $K_{\tau+1}$ as

$$K_{\tau+1} = \prod_{s=0}^{\infty}\left(\frac{1}{2+\rho}\right)^{(1-\alpha)^s}\alpha^{(1-\alpha)^s}TFP_{\tau-s}^{(1-\alpha)^s}L^{\alpha(1-\alpha)^s}.$$

With this result, final goods production can be written as

$$Y_{\tau+1} = TFP_{\tau+1}L^\alpha\prod_{s=1}^{\infty}\left(\frac{1}{2+\rho}\right)^{(1-\alpha)^s}\alpha^{(1-\alpha)^s}TFP_{\tau+1-s}^{(1-\alpha)^s}L^{\alpha(1-\alpha)^s}.$$

The growth factor of final goods production $\gamma_{Y,\tau+1} = \frac{Y_{\tau+1}}{Y_\tau}$ is therefore

$$\gamma_{Y,\tau+1} = \prod_{s=0}^{\infty}\left(\frac{TFP_{\tau+1-s}}{TFP_{\tau-s}}\right)^{(1-\alpha)^s} = \prod_{s=0}^{\infty}\gamma_{TFP,\tau+1-s}^{(1-\alpha)^s}.$$

where $\gamma_{TFP,\tau}$ is the growth factor of total factor productivity which can take on the values $\gamma_{TFP,1}$, $\gamma_{TFP,2}$, $\gamma_{TFP,3}$ or $\gamma_{TFP,4}$ as defined in the preceding section. The continuously compounded growth rate of final goods production $g_{Y,\tau+1} = \ln\gamma_{\tau+1}$ then follows the process

$$g_{Y,\tau+1} = g_{TFP,\tau+1} + (1-\alpha)g_{TFP,\tau} + (1-\alpha)^2 g_{TFP,\tau-1} + (1-\alpha)^3 g_{TFP,\tau-2} + ..., \tag{28}$$

where $g_{TFP,\tau} = \ln\gamma_{TFP,\tau}$. Remember from the preceding section on the growth behavior of total factor productivity that $\gamma_{TFP,\tau}$ is a random variable which can potentially take on four different values with corresponding unconditional probabilities, i.e.

$$g_{TFP,\tau} = \begin{cases} \ln\gamma_{TFP,1} & \text{with probability } p_0^2, \\ \ln\gamma_{TFP,2} & \text{with probability } p_0(1-p_0), \\ \ln\gamma_{TFP,3} & \text{with probability } p_0(1-p_0), \\ \ln\gamma_{TFP,4} & \text{with probability } (1-p_0)^2. \end{cases}$$

Therefore equation (28) defines a moving average process of order infinity with geometrically declining coefficients. From the duality of autoregressive and moving average processes, the growth rate of final goods production has an autoregressive representation of order one given by

$$g_{Y,\tau} = (1-\alpha)g_{Y,\tau-1} + g_{TFP,\tau}.$$

Note that the $g_{TFP,\tau}$ are not distributed independently because their possible realizations depend on whether in $\tau-1$ a new technology has been created and integrated or not. They follow the Markov chain given in the preceding section. Thus their are additional dynamics present compared to a usual AR(1) process.

The reason for this result is first, the dependence of the capital stock on last periods savings, second, the independence of saving from the interest rate due to log preferences and third, the randomized occurrence of new technology sectors for the production technology of final goods which gives rise to shocks in the total factor productivity.

The result for the behavior of output growth crucially depends on the preferences of consumers. To illustrate this, assume that utility is not given by log preferences, but by linear preferences

$$U_\tau = E_\tau\left(c_\tau + \frac{1}{1+\rho}c_{\tau+1}\right).$$

The optimality condition for the households is then

$$E_\tau(r_{\tau+1}) = \rho.$$

Since expectations are built on information available in period τ, there are two cases to be considered.

If there is no integration of a new technology sector in period τ, then it follows that

$$\rho+\delta = E_\tau(r_{t+1}) + \delta = (1-\alpha)\left(\frac{L}{K_{1,\tau+1}}\right)^\alpha (p_0\gamma_{TFP,1} + (1-p_0)\gamma_{TFP,3})TFP_\tau,$$

where TFP_τ is total factor productivity in period τ and $K_{1,\tau+1}$ is the capital stock built up through saving in period τ conditional on this case.

If a new technology sector has been integrated in period τ, then

$$\rho+\delta = E(\tau)(r_{t+1}) + \delta = (1-\alpha)\left(\frac{L}{K_{2,\tau+1}}\right)^\alpha (p_0\gamma_{TFP,2} + (1-p_0)\gamma_{TFP,4})TFP_\tau$$

Thus, there a two possible states for the capital stock, $K_{1,\tau+1}$ and $K_{2,\tau+1}$ in period $\tau+1$, depending on what happens in period τ

$$K_{1,\tau+1} = (\rho+\delta)^{-\frac{1}{\alpha}}(1-\alpha)^{\frac{1}{\alpha}}(p_0\gamma_{TFP,1}+(1-p_0)\gamma_{TFP,3})^{\frac{1}{\alpha}}TFP_\tau^{\frac{1}{\alpha}}L,$$
$$K_{2,\tau+1} = (\rho+\delta)^{-\frac{1}{\alpha}}(1-\alpha)^{\frac{1}{\alpha}}(p_0\gamma_{TFP,2}+(1-p_0)\gamma_{TFP,4})^{\frac{1}{\alpha}}TFP_\tau^{\frac{1}{\alpha}}L.$$

For the growth factor of final goods production it follows that it can take on four possible values as the growth factor of total factor productivity. If in period τ no new technology sector has not been integrated, $K_{1,\tau+1}$ is relevant and output is given either by

$$Y_{1,\tau+1} = (\rho+\delta)^{-\frac{1-\alpha}{\alpha}}(1-\alpha)^{\frac{1-\alpha}{\alpha}}(p_0\gamma_{TFP,1}+(1-p_0)\gamma_{TFP,3})^{\frac{1-\alpha}{\alpha}}\gamma_{TFP,1}TFP_\tau^{\frac{1}{\alpha}},$$

or

$$Y_{3,\tau+1} = (\rho+\delta)^{-\frac{1-\alpha}{\alpha}}(1-\alpha)^{\frac{1-\alpha}{\alpha}}(p_0\gamma_{TFP,1}+(1-p_0)\gamma_{TFP,3})^{\frac{1-\alpha}{\alpha}}\gamma_{TFP,3}TFP_\tau^{\frac{1}{\alpha}},$$

depending on whether no or one new technology is integrated in period $\tau+1$.
For the other two cases where $K_{2,\tau+1}$ is relevant, the corresponding figures for final output are given by

$$\begin{aligned}
Y_{2,\tau+1} &= (\rho+\delta)^{-\frac{1-\alpha}{\alpha}}(1-\alpha)^{\frac{1-\alpha}{\alpha}}(p_0\gamma_{TFP,2}+(1-p_0)\gamma_{TFP,4})^{\frac{1-\alpha}{\alpha}}\gamma_{TFP,2}TFP_\tau^{\frac{1}{\alpha}},\\
Y_{4,\tau+1} &= (\rho+\delta)^{-\frac{1-\alpha}{\alpha}}(1-\alpha)^{\frac{1-\alpha}{\alpha}}(p_0\gamma_{TFP,2}+(1-p_0)\gamma_{TFP,4})^{\frac{1-\alpha}{\alpha}}\gamma_{TFP,4}TFP_\tau^{\frac{1}{\alpha}},
\end{aligned}$$

The four growth factors for final output production can be written as

$$\begin{aligned}
\gamma_{Y,1,\tau+1} &= (p_0\gamma_{TFP,1}+(1-p_0)\gamma_{TFP,3})^{\frac{1-\alpha}{\alpha}}\gamma_{TFP,1}\left(\frac{r_\tau+\delta}{\rho+\delta}\right)^{\frac{1-\alpha}{\alpha}},\\
\gamma_{Y,3,\tau+1} &= (p_0\gamma_{TFP,1}+(1-p_0)\gamma_{TFP,3})^{\frac{1-\alpha}{\alpha}}\gamma_{TFP,3}\left(\frac{r_\tau+\delta}{\rho+\delta}\right)^{\frac{1-\alpha}{\alpha}},\\
\gamma_{Y,2,\tau+1} &= (p_0\gamma_{TFP,2}+(1-p_0)\gamma_{TFP,4})^{\frac{1-\alpha}{\alpha}}\gamma_{TFP,2}\left(\frac{r_\tau+\delta}{\rho+\delta}\right)^{\frac{1-\alpha}{\alpha}},\\
\gamma_{Y,4,\tau+1} &= (p_0\gamma_{TFP,2}+(1-p_0)\gamma_{TFP,4})^{\frac{1-\alpha}{\alpha}}\gamma_{TFP,4}\left(\frac{r_\tau+\delta}{\rho+\delta}\right)^{\frac{1-\alpha}{\alpha}}.
\end{aligned}$$

With the corresponding results for the two possible states of the capital stock in period τ, $K_{1,\tau}$ and $K_{2,\tau}$, given above, there are two possible general states for the interest rate

$$
\begin{aligned}
r_{1,\tau} &= \frac{TFP_\tau}{TFP_{\tau-1}} \frac{1}{p_0 \gamma_{TFP,1} + (1-p_0)\gamma_{TFP,3}}(\rho+\delta) - \delta, \\
r_{2,\tau} &= \frac{TFP_\tau}{TFP_{\tau-1}} \frac{1}{p_0 \gamma_{TFP,2} + (1-p_0)\gamma_{TFP,4}}(\rho+\delta) - \delta,
\end{aligned}
$$

where $r_{1,\tau}$ realizes if in period $\tau - 1$ no new technology sector has been integrated and $r_{2,\tau}$ otherwise. $\frac{TFP_\tau}{TFP_{\tau-1}}$ is therefore given by either $\gamma_{TFP,1}$ or $\gamma_{TFP,3}$ in the case of $r_{1,\tau}$ or by $\gamma_{TFP,2}$ and $\gamma_{TFP,4}$ in the case of $r_{2,\tau}$ depending on what happens in period τ. We therefore end up with four possible values for the net interest rate which are time invariant and only the realization in period τ is relevant for the growth factor of output. Which growth factor emerges can be seen from the concordance below

$$
\begin{aligned}
\gamma_{Y,1,\tau+1} &\Leftrightarrow r_\tau = \tfrac{\gamma_{TFP,1}}{p_0\gamma_{TFP,1}+(1-p_0)\gamma_{TFP,3}}(\rho+\delta) - \delta, \\
\gamma_{Y,3,\tau+1} &\Leftrightarrow r_\tau = \tfrac{\gamma_{TFP,3}}{p_0\gamma_{TFP,1}+(1-p_0)\gamma_{TFP,3}}(\rho+\delta) - \delta, \\
\gamma_{Y,2,\tau+1} &\Leftrightarrow r_\tau = \tfrac{\gamma_{TFP,2}}{p_0\gamma_{TFP,2}+(1-p_0)\gamma_{TFP,2}}(\rho+\delta) - \delta, \\
\gamma_{Y,4,\tau+1} &\Leftrightarrow r_\tau = \tfrac{\gamma_{TFP,4}}{p_0\gamma_{TFP,2}+(1-p_0)\gamma_{TFP,2}}(\rho+\delta) - \delta,
\end{aligned} \tag{29}
$$

In every period there are four different states for the growth factor of final output which are time invariant. Which one realizes is of course depending on the past integration policy of new technologies by final output producers. The behavior of this growth factor can be described by a first order homogenous and ergodic Markov chain with the same transition matrix (26) as given in the section on growth in total factor productivity.

The crucial difference, compared with the case of log preferences, is that there are now less dynamics in the process for the growth factor of final goods production. In particular the autoregressive structure is now missing. The reason for this is that in the case of log preferences, households try to smooth the consumption path since they have a finite elasticity of intertemporal substitution. In the case of linear preferences, this argument is missing because the intertemporal elasticity of substitution is infinity and there is no gain in utility from consumption smoothing.

4 Conclusion

The aim of the paper was to develop a growth model that takes account of several aspects which seem to be partially less recognized in the modern growth theory. First, new combinations of production factors should work as one engine of growth in the model. This reflects modern as well as older arguments in the growth debate. Second, one aim was also to show that growth can take place in the presence of "fishing out" effects in the traditional dimensions of growth and how a "standing on shoulders of giants" effect can be modelled in a less ad hoc and more endogenous way. New combinations of new technologies with existing ones is a partially way out of this problem as well as the possibility that new technologies with high growth potential can compensate for old technologies with low growth potential.

The model developed above belongs to the class of endogenous growth models. This can be seen by introducing an additional policy parameter into the model which might be interpreted as a proportionate R&D subsidy financed by a lump sum tax. Imagine that only a fraction $1-\beta$ of the fixed costs in the cost functions (6) have to be incurred by the innovating firms. This directly influences the growth factor of total factor productivity which is then given by

$$\begin{aligned}
\gamma_{TFP,1} &= \prod_{s=1}^{\bar{m}} \left(\frac{1}{\mu} \frac{(1-\delta_\alpha)^{s-1}\delta_\alpha(1-\alpha)}{1-(1-\delta_\alpha)^{s-1}\delta_\alpha(1-\alpha)} \right)^{(1-\delta_\alpha)^{s-1}\delta_\alpha(1-\alpha)}, \\
\gamma_{TFP,2} &= \left(\frac{\gamma-1}{\gamma} \frac{1}{(1-\beta)\eta e^{\frac{\delta_\alpha(1-\alpha)}{1-\delta_\alpha(1-\alpha)}}} \right)^{1-\delta_\alpha(1-\alpha)} \gamma_{TFP,1}, \\
\gamma_{TFP,3} &= \left(\frac{\gamma-1}{\gamma} \frac{1}{(1-\beta)\eta} \right)^{\delta_\alpha(1-\alpha)} e^{\delta_\alpha(1-\alpha)-\mu} \gamma_{TFP,1}, \\
\gamma_{TFP,4} &= \left(\frac{\gamma-1}{\gamma} \frac{1}{(1-\beta)\eta e^{\mu}} \right) \gamma_{TFP,1}.
\end{aligned}$$

With such a subsidy policy the growth behavior of the economy can be influenced directly. Besides the influence on the above growth factors, there is an additional level effect on final output production. If the subsidy policy is introduced, fixed costs for all established technology sectors falls. This leads to a one time rise in the number of intermediate input factor producers, shifting up production.

The frequency of occurrence of new technology sectors can be influenced by an analogous subsidy of R&D costs of firms engaged in creating new technologies. This subsidy does not influence the degree of horizontal differentiation as this is always equal to one, but more firms will compete in creating new technologies due to lower fixed costs. This rises the probability that at least one new technology is created in each period and hence reduces the average time interval between the creation of two new technologies.

As has been shown, the model develops dynamics in both the growth rate of factor productivity and final output. The dynamics with respect to total factor productivity are totally determined by the production side of the economy, whereas the dynamics of final output growth are also influenced by consumer preferences. In particular, if we have log preferences and AR(1)-process with non independent shocks emerges for this growth rate.

A task that is left for future research is the determination of the optimal growth rate in the economy. There are several externalities in the model above which are neglected by the market participants. Therefore it is very likely that the resulting growth rate is not optimal. First, firms producing intermediate input factors neglect their influence on the average quality level. Not doing so would give an additional incentive to invest in the quality level. Second, final output producers are not able to maximize total factor productivity in the long run because of the market environment. Third, creators of new technology sectors capture the gains from this innovation only for one period and leave their knowledge as a public good for future periods. Finally, monopoly power plays an important role in forming incentives for innovative behavior and by the same time lets prices to deviate from marginal costs. Taking account of these facts in the model would give the long run optimal growth rate.

The short run behavior of agents in the above model is clearly a point that can be criticized and there might be market environments in the real world where the underlying assumptions are not fulfilled. Also criticized can be the assumption of a constant depreciation rate of output elasticities which was used to solve the model.

This assumption should give a picture of a world where the newest technologies are the ones which are most important. As technologies get older they are more and more replaced in the production technology by newer ones. Clearly there might be different environments where this idea does not hold. But this work is left for future research.

5 Literature

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