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Abstract

The main contribution of this paper is to place smooth infinite economies in the setting of the equilibrium manifold and the natural projection map à la Balasko. We show that smooth infinite economies have an equilibrium set that has the structure of a Banach manifold and that the natural projection map is smooth. We define regular and critical economies, and regular and critical prices, and we show that the set of regular economies coincides with the set of economies whose excess demand function has only regular prices. Generic determinacy of equilibria follows as a by-product.
Keywords: General equilibrium, infinite economies, intertemporal choice, uncertainty.
JEL Classification: D5, D50, D51, D80, D90.

Resumen

La principal contribución de este trabajo es ubicar a las economías infinitas suaves en el marco de la variedad de equilibrio y la proyección natural à la Balasko. Demostramos que las economías infinitas suaves tienen un conjunto de equilibrio que tiene la estructura de una variedad diferencial de Banach y que la proyección natural es suave. Definimos economías regulares y críticas, y precios regulares y críticos, y demostramos que el conjunto de economías regulares coincide con el conjunto de economías cuya función de exceso de demanda sólo contiene precios regulares. Determinación genérica de equilibrios se obtiene como corolario.
Palabras Clave: Equilibrio general, economías infinitas, elección intertemporal, incertidumbre.

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1 Introduction

For pure exchange economies with a finite number of agents and a finite number of commodities, it is well known that all the initial endowments that define an economy have a competitive equilibrium, and that *almost all* initial endowments give rise to a finite number of equilibria. Furthermore, the structure of the equilibrium set has been studied in great detail and, together with a systematic study of the natural projection map (Balasko, 1988), it is known that the equilibrium set is connected, simply-connected, a smooth manifold, it is diffeomorphic to the space of initial endowments, and so on. There are however many examples in the economic literature where the consumption space is infinite dimensional; usually these models arise when consumption is a function of a parameter $m \in M$, where M might stand for an infinite discrete time ($M = \mathbb{N}$), continuous time ($M = [0, T]$), states of nature ($M = [0, 1]$), spatial location ($M = \mathbb{R}^3$), product characteristics (M a compact set), etc.

In an attempt to study infinite economies the literature has been presented with challenges which seem to come in at least four varieties:[1] (i) strictly monotonic preferences may not be continuous or they may fail to be represented in many consumption spaces; (ii) demand functions do not exist or they are not continuous unless a specific consumption set is chosen; (iii) the price space is unmanageably large; and (iv) the consumption space or the

[1]For an older survey of infinite economies we refer the reader to Mas-Colell & Zame (1991).

price space (or both) have the property of having an empty interior which makes impossible the use of tools of differential topology.

For instance, recent work of Hervés-Beloso and Monteiro (2009) has shown that if we consider representable or continuous strictly monotonic preferences on a consumption space with a continuum of commodities, the consumption set should be a subset of the space of continuous functions (or of integrable functions).

For individual demand functions, Araujo (1988) shows that when the commodity set is a general Banach space a demand function will exist if and only the commodity space is reflexive. He also shows that even if the demand function exists, it will be C^1 if and only if the commodity space is actually a Hilbert space. These results suggest that unless we use ℓ_2 or L_2 as the consumption space there is little hope of studying determinacy in a general setting.

Another possibility, as is done by Kehoe et al (1989), is to study determinacy of equilibria in economies with a double infinity of agents and goods where the commodity set is chosen to be a Hilbert space. The disadvantage of this approach, as they put it, is that the price domain (and implicitly the consumption set) has an empty interior. This means that they are allowing, to some extent, negative prices and consumption.

A further approach is to assume separable utilities. In a way, allowing separable utilities is equivalent to decomposing an infinite-dimensional optimization problem into an infinite sequence but of finite-dimensional problems. The advantage is that with separability only a small subset of the entire price space can support equilibria and, hence, there is no real loss of information from discarding those elements of the price space that do not support equilibria anyway. This approach has been followed, for instance by Mas-Colell (1991), Chichilnisky and Zhou (1998) and Crès et al (2009).

Yet another approach is to use the "Negishi method". Loosely speaking, it consists of substituting the study of price equilibria (which take values in an infinite-dimensional set) by the welfare weights associated with the equilibrium allocations (which take values in a finite-dimensional set if there are finitely many agents). This approach has been used for instance by Balasko (1997a, 1997b, 1997c) to study the infinite-horizon model. The state-of-the-art approach consists in using the Negishi method with a weakened version of differentiability. Shannon (1999) and Shannon and Zame (2002) introduce the notion of quadratic concavity and demonstrate that Lipschitz continuity of the excess spending map is sufficient to yield generic determinacy. Because the nature of regularity for Lipschitz functions is weaker than for smooth economies, the set of regular economies is not open nor is it the intersection of a countable family of open sets. Instead they use a measure-theoretic analogue of full Lebesgue measure for infinite dimensional spaces[2].

In this paper we propose to set smooth infinite economies in a setting *á la Balasko.* This is, as in finite dimensions, we study the entire equilibrium

[2]The Negishi method does however have a caveat: it requires the first welfare theorem to hold, which means that it cannot be applied to economic models that do not lead to optimality situations such as economies with incomplete financial markets.

set, showing that it actually is a manifold, and combining it with a study of the natural projection map. Although Shannon and Zame (2002) consider more general commodity spaces and preferences, our framework will allow us to study more than just determinacy by allowing us to study the entire equilibrium manifold. The technical reason behind this come from discarding a large subset of the price space that cannot support equilibria anyway. This will allow us to use the analogues in infinite dimensions of results from multivariable calculus such as the inverse and implicit function theorems, the regular value theorem, and Sard's theorem. Hence, we can compare the price equilibria of economies that vary and our approach allows us to, among other things, show that the equilibrium set is a manifold, define the concepts of regular and critical economies, and regular and critical prices, and relate these two concepts with each other and with the finite-dimensional case. Determinacy follows as a by-product.

This paper is structured as follows. In section 2, we mention a couple of examples that lead to infinite economies. The first example (with $M = [0,1]$) is an exchange economy with uncertainty, the second ($M = [0,T]$) is an economy with continuous time. In section 3 we review some basic material of Fredholm theory. *Fredholmness* is a property that functions need to satisfy in order for results of infinite-dimensional calculus and topology to hold. In section 4 we define the market and study properties of preferences, consumption, prices, and individual demand functions. Then in section 5 we study properties of aggregate excess demand functions and show two technical results: that the excess demand function is a vector field on the (infinite-dimensional) normalized price space and that it is a Fredholm map.

Sections 6 and 7 include the main results where we show that there is an almost perfect parallel between finite and smooth infinite-dimensional economies. Here we show that the equilibrium set has the structure of a manifold, we define regular and critical economies, critical and regular prices and study the relation between these concepts. We finally show as a by-product determinacy of equilibria.

2 Examples of economies with an infinite dimensional consumption space

To fix ideas, we wish to describe in this section two examples that lead naturally to consumption spaces with infinite dimensions. In section 4 we will explain how these examples are encapsulated in a more general setting in which we study regular and critical economies, and regular and critical prices. Further examples can be seen in Mas-Colell and Zame (1991).

2.1 An example of economies with uncertainty

The following example is a particular case of both Mas-Colell (1991) and Crès et al (2009) where we consider a two-time period $t = 0, 1$ economy with complete financial markets and uncertainty at the second time period. The set of states is $M = [0, 1]$ and the C^1 map $\pi : M \to \mathbb{R}_+$ is the density of the set of states M. We suppose there is a finite number $i = 1, \ldots, I$ of consumers and a finite number n of goods at each time period and at each state. A consumption bundle is a pair $x_i = (x_i^0, x_i^1)$ where at $t = 0$ consumption is a vector $x_i^0 \in \mathbb{R}^n_{++}$ and at $t = 1$ it is a C^1 map $x_i^1 : M \to \mathbb{R}^n_{++}$. We suppose that agents are equipped with a $t = 0$ endowment $\omega_i^0 \in \mathbb{R}^n_{++}$ and a C^1 initial endowment at $t = 1$ of the form $\omega_i^1 : M \to \mathbb{R}^n_{++}$. Preferences are represented by a time- and state-dependent utility of the form

$$U_i(x_i) = u_i(x_i^0) + \int_M u^i\left(x_i^1(s)\right) \pi(s)\, ds.$$

It is shown in Mas-Colell (1991) and Crès et al (2009) that if $(p, x_1, \ldots x_I)$ is an equilibrium, then p and x_i for each i are all continuous maps from M to $\mathbb{R}^n_{++}$. In other words, prices, consumption and endowments are all elements of the same space $C(M, \mathbb{R}^n_{++})$.

2.2 A continuous-time economy

Suppose that in an economy the consumption of n goods is done continuously through time $t \in [0, T]$. Then, a continuous function $x^i : [0, T] \to \mathbb{R}^n_{++}$ represents the consumption of the n goods by agent i at time t. Alternatively, $x(t)$ may represent a continuous instantaneous rate of consumption.

3 Fredholm Index Theory

Since Fredholm theory is not widely used in the economic literature, in this section we provide some basic definitions, where the classical reference is the paper of Smale (1965). Before presenting the formal definitions, we will aim to clarify, rather informally, the motivation.

3.1 Motivation

Suppose that we consider a linear map T between any two vector spaces V and W. We may ask ourselves, what conditions would T need to satisfy in order for it to be a bijection, that is, a map that is both injective and surjective? If T were a bijection, this would also mean that T is invertible.

There are two basic results of linear algebra that would answer this question. First, recall that the kernel of T, or $\ker T$, consists of those points of V that are mapped into zero in W under T. In order for T to be injective, we would require that $\ker T = \{0\}$. Similarly, recall that the range of T, or $\operatorname{range} T$, consists of all those points that are in the image under T in W. For T to be surjective, we would require that $\operatorname{range} T = W$.

As it happens, these two conditions are rather restrictive. Fredholm operators were introduced since, loosely speaking, they are "almost invertible": they are "almost injective" and "almost surjective". By this we mean that $\ker T$ is a finite-dimensional subspace of V (not just the point $\{0\}$ but also not an infinite-dimensional set) and the range of T "misses" the entire set

W only by a finite-dimensional subspace.

Expanding further these notions, two linear maps $T : V \to W$ and $S : W \to V$ are "pseudo-inverses" to each other if $ST = I + G_1$ and $TS = I + G_2$, where I is the identity and G_1 and G_2 are two maps with finite-dimensional range. In other words, while ST and TS are not the identity, they fail to be so only by a "compact perturbation" of the identity. It can be shown that $T : V \to W$ will have a pseudo-inverse if and only if T is a Fredholm operator. Fredholm maps are the nonlinear notion of a Fredholm operator.

3.2 Definitions

If V and W are linear spaces and $T : V \to W$ is a linear map, we define the **kernel** of T, denoted $\ker T$, to be the set of points in V mapped into zero and the **range** is the image of V under T in W. Also, if Y is a linear subspace of W, we say that two points w_1 and w_2 of W are **equivalent modulo** Y, denoted $w_1 = w_2 (\text{mod} Y)$ if $w_1 - w_2 \in Y$. We denote by W/Y the set of equivalence classes. When equipped with a linear structure we call it the **quotient space** and define $\text{codim} Y = \dim W/Y$.

A linear **Fredholm operator** is a continuous linear map $L : E_1 \to E_2$ from one Banach space to another with the properties that:

1. $\dim \ker L < \infty$;

2. range L is closed;

3. coker $L = E_2/\text{range} L$ has finite dimension.

The **index of a Fredholm operator** L is an integer given by dim ker$L -$ dim coker$L$. Fredholm operators of index zero are of particular relevance since compact perturbations of the identity are Fredholm operators of index zero, and conversely, any Fredholm operator of index zero differs from a compact perturbation of the identity only by a linear homeomorphism.

A **Fredholm map** is a C^1 map $f : M \to V$ between differentiable manifolds locally like Banach spaces such that for each $x \in M$ the derivative $Df(x) : T_xM \to T_{f(x)}V$ is a Fredholm operator. The **index of a Fredholm map** f at the point $x \in M$ is defined to be the index of $Df(x)$. It can be shown that if M is connected, this definition does not depend on x. Again, Fredholm maps of index zero are of particular interest since any diffeomorphism between Banach spaces is a Fredholm map of index zero.

A **left Fredholm map**[3] is a map of Banach manifolds of class at least C^1 whose derivative at each point has closed image and finite dimensional kernel.

A map is $\sigma-$**proper** if its domain is the countable union of sets, restricted to each of which the function is proper.

[3]Some authors call it a **semi-Fredholm map.**

4 The Market

4.1 Preferences and consumption

Following on the examples of section 2, we assume that the **commodity space** is $C(M, \mathbb{R}^n)$ where M, the parameter space, is a compact subset of some $\mathbb{R}^m$. The **consumption set** $X = C^{++}(M, \mathbb{R}^n)$ is then the positive cone of $C(M, \mathbb{R}^n)$. It consists of the functions in $C(M, \mathbb{R}^n)$ that have an image in $\mathbb{R}^n_{++}$. Notice that X has a nonempty interior.

We suppose that there are $i = 1, \ldots, I$ agents and that their preferences are fixed and represented by a separable utility function

$$U_i(x) = \int_M u^i(x(t), t)dt$$

where $u^i(x(t), t) : \mathbb{R}^n_{++} \times M \to \mathbb{R}$ is a strictly monotonic, concave, C^2 function where $\{y \in R^n_{++} : u^i(y, t) \geq u^i(x, t)\}$ is closed. This implies that $U_i(x)$ is strictly monotonic, concave and twice Fréchet differentiable.

Two comments are in order. The first, is that Mas-Colell (1991) has a similar framework to the one that we propose and he pointed out then, and so do we here, that non-separability was one of the main stumbling blocks for a general theory of regular economies with infinitely many commodities. While Shannon and Zame (2002) overcame this difficulty for the question of determinacy, it still remains problematic for the study of the equilibrium manifold. Indeed, the main contributions in our understanding of the infinite equilibrium manifold (e.g., Balasko 1997a, 1997c) assumes separability. The second comment is that a double infinity of commodities and agents is the cause of strong indeterminacy results. While purely speculative at this stage, it might be possible to extend this paper into considering a continuum of agents, analogous to Kehoe et al (1989). The approach might include understanding which conditions we would need in order to guarantee that the "mean excess demand function" is Fredholm.

4.2 Prices

Strictly speaking, a price $p : C(M, \mathbb{R}^n) \to \mathbb{R}$ is a bounded and linear real-valued function on $C(M, \mathbb{R}^n)$ which gives non-negative values to any element of $C^{++}(M, \mathbb{R}^n)$. In other words, a price is an element of the positive cone of the dual space of the commodity set. However, it can be shown that with separable utilities, if a price p supports equilibria then $p \in C^{++}(M, \mathbb{R}^n)$, i.e., equilibrium prices, consumption and initial endowments are all elements of the same space X. See for instance Mas-Colell (1991), Chichilnisky and Zhou (1988) or Crés et al (2009).

If f and g are two elements of $C(M, \mathbb{R}^n)$, the inner product on $C(M, \mathbb{R}^n)$ is given by $\langle f, g \rangle = \int_M \langle f(t), g(t) \rangle \, dt$, so that if p and x denote price and consumption respectively, the **value** of x is given by

$$\langle p, x \rangle = \int_M \langle p(t), x(t) \rangle \, dt.$$

Finally, as in finite dimensions, we normalize prices and so define the **price space** to be

$$S = \left\{ p \in C^{++}(M, \mathbb{R}^n) : \|p\|^2 = \langle p, p \rangle = 1 \right\}.$$

4.3 Individual Demand Functions

The individual demand function of agent i is a map $f_i : S \times (0, \infty) \to X$ where $f_i(p, y)$ is the unique solution to the optimization problem

$$\max_{\langle p, x \rangle = y} U_i(x)$$

Denote by u_x the partial derivative of u with respect to x. It is shown in Chichilnisky and Zhou (1998) that given the assumptions about utility functions made in 4.1, the individual demand functions of all agents satisfy the following properties:

1. $\langle p, f_i(p, y) \rangle = y$ for any $p \in S$ and for any $y \in (0, \infty)$;
2. $u_x^i(f_i(p, y), t) = \lambda p$ for some $\lambda > 0$;
3. $f_i : S \times (0, \infty) \to X$ is a diffeomorphism (i.e., both f_i and its inverse are continuously differentiable); as such, $f_i : S \times (0, \infty) \to X$ is a Fredholm map of index zero.

5 Two Properties of Aggregate Excess Demand Functions

Recall that we have fixed preferences so the only parameters defining an economy are the initial endowments. We then denote an economy by $\omega = (\omega_1, \ldots, \omega_I) \in \Omega = X^I$. For a fixed economy $\omega \in \Omega$ its **aggregate excess demand function** is a map $Z_\omega : S \to C(M, \mathbb{R}^n)$ defined by

$$Z_\omega(p) = \sum_{i=1}^{I} \left(f_i(p, \langle p, \omega_i \rangle) - \omega_i \right).$$

We also define $Z : \Omega \times S \to C(M, \mathbb{R}^n)$ by the evaluation

$$Z(\omega, p) = Z_\omega(p).$$

Definition 1. *We say that $p \in S$ is an* ***equilibrium*** *of the economy $\omega \in \Omega$ if $Z_\omega(p) = 0$. We denote the* ***equilibrium set*** *by*

$$\Gamma = \left\{ (\omega, p) \in \Omega \times S : Z(\omega, p) = 0 \right\}.$$

In order to explore the structure of aggregate excess demand functions, we first show the well-known result that the excess demand defines a vector field on the price space[4].

[4]In the language of vector bundles, if we denote by TS the tangent bundle of S and TS_0 its zero section, Theorem 1 says that we can interpret Z_ω as a section of TS and an equilibrium as a point where this section intersects TS_0.

Theorem 1. *The excess demand function* $Z_\omega : S \to C(M, \mathbb{R}^n)$ *of economy* $\omega \in \Omega$ *is a vector field on* S*.*

Proof. Since f_i satisfies the property that $\langle p, f_i(p, y)\rangle = y$ for any $p \in S$ and for any $y \in (0, \infty)$, then

$$\begin{aligned}
\langle p\,,\, Z_\omega(p)\rangle &= \langle p\,,\, \sum_{i=1}^{I}(f_i(p, \langle p, \omega_i\rangle) - \omega_i)\rangle \\
&= \sum_{i=1}^{I}\langle p\,,\, f_i(p, \langle p, \omega_i\rangle)\rangle - \sum_{i=1}^{I}\langle p, \omega_i\rangle \\
&= \sum_{i=1}^{I}\langle p, \omega_i\rangle - \sum_{i=1}^{I}\langle p, \omega_i\rangle \\
&= 0.
\end{aligned}$$

□

In order to use techniques of differential topology in infinite dimensions, we require our maps to be Fredholm. We now show that this is the case for the excess demand function.

Theorem 2. *The excess demand function* $Z_\omega : S \to C(M, \mathbb{R}^n)$ *of economy* $\omega \in \Omega$ *is a Fredholm map of index zero.*

The proof of Theorem 2 is rather computational and so we leave it to the appendix. We can mention, however, that the proof consists of two parts. The first is to show that if Df_i denotes the Fréchet derivative of

$f_i : S \times (0, \infty) \to X$, then Df_i can be written as the sum of an invertible operator plus a finite rank operator and hence it is a Fredholm map of index zero. The second part consists of explicitly writing the Fréchet derivative of the excess demand function $Z_\omega : S \to C(M, \mathbb{R}^n)$, denoted DZ_ω, in terms of the Df_i's and once again showing that it can be written as the sum of an invertible operator plus a finite rank operator.

6 The equilibrium set

6.1 Regular values of Z

In this section we wish to show that the equilibrium set Γ is a manifold. Our result is an extension of Balasko's work (1988) to infinite dimensions. We will show that Γ is a manifold in two steps: first, in Theorem 3, we show that 0 is a regular value of the excess demand function Z. We will then use this fact in Theorem 4 to show that Γ is indeed a manifold (actually a Banach manifold), and also that the projection map from the equilibrium set to the parameter space Ω is smooth.

Theorem 3. *Let TS denote the tangent bundle to the price space S. Then, the derivative of the map $Z : \Omega \times S \to TS$ is a surjective map. In particular, it has 0 as a regular value.*

We also leave the proof to the appendix since it is rather computational.

6.2 Transversality

We need two final definitions in order to show that Γ is a manifold. But first recall that the "components" of a topological space are the "pieces" that the space can be broken into. Precisely, given a topological space $\mathcal{T}$, one defines an equivalence relation by setting $t_1 \sim t_2$ if there is a connected subspace of $\mathcal{T}$ containing both t_1 and t_2. The equivalence classes are called the **components** of $\mathcal{T}$.

Additionally, the closed subspace F of a Banach space E is said to **split**, if there is a closed subspace $G \subset E$ such that $E = F \oplus G$.

Definition 2. *(Abraham and Robbin, 1967, p.45) Let X and Y be C^1 manifolds, $f : X \to Y$ a C^1 map, and $W \subset Y$ a submanifold. We say that f is **transversal to W at a point** $x \in X$, in symbols $f \pitchfork_x W$, iff, where $y = f(x)$, either $y \notin W$ or $y \in W$ and*

1. *the inverse image $(T_x f)^{-1}(T_y W)$ splits; and,*

2. *the image $(T_x f)(T_x X)$ contains a closed component to $T_y W$ in $T_y Y$.*

*We say f is **transversal to** W, in symbols $f \pitchfork W$, iff $f \pitchfork_x W$ for every $x \in X$.*

Definition 3. *(Quinn, 1970) A C^∞ **representation of maps** $\rho : A : M \to N$ consists of Banach manifolds A, M, N together with a function*

$\rho : A \to C^\infty(M, N)$ *such that the evaluation map*

$$Ev_\rho : A \times M \to N, \qquad\qquad (a, m) \mapsto \rho_a(m)$$

is C^∞*.*

The relevance of these two notions is because Quinn (1970) shows that if we have a C^∞ map $F : W \to N$ which is transversal to Ev_ρ and if we form the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & W \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle F}\big\downarrow \\ A \times M & \xrightarrow{Ev_\rho} & N \\ {\scriptstyle \pi_A}\big\downarrow & & \\ A & & \end{array} \tag{1}$$

where $P = (Ev_\rho \times F)^{-1}(\Delta_N)$ and Δ_N denotes the diagonal in $N \times N$, then P is a C^∞ Banach manifold, and $\pi_A \circ h$ is a C^∞ map.

6.3 The equilibrium set is a Banach manifold

We are finally ready to show that the equilibrium set Γ is indeed a Banach manifold.

Theorem 4. *The equilibrium set* Γ *is a* C^∞ *Banach manifold. We shall call it the* ***equilibrium manifold****. Furthermore the natural projection map* $pr_\Omega : \Omega \times S|_\Gamma \to \Omega$ *is a* C^∞ *map.*

Proof. We start then by noticing that $Z : \Omega \times S \to TS$ is a C^∞ representation of maps as defined above. Notice also that the inclusion $T_0S \to TS$ is a C^∞ map. We also know from Theorem 3 that DZ is surjective, so it has 0 as a regular value. Then, we can form the pullback diagram

$$
\begin{array}{ccc}
\Gamma & \longrightarrow & T_0S \\
\downarrow & & \downarrow \\
\Omega \times S & \xrightarrow{Z} & TS \\
{\scriptstyle pr_\Omega}\downarrow & & \\
\Omega & &
\end{array}
$$

and as in diagram (1) of definition 3 we get that Γ is a C^∞ Banach manifold and the natural projection map is a C^∞ map. $\square$

7 Regular and Critical Economies

In this section we define the notion of regular and critical infinite economies, and regular and critical infinite prices. Recall that if f is a C^1 map from an open connected subset of a Banach space X to another Banach space Y, and Df denoted the Fréchet derivative of f, then $x \in X$ is is a **regular point** for f if $Df(x)$ if a surjective linear mapping. If $x \in X$ is not regular, x is then called a **singular point**.

Similarly, singular values and regular values y of f are defined by considering the sets $f^{-1}(y)$. If $f^{-1}(y)$ has a singular point, y is called a **singular value**, otherwise y is a **regular value**.

Definition 4. *We say that a smooth infinite economy is* ***regular*** *(resp.* ***crit-***

ical) *if and only if* ω *is a regular (resp. critical) value of the projection* $pr : \Gamma \to \Omega$.

Definition 5. *Let* Z_ω *be the excess demand of economy* ω*. A price system* $p \in S$ *is a* ***regular equilibrium price*** *system if and only if* $Z_\omega(p) = 0$ *and* $DZ_\omega(p)$ *is surjective.*

We would like to compare the set of regular economies with those economies whose excess demand function has only regular prices. In finite dimensions these two sets are equal. Quinn (1970) will tell us that these two sets coincide; precisely, in diagram (1) of definition 3, $\rho_a \pitchfork F$ if and only if a is a regular value of $\pi_A \circ h$. And so we get,

Theorem 5. *The economy* $\omega \in \Omega$ *is regular if and only if all equilibrium prices of* Z_ω *are regular.*

Proof. Consider the diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & T_0S \\ \big\downarrow & & \big\downarrow \\ \Omega \times S & \xrightarrow{Z} & TS \\ {\scriptstyle pr_\Omega}\big\downarrow & & \\ \Omega & & \end{array}$$

Quinn's result says that the excess demand Z_ω is transversal to zero if and only if 0 is a regular value of pr_Ω. □

7.1 Determinacy

We would now like to understand how big is the set of economies that give an excess demand function with all equilibrium prices being regular. For that, we need a result of Quinn who has also proved that a transversal density theorem holds in infinite dimensions.

Theorem 6. *(Quinn, 1970) Let $\rho : A : M \to N$ be a C^∞ representation of left Fredholm maps, M separable, and $F : W \to N$ a C^∞ σ-proper left Fredholm map. If further*

1. *F is transversal to Ev_ρ; and,*
2. *each ρ_a satisfies that for each $m \in M$ and $w \in W$ such that $\rho_a(m) = F(w)$, then $(imT_m\rho_a) \cap (imT_wF)$ is finite dimensional*

then the set of a with $\rho_a \pitchfork F$ is residual in A.

The infinite-dimensional transversal density theorem can be used to give us an alternative proof that a generic economy is regular.

Theorem 7. *The set of regular economies is residual in Ω. That is, the set of economies $\omega \in \Omega$ that give rise to an excess demand function Z_ω with only regular equilibrium prices, are residual in Ω.*

Proof. Observe that the inclusion $T_0S \to TS$ given by $p' \mapsto (p', 0)$ is σ-proper since its domain consists of one set restricted to which the inclusion is proper since the inclusion map is continuous. Now, $T_0S \to TS$ is also left Fredholm since the derivative of the inclusion map is again the inclusion map, so it is continuous (and so has a closed image) and has finite dimensional kernel.

We also know that $Z(\omega, p)$ has 0 as a regular value since $DZ(\omega, p)$ is surjective.

All that we need to show is that for each $p \in S$ and $p' \in T_0S$ such that $Z_\omega(p) = I(p')$, where $I : T_0S \to TS$ is the inclusion map, we have

$$(\mathrm{im} T_p Z_\omega) \cap (\mathrm{im} T_x I)$$

is finite-dimensional. But this follows immediately if we notice that $Z_\omega(p) = I(p')$ whenever p is an equilibrium, i.e. a zero of the vector field Z_ω. In this case $(\mathrm{im} T_p Z_\omega) = 0$ and $(\mathrm{im} T_x I) = 0$. Therefore, Theorem 6 implies the result. $\square$

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Appendix

Proof of Theorem 2

Proof. Recall that the consumers' problem is given by

$$\max_{x \in X} U_i(x) \quad \text{s.t.} \quad \langle p, x \rangle = w_i$$

where

- $X = C^{++}(M, \mathbb{R}^n)$;
- $U_i : X \to \mathbb{R}$ is given by $U_i(x) = \int_M u^i(x(t), t)\, dt$;
- $u^i : \mathbb{R}^n_{++} \times M \to \mathbb{R}$ is, for each i, strictly monotonic, concave, C^2 function where $\{y \in \mathbb{R}^n_{++} : u^i(y, t) \geq u^i(x, t)\}$ is closed;
- In principle, p is an element of the positive cone of the dual of $C(M, \mathbb{R}^n)$. However, we have explained that with separable utilities, actually p is an element of $X = C^{++}(M, \mathbb{R}^n)$;
- Furthermore, we normalize so that $p \in S = \{p \in C^{++}(M, \mathbb{R}^n) : \|p\| = 1\}$;
- $w_i = \langle p, \omega_i \rangle \in (0, \infty)$.

Notice that $p \in S$ and $\omega_i \in X$, for each i, are independent (i.e., exogenously determined) variables of the problem.

Now, because of the assumptions that we have placed on the utility functions u^i (smoothness, concavity, monotonicity), this implies that for each

$p \in S$ and for each $w_i \in (0, \infty)$ the optimization problem has a unique solution that we will denote by $f_i(p, w_i)$ where $f_i : S \times (0, \infty) \to X$.

The first order optimality conditions can then be written as:

$$w_i = \langle p, f_i(p, w_i) \rangle \tag{2}$$

$$DU_i(f_i(p, w_i)) = \lambda_i(p, w_i) \cdot p \tag{3}$$

where DU_i denotes the Fréchet derivative of $U_i : X \to \mathbb{R}$ and $\lambda_i : S \times (0, \infty) \to \mathbb{R}$ is a Lagrange multiplier.

The strategy is to calculate the total derivatives of equations (2) and (3) and solve for $Df_i(p, w_i)$. We will exploit the simplicity of $U_i(x)$ written in terms of u^i. Hence, we first write equations (2) and (3) as

$$w_i = \langle p, f_i(p, w_i) \rangle \tag{4}$$

$$u^i_x(f_i(p, w_i), t) = \lambda_i(p, w_i) \cdot p \tag{5}$$

Taking total derivatives on both sides of equations (4) and (5) we get

$$Dw_i = f_i(p, w_i) + \langle p, Df_i(p, w_i) \rangle$$

$$u^i_{xx}(f_i(p, w_i), t) \cdot Df_i(p, w_i) = \lambda_i(p, w_i) + p \cdot D\lambda_i(p, w_i)$$

where we write $\langle p, Df_i(p, w_i) \rangle$ to denote the linear transformation Df_i composed with the linear transformation p.

Simplifying, and remembering that since $u^i(x)$ is concave, the linear transformation (u^i_{xx}) is negative definite and hence (u^i_{xx}) is invertible for each t, we now have

$$Dw_i = f_i(p, w_i) + \langle p, Df_i(p, w_i)\rangle \tag{6}$$

$$Df_i(p, w_i) = \lambda_i(p, w_i)\,(u^i_{xx})^{-1} + (u^i_{xx})^{-1}\,p \cdot D\lambda_i(p, w_i) \tag{7}$$

Making a substitution of the expression of Df_i found in (7) into Dw_i of equation (6), and remembering that p is linear, we get

$$\begin{aligned} Dw_i &= f_i(p, w_i) + \langle p, Df_i(p, w_i)\rangle \\ &= f_i(p, w_i) + \langle p, \lambda_i(p, w_i)\,(u^i_{xx})^{-1} + D\lambda_i(p, w_i)\,(u^i_{xx})^{-1}\,p\rangle \\ &= f_i(p, w_i) + \langle p, \lambda_i(p, w_i)\,(u^i_{xx})^{-1}\rangle + \langle p, D\lambda_i(p, w_i)\,(u^i_{xx})^{-1}\,p\rangle \\ &= f_i(p, w_i) + \lambda_i(p, w_i)\,(u^i_{xx})^{-1}p + D\lambda_i(p, w_i)\,\langle p, (u^i_{xx})^{-1}\,p\rangle. \end{aligned}$$

Therefore,

$$D\lambda_i(p, w_i) = \frac{1}{\langle p, (u^i_{xx})^{-1}\,p\rangle}\left[Dw_i - f_i(p, w_i) - \lambda_i(p, w_i)\,(u^i_{xx})^{-1}p\right] \tag{8}$$

where the denominator $\langle p, (u^i_{xx})^{-1}\,p\rangle$ does not vanish since p and $(u^i_{xx})^{-1}$ are positive operators.

We substitute the expression of $D\lambda_i$ found in (8) into (7) to get,

$$\begin{aligned} Df_i(p,w_i) &= \lambda_i(p,w_i)\,(u^i_{xx})^{-1} + D\lambda_i(p,w_i)\,(u^i_{xx})^{-1}\,p \\ &= \lambda_i(p,w_i)\,(u^i_{xx})^{-1} + \\ &+ \frac{(u^i_{xx})^{-1}\,p}{\langle p,(u^i_{xx})^{-1}\,p\rangle}\left[Dw_i - f_i(p,w_i) - \lambda_i(p,w_i)\,(u^i_{xx})^{-1}p\right]. \end{aligned}$$

What we have shown is that $Df_i(p,w_i)$ can be written as the sum of the invertible operator

$$\lambda_i(p,w_i)\,(u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\,p}{\langle p,(u^i_{xx})^{-1}\,p\rangle}Dw_i$$

and the finite rank operator

$$-\frac{(u^i_{xx})^{-1}\,p}{\langle p,(u^i_{xx})^{-1}\,p\rangle}\left[f_i(p,w_i) + \lambda_i(p,w_i)(u^i_{xx})^{-1}\,p\right].$$

Now, let $w_i = \langle p,\omega_i\rangle$ and recall that $Z_\omega : S \to C(M,\mathbb{R}^n)$ is given by

$$Z_\omega(p) = \sum_{i=1}^{I}\left(f_i(p,\langle p,\omega_i\rangle) - \omega_i\right)$$

and so $DZ_\omega : TS \to TC(M, \mathbb{R}^n)$ is given by

$$
\begin{aligned}
DZ_\omega(p) &= \sum_{i=1}^{I} Df_i(p, w_i) \\
&= \sum_{i=1}^{I} \left\{ \lambda_i(p, w_i)\, (u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\, p}{\langle p, (u^i_{xx})^{-1}\, p\rangle} Dw_i \right\} + \\
&+ \sum_{i=1}^{I} \left\{ -\frac{(u^i_{xx})^{-1}\, p}{\langle p, (u^i_{xx})^{-1}\, p\rangle} \left[f_i(p, w_i) + \lambda_i(p, w_i)(u^i_{xx})^{-1}\, p \right] \right\}.
\end{aligned}
$$

Finally, noticing again that since $u^i(x)$ is concave, the linear transformation (u^i_{xx}) is negative definite and hence (u^i_{xx}) is invertible. Additionally, the sum of negative-definite linear transformations is again negative-definite. Hence,

$$
\sum_{i=1}^{I} \left\{ -\frac{(u^i_{xx})^{-1}\, p}{\langle p, (u^i_{xx})^{-1}\, p\rangle} \left[f_i(p, w_i) + \lambda_i(p, w_i)(u^i_{xx})^{-1}\, p \right] \right\}
$$

has finite rank, and

$$
\sum_{i=1}^{I} \left\{ \lambda_i(p, w_i)\, (u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\, p}{\langle p, (u^i_{xx})^{-1}\, p\rangle} Dw_i \right\}
$$

is invertible. Therefore, DZ_ω is written as the sum of an invertible operator and an operator of finite rank which in turn implies that Z_ω is a Fredholm map of index zero. □

Proof of Theorem 3

Proof. Notice that because of the properties of individual demand functions studied in section 4.3, we get that Z is differentiable. We need to compute the derivative

$$DZ : T(\Omega \times S) \to T(TS).$$

Linearizing $Z(\omega, p)$ to first order in ϵ, dropping the $O(\epsilon^2)$ terms, and letting $w_i = \langle p, \omega_i \rangle$, we get

$$\begin{aligned}
&Z(\omega_1 + \epsilon k_1, \ldots, \omega_I + \epsilon k_I, p + \epsilon h) \\
&= \sum f_i(p + \epsilon h, \langle p + \epsilon h, \omega_i + \epsilon k_i \rangle) - \sum (\omega_i + \epsilon k_i) \\
&= \sum f_i(p + \epsilon h, \langle p, \omega_i \rangle + \epsilon \langle p, k_i \rangle + \epsilon \langle h, \omega_i \rangle) - \sum \omega_i - \epsilon \sum k_i \\
&= \sum [f_i(p, \langle p, \omega_i \rangle) + \epsilon (D_{w_i} f_i)_{(p, \langle p, \omega_i \rangle)}(\langle p, k_i \rangle) + \\
&+ \epsilon (D_{w_i} f_i)_{(p, \langle p, \omega_i \rangle)}(\langle h, \omega_i \rangle) + \epsilon (D_p f_i)_{(p, \langle p, \omega_i \rangle)}(h)] - \sum \omega_i - \epsilon \sum k_i \\
&= Z(\omega_1, \ldots, \omega_I, p) + \\
&+ \epsilon \sum [(D_p f_i)_{(p, \langle p, \omega_i \rangle)}(h) + (D_{w_i} f_i)_{(p, \langle p, \omega_i \rangle)}(\langle p, k_i \rangle + \langle h, \omega_i \rangle) - k_i]
\end{aligned}$$

Alternatively, in matrix form, $DZ_{(\omega,p)} =$

$$\left(\begin{array}{c|c}
\overbrace{0 \ldots 0}^{i=1,\ldots,I} & 1 \\
\hline
\underbrace{(D_{w_i} f_i)_{(p, \langle p, \omega_i \rangle)}(\langle p, - \rangle) - Id}_{i=1,\ldots,I} & \sum_i (D_p f_i)_{(p, \langle p, \omega_i \rangle)}(-) + \\
 & + \sum_i (D_{w_i} f_i)_{(p, \langle p, \omega_i \rangle)}(\langle -, \omega_i \rangle)
\end{array}\right)$$

where the *dashes* simply indicate that the left side of the matrix acts on $(k_1, \ldots, k_I)$ while the right acts on h .

To compute the cokernel let

$$DZ_{(\omega,p)}(k_1, \ldots, k_I, h) = (Q, \dot{Q}) \in T(TS).$$

We need to solve for $(k_1, \ldots, k_I, h)$. We first observe that $h = Q$. The second row would then be,

$$\sum \{[(D_{w_i} f_i)(\langle p, k_i \rangle) - (k_i)] + [(D_p f_i)(Q)] + [(D_{w_i} f_i)(\langle Q, \omega_i \rangle)]\} = \dot{Q}.$$

Then,

$$\sum [(D_{w_i} f_i)(\langle p, k_i \rangle) - (k_i)] = H(Q, \dot{Q}) \tag{9}$$

where

$$H(Q, \dot{Q}) = \dot{Q} - \sum \{[(D_p f_i)(Q)] + [(D_{w_i} f_i)(\langle Q, \omega_i \rangle)]\}.$$

But for every $i = 1, \ldots, I$, the map

$$k_i \mapsto (D_{w_i} f_i)(\langle p, k_i \rangle) - (k_i)$$

is onto. And, therefore, so is DZ. □