DOCUMENTS DE TREBALL

DE LA FACULTAT D'ECONOMIA I EMPRESA

Col.lecció d'Economia

E10/238

Multi-sided Böhm-Bawerk assignment markets: the core

Oriol Tejada

Department of Actuarial, Financial and Economic Mathematics
Universitat de Barcelona, Av.Diagonal, 690, E-08034 Barcelona, Spain
e-mail: oriol.tejada@ub.edu

Agraïments: I am very grateful to Carles Rafels for his guidance and helpful comments. I would also like to thank Josep Maria Izquierdo, Xavier Martínez de Albéniz and Marina Núñez for their suggestions. Support from grant Programa FPU of Ministerio de Educación y Ciencia of the Spanish Government and from the Barcelona Economics Program of CREA, Ministerio de Educación y Ciencia and FEDER, under grant ECO2008-02344/ECON is acknowledged

Multi-sided Böhm-Bawerk assignment markets: the core

Abstract: We introduce the class of multi-sided Böhm-Bawerk assignment games, which generalizes the well-kown two-sided Böhm-Bawerk assignment games to situations with an arbitrary number of sectors. We reach the extreme core allocations of any multi-sided Böhm-Bawerk assignment game by means of an associated convex game defined on the set of sectors instead of the set of sellers and buyers. We also study when the core of these games is stable in the sense of von Neumann-Morgenstern.

Keywords: Assignment games, multi-sided markets, homogeneous goods, core, extreme points

JEL Classification: C70, C78

Resum: En aquest treball introduïm la classe de multi-sided Böhm-Bawerk assignment games, que generalitza la coneguda classe de jocs d'assignació de Böhm-Bawerk bilaterals a situacions amb un nombre arbitrari de sectors. Trobem els extrems del core de qualsevol multi-sided Böhm-Bawerk assignment game a partir d'un joc convex definit en el conjunt de sectors enlloc del conjunt de venedors i compradors. Addicionalment estudiem quan el core d'aquests jocs d'assignació és estable en el sentit de von Neumann-Morgenstern.

1 Introduction

Consider a market with two different goods, for instance software and hardware products. In this market there are n_S owners of one unit of software and n_H owners of one unit of hardware. All of them want to sell their goods. There are also n_B buyers, which want to buy at most one unit of software and one unit of hardware and have no utility on buying separately either one. The i^{th} software seller values her good at c_i^S dollars, the j^{th} hardware seller values her good at c_j^H dollars and the k^{th} buyer values the bundle formed by the software and the hardware goods of the former sellers at w_{ij}^k dollars.

In this market, a transaction can only be carried out when a buyer pays for exactly one unit of software and one unit of hardware. Let p_i and q_j be the prices that the k^{th} buyer pays for the goods of the i^{th} software seller and the j^{th} hardware seller, respectively. At these prices, her utility is given by $w_{ij}^k - p_i - q_j$, whereas the benefit of the software seller is $p_i - c_i^S$ and the benefit of the hardware seller is $q_j - c_j^H$. If we assume that the utility of the agents is monetary and transferable the total surplus generated by this transaction is $\left(w_{ij}^k - p_i - q_j\right) + \left(p_i - c_i^S\right) + \left(q_j - c_j^H\right) = w_{ij}^k - c_i^S - c_j^H$. If $w_{ij}^k - c_i^S - c_j^H < 0$ we assume that no transaction will be carried out since no prices favorable to all parts exist. Let $a_{ijk} = \max\left\{0, w_{ij}^k - c_i^S - c_j^H\right\}$ be the total gain generated when the i^{th} software seller, the j^{th} hardware seller and the k^{th} buyer make a transaction. The above market is completely determined by giving the sets of buyers and owners (or sellers) and the set of parameters a_{ijk}.

In this paper we study a particular case of the above market obtained when all software goods and hardware goods are respectively *homogeneous*, i.e. the valuation of each buyer does not depend on which sellers she buys the two goods from. Therefore we can denote $w_{ij}^k = w^k$, and thus the profit generated by the i^{th} software seller, the j^{th} hardware seller and the k^{th} buyer is $a_{ijk} = \max\left\{0, w^k - c_i^S - c_j^H\right\}$. This latter type of markets generalizes the bilateral *Böhm-Bawerk horse markets* (Böhm-Bawerk, 1923) to a multilateral situation, and hence will be called *multi-sided Böhm-Bawerk markets*.

We analyze multi-sided Böhm-Bawerk markets within the framework of multi-sided assignment games, which are introduced by Quint (1991) as the generalization of two-sided assignment games (Shapley and Shubik, 1972), and they have been studied also in Lucas (1995), Stuart (1997), Sherstyuk (1998, 1999), Brânzei et al. (2007) and Tejada and Rafels (2010). The two-sided Böhm-Bawerk market has also been reinterpreted as an auction game in Schotter (1974) and Muto (1983). A game-theoretical study of the two-sided Böhm-Bawerk market can be found in Shapley and Shubik (1972), Moulin (1995), Osborne (2004) and Nuñez and Rafels (2005).

The main objective of this paper is to study the set of extreme core allocations of multi-sided Böhm-Bawerk assignment games. To do so, to each m-sided Böhm-Bawerk assignment game we associate a nonnegative convex game of m players, which are fictitious agents that correspond to the m sectors of the market, and hence will be called the *sectors game*. We prove that the core and the set of extreme core allocations of this latter game are strongly related with those of the former. As a consequence, we show that all extreme core allocations of a multi-sided Böhm-Bawerk assignment game are marginal worth vectors, generalizing a property that holds for all two-sided assignment games (Hamers et al., 2002). We also give attainable bounds for the number of extreme core allocations of a multi-sided Böhm-Bawerk assignment game and we study when the core of these games is stable.

The rest of the paper is organized as follows. In Section 2 we introduce the notation and we describe the multi-sided assignment model. In Section 3 we introduce the class of multi-sided Böhm-Bawerk problems and we present the results of the paper.

2 Preliminaries and notation

A cooperative game is a pair (N, v), where N is the finite set of players and $v(S) \in \mathbb{R}$ for any coalition $S \subseteq N$, being $v(\varnothing) = 0$. The core of a game is the set of allocations that cannot be improved upon by any coalition on its own. Formally, given (N, v), the core is the set $C(v) := \{x \in \mathbb{R}^n : x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\}$, where as usual $x(S) := \sum_{i \in S} x_i$ and $x(\varnothing) = 0$. A game is *balanced* if the core is nonempty. A *subgame* of (N, v) is any game (N', v') where $\varnothing \subsetneq N' \subseteq N$ and v' is the restriction of v to subsets of N'. A game is *totally balanced* if the core of any subgame is nonempty.

Given a finite set N, an ordering θ of N is a bijection from N to $\{1, ..., |N|\}$, where $|N|$ denotes the cardinality of N. Let $\Theta(N)$ be the set of all orderings of N. Given (N, v), the *marginal worth vector* $m^{\theta}(v) \in \mathbb{R}^n$ associated with θ is defined (see Shapley, 1972) by $m_i^{\theta} = v(\{j \in N : \theta(j) \leq \theta(i)\}) - v(\{j \in N : \theta(j) < \theta(i)\})$, for all $i \in N$. A game (N, v) is *convex* if for all $i \in N$ and for all $S \subseteq T \subseteq N \backslash \{i\}$ we have $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. It is well-known (Shapley, 1972, and Ichiishi, 1981) that a game is convex if and only if its core coincides with the convex hull of all marginal worth vectors.

Given a cooperative game (N, v), a vector $x \in \mathbb{R}^n$ is efficient if $\sum_{i=1}^n x_i = v(N)$. The set of *imputations* is the set of individually rational efficient vectors, i.e. $I(v) = \{x \in \mathbb{R}^n : x_i \geq v(\{i\}), \sum_{i=1}^n x_i = v(N)\}$. An imputation x dominates another imputation y via coalition $S \subseteq N$ if $x(S) \leq v(S)$ and $x_i > y_i$ for all $i \in S$. Then, a binary relation is defined on the set of imputations: given $x, y \in I(v)$, we say x *dominates* y if it does so via some coalition. With this definition, the core $C(v)$, whenever it is nonempty, is proved to coincide with the set of undominated imputations. This means that all allocations outside the core are dominated, although not necessarily dominated by a core allocation.

A subset V of imputations is a *stable set* (von Neumann and Morgenstern, 1944) if it is *internally stable* (for all $x, y \in I(v)$, x does not dominate y) and *externally stable* (for all $y \in I(v) \backslash V$, there exists $x \in V$ such that x dominates y). Since the core is the set of undominated imputations, all the stable sets of a given game (N, v) contain its core. And when the core is a stable set, then it is the unique stable set.

An *m-sided assignment problem* (m-SAP) denoted by $(N^1, ..., N^m; A)$, is given by $m \geq 2$ different nonempty finite sets (or sectors) of agents $N^1, ..., N^m$ and a nonnegative m-

dimensional matrix $A = (a_E)_{E \in \prod_{k=1}^{m} N^k}$. With some abuse of notation, let $N^k = \{1, 2, ..., n_k\}$ for all k, $1 \leq k \leq m$. We shall refer to the i^{th} agent of type k as $i \in N^k$. We name any m-tuple of agents $E \in \prod_{k=1}^{m} N^k$ an *essential coalition*. Each entry $a_E \geq 0$ represents the profit associated to the essential coalition E. As an abuse of notation, we also use E to denote the set of agents that form the essential coalition. An m-SAP is *square* if $n_1 = ... = n_m$.

A *matching* among $N^1, ..., N^m$ is a set of essential coalitions $\mu = \{E^r\}_{r=1}^{t}$ where $t = \min_{1 \leq k \leq m} |N^k|$ and any agent belongs at most to one of the essential coalitions $E^1, ..., E^t$. We denote by $\mathcal{M}(N^1, ..., N^m)$ the set of all matchings among $N^1, ..., N^m$. An agent $i \in N^k$, for some $k = 1, ..., m$, is *unmatched* under μ if it does not belong to any of its essential coalitions. A matching is *optimal* if it maximizes $\sum_{E \in \mu} a_E$ in $\mathcal{M}(N^1, ..., N^m)$. We denote by $\mathcal{M}_A^*(N^1, ..., N^m)$ the set of all optimal matchings of $(N^1, ..., N^m; A)$.

For each multi-sided assignment problem $(N^1, ..., N^m; A)$, the associated *multi-sided assignment game* (m-SAG) is the cooperative game (N, ω_A) with set of players $N = \cup_{k=1}^{m} N^k$ composed of all agents of all types and characteristic function

$$(1) \qquad \omega_A(S) = \max_{\mu \in \mathcal{M}(N^1 \cap S, ..., N^m \cap S)} \left\{ \sum_{E \in \mu} a_E \right\}, \text{ for any } S \subseteq N,$$

where the summation over the empty set is zero.

Given (N, ω_A), its core, $C(\omega_A)$, coincides with the set of nonnegative vectors $x = (x_{11}, ..., x_{1n_1}; ...; x_{m1}, ..., x_{mn_m})$, where x_{ki} stands for the payoff to agent $i \in N^k$, that satisfy $a_E - \sum_{k=1}^{m} x_{ki_k} \leq 0$ for any $E = (i_1, ..., i_m) \in \prod_{k=1}^{m} N^k$, where the inequality must be tight if E belongs to some optimal matching, and $x_{ki} = 0$ if agent $i \in N^k$ is unmatched under some optimal matching. The two latter conditions guarantee the efficiency of the core allocations.

In the case of only two sectors, i.e. $m = 2$, the above setting reduces to the classic Shapley-Shubik assignment market (Shapley and Shubik, 1972). It is well-known that two-sided assignment games are totally balanced. However, for more than two sides the core of a m-SAG may be empty -see Kaneko and Wooders (1982) in a more general framework or Quint (1991)- and also balanced m-SAGs might not be totally balanced (Quint, 1991).

3 The Böhm-Bawerk model

The three-sided markets described in the Introduction can be easily generalized to include arbitrary m-sided markets with $m-1$ different types of homogeneous goods. When $m=2$ the setting reduces to the classical bilateral Böhm-Bawerk assignment market, a celebrated model that has received wide attention in the literature (see Introduction). Observe that each buyer or seller in these markets is characterized by a single arbitrary nonnegative valuation, and the set of all these valuations are the basic data of the market. As in the case of Shapley and Shubik bilateral market, no restrictions are placed on communication, on transfers of money, or on transfers of goods.

The basic problem is to decide how the profitability of the market that comes from the differences in subjective valuations is going to be shared among sellers and buyers. In this market, a profit can only be reached through trades among agents in the market, i.e. assigning buyers to sellers and forming matchings. Hence, the situation fits into the framework of multi-sided assignment games. Thus, to analyze the problem we define a multi-sided assignment game based on the set of valuations of buyers and sellers and we study its core. In the particular case of two-sided Böhm-Bawerk assignment games, it is well-known that the core is a segment, whose extremes are the buyers-optimal and the sellers-optimal allocations. In this paper we generalize these results.

Let us introduce a multi-sided Böhm-Bawerk market (or problem) with an arbitrary number of sectors.

Definition 1 *An m-sided Böhm-Bawerk market (or problem) is a pair $(\mathbf{c}; w)$ where $\mathbf{c} = (c_1, ..., c_{m-1}) \in \mathbb{R}^{N_1} \times ... \times \mathbb{R}^{N_{m-1}}$ are the sellers' valuations and $w = (w_1, ..., w_{n_m}) \in \mathbb{R}^{N_m}$ are the buyers' valuations.*

From now on, in order to simplify the analysis of the model we will assume that valuations of the sellers of each sector are arranged in a nondecreasing way and valuations of the buyers are arranged in a nonincreasing way, i.e.

$$c_{k1} \leq ... \leq c_{kn_k}, \text{ for all } k = 1, ..., m-1 \text{ and } w_1 \geq ... \geq w_{n_m}. \tag{2}$$

Given an m-sided Böhm-Bawerk problem $(\mathbf{c}; w)$, we denote by $A(\mathbf{c}; w)$ the m-dimensional matrix defined by

$$a_E = \max\left\{0, w_{i_m} - \sum_{k=1}^{m-1} c_{ki_k}\right\}, \text{ for all } E = (i_1, ..., i_m) \in \prod_{k=1}^{m} N^k. \tag{3}$$

Notice that, by (2), for all $E, E' \in \prod_{k=1}^{m} N^k$,

$$E \leq E' \Longrightarrow a_E \geq a_{E'}. \tag{4}$$

When no confusion may arise, we will write simply A instead of $A(\mathbf{c}; w)$. Example 1 below is based on an example from both the paper of Shapley and Shubik (1972) and Böhm-Bawerk's (1923) book, modified in such a way that each seller has been unsymmetrically split up into two different sellers. That is, eight individuals each have one software good for sale and eight other individuals each have one hardware good for sale. Also ten other individuals each wish to buy exactly one software good and one hardware good. Although all software goods are alike and all hardware goods are alike, traders (either buyer or sellers) have different subjective valuations. This numerically specific market will be used through the paper and it will translate into a 26-person cooperative game (instead of an 18-person game in the case of Shapley and Shubik). We use $S_1, ..., S_8$, $H_1, ..., H_8$ and $B_1, ..., B_{10}$ to denote respectively the software sellers, the hardware sellers and the buyers.

Example 1 *A three-sided market with the following agents' valuations:*

Software sellers	Hardware sellers	Buyers
S_1 values her good at \$5	H_1 values her good at \$5	B_1 values a pair at \$30
S_2 values her good at \$5	H_2 values her good at \$6	B_2 values a pair at \$28
S_3 values her good at \$7	H_3 values her good at \$8	B_3 values a pair at \$26
S_4 values her good at \$8	H_4 values her good at \$9	B_4 values a pair at \$24
S_5 values her good at \$**11**	H_5 values her good at \$**9**	B_5 values a pair at \$**22**
S_6 values her good at \$12.7	H_6 values her good at \$10.3	B_6 values a pair at \$21
S_7 values her good at \$13	H_7 values her good at \$12	B_7 values a pair at \$20
S_8 values her good at \$13	H_8 values her good at \$13	B_8 values a pair at \$18
		B_9 values a pair at \$17
		B_{10} values a pair at \$15

Given $(\mathbf{c}; w)$ an m-sided Böhm-Bawerk market, $(N, \omega_{A(\mathbf{c};w)})$ is the associated multi-sided assignment game -see (1)-, which we will call a *multi-sided Böhm-Bawerk assignment game*, where N is composed of all sellers and buyers and $\omega_{A(\mathbf{c};w)}$ is defined by (1) and (3).

For all $i \in \mathbb{N}$, we introduce the notation $E^i := (i, ..., i) \in \mathbb{R}^m$. By (2), the diagonal matching $\{E^i : 1 \leq i \leq n\}$ is an optimal matching (in general it is not the unique optimal matching), where $n := \min_{1 \leq k \leq m} n_k$. In this paper we study the core $C(\omega_{A(\mathbf{c};w)})$ of $(N, \omega_{A(\mathbf{c};w)})$, which coincides with the following set:

$$(5) \qquad \left\{ x \in \mathbb{R}_+^{N^1} \times ... \times \mathbb{R}_+^{N^m} \;\middle|\; \begin{array}{l} x(E^i) = a_{E^i} \text{ for all } 1 \leq i \leq n, \\ x(E) \geq a_E \text{ for all } E \in \prod_{k=1}^m N^k \text{ and} \\ x_{ki} = 0 \text{ for all } i \in N^k, k \in M \text{ and } i > n. \end{array} \right\},$$

which is a polyhedral in $\mathbb{R}_+^{N^1} \times ... \times \mathbb{R}_+^{N^m}$ and hence it has a finite number of extreme points[1]. From Quint (1991), we know that $(N, \omega_{A(\mathbf{c};w)})$ is a totally balanced game, which implies that $C(\omega_{A(\mathbf{c};w)})$ is always nonempty.

As seen in the Introduction and using (3), only when $a_E > 0$ there exist prices that

[1] $x \in C(\omega_A)$ is an *extreme point* of $C(\omega_A)$ if $x = \frac{1}{2}x' + \frac{1}{2}x''$ where $x', x'' \in C(\omega_A)$ implies $x = x' = x''$.

support a trade between the agents that form the essential coalition E and imply a Pareto improvement with respect to the non-trade situation. A matching among $N^1, ..., N^m$ assigns agents to agents and form essential coalitions and singletons, and hence can be interpreted as a collection of trades. We define r as the highest number of Pareto-improving trades that can take place simultaneously:

$$r = \max_{1 \leq i \leq n} \{i : a_{E^i} > 0\}, \tag{6}$$

with the convention that $r = 0$ if all entries of $A(\mathbf{c}; w)$ are zero. We say that $i \in N^k$, for some $k = 1, .., m$, is *active* if $1 \leq i \leq r$. Otherwise we say that $i \in N^k$ is *inactive*.

We also introduce a vector $t_{\mathbf{c};w} \in \mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m}$ which includes the nonnegative differences in valuations of either sellers or buyers with respect to the corresponding r^{th} seller of the same sector or r^{th} buyer, respectively. As we show in Theorem 1, allocations of $C(\omega_{A(\mathbf{c};w)})$ can be decomposed into two terms, one variable term and one constant term given precisely by t. The *translation vector* $t_{\mathbf{c};w} = (t_{11}, ..., t_{1n_1}; ...; t_{m1}, ..., t_{mn_m}) \in \mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m}$ is defined by

$$\begin{aligned} t_{ki} &= \max\{0, c_{kr} - c_{ki}\} \quad \text{for all } 1 \leq k \leq m-1 \text{ and } 1 \leq i \leq n_k, \\ t_{mi} &= \max\{0, w_i - w_r\} \quad \text{for all } 1 \leq i \leq n_m. \end{aligned} \tag{7}$$

In Example 1, we have $r = 5$ (it is marked in bold in Example 1) and

$$t_{\mathbf{c};w} = (6, 6, 4, 3, 0, 0, 0, 0; 4, 3, 1, 0, 0, 0, 0, 0; 8, 6, 4, 2, 0, 0, 0, 0, 0, 0).$$

In the following, to any multi-sided Böhm-Bawerk assignment game we associate another game defined on the *set of sectors* $M = \{1, ..., m\}$. Below we discuss why we call these fictitious players as sectors. To define this new game we only take into account both the r^{th} and the $r+1^{th}$ agents (if exist) from each sector of the original multi-sided Böhm-Bawerk assignment game. Notice that a natural way to identify coalitions of the set of sectors M with essential coalitions of the set of agents N arises: for any $S \subseteq M$ we define the notation $E^S := r\mathbf{1}_S + (r+1)\,\mathbf{1}_{M \setminus S} \in \mathbb{R}^m$, where, for each $T \subseteq M$, $\mathbf{1}_T \in \mathbb{R}^m$ is the vector such that

$\mathbf{1}_T(k) = 1$ if $k \in T$ and $\mathbf{1}_T(k) = 0$ if $k \notin T$. The case in which there is no $r+1^{th}$ agent for some of the sectors in $M \backslash S$ must be treated separately, because in this case $E^S \in \mathbb{R}^m$ can still be defined but it is not an essential coalition of N, i.e. $E^S \notin \prod_{k=1}^m N^k$.

Definition 2 *Given an m-sided Böhm-Bawerk assignment game $(N, \omega_{A(\mathbf{c};w)})$, the associated sectors game $(M, v^M_{\mathbf{c};w})$ is the cooperative game with set of players $M = \{1, ..., m\}$ composed of all sectors and characteristic function defined, for each $S \subseteq M$, by*

$$v^M_{\mathbf{c};w}(S) = \begin{cases} a_{E^S} & \textit{if } E^S \in \prod_{k=1}^m N^k \\ 0 & \textit{if } E^S \notin \prod_{k=1}^m N^k \end{cases}$$

if $r > 0$ and $v^M_{\mathbf{c};w}(S) = 0$ if $r = 0$.

If $v^M_{\mathbf{c};w}(S) > 0$ then necessarily $E^S \in \prod_{k=1}^m N^k$. By (6), if $r > 0$ we always have $v^M_{\mathbf{c};w}(M) = a_{E^M} > 0$ and $v^M_{\mathbf{c};w}(\varnothing) = 0$. When no confusion may arise we will write simply v^M instead of $v^M_{\mathbf{c};w}$.

In Theorem 1 below we show that the core and the extreme core allocations of the sectors game $(M, v^M_{\mathbf{c};w})$ are strongly related to the core and the extreme core allocations respectively of the multi-sided Böhm-Bawerk assignment game $(N, \omega_{A(\mathbf{c};w)})$. Specifically, we prove that for each allocation $\overline{x} \in C(v^M_{\mathbf{c};w})$, there is a unique core allocation $x \in C(\omega_{A(\mathbf{c};w)})$ such that the variable part of the payoffs to agents of the k^{th} sector at x coincides with the payoff to sector $k \in M$ at $\overline{x}$, and vice versa, hence giving sense to call these fictitious players as sectors. Since payoffs in both games belong to different spaces ($\mathbb{R}^M$ versus $\mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m}$), we need to define a function to map payoffs in the sectors game to payoffs in the multi-sided Böhm-Bawerk game. Given an m-sided Böhm-Bawerk assignment game $(N, \omega_{A(\mathbf{c};w)})$, we introduce the *replica operator* $\mathcal{R}_{\mathbf{c};w}$ defined by

$$\begin{array}{lcccc} \mathcal{R}_{\mathbf{c};w}: & \mathbb{R}^M & \longrightarrow & \mathbb{R}^{N^1} \times \quad ... \quad \times \mathbb{R}^{N^m} \\ & (\overline{x}_1, ..., \overline{x}_m) & \longrightarrow & (\overbrace{\overline{x}_1, ..., \overline{x}_1}^{r}, \overbrace{0, ..., 0}^{n_1 - r}; \; ... \; ; \overbrace{\overline{x}_m, ..., \overline{x}_m}^{r}, \overbrace{0, ..., 0}^{n_m - r}) \end{array} \tag{8}$$

Notice that $\mathcal{R}_{\mathbf{c};w}$ is an injective linear function. In the case of Example 1,

$$\mathcal{R}_{\mathbf{c};w}(\overline{x}_1, \overline{x}_2, \overline{x}_3) =$$

$$(\overline{x}_1, \overline{x}_1, \overline{x}_1, \overline{x}_1, \overline{x}_1, 0, 0, 0; \overline{x}_2, \overline{x}_2, \overline{x}_2, \overline{x}_2, \overline{x}_2, 0, 0, 0; \overline{x}_3, \overline{x}_3, \overline{x}_3, \overline{x}_3, \overline{x}_3, 0, 0, 0, 0, 0).$$

Before proving the next theorem we introduce further some notation. Given $t \in \mathbb{R}^l$ and $B \subset \mathbb{R}^l$, let $t + B := \{x \in \mathbb{R}^l : x = t + x' \text{ and } x' \in B\}$ denote the translated set B by the vector t.

Theorem 1 *Let $(N, \omega_{A(\mathbf{c};w)})$ be an m-sided Böhm-Bawerk assignment game and let $(M, v^M_{A(\mathbf{c};w)})$ be the associated sectors game. Then,*

1. $C(\omega_{A(\mathbf{c};w)}) = t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(C\left(v^M_{\mathbf{c};w}\right))$.
2. $Ext\left\{C(\omega_{A(\mathbf{c};w)})\right\} = t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(Ext\left\{C\left(v^M_{\mathbf{c};w}\right)\right\})$.

Proof. We assume $r > 0$ to avoid (N, ω_A) and (M, v^M) being the null game, where both statements can be easily verified. We start proving Part 1.

First we show that $C(\omega_A) \subseteq t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(C\left(v^M\right))$. Let $i \in N^k$ for some $k \in M$. We introduce the notation $E^{i,k} := i\mathbf{1}_{\{k\}} + r\mathbf{1}_{M\setminus\{k\}}$ and $\overline{E}^{i,k} := r\mathbf{1}_{\{k\}} + i\mathbf{1}_{M\setminus\{k\}}$. If $i \leq r$ then $E^{i,k}$ and E^M belong to an optimal matching, since

$$a_{E^{i,k}} + a_{\overline{E}^{i,k}} = a_{E^M} + a_{E^i}, \tag{9}$$

where $E^M = E^r = r\mathbf{1}_M$ and $E^i = i\mathbf{1}_M$. Next, we consider any $x \in C(\omega_A)$. By core conditions, $x(E^{i,k}) = a_{E^{i,k}} > 0$ and $x(E^M) = a_{E^M} > 0$, where the positivity holds by (4) and (6). Applying (3) and Definition 2, if we subtract these two latter expressions we obtain, given $i \in N^k$ such that $i \leq r$,

$$\begin{aligned} x_{ki} - x_{kr} &= c_{kr} - c_{ki} \quad \text{if } 1 \leq k < m \text{ and} \\ x_{ki} - x_{kr} &= w_i - w_r \quad\; \text{if } k = m. \end{aligned} \tag{10}$$

If $i > r$ there are two possibilities. Either $i \in N^k$ is not assigned under diagonal matching or it is assigned, which by core conditions implies $x(E^i) = 0$. In both cases, given $i \in N^k$ such that $i > r$, we obtain

$$x_{ki} = 0, \text{ for all } k \in M. \tag{11}$$

Observe that (10) and (11), together with (7), imply that, given $k \in M$ and $i \in N^k$, then $x_{ki} + t_{ki} = x_{ki} + c_{ki} - c_{kr} = x_{kr}$ if $i \leq r$ and $x_{ki} + t_{ki} = 0$ if $i > r$. Let $\overline{x} = (\overline{x}_1, ..., \overline{x}_m) := (x_{1r}, ..., x_{mr}) \in \mathbb{R}^m$. By (8), $x = t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\overline{x})$. It remains to show that $\overline{x} \in C\left(v^M\right)$. On the one hand, $\overline{x}(M) = x(E^M) = a_{E^M} = v^M(M)$ since E^M belongs to an optimal matching. Thus, $\overline{x}$ is an efficient allocation. On the other hand, let $S \subseteq M$ be an arbitrary coalition of sectors. If $v^M(S) = 0$, we trivially have $\overline{x}(S) \geq 0 = v^M(S)$. Hence, assume $v^M(S) > 0$. In this case, $\overline{x}(S) = x(E^S) \geq a_{E^S} = v^M_{\mathbf{c};w}(S)$ where the first equality holds by (11) and the inequality holds since x belongs to $C(\omega_{A(\mathbf{c};w)})$. In conclusion, $x \in t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(C\left(v^M_{\mathbf{c};w}\right))$.

Second we show that $C(\omega_A) \supseteq t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(C\left(v^M\right))$. Consider $\overline{x} \in C\left(v^M\right)$ and let $x = t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\overline{x}) \in \mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m}$, which by construction has nonnegative components. We start proving that x is an efficient vector. We apply (7) and (8). On the one hand, for all $1 \leq i \leq r$, we have $x(E^i) = \sum_{k=1}^{m-1} (\overline{x}_k + (c_{kr} - c_{ki})) + (\overline{x}_m + (w_i - w_r)) = \overline{x}(M) + a_{E^i} - a_{E^M} = a_{E^i}$, where the last equality holds since $\overline{x} \in C\left(v^M\right)$. On the other hand, for all $r+1 \leq i \leq n$, where $n = \min_{k \in M} n_k$, we have $\sum_{k=1}^{m} x_{ki} = 0 = a_{i...i}$. Lastly, if $i \in N^k$ for some $k \in M$ is unassigned under the diagonal matching, then $i > r$ and we have $x_{ki} = 0$.

It remains to check that, at x, no essential coalition can improve on their own. Let $E = (i_1, ..., i_m) \in \prod_{k=1}^{m} N^k$ be an arbitrary essential coalition and let $S_E := \{k \in M : 1 \leq i_k \leq r\}$. We distinguish two cases.

- *Case 1:* $m \in S_E$.

  If $a_E = 0$, we trivially have $x(E) \geq 0 = a_E$. Hence, we assume $a_E > 0$. Let $E' :=$

$\sum_{k\in S_E} i_k \mathbf{1}_{\{k\}} + (r+1)\,\mathbf{1}_{M\setminus S_E}$. Then, by construction of x,

$$\begin{aligned}
x(E) &= \overline{x}_m + (w_{i_m} - w_r) + \sum_{k\in S_E\setminus\{m\}} (\overline{x}_k + (c_{kr} - c_{ki_k})) \\
&= \overline{x}(S_E) + \sum_{k\in S_E\setminus\{m\}} (c_{kr} - c_{ki_k}) + (w_{i_m} - w_r) \\
&\geq v^M(S_E) + \sum_{k\in S_E\setminus\{m\}} (c_{kr} - c_{ki_k}) + (w_{i_m} - w_r) \\
&= w_r - \sum_{k\in S_E\setminus\{m\}} c_{kr} - \sum_{k\notin S_E} c_{k(r+1)} + \sum_{k\in S_E\setminus\{m\}} (c_{kr} - c_{ki_k}) + (w_{i_m} - w_r) \\
&= w_{i_m} - \sum_{k\in S_E\setminus\{m\}} c_{ki_k} - \sum_{k\notin S_E} c_{k(r+1)} = a_{E'} \geq a_E,
\end{aligned}$$

where the first inequality holds since $\overline{x} \in C\left(v^M\right)$, the last inequality holds by (4) and the last two equalities are obtained applying (3).

- *Case 2: $m \notin S$.*

  The proof is similar to that of the above case and it is left to to the reader.

Finally we prove Part 2. Since a translation does not change the extreme points of a polytope, by Part 1 it is enough to prove that $\overline{x} \in Ext\left\{C\left(v^M\right)\right\}$ if and only if $\mathcal{R}_{\mathbf{c};w}(\overline{x}) \in Ext\left\{\mathcal{R}_{\mathbf{c};w}(C\left(v^M\right))\right\}$. This equivalence comes from observing that $\mathcal{R}_{\mathbf{c};w}(\overline{x}) = \frac{1}{2}\mathcal{R}_{\mathbf{c};w}(\overline{x}') + \frac{1}{2}\mathcal{R}_{\mathbf{c};w}(\overline{x}'')$ if and only if $\overline{x} = \frac{1}{2}\overline{x}' + \frac{1}{2}\overline{x}''$, since $\mathcal{R}_{\mathbf{c};w}$ is an injective linear function. ■

The above result shows that each core allocation x of a Böhm-Bawerk multi-sided assignment game $(N, \omega_{A(\mathbf{c};w)})$, which is given by (5), can be decomposed into two terms: one constant term given by $t_{\mathbf{c};w}$ which is different for any agent, and one common term (for all agents of the same sector) given by the unique allocation $\overline{x} \in C(v^M_{\mathbf{c};w})$ associated to x.

Next we turn into the study of the sectors game and exploit its features to prove further results of the original multi-sided Böhm-Bawerk assignment game. We start with the sectors game associated to Example 1, which is shown below.

$v^M(\{1\}) = a_{566} = 0 \qquad v^M(\{1,2\}) = a_{556} = 1$

$v^M(\{2\}) = a_{656} = 0 \qquad v^M(\{1,3\}) = a_{565} = 0.7 \qquad v^M(\{1,2,3\}) = a_{555} = 2$

$v^M(\{3\}) = a_{665} = 0 \qquad v^M(\{2,3\}) = a_{655} = 0.3$

The core of the above game is depicted in Figure 1.

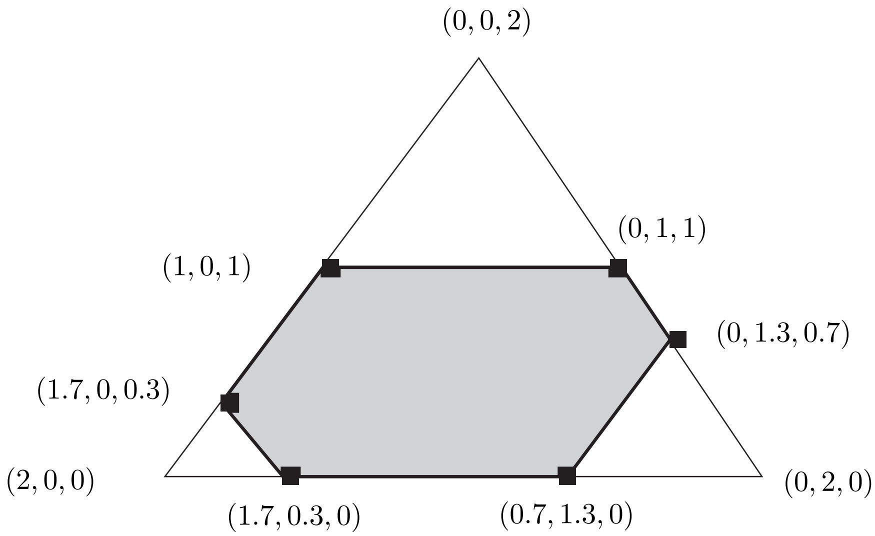

Figure 1: The core of the sectors game associated to a three-sided Böhm-Bawerk assignment game

The following proposition proves that the sectors game associated to a multi-sided Böhm-Bawerk assignment game has a special structure, it is a convex game.

Proposition 1 *Let $(N, \omega_{A(\mathbf{c};w)})$ be an m-sided Böhm-Bawerk assignment game. Then, the associated sectors game $(M, v^M_{\mathbf{c};w})$ is convex.*

Proof. First of all, observe that, by definition of the sectors game and (4), v^M is monotone, i.e. $v^M(S) \le v^M(T)$ for $S \subseteq T \subseteq M$. Next, let $k \in M$ and $S \subseteq T \subseteq M\backslash\{k\}$. We want to show that

$$v^M(S \cup \{k\}) - v^M(S) \le v^M(T \cup \{k\}) - v^M(T). \tag{12}$$

We distinguish some cases.

- *Case 1:* $v^M(S) > 0$.

  By monotonicity of v^M, we have $v^M(S \cup \{k\}), v^M(T \cup \{k\}), v^M(T) > 0$. Since $v^M(S) > 0$, there exists the $r+1^{th}$ agent (either seller or buyer) of the k^{th} sector. Applying the definition of the sectors game, the two terms of (12) are equal to $w_r - w_{r+1}$ if $k = m$ or $c_{k(r+1)} - c_{kr}$ if $1 \leq k < m$. Thus (12) holds.

- *Case 2:* $v^M(S) = 0$ and $v^M(S \cup \{k\}) > 0$.

  By monotonicity of v^M, we have $v^M(T \cup \{k\}) > 0$. Since $v^M(S \cup \{k\}) > 0$, there exists the $r+1^{th}$ agent (either seller or buyer) for each of the sectors in $M \setminus (S \cup \{k\})$. Suppose $k = m$ (the other cases are analogous and they are left to the reader). On the one hand, if $v^M(T) > 0$ then (12) reduces to $w_r - \sum_{l \in S} c_{lr} - \sum_{l \in M \setminus S \cup \{k\}} c_{l(r+1)} \leq w_r - w_{r+1}$, which is equivalent to $v^M(S) = 0$. On the other hand, if $v^M(T) = 0$ then (12) trivially holds by monotonicity of v^M.

- *Case 3:* $v^M(S) = v^M(S \cup \{k\}) = 0$.

  In this case (12) trivially holds by monotonicity of v^M.

■

As a consequence of Proposition 1 and Theorem 1 we provide a method to find all the extreme core allocations of a multi-sided Böhm-Bawerk assignment game (see Corollary 1). In words, for each extreme point of the core of an m-sided Böhm-Bawerk assignment game there is a permutation of the set of sectors such that when we replicate and translate the marginal worth vector associated to this latter permutation we obtain the former vector, and vice versa.

Corollary 1 *Let $(N, \omega_{A(\mathbf{c};w)})$ be an m-sided Böhm-Bawerk assignment game and let $(M, v^M_{\mathbf{c};w})$ be the associated sectors game. Then,*

$$Ext\left\{C(\omega_{A(\mathbf{c};w)})\right\} = \left\{t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}\left(m_\theta(v^M_{\mathbf{c};w})\right)\right\}_{\theta \in \Theta(M)}.$$

To illustrate the above result we study the extreme core allocations of the three-sided Böhm-Bawerk assignment game that corresponds to Table 1. The set $Ext(C(\omega_{A(\mathbf{c};w)}))$ is obtained from the six possible marginal worth vectors of $(M, v^M_{\mathbf{c};w})$:

θ	$m^\theta(v^M_{\mathbf{c};w})$	$t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}\left(m_\theta(v^M_{\mathbf{c};w})\right)$
(1,2,3)	(0,1,1)	(6,6,4,3,0,0,0,0;5,4,2,1,1,0,0,0;9,7,5,3,1,0,0,0,0,0)
(1,3,2)	(0,1.3,0.7)	(6,6,4,3,0,0,0,0;5.3,4.3,2.3,1.3,1.3,0,0,0;8.7,6.7,4.7,2.7,0,0,0,0,0,0)
(2,1,3)	(1,0,1)	(7,7,5,4,1,0,0,0;4,3,1,0,0,0,0,0;9,7,5,3,1,0,0,0,0,0)
(2,3,1)	(1.7,0,0.3)	(7.7,7.7,5.7,4.7,1.7,0,0;4,3,1,0,0,0,0,0;8.3,6.3,4.3,2.3,0.3,0,0,0,0,0)
(3,1,2)	(0.7,1.3,0)	(6.7,6.7,4.7,3.7,0.7,0,0;5.3,4.3,2.3,1.3,1.3,0,0,0;8,6,4,2,0,0,0,0,0,0)
(3,2,1)	(1.7,0.3,0)	(7.7,7.7,5.7,4.7,1.7,0,0;4.3,3.3,1.3,0.3,0.3,0,0,0;8,6,4,2,0,0,0,0,0,0)

Also as a consequence of Theorem 1 and Proposition 1, the attainable lower and upper bounds for the core payoffs of any active agent $i \in N^k, k \in M$ are respectively $t_{ki} + v^M(\{k\})$ and $t_{ki} + v^M(M) - v^M(M \backslash \{k\})$. Moreover, for any ordering $\theta \in \Theta(M)$ of the set of sectors, the corresponding extreme core allocation $t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}\left(m_\theta(v^M_{\mathbf{c};w})\right)$ of the m-sided Böhm-Bawerk assignment game $(N, \omega_{A(\mathbf{c};w)})$ can be interpreted as follows: agents of the first sector in θ receive their best payoff, agents of the second sector in θ receive their best possible payoff provided that agents of the first sector receive their best payoff, and so on and so forth, hence generalizing the interpretation of the only two extreme core allocations in the case of two-sided Böhm-Bawerk assignment games as the buyers-optimal and the sellers-optimal.

Corollary 1 implies that the core of an m-sided Böhm-Bawerk game has at most $m!$ extreme core allocations. The next result shows that this bound is only attainable for multi-sided Böhm-Bawerk assignment games with at most three sectors, i.e. $m \leq 3$, (see for instance Figure 1).

Proposition 2 *Let $(N, \omega_{A(\mathbf{c};w)})$ be an m-sided Böhm-Bawerk assignment game. Then,*

- $1 \leq \left|Ext\left\{C(\omega_{A(\mathbf{c};w)})\right\}\right| \leq m\binom{m-1}{m/2}$ *if m is even,*
- $1 \leq \left|Ext\left\{C(\omega_{A(\mathbf{c};w)})\right\}\right| \leq m\binom{m-1}{(m-1)/2}$ *if m is odd,*

and the bounds are attainable.

Proof. The lower bound is attained, for instance, when $(N, \omega_{A(\mathbf{c};w)})$ is the null game. By Corollary 1, to calculate which is the number of extreme points of the core of $(N, \omega_{A(\mathbf{c};w)})$ it suffices to count how many different marginal worth vectors of the associated sectors game (M, v^M) there are. Let $\theta \in \Theta(M)$ be an arbitrary ordering of the set of sectors. Given $k \in M$, let $P_{\theta,k} := \{l \in M : \theta(l) < \theta(k)\}$ be the set of predecessors of k w.r.t. θ, and $F_{\theta,k} := \{l \in M : \theta(l) < \theta(k)\}$ be the set of followers of k w.r.t. θ.

Since (M, v^M) is monotone, we can define t^*_θ as the lowest integer $t \in \{1, ..., m\}$ such that $v^M(\{k : \theta(k) \leq t\}) > 0$. Then, let $k^*_\theta = \theta^{-1}(t^*_\theta)$ be the agent that appears in the k^*_θ-th position in the ordering θ. Applying (3) and Definition 2, it can be checked that any marginal worth vector of (M, v^M) has the following description

(13)

$$m^\theta_k(v^M) = \begin{cases} 0 & \text{if } 0 \leq \theta(k) < t^*_\theta \\ a_E & \begin{array}{c} \text{if } \theta(k) = t^*_\theta, \\ \text{where } E = r\mathbf{1}_{P_{\theta,k^*_\theta}} + r\mathbf{1}_{\{k^*_\theta\}} + (r+1)\,\mathbf{1}_{F_{\theta,k^*_\theta}} \end{array} \\ c_{k(r+1)} - c_{kr} & \text{if } t^*_\theta < \theta(k) \leq m \text{ and } k \neq m \\ w^r - w^{r+1} & \text{if } t^*_\theta < \theta(k) \leq m \text{ and } k = m \end{cases}$$

Moreover, (13) reveals that, given θ, if we permute either the set of the predecessors or the set of followers of $k^*_\theta = \theta^{-1}(t^*_\theta)$, the marginal worth vector (given by (13)) remains invariant. That is, $m^\theta(v^M) = m^{\theta'}(v^M)$ if $P_{\theta,k^*_\theta} = P_{\theta',k^*_{\theta'}}$ and $F_{\theta,k^*_\theta} = F_{\theta',k^*_{\theta'}}$. For each order θ there are $(t^*_\theta - 1)!\,(m - t^*_\theta)!$ different orderings that are obtained permuting either the set of the predecessors or the set of followers of k^*_θ and thus give rise to the same marginal worth vector. Observe that the more equidistant t^*_θ is with respect to 1 and m, the smaller $(t^*_\theta - 1)!\,(m - t^*_\theta)!$ is. Thus, it is not difficult to check that

$$(14) \qquad \min_{t^*_\theta \in \{1,...,m\}} (t^*_\theta - 1)!\,(m - t^*_\theta)! \; = \begin{cases} (m/2)!\,(m/2 - 1)! & \text{if } m \text{ is even} \\ ((m-1)/2)!\,((m-1)/2)! & \text{if } m \text{ is odd} \end{cases}$$

When, for all $\theta \in \Theta(M)$, t^*_θ is the minimum value given by (14), we obtain the following upper bounds for the number of extreme points of $C(\omega_A)$:

- $|Ext\,\{C(\omega_A)\}| \leq \frac{m!}{(m/2)!(m/2-1)!} = m\binom{m-1}{m/2}$ if m is even,
- $|Ext\,\{C(\omega_A)\}| \leq \frac{m!}{((m-1)/2)!((m-1)/2)!} = m\binom{m-1}{(m-1)/2}$ if m is odd.

To prove that these bounds are attainable we consider some specific m-sided Böhm-Bawerk market with two agents (either sellers or buyers) for each sector. We need to distinguish two cases.

- *Case 1:* m is even.

  Let us introduce the m-sided Böhm-Bawerk market $(\mathbf{c}; w)$ where $\mathbf{c} = (c_1, ..., c_{m-1}) \in \mathbb{R}^{2(m-1)}$ is given by $c_k = (1-\varepsilon, 2)$ for all $1 \leq k \leq m-1$, and $w = (3m/2-1+\varepsilon, 3m/2-2)$, for some small enough $\varepsilon > 0$. By the symmetry of the problem, it is easy to check that

$$v^M(S) = \begin{cases} 0 & \text{if } |S| < m/2 \\ \varepsilon m/2 & \text{if } |S| = m/2 \\ |S|\,(1+\epsilon) - m/2 & \text{if } |S| > m/2 \end{cases} \tag{15}$$

  If $\varepsilon > 0$ satisfies $\varepsilon(m/2-1) < 1$ then $0 < \varepsilon m/2 < 1+\varepsilon$. If we plug (15) into (13) we realize that all marginal worth vectors $m^\theta(v^M)$ have the same structure: they pay 0 to the first $m/2-1$ sectors in θ, $\varepsilon m/2$ to the $m/2^{th}$ sector in θ and $1+\varepsilon$ to the last $m/2-1$ sectors in θ. Hence, to construct one marginal worth vector we proceed as follows: we pick one sector $k \in M$ and plug it into the 'central' position $m/2+1$. Then, among the remaining $m-1$ sectors we pick $m/2$ to be the predecessors of k. All marginal worth vectors that are constructed like this are different. Furthermore, there are exactly $m\binom{m-1}{m/2}$ such vectors. Therefore the upper bound for the number of extreme points is attainable.

- *Case 2:* m is odd.

  It is analogous to the above case by taking the m-sided Böhm-Bawerk market $(\mathbf{c};w)$ where $\mathbf{c}=(c_1,...,c_{m-1})\in\mathbb{R}^{2(m-1)}$ is given by $c_k=(1-\varepsilon,2)$ for all $1\leq k\leq m-1$, and $w=(3\,(m-1)\,/2,3\,(m-1)\,/2-1-\varepsilon)$, for some small enough $\varepsilon>0$, and hence it is left to the reader.

■

Observe that Proposition 2 tells that, as it is already known, a two-sided Böhm-Bawerk assignment game ($m=2$) has at most 2 extreme core allocations. Notice also that, for instance, the maximum number of extreme core allocations is respectively $6,12,30,60$ for $m=3,4,5,6$ respectively. We want to stress that the number of extreme core allocations of an m-sided Böhm-Bawerk assignment game does not depend on the number of buyers or sellers of each sector but only on the number m of sectors.

Hamers et al. (2002) prove that classical bilateral assignment games satisfy the *CoMa-property*, i.e. any extreme core allocation is a marginal worth vector. We next show that this property also holds for multi-sided Böhm-Bawerk games.

Theorem 2 *Multi-sided Böhm-Bawerk assignment games satisfy the CoMa-property.*

Proof. We prove that for each extreme core allocation x of $(N,\omega_{A(\mathbf{c};w)})$ there is an ordering $\theta^N\in\Theta(N)$ of the set of agents N (composed of all buyers and sellers) such that x is the marginal worth vector associated to θ^N. By Corollary 1, we know that every extreme core allocation of $(N,\omega_{A(\mathbf{c};w)})$ is the translation and replica of one marginal worth vector of the sectors game $(M,v^M_{\mathbf{c};w})$. Taking advantage of the above facts, the proof of the theorem consists on associating to each ordering of sectors $\theta^M\in\Theta(M)$ an ordering of agents $\theta^N\in\Theta(N)$ such that $m^{\theta^N}(\omega_A)=t_{\mathbf{c};w}+\mathcal{R}_{\mathbf{c};w}(m^{\theta^M}(v^M))$.

Without loss of generality let $\theta^M=(1,2,...,m)$ be the natural ordering of the set of sectors. Then, let $\theta^N\in\Theta(N)$ be any ordering of the set of agents (defined from θ^M) constructed as follows: all inactive agents appear (arbitrarily ordered) before all active agents, i.e. $\theta^N(i') < \theta^N(i'')$ for all $i' \in \{i \in N^k : k \in M, i > r\}$ and $i'' \in \{i \in N^k : k \in M, i \leq r\}$, and all active agents are ordered as follows,

$$(r \in N^1, ..., r \in N^m, r-1 \in N^1, ..., r-1 \in N^m, \dots\dots, 1 \in N^1, ..., 1 \in N^m).$$

Notice that we are implicitly using the natural ordering θ^M in the restriction of θ^N to the set of active agents since, for any index $i \in \{1, ..., r\}$, agents $i \in N^1, ..., i \in N^m$ are ordered following the ordering of sectors $\theta^M = (1, 2, ..., m)$.

For notational convenience, we use $m_{ki}^{\theta^N}(\omega_A)$ to denote the payoff to agent $i \in N^k, k \in M$, according to the marginal worth vector $m^{\theta^N}(\omega_A)$. By (2) and (6), we have $\omega_A(S) = 0$ for all $S \subseteq N$ composed only of inactive agents. Hence, for all $i \in N^k$ and $k \in M$ such that $i > r$, we have $m_{ki}^{\theta^N} = 0$, which coincides with the payoff to agent $i \in N^k$ in the translation and replica of $m_k^{\theta^M}(v^M)$.

Next assume that $i \in N^k$ and $k \in M$ such that $i \leq r$. By the definition of the characteristic function ω_A in the case of a multi-sided Böhm-Bawerk assignment game, $\omega_A(S)$ is obtained as follows. The buyer in S with higher valuation (if exists) and the sellers of each sector in S with lower valuations (if exist) are arranged in an essential coalition $E \in \prod_{k=1}^m N^k$. If $a_E = 0$ or E cannot be formed we stop. If not, we keep repeating the above procedure with the remaining agents, until either the new essential coalition has zero worth or no essential coalition can be formed. Lastly $\omega_A(S)$ is obtained adding up all the worths associated to essential coalitions constructed. In other words, to obtain $\omega_A(S)$ we partition S into 'ranking-ordered' coalitions and add up their corresponding worths.

In the case in which $S = P_{\theta^N,i} \cup \{i\}$, where $P_{\theta^N,i}$ denotes the set of predecessors of $i \in N^k$ w.r.t. θ^N, it is easy to check that the essential coalitions constructed by the above procedure are (if exist)

$$\widehat{E}^i := (\overbrace{i, ..., i}^{k}, \overbrace{i+1, ...i+1}^{m-k}), ..., \widehat{E}^r := (\overbrace{r, ..., r}^{k}, \overbrace{r+1, ...r+1}^{m-k}).$$

Similarly, in the case in which $S = P_{\theta^N,i}$, the essential coalitions constructed by the above procedure are (if exist)

$$\widetilde{E}^i := (\overbrace{i,...,i}^{k-1},\overbrace{i+1,...i+1}^{m-k+1}),...,\widetilde{E}^r := (\overbrace{r,...,r}^{k-1},\overbrace{r+1,...r+1}^{m-k+1}) \ .$$

Therefore, by (2) and (6),

$$\begin{aligned} m_{ki}^{\theta^N}(\omega_A) &= \omega_A(P_{\theta^N,i\in N^k}\cup\{i\})-\omega_A(P_{\theta^N,i\in N^k}) \\ &= \left(\sum_{l=i}^{r-1} a_{\widehat{E}^l}+v^M(\{1,...,k-1,k\})\right)-\left(\sum_{l=i}^{r-1} a_{\widetilde{E}^l}+v^M(\{1,...,k-1\})\right) \\ &= \left\{ \begin{array}{l} c_{kr}-c_{ki}+v^M(\{1,...,k-1,k\})-v^M(\{1,...,k-1\}), \ \text{ if } k<m \\ w_i-w_r+v^M(\{1,...,k-1,k\})-v^M(\{1,...,k-1\}), \ \text{ if } k=m \end{array} \right\} \\ &= t_{ki}+m_k^{\theta^M}(v^M), \end{aligned}$$

where the third equality holds applying (3) to all entries of the matrix A in the sum, which by (6) are strictly positive, and the last equality is obtained applying (7) and (8). Observe that in the case where either $\widehat{E}^r \notin \prod_{k=1}^m N^k$ or $\widetilde{E}^r \notin \prod_{k=1}^m N^k$ we have $v^M(\{1,...,k-1,k\})=0$ and $v^M(\{1,...,k-1\})=0$, respectively.

In conclusion, $m^{\theta^N}(\omega_{A(\mathbf{c};w)}) = \overrightarrow{t}_{\mathbf{c};w}+\mathcal{R}_{\mathbf{c};w}(m^{\theta^M}(v^M_{\mathbf{c};w}))$ and hence $(N,\omega_{A(\mathbf{c};w)})$ satisfies the CoMa-property. ■

In the final part of this paper we investigate when the core of a multi-sided Böhm-Bawerk assignment game is a stable set in the sense of von Neumann-Morgestern. Given a multi-sided Böhm-Bawerk assignment game, we say that agent $i\in N^k, k\in M$ is a *null player* if $a_E=0$ for all $E\in\prod_{k=1}^m N^k$ such that $i\in E$. An special subclass of multi-sided Böhm-Bawerk assignment games is the class of multi-sided assignment games with a constant matrix, i.e. with all entries equal, which are called *multi-sided glove markets* (or *T-markets*, Brânzei et al., 2007). A multi-sided glove market is therefore obtained when all buyers' valuations are the same and, for each other sector, all sellers' valuations coincide.

Our next result identifies necessary and sufficient conditions which guarantee that the core of a multi-sided Böhm-Bawerk assignment game is stable, and it generalizes the result known for the two-sided case. Nevertheless, the proof presented here is not parallel to that of the two-sided case since, unlike for this latter case, in the general case there is not known yet a necessary and sufficient condition for the core of an arbitrary multi-sided assignment game to be stable (see Solymosi and Raghavan, 2001).

Proposition 3 *Given an m-sided Böhm-Bawerk assignment game* $(N, \omega_{A(\mathbf{c};w)})$ *without null players, the following statements are equivalent:*

(a) $C(\omega_{A(\mathbf{c};w)})$ *is a stable set.*

(b) $(N, \omega_{A(\mathbf{c};w)})$ *is an square m-sided glove market.*

Proof. Since there are no null players, we necessarily have $r > 0$. First we prove that (a) implies (b). Consider the allocation

$$y = (\overbrace{a_{E^1}, ..., a_{E^r}}^{r}, \overbrace{0, ...0}^{n_1 - r}; \overbrace{0, ..., 0}^{n_2}; ...; \overbrace{0, ..., 0}^{n_m}) \in \mathbb{R}^{N^1} \times ... \times \in \mathbb{R}^{N^m}. \tag{16}$$

Suppose that $y \notin C(\omega_A)$. Since $C(\omega_A)$ is stable, there must be $x \in C(\omega_A)$ such that x dominates y via coalition $T \subseteq M$ and $\omega_A(T) > 0$. Let $E = (i_1, ..., i_m) \subseteq T$ be some essential coalition such that $a_E > 0$. Then, $x_{1i_1} > y_{1i_1} = a_{E^{i_1}} = x(E^{i_1}) \geq x_{1i_1}$,where the strict inequality holds by the domination conditions, the first equality holds by (16), the second equality and the last inequality hold by (5). Hence, we have a contradiction and thus $y \in C(\omega_A)$. Analogously,

$$z = (\overbrace{0, ..., 0}^{n1}; \overbrace{a_{E^1}, ..., a_{E^r}}^{r}, \overbrace{0, ...0}^{n_2 - r}; ; ...; \overbrace{0, ..., 0}^{n_m}) \in C(\omega_A). \tag{17}$$

Next we prove that (N, ω_A) is square. Suppose not, i.e. there is at least one inactive agent. We can assume without loss of generality that $r + 1 \in N^1$ exists. Let $E = (i_1, ..., i_m) \in \{r + 1\} \times N^2 \times ... \times N^m$ be any essential coalition containing agent $r + 1 \in N^1$. Then, $y(E) = y_{1i_1} = a_{E^{i_1}} = 0 \geq a_E \geq 0$, where the first two equalities hold by (16), the third equality holds by (6) and the first inequality holds since $y \in C(\omega_A)$. Therefore, $a_E = 0$ for all $E \in \{r + 1\} \times N^2 \times ... \times N^m$ such that $r + 1 \in E$, which contradicts (N, ω_A) has not null players.

Lastly, we prove that A is a constant matrix. By (4) we have

$$a_{E^1} \geq a_E \geq a_{E^r} \text{ for all } E \in \prod_{k=1}^{m} N^k = \{1, ..., r\}^m. \tag{18}$$

Moreover, $a_{E^1} = y(E') = a_{E'} = z(E') = a_{E^r}$, where $E' = (1, r, ..., r)$, the first equality holds by (16), the second and third equalities hold since $y, z \in C(\omega_A)$ and, by (9), E' belongs to some optimal matching, and the last equality holds by (17). Therefore, (18) reduces to a chain of equalities and thus $(N, \omega_{A(\mathbf{c};w)})$ is an square m-sided glove market.

Second, we prove that (b) implies (a). Let (N, ω_A) be an square multi-sided glove market and let $y \in I(\omega_A) \backslash C(\omega_A)$. By (5), there must be an essential coalition $E = (i_1, ..., i_m) \in \prod_{k=1}^{m} N^k$ such that $y(E) < C$. Consider the vector $\overline{x} \in \mathbb{R}^M$ defined by $\overline{x}_k := y_{ki_k} + \delta/m$, for all $k \in M$, where $\delta := C - x(E) > 0$. It is straightforward to check that $\overline{x} \in C(v^M)$, since $v^M(M) = C$ and $v^M(S) = 0$ for all $S \subsetneq M$. Then, $x = t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\overline{x}) \in \mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m}$ belongs to $C(\omega_A)$. Furthermore, we have $y_{ki_k} < x_{ki_k}$ for all $k \in M$. Then x dominates y via E and $C(\omega_A)$ is a stable set. ■

In the general case in which there might be null players, a multi-sided Böhm-Bawerk assignment game $(N, \omega_{A(\mathbf{c};w)})$ has an stable core if and only if the square $r \times \overbrace{\cdots}^{m} \times r$ submatrix given by the active agents of each sector -where r is defined in (6)- is constant and the remaining entries are null.

We conclude with two final remarks. On the one hand, one may be tempted to think that the cooperative analysis of a multi-sided Böhm-Bawerk market made throughout this paper can be simplified to the analysis of a (classical) two-sided Böhm-Bawerk assignment game, by clustering the $(m-1)$-tuples of sellers (one of each sector) into single sellers. However, merging sellers presents two main drawbacks. First, there is not a unique way to merge sellers. In fact, if active sellers are merged with inactive sellers, the profitability of the market, i.e. the value of the grand coalition in the corresponding cooperative game, may decrease. Second, even if active sellers are merged with active sellers and inactive sellers are merged with inactive sellers, the merging of the extreme points (or other solution concepts, e.g. the nucleolus) of an m-sided Böhm-Bawerk assignment game may not coincide respectively with the extreme points (or the nucleolus) of the merging of the m-sided Böhm-Bawerk assignment game.

On the other hand, most markets are interesting precisely when the worths of coalitions are not additively separable in individual agents' contributions. In fact, this is the case in our problem because of three potentially non-trivial aspects: (1) valuations are arbitrarily nonnegative, (2) worths of essential coalitions are truncated at zero making this case distinct from the purely additive case and (3) the number of firms and buyers may be arbitrarily different.

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