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Abstract

Most, if not all, production technologies are stochastic. This article demonstrates how data envelopment analysis (DEA) methods can be adapted to accommodate stochastic elements in a state-contingent setting. Specifically, we show how observations on a random input, not under the control of the producer and not known at the time that variable input decisions are made, can be used to partition the state space in a fashion that permits DEA models to approximate an event-specific production technology. The approach proposed in this article uses observed data on random inputs and is easy to implement. After developing the event-specific DEA representation, we apply it to a data set for Western Australian wheat farmers. Our results highlight the need for acknowledging stochastic elements in efficiency analysis.

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Agricultural production technologies are inherently uncertain. Unpredictable climatic variables such as rainfall are essential to production and farmers must plan for a range of contingencies when making production decisions. However, with few exceptions, data envelopment models and methods used in making efficiency comparisons rely on the assumption that the underlying technology is deterministic, with any stochastic component being confined to an error term. O'Donnell, Chambers and Quiggin (2006) have shown that efficiency analysis, whether based on stochastic frontier or data envelopment models, can be seriously biased if methods developed for nonstochastic technologies are applied to data sets generated by firms facing truly stochastic technologies and decision environments. The purpose of this article is to demonstrate how data envelopment analysis (DEA) methods can be adapted to accommodate stochastic elements in a state-contingent setting. Specifically, we show how observations on a random input, not under the control of the producer and not known at the time that variable input decisions are made, can be used to partition the state space in a fashion that permits DEA models to approximate an event-specific production technology. After developing the event-specific DEA representation, we apply it to a data set from Western Australia to illustrate the differences in efficiency calculations that can emerge when the stochastic nature of the technology is taken into account. For our data set, allowing for the event-specific nature of the data set has dramatic consequences in calculated efficiency scores.

In what follows, we first define a stochastic production technology. Then we show how information on a random input can be used to define a partition of the state-space that permits specification of an event-specific version of the technology, and we show how that specification can be implemented in a DEA framework. We discuss our data set next, and then we apply our method to that data set, discuss our findings, and then the article concludes.

The Stochastic Technology

The stochastic setting is represented by a measurable space (S, Ω) where S is the state space and Ω are its measurable subsets (events). In this setting random variables are treated as measurable maps from S to the reals. Thus, random variable, $\tilde{f}$, can be thought of as the element of $\mathbb{R}^S$, defined by

$$\tilde{f} = \{f(s) : s \in S\},$$

where $f : S \to \mathbb{R}$ is the map defining the random variable, and it is required that $\{s : f(s) = v\}$ belongs to Ω for all $v \in \mathbb{R}$. Random variables will always be distinguished from their *ex post* values by a tilde $(\tilde{\ })$. Hence, $\tilde{f} \in \mathbb{R}^S$ represents the random variable, and $f(s)$ denotes the *ex post* (observed) outcome associated with realization s of S. Denote by $\tilde{1}$ the degenerate (constant) random variable whose outcome equals one for all $s \in S$.

The stochastic production technology uses multiple non-stochastic inputs to produce a single stochastic output.[1] That stochastic output is represented by the random variable $\tilde{z} \in \mathbb{R}_+^S$. The technology is represented by a set $T \subset \mathbb{R}_+^S \times \mathbb{R}_+^N$, where N represents the number of inputs that are under the direct control of the producer and that are applied prior to the resolution of uncertainty. T is defined by

$$T = \{(\tilde{z}, \mathbf{x}) : \mathbf{x} \text{ can produce } \tilde{z}\},$$

where $\tilde{z} \in \mathbb{R}_+^S$ denotes the stochastic output, and $\mathbf{x} \in \mathbb{R}_+^N$ denotes the nonstochastic inputs. We assume that T is nonempty, exhibits free disposal of inputs and outputs, and is convex. The interpretation of the technology is as follows. Before the producer knows the realization $s \in \Omega$, he or she picks $(\tilde{z}, \mathbf{x})$ from within T. If the realized state is $s \in S$, then realized output is $z(s)$, while if $s' \neq s$ is realized, then *ex post* output is $z(s')$.

We now consider a comprehensive partition of the state space, S, into mutually exclusive events. Call that partition $\hat{\Omega}$ and denote a typical element of it by ω. These events are mutually exclusive

$$\omega \neq \omega' \Rightarrow \omega \cap \omega' = \varnothing,$$

and the partition is comprehensive

$$\bigcup_{\omega \in \hat{\Omega}} = S.$$

If one only has data on ex post output realizations, then empirical approximation of T requires an identifying restriction on T. To that end, we assume that T can be represented in terms of a family of event-specific stochastic production functions so that

$$T = \left\{ (\tilde{z}, \mathbf{x}) : z(s) \leq g_{\omega}(\mathbf{x}, s), \omega \in \hat{\Omega}, s \in \omega \right\},$$

where each g_{ω} is a nondecreasing and concave function of the nonstochastic inputs. In what follows, g_{ω} is termed the *event-specific production function* for the event ω. The basic idea behind an event-specific representation of the technology is that the occurence of different events fundamentally changes the *ex post* conditions under which stochastic production takes place. An obvious special case of an event-specific production function is the state-contingent production function that has been axiomatically studied by Chambers and Quiggin (2000). In that case, $\hat{\Omega} = S$, and

$$T = \{ (\tilde{z}, \mathbf{x}) : z(s) \leq g_{s}(\mathbf{x}, s), s \in S \}.$$

An event-specific technology has a number of advantages for applied work. Most importantly, as noted above and as we show below, it allows one to use *ex post* observations on output in the construction of empirical approximations of the technology. Thus, choosing an event-specific representation represents an important identifying restriction. However, it comes with costs. In particular, as O'Donnell, Chambers, and Quiggin (2006) have shown through simulation analysis, if the true technology is not event-specific then empirical representations of the technology based upon this identifying restriction can lead to serious errors and biases in approximating the frontier of the technology and in measuring efficiency. Theoretically, as Chambers and Quiggin (2000) have demonstrated, event-specific technologies place strong a priori assumptions on the degree of substitutability between ex post realizations of the stochastic output.

The special case of the event-specific technology, known as the state-contingent production function, is decidedly the most common empirical representation of stochastic technologies. It forms the basis for the standard representation of most stochastic frontier representation of technologies. As a general rule in applied econometric work, however, the practical specification of S is predicated more upon econometric and empirical convenience than it is on capturing the actual decision environment that the decision maker faces. More specifically, S is usually viewed as an 'error' space that arises from problems in measuring inputs and outputs and simple, although econometrically convenient, stochastic errors by producers who face a nonstochastic decision environment.

But in the truly stochastic decision environment in which most firms operate, S is not an 'error space'. Rather, S provides a comprehensive and mutually exclusive description of all possible states of the world that the producer can face after he or she makes his or decision about the nonstochastic inputs $\mathbf{x}$ and the stochastic output $\tilde{z}$. For many practical instances, S can be relatively narrowly defined. In what follows, we assume that it can be defined by the possible realizations of a real-valued, random input, which with an abuse of notation we denote as s, to the production process whose realization occurs after $(\tilde{z}, \mathbf{x})$ is chosen. Hence, in what follows $S \subset \mathbb{R}_+$ corresponds to the support of that random input. The partition of S given by $\hat{\Omega}$ is then given by consecutive subintervals of the positive reals.

The choice of $\hat{\Omega}$ is motivated by the need to represent production uncertainty in a relatively compact and empirically tractable fashion. For practical purposes, this requires that the number of elements of the partition $\hat{\Omega}$ should be small. It does not mean, however, that our method can only be applied if there is only one random input to the production process. Suppose that there were two. Then S could be defined as a subset of $\mathbb{R}^2_+$, and events could be defined by appropriate partitions of that set.

A DEA model of the event-specific technology

Our theoretical model relates *ex post* output to realizations of the random input, s, according to

$$z\left(s\right) \leq g_{\omega}\left(\mathbf{x},s\right),$$

$\omega \in \hat{\Omega}, s \in \omega$. Thus, in terms of a DEA technology, one can legitimately think in terms of a technology that characterizes the interaction between nonstochastic inputs, the stochastic input s, and realized output. A standard (VRS) DEA representation of such a technology, assuming free disposability, would be:

$$T^D = \Big\{ (z, \mathbf{x}, s) : z \le \sum_{k=1}^{K} \lambda_k z^k, \ \mathbf{x} \ge \sum_{k=1}^{K} \lambda_k \mathbf{x}^k, \\ s \ge \sum_{k=1}^{K} \lambda_k s^k, \ \sum_{k=1}^{K} \lambda_k = 1, \\ \lambda_k \ge 0, \ k = 1, .., K \Big\},$$

where $\left(z^k, \mathbf{x}^k, s^k\right)$ corresponds to the kth observations on the *ex post* output, the nonstochastic inputs, and the observed random input, and where $\lambda = (\lambda_1, ..., \lambda_K)$ are the DEA activity variables.

Notice that while T^D accounts for the presence of the random input, it is not a properly event-specific technology because it presumes that the same production frontier applies across all events $\omega \in \hat{\Omega}$. A technology that accommodates both the presence of the random input and the event-specific nature of the technology can be constructed by using the K *ex post* values of the random input to partition the data into subsets that correspond to each of the events ω defined by the partition $\hat{\Omega}$ of the state space. Denote the number of observations falling into the event ω by $K(\omega)$ and the kth observation falling into that event by $\left(z^k(\omega), \mathbf{x}^k(\omega), s^k(\omega)\right)$. Then the event-specific DEA frontier associated with those observations and with event ω is given by

$$T^\omega = \Big\{ (z, \mathbf{x}, s) : \ z \le \sum_{k=1}^{K(\omega)} \lambda_k z^k(\omega), \ \mathbf{x} \ge \sum_{k=1}^{K(\omega)} \lambda_k \mathbf{x}^k(\omega), \\ s \ge \sum_{k=1}^{K(\omega)} \lambda_k s^k(\omega), \ \sum_{k=1}^{K(\omega)} \lambda_k = 1, \\ \lambda_k \ge 0, \ k = 1, .., K(\omega) \Big\}$$

and the (VRS) DEA approximation to T is given by

$$T^{\hat{\Omega}} = \Big\{ (z, \mathbf{x}, s) : (z, \mathbf{x}, s) \in T^\omega, \omega \in \hat{\Omega} \Big\}.$$

An Application

To illustrate how a DEA approximation to an event-specific technology can be constructed and the difference that it can make in actual efficiency calculations, we apply our methodology to a data set on crop yields derived from experimental field trial data. The data were obtained from the Crop Variety Testing (CVT) program of the West Australian Department of Agriculture (Hunter, 2005). The data relate yields on barley production to three fertilizer inputs (Nitgrogen, Phosphorus and Sulphur) that were under the direct control of the experimenter and two inputs (pre- and post-sowing rainfall) that were not under the control of the experimenters at the time that fertilizer applications were made. Thus, for the purposes of our analysis, we take the random inputs, s, defining the state space to be rainfall as measured in millimeters.

There are several reasons why agricultural field trial data provide a particularly convenient framework in which to illustrate our methodology. First, because these data emerge from experiments by professional agronomists who are presumably well-acquainted with the most modern and advanced production methods, it is hard to imagine that there should be any inherent efficiency differences across observations, other than those that emerge from truly random effects and observation error. Thus, in principle, one would expect most such observations to be relatively close to the ideal frontier. This is not the case, for example, in data that are gathered under less controlled circumstances, where true differences in ability and in "human capital" can explain observed efficiency differences. Second, these data contain inputs that are both under the direct control of the experimenters (fertilizer levels) and inputs that are controlled by Nature. Hence, they seem to offer an ideal framework in which to investigate how apparent efficiency differences can emerge across observations not from any inherent different in knowledge or true efficiency but from the truly stochastic nature of such technologies.

Figure 1 presents the empirical distribution for rainfall over the farms in the sample. On the basis of this empirical distribution, we have split the rainfall state space into three events: *low rainfall* (below 277.2 mm per annum), *medium rainfall* (between 277.2 and 426.8 mm per annum) and *high rainfall* (above 426.8 mm per annum). These three groups have, respectively, 82, 91, and 97 observations in them.

Figure 1 about here.

In the empirical analysis, input- (TEx) and output-oriented (TEy) technical efficiency scores were computed for the observations. First, we calculated the efficiency scores using representation T^D above that presumed that all observations come from a common technology (i.e. a combined frontier). Then we calculated efficiency scores from frontiers calculated from the data partitioned according to the three rainfall events {low, medium, high} using the representation T^{ω}. In all cases, we presumed that the technology exhibited variable returns to scale and free disposability of inputs and outputs. Summary of these estimates are presented in table 1.

Table 1 about here.

We then compared the resulting efficiency scores that emerged from these two distinct methods using two different test statistics: the Kolmogorov-Smirnov nonparametric test and Banker tests. The Kolmogorov-Smirnov test statistic is a general distribution-free nonparametric test which quantifies differences in both location and shape of empirical cumulative distribution functions. Banker's test (Banker, 1993), on the other hand, uses F-statistics that can be constructed from the TE estimates under the assumption of normal or exponential distributions for the efficiency terms (Banker 1993 and Banker and Chang 1995), under the null hypothesis that T^D and $T^{\hat{\Omega}}$ are the same.[2]

According to the Kolmogorov-Smirnov test results (please see table 2), the input- and output-oriented efficiency scores calculated relative to T^{ω} for ω equal to high rainfall are significantly higher (at 99% confidence level) than those calculated relative to T^D. Similar results are obtained for ω equal to medium rainfall. The Banker test results reported in table 3 confirm these findings except in the case of the output-oriented scores for the medium rainfall group.

Tables 2 and 3 about here.

The input- and output-oriented efficiency scores calculated relative to T^{ω} for ω equal to low rainfall, however, are found to be similar to those calculated for these observations from T^D. Thus, on the basis of these results, we are led to conclude that T^D does a relatively good job of capturing the stochastic technology for low rainfall observations, but fails to capture the event-specific nature of the technology for increased levels of rainfall. A closer look at these numbers reveals some interesting patterns.

First, as the test results above indicate, for the medium and high rainfall groups, efficiency scores are higher when the DEA frontier includes only observations from the group. The magnitude and proportion of efficiency score changes are most pronounced for the high rainfall group. All the input-oriented efficiency scores and 78% of the output-oriented scores for the high rainfall observations are strictly higher when efficiency is calculated relative to T^{ω} rather than T^{D}. See table 4. The corresponding figures for the medium rainfall group are 90% and 71%. These changes are less frequent in the case of the low rainfall group but are virtually nill in magnitude as the figures in table 1 show. These ratios of input-oriented efficiency scores from separate and combined frontiers (TEx ratios)are plotted against rainfall measurements in figures 2 and 3.

Table 4 about here.

Figure 2 about here.

Second, the frequency and level of disparity between efficiency scores is greater for the input-based scores than for output-oriented scores (for both medium and high rainfall groups). For the high rainfall group, the TE ratios of the input-oriented scores from separate and combined frontiers have a mean (and also median) value of 2.25; these mean and median values are lower (1.08 and 1.28, respectively) for the output-oriented scores. The pattern is the same for the medium rainfall group with the TE ratios from the input-oriented frontier being higher. However, the degree of efficiency understatement from T^{D} increases with rainfall in the case of input-oriented measures but not in the case of the output-oriented scores.

Figure 3 about here.

The observed pattern of efficiency underestimation associated with T^{D} can be explained as follows. When T^{D} is used, observations from the high rainfall category are dominated by those from the other two categories. In fact, for both input- and output-oriented frontiers, none of the high rainfall observations are included as members of the best practice frontier for T^D. The pattern is less pronounced, but still observable, for the middle rainfall groups. This suggests that medium to high levels of rainfall fundamentally alter the production relationships between the inputs under producer control and rainfall variables. As we have noted above, we have imposed free disposability of inputs in the construction of DEA frontiers. However, it is very obvious that, in the extreme, very high levels of rainfall on a fixed plot of land can lead to a downward shift in productivity frontiers as the land becomes increasingly waterlogged. But even less dramatically, as rainfall reaches medium levels, there appears to be a levelling of the yield frontier associated with rainfall (in physical production terms, a von Liebig effect (Paris, 1992; Chambers and Lichtenberg, 1996)). Empirically, outliers in the data for rainfall levels that are low but not low enough to severely damage crop growth dominate observations from the high rainfall and medium rainfall groups that enjoy higher rainfall levels without correspondingly higher yield levels. Although rainfall levels in Western Australia are not very high, the yield plateau associated with von Liebig effects occurs within the range of the data. For soils with low water holding capacity, common in the West Australian Wheatbelt, additional rain mainly contributes to increased drainage (Asseng, Turner and Keating, 2001). A levelling of the yield frontier due to a von Liebig type effect would naturally be associated with greater measured input inefficiency than measured output inefficiency.

Conclusion

When stochastic elements alter the nature of the underlying technology, efficiency measures computed from models that ignore stochasticity can lead to misguided management actions. The standard approaches to efficiency measurement do not allow for the stochastic nature of technologies. This is true of both deterministic approaches, such as data envelopment analysis (DEA), and stochastic frontier (SFA) formulations, which incorporate stochastic errors merely as representations of measurement problems or omitted variables rather than as an explicit recognition of the stochastic nature of the underlying technology. Applying these models to data sets generated by a stochastic technology can lead to biased or erroneous estimates of efficiency performance. The purpose of this article is to show how event-specific representation of the production technology can be specified and then implemented within a data envelopment analysis framework.

The article started by describing how the state space can be partitioned to define event-specific production relationships that approximate the underlying stochastic technology. The purpose of these event-specific technologies is to provide empirical representations of the underlying technology that reflect the fact that the structure of the production technology might be shaped differently by different events. The article then shows how the event-specific representations can be implemented in a data envelopment analysis framework using the realized values of a random input to partition the data into comprehensive and exclusive subsets.

The event-specific DEA models are applied to agricultural field trial data and the results compared with those obtained from a standard DEA model that ignores the stochastic nature of the data. These field trial data provide an excellent opportunity for demonstrating the benefits of the event-specific formualtion. First, the trial data involve the use of inputs that are under the direct control of the agronomist or the experimenter as well as inputs such as rainfall that are stochastic or under the control of Nature. Second, the experimental nature of the data imples that there is very little besides stochastic or natural events that would be responsible for observed efficiency differences. Rainfall data is used to partition the state space into low, medium and high rainfall events. Both input-oriented and output-oriented efficiency scores were calculated for the comparison of the alternative DEA models.

We find that estimates of efficiency performance change dramatically when an event-specific technology representation is adopted. This is particularly true for data points relating to medium and high rainfall events. For the data set used in the article, the calculations indicate that input-oriented efficiency scores were underestimated, on average, by 50% or more in the case of high rainfall event data. The results highlight the degree to which our understanding of efficiency levels can be distorted when models that do not recognize the stochastic nature of the production process are used.

Notes

[1]We concentrate on a single output technology for the sake of simplicity. It is apparent, however, that our method can be easily extended to multiple-output stochastic technologies following the lines developed in Chambers and Quiggin (2000).

[2]These tests are based on analytical results obtained for the single output case. For multiple output efficiency scores, such analytically based statistical tests are not available and one has to rely on bootstrapping methods (Simar and Wilson, 2000).

References

Asseng, S., Turner, N. C. and Keating, B. A. (2001), 'Analysis of water- and nitrogen-use efficiency of wheat in a mediterranean climate', *Plant and Soil*, **233**: 127–43.

Banker, R.D. (1993), 'Maximum likelihood, consistency and data envelopment analysis: A statistical foundation', *Management Science*, pp. 1265–73.

Chambers, Robert G. and Lichtenberg, Erik (1996), 'A nonparametric approach to the von liebig-paris technology', *American Journal of Agricultural Economics*, **78**: 373–86.

Hunter, Rod (2005), 'Crop variety testing database: Background', South Perth, Western Australia.

O'Donnell, C., Chambers, Robert G. and Quiggin, J. (2006), 'Efficiency analysis in the presence of uncertainty', Technical report, Risk and Sustainable Mangement Group, Schools of Economics and Political Science, University of Queensland, Brisbane.

Paris, Quirino (1992), 'The von liebig hypothesis', *American Journal of Agricultural Economics*, **74**(4): 1019–28.

Simar, L. and Wilson, P. W. (2000), 'Statistical inference in nonparametric frontier models: The state of the art', *Journal of Productivity Analysis*, **13**: 49–78.

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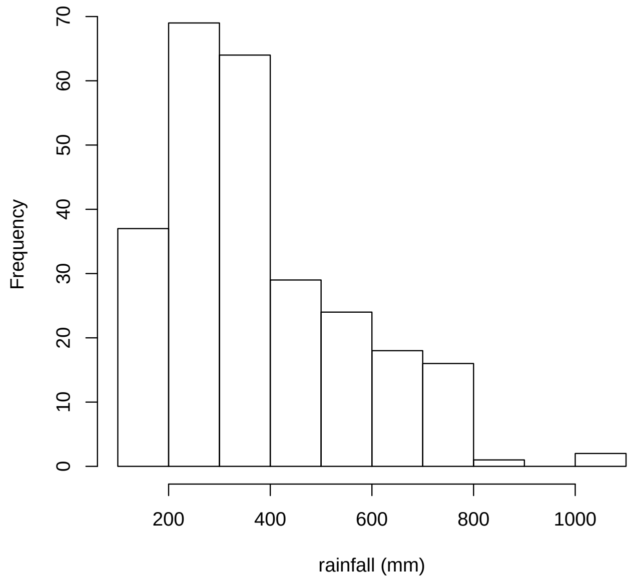

Figure 1: Empirical distribution of rainfall

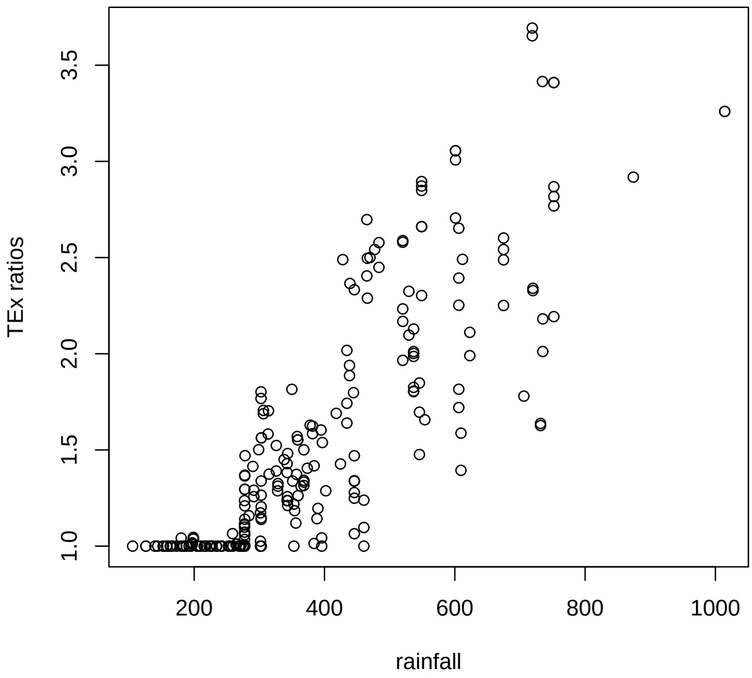

Figure 2: Ratios of input-oriented technical efficiency measures from separate and combined frontiers plotted against rainfall

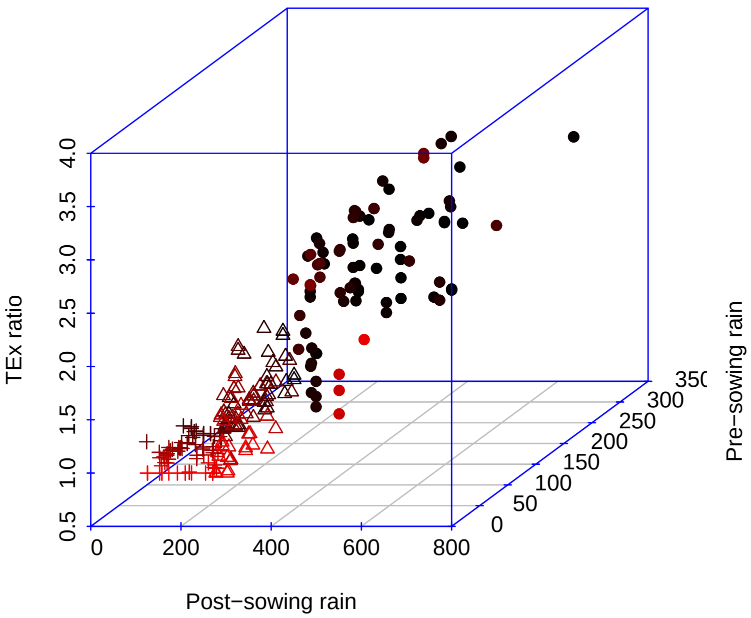

Figure 3: Ratios of input-oriented technical efficiency measures from separate and combined frontiers plotted against pre- and post-sowing rainfall

Table 1: A comparison of technical efficiency estimates from separate and combined technology frontiers

	Input-oriented TE					
	low rainfall group		medium rainfall group		high rainfall group	
	combined frontier	separate frontier	combined frontier	separate frontier	combined frontier	separate frontier
mean	0.88	0.88	0.72	0.89	0.44	0.90
median	0.90	0.90	0.68	0.92	0.40	0.96
	Output-oriented TE					
	low rainfall group		medium rainfall group		high rainfall group	
	combined frontier	separate frontier	combined frontier	separate frontier	combined frontier	separate frontier
mean	0.75	0.76	0.63	0.72	0.53	0.66
median	0.73	0.75	0.62	0.73	0.52	0.65

Table 2: Kolmogorov-Smirnov tests of technical efficiency estimates from separate and combined frontiers (p-values for two-sided and one-sided tests)

	Input-oriented/VRS		
Null Hypothesis	low rainfall group	medium rainfall group	high rainfall group
TEs from combined and seprate frontiers are same	0.998	0.0000	0.0000
TEs from combined frontiers are not smaller	0.7372	0.0000	0.0000
	Output-oriented/VRS		
Null Hypothesis	low rainfall group	medium rainfall group	high rainfall group
TEs from combined and seprate frontiers are same	0.998	0.0247	0.0077
TEs from combined frontiers are not smaller	0.7372	0.0123	0.0038

Table 3: Banker test difference in efficiency scores from separate and combined technology frontiers (Note: figures indicate distribution area beyond ctitical statistic value, i.e. P[X > x])

	under normal distribution assumption for efficiency terms		
	lower rainfall group	medium rainfall group	high rainfall group
TEx	0.46	0.00	0.00
TEy	0.34	0.40	0.10
	under exponential distribution assumption for efficiency terms		
	lower rainfall group	medium rainfall group	high rainfall group
TEx	0.44	0.00	0.00
TEy	0.36	0.41	0.02

Table 4: Efficiency change count: proportion of TE scores that are strictly higher for separate than for combined frontiers

	lower rainfall group	medium rainfall group	high rainfall group
TEx	25.61	90.11	100.00
TEy	51.22	71.43	78.16