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A Goodness-of-fit Test for Copulas *

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Abstract

We propose a new rank-based goodness-of-fit test for copulas. It uses the information matrix equality and so relates to the White (1982) specification test. The test avoids parametric specification of marginal distributions, it does not involve kernel weighting, bandwidth selection or any other strategic choices, it is asymptotically pivotal with a standard distribution and simple to compute compared to available alternatives. The finite-sample size of this type of tests is known to deviate from their nominal size based on asymptotic critical values, and bootstrapping critical values could be a preferred alternative. A power study shows that, in a bivariate setting, the test has reasonable properties compared to its competitors. We conclude with an application in which we apply the test to two stock indices.

JEL Classification: C13

Keywords: Copula, MLE, Information Matrix, Goodness-of-fit

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1 Introduction

Copulas are functions that allow modeling dependence between random variables separately from their marginal distributions. Consider two continuous random variables X_1 and X_2 with cdf's $F_1(x_1)$ and $F_2(x_2)$ and pdf's $f_1(x_1)$ and $f_2(x_2)$, respectively. Suppose the joint cdf of (X_1, X_2) is $H(x_1, x_2)$ and the joint pdf is $h(x_1, x_2)$. A copula is a function $C(u, v)$ such that $H(x_1, x_2) = C(F_1(x_1), F_2(x_2))$ or, in densities if they exist, $h(x_1, x_2) = c(F_1(x_1), F_2(x_2)) f_1(x_1) f_2(x_2)$. [For notational simplicity we will often write $H = C(F_1, F_2)$ and $h = c(F_1, F_2) f_1 f_2$]. The marginal densities f_1 and f_2 are now "extracted" from the joint density and the copula density c captures the dependence between X_1 and X_2. Sklar (1959) showed that given H, F_1 and F_2 of continuous variables, there exists a unique C. So, given F_1 and F_2, the choice when constructing a joint distribution is which copula C to use.

Let C_θ denote the chosen copula family with dependence parameter(s) θ. Numerous papers have used different copula families in applications from finance (e.g., Patton, 2006; Breymann et al., 2003), from risk management (e.g., Embrechts et al., 2003, 2002) and from health and labor economics (Smith, 2003; Cameron et al., 2004). Theoretical results on parametric and semiparametric estimation of copula-based models are contained in Genest et al. (1995); Joe (2005); Chen and Fan (2006b); Prokhorov and Schmidt (2009); among others. But the issue of copula specification testing – clearly relevant in any copula-based application – has not received as much attention in the literature as the estimation problem.

A copula family is correctly specified if, for some θ_o, $C_{\theta_o}(F_1, F_2) = H$. In this paper, we wish to construct a goodness of fit test for copulas using this definition. It would be desirable if such a goodness of fit test did not involve parametric specification of the marginal distributions because if it does, it essentially tests a joint hypothesis of correct copula *and* marginal specifications. It is also desirable that this test be applicable to any copula family without requiring any strategic choices and arbitrary parameters, e.g., the choice of a kernel and a bandwidth. Genest et al. (2009) call tests that have these desirable properties "blanket" goodness of fit tests.

There exist a number of copula goodness-of-fit tests (see Genest et al., 2009; Berg, 2009, for recent surveys). However, only a few are "blanket". For example, Klugman and Parsa

(1999) propose tests that involve ad hoc categorization of the data; Fermanian (2005) and Scaillet (2007) propose tests that are based on kernels, weight functions and use the associated smoothing parameters; Panchenko (2005) proposes a test based on a V-statistic, whose asymptotic distribution is unknown and depends on the choice of bandwidth; Prokhorov and Schmidt (2009) propose a conditional moment test for whether the copula-based score function has zero mean, which depends on parametric marginals and does not distinguish between the correct copula and any other copula that has a zero mean score function. All these tests do not qualify as "blanket".

Genest et al. (2009) report five testing procedures that qualify as "blanket" tests. These tests are based on empirical copula and on Kendall's and Rosenblat's probability integral transformation of the data as in, e.g., Dobrić and Schmid (2007); Breymann et al. (2003); Genest et al. (2006); Genest and Rémillard (2008). Recently Mesfioui et al. (2009) proposed one more "blanket" test based on a sample equivalent of Spearman's dependence function. All of these tests are substantially more difficult computationally than the "blanket" test we propose. Moreover, unlike our test, these tests are not asymptotically pivotal and require a procedure such as parametric bootstrap to obtain approximate p-values.

The test we propose is based on the information matrix equality which equates the copula Hessian and the outer-product of copula score. In essence this is the White (1982) specification test adapted to the first-step nonparametric estimation of marginal distributions. The first stage affects the asymptotic variance of the estimated Hessian and estimated outer-product in a nontrivial way. In Section 3 we show that our test statistic asymptotically has a χ^2 distribution and in the Appendix we provide the necessary adjustments for the first-stage rank estimation. Section 2 sets the stage by discussing the connection between copulas and the information matrix equality. In Section 4, we conduct a power study of the new test. As an illustration, Section 5 tests the goodness-of-fit of the Gaussian copula in a model with two stock indices. Section 6 concludes.

2 Copulas and Information Matrix Equivalence

Consider an N-dimensional copula $C(u_1, \ldots, u_N)$ and N univariate marginals $F_n(x_n)$, $n = 1, \ldots, N$. Then, by Sklar's theorem, the joint distribution of $(X_1, \ldots, X_N)$ is given by

$$H(x_1, \ldots, x_N) = C(F_1(x_1), \ldots, F_N(x_N)).$$

Assume F_n is continuous, $n = 1, \ldots, N$, so $C(u_1, \ldots, u_n)$ is unique. Assume further that the copula density exists, then the joint density of $(X_1, \ldots, X_N)$ is

$$\begin{aligned} h(x_1, \ldots, x_N) &= \left. \frac{\partial^N C(u_1, \ldots, u_N)}{\partial u_1 \ldots \partial u_N} \right|_{u_n = F_n(x_n), n=1,\ldots,N} \prod_{n=1}^{N} f_n(x_n) \\ &= c(F_1(x_1), \ldots, F_N(x_N)) \prod_{n=1}^{N} f_n(x_n), \end{aligned}$$

where $c(u_1, \ldots, u_N)$ is the copula density.

We are interested in goodness-of-fit testing of parametric copula families, so our copulas are parametric. For example, the N-variate Gaussian copula with $\frac{N(N-1)}{2}$ parameters can be written as follows

$$\Phi_N(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N); R),$$

where Φ_N is the joint distribution function of N standard normal covariates with a given correlation matrix R and Φ^{-1} is the inverse of the standard normal cdf. For Gaussian copulas, the copula parameters are the distinct elements of R. (See Nelsen, 2006; Joe, 1997, for examples of other copula families).

Let subscript θ denote the dependence parameter vector of a copula function and let p denote its dimension. It is well known that if there exists a value θ_o such that $H(x_1, \ldots, x_N) = C_{\theta_o}(F_1(x_1), \ldots, F_N(x))$ then we have a correctly specified likelihood model and, under regularity conditions, the MLE is consistent for θ_o. Moreover, in this case White's (1982) information matrix equivalence theorem holds: the Fisher information matrix can be equivalently calculated as minus the expected Hessian or as the expected outer product of the score function.

We wish to apply the information matrix equivalence theorem to copulas. Assume that the copula-based likelihood is three times continuously differentiable and the relevant expectations exist. Differentiability three times is required since, aside from the Hessian used in calculating the statistics, there is also an asymptotic variance expression involving the third derivative of the log-copula density. Let $\mathbb{H}(\theta)$ denote the expected Hessian matrix of $\ln c_\theta$ and let $\mathbb{C}(\theta)$ denote the expected outer product of the corresponding score function [not to confuse with copula C]. Then,

$$\begin{aligned}\mathbb{H}(\theta) &= \mathbb{E}\nabla^2_\theta \ln c_\theta(F_1(x_1),\dots,F_N(x_N)) \\ \mathbb{C}(\theta) &= \mathbb{E}\nabla_\theta \ln c_\theta(F_1(x_1),\dots,F_N(x_N))\nabla'_\theta \ln c_\theta(F_1(x_1),\dots,F_N(x_N)),\end{aligned}$$

where "∇_θ" denotes derivatives with respect to θ and expectations are with respect to the true distribution H.

White's (1982) information matrix equivalence theorem essentially says that, under correct copula specification,

$$-\mathbb{H}(\theta_o) = \mathbb{C}(\theta_o).$$

Our copula misspecification test uses this equality. Specifically, we will test

$$\mathcal{H}_0 : \mathbb{H}(\theta_o) + \mathbb{C}(\theta_o) = 0 \text{ against } \mathcal{H}_1 : \mathbb{H}(\theta_o) + \mathbb{C}(\theta_o) \neq 0 \tag{1}$$

3 Test

In practice, θ_o is not observed. Moreover, the matrices $\mathbb{H}(\theta)$ and $\mathbb{C}(\theta)$ contain the marginals F_n which are usually unknown. However, these quantities are easily estimated. In particular, it is common to use the empirical distribution function $\hat{F}_n$ in place of F_n, a consistent estimate $\hat{\theta}$ in place of θ_o, the sample averages $\bar{\mathbb{H}}$ and $\bar{\mathbb{C}}$ in place of the expectations $\mathbb{H}$ and $\mathbb{C}$.

Given T observations $(x_1,\dots,x_N)$, the empirical distribution function is given by

$$\hat{F}_n(s) = T^{-1}\sum_{t=1}^{T} I\{x_{nt} \le s\},$$

where $I\{\cdot\}$ is the indicator function and s takes values in the observed set of x_n. Then, $\hat{\theta}$ – a consistent estimator of θ_o sometimes called the Canonical Maximum Likelihood estimator

(CMLE) – is the solution to

$$\max_{\theta} \sum_{t=1}^{T} \ln c_{\theta}(\hat{F}_1(x_{1t}), \ldots, \hat{F}_N(x_{Nt})).$$

The following new notation is used for the sample counterparts:

$$\begin{aligned}
\hat{\mathbb{H}}_t(\theta) &= \nabla^2_{\theta} \ln c_{\theta}(\hat{F}_1(x_{1t}), \ldots, \hat{F}_N(x_{Nt})), \\
\hat{\mathbb{C}}_t(\theta) &= \nabla_{\theta} \ln c_{\theta}(\hat{F}_1(x_{1t}), \ldots, \hat{F}_N(x_{Nt}))\nabla'_{\theta} \ln c_{\theta}(\hat{F}_1(x_{1t}), \ldots, \hat{F}_N(x_{Nt})).
\end{aligned}$$

Then, the sample equivalents of $\mathbb{H}(\theta)$ and $\mathbb{C}(\theta)$ for arbitrary θ are

$$\begin{aligned}
\bar{\mathbb{H}}(\theta) &= T^{-1} \sum_{t=1}^{T} \hat{\mathbb{H}}_t(\theta), \\
\bar{\mathbb{C}}(\theta) &= T^{-1} \sum_{t=1}^{T} \hat{\mathbb{C}}_t(\theta).
\end{aligned}$$

The test we propose is based on distinct elements of the testing matrix $\bar{\mathbb{H}}(\hat{\theta}) + \bar{\mathbb{C}}(\hat{\theta})$. Given that the dimension of θ is p, there are $p(p+1)/2$ such elements. Under correct copula specification, these are all zero. So our test is in essence a variant of the likelihood misspecification test of White (1982). What distinguishes our test is that we deal with a semiparametric likelihood specification – a parametric copula and nonparametric marginals – while White (1982) deals with a full but possibly incorrect parametric log-density. Correspondingly, the elements of the White (1982) testing matrix (he calls them "indicators") do not contain empirical marginal distributions as arguments and this precludes direct application of his test statistic in our setting.

White (1982) points out that it is sometimes appropriate to drop some of the indicators because they are identically zero or represent a linear combination of the others. When $p = 1$ – the case of bivariate one-parameter copula – this problem does not arise. Whether it arises in higher dimensional models is a copula-specific question that we do not address in this paper. Assume that no indicators need be dropped.

Following White (1982), define

$$d_t(\theta) = vech(\mathbb{H}_t(\theta) + \mathbb{C}_t(\theta))$$

and

$$\hat{d}_t(\theta) = vech(\hat{\mathbb{H}}_t(\theta) + \hat{\mathbb{C}}_t(\theta))$$

where *vech* denotes vertical vectorization of the lower triangle of a matrix. Note that, in our setting, $d_t(\theta)$ depends on the unknown marginals while $\hat{d}_t(\theta)$ uses their empirical counterparts $\hat{F}_n, n = 1, \ldots, N$. Define the indicators of interest

$$\bar{D}_\theta \equiv \bar{D}(\theta) \equiv T^{-1} \sum_{t=1}^{T} \hat{d}_t(\theta).$$

Let $\bar{D}_{\hat{\theta}} = \bar{D}(\hat{\theta})$ and $D_\theta = Ed_t(\theta)$. Also note that, under correct specification, $D_{\theta_o} \equiv Ed_t(\theta_o) = 0$.

What is different in the present setting from White (1982) is that nonparametric estimates of the marginals are used to construct the joint density. It is well known that the empirical distribution converges to the true distribution at the rate $\sqrt{T}$ so the CMLE estimate $\hat{\theta}$ that uses empirical distributions $\hat{F}_n$ is still $\sqrt{T}$-consistent. The rate of convergence of the CMLE follows from Proposition 2.1 of Genest et al. (1995), which, along with everything that follows, is subject to regularity conditions.[1]

The asymptotic variance matrix of $\sqrt{T}\hat{\theta}$ will be affected by the nonparametric estimation of marginals. Therefore, the asymptotic variance of $\sqrt{T}\bar{D}_{\hat{\theta}}$ will also be affected. To derive the proper adjustments to the variance matrix we use the results on semiparametric estimation of Newey (1994) and Chen and Fan (2006b). Specifically, Chen and Fan (2006b) derive the distribution of $\hat{\theta}$ given the empirical estimates $\hat{F}_n, n = 1, \ldots, N$. Our setting is complicated by the fact that the test statistic is a function of both $\hat{\theta}$ and $\hat{F}_n, n = 1, \ldots, N$. The main result is given in the following proposition while the derivation of the asymptotic distribution is deferred to the Appendix.

[1] The regularity conditions can be found in many papers on semiparametric copula estimation (see, e.g., Genest et al., 1995; Shih and Louis, 1995; Chen and Fan, 2006b,a; Hu, 1998). They include compactness of the parameter set, smoothness of the marginals, existence and continuity of the relevant log-density derivatives. Verification of these conditions for commonly used copula families is beyond the scope of this paper. For many copulas, including those we use, this has been done elsewhere (see, e.g., Hu, 1998, Chapter 5).

Proposition 1 *Under correct copula specification and suitable regularity conditions, the information matrix test statistic*

$$\mathscr{I} = T\bar{D}'_{\hat{\theta}} V_{\theta_o}^{-1} \bar{D}_{\hat{\theta}},$$

where V_{θ_o} is given in (3) in Appendix, is distributed asymptotically as $\chi^2_{p(p+1)/2}$.

The test statistic has a similar structure and identical asymptotic distribution to that of the White (1982) test. Indeed it is a variant of that test adjusted for the first step estimation of the marginals. It is known that the White (1982) test statistic goes to infinity almost surely when the H_o does not hold (see, e.g., Golden et al., 2010). So we may expect our test to be consistent, too, but we do not pursue this point further in this paper.[2]

In practice, a consistent estimate of V_{θ_o} will be used. Under correct copula specification, such an estimate can be obtained by replacing θ_o and F_{nt} in (3) by their consistent estimates $\hat{\theta}$ and $\hat{F}_{nt}$.

Unlike available alternatives, this test statistic is simple, easy to compute and has a standard asymptotically pivotal distribution. It involves no strategic choices such as the choice of a kernel and associated smoothing parameters or any arbitrary categorization of the data. Essentially this is White's information equivalence test with the complication of a first-step empirical distribution estimation. However, as such, it also inherits a number of drawbacks. One complication is the need to evaluate the third derivative of the log-copula density function. Lancaster (1984) and Chesher (1983) show how to construct simplified versions of the test statistic, which are asymptotically equivalent to White's original statistic but do not use the third order derivatives. Probably the simplest form of the test is TR^2, where R^2 comes from the regression of a vector of ones on

$$\nabla_{\theta_j} \ln c_\theta(\hat{F}_1(x_{1t}), \ldots, \hat{F}_N(x_{Nt})), \quad j = 1, \ldots, p$$

and

$$\nabla^2_{\theta_j\theta_k} \ln c_\theta(\hat{F}_1(x_{1t}), \ldots, \hat{F}_N(x_{Nt})) + \nabla_{\theta_j} \ln c_\theta(\hat{F}_1(x_{1t}), \ldots, \hat{F}_N(x_{Nt})) \nabla_{\theta_k} \ln c_\theta(\hat{F}_1(x_{1t}), \ldots, \hat{F}_N(x_{Nt})),$$

[2] For test consistency, it is important to differentiate between the H_o as stated in (1) and the null of a specific copula family. The test may not be consistent against false copula densities such that $\mathbb{H}(\theta_o) + \mathbb{C}(\theta_o) = 0$. This seems to be a feature of all information matrix based tests. We thank a referee for pointing this out to us.

$$j = 1, \ldots, p, \quad k = 1, \ldots, p,$$

evaluated at $\hat{\theta}$.

An important problem is the well-documented poor finite sample properties of the test, especially of the TR^2 form (see, e.g., Taylor, 1987; Hall, 1989; Chesher and Spady, 1991; Davidson and MacKinnon, 1992). Horowitz (1994), for example, points out to large deviations of the finite-sample size of various forms of the White test from their nominal size based on asymptotic critical values and suggests using bootstrapped critical values instead. Of course our test will inherit this problem.

4 Power Study

In this section, we study the size and power properties of the test statistic we derived in Proposition 1. We remark on how this test compares with other copula goodness-of-fit tests discussed in Genest et al. (2009) but we do not compare here the various alternative forms of the test statistic such as the TR^2 form. We start by plotting size-power curves under various copula families (see, e.g., Davidson and MacKinnon, 1998, for a comparison of this and other graphical ways of studying test properties). We generate K realizations of the test statistic $\mathscr{I}$ using a data-generating process (DGP). Denote these simulated values by $\mathscr{I}_j$, $j = 1, \ldots, K$. Our size-power curves are based on the empirical distribution function (EDF) of the simulated p-value of $\mathscr{I}_j$, $p_j \equiv p(\mathscr{I}_j)$, i.e. the probability that $\mathscr{I}$ is greater than or equal to $\mathscr{I}_j$ according to its simulated distribution. At any point y in the (0, 1) interval, the EDF of the p-values is defined by

$$\hat{F}(y) \equiv \frac{1}{K} \sum_{j=1}^{K} I(p_j \leq y).$$

We choose the following values for y_i, $i = 1, \ldots, m$:

$$y_i = 0.001,\ 0.002,\ \ldots,\ 0.010,\ 0.015,\ \ldots,\ 0.990,\ 0.991,\ \ldots,\ 0.999 \quad (m = 215),$$

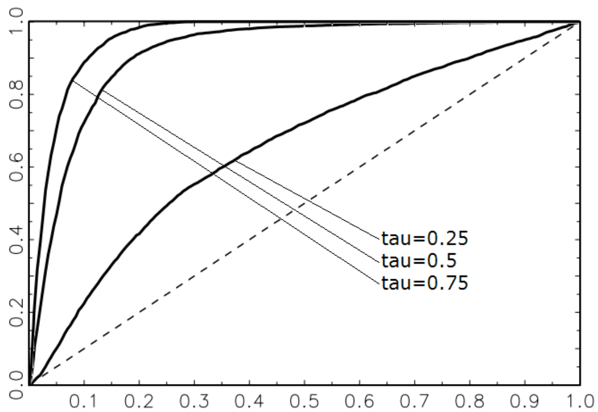
where we follow Davidson and MacKinnon (1998) and use a smaller grid near 0 and 1 in order to study the tail behavior more closely.

The point of drawing size-power curves is to plot power against true, rather than nominal, size. Given the well-documented poor finite sample size property of the information matrix test, this is useful because we can display the test power in situations when the nominal size is definitely incorrect. Two values of the test statistic are computed: one under the null DGP (H_0) and the other under the alternative DGP (H_1). Let $F(y)$ and $F^*(y)$ be the probabilities of getting a p-value less than y under the null and the alternative, respectively, and let $\hat{F}(y)$ and $\hat{F}^*(y)$ be their empirical counterparts. Given the sample size T, the number of simulation replications K and the grid of size m, a size-power curve is the set of points $(\hat{F}(y_i), \hat{F}^*(y_i))$, $i = 1, \ldots, m$, on the unit square where the horizontal axis measures size and the vertical axis measures power.

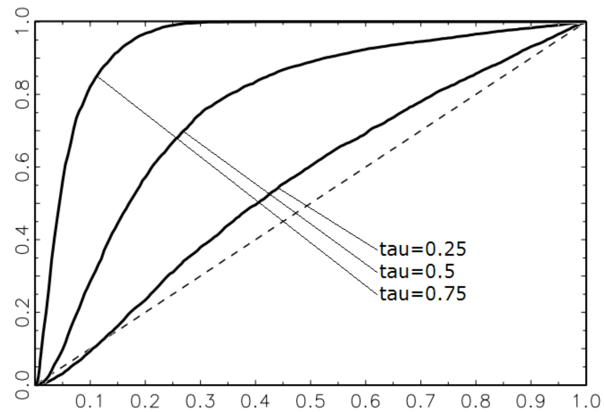
We keep the grid the same, set $K = 10,000$, and vary the sample size T and the strength of dependence in the various null and alternative DGPs we consider. The various null and alternative copula families are selected from the list used by Genest et al. (2009) in a large scale Monte Carlo study and, as usual, the dependence strength is measured by Kendall's τ, where $\tau = 4\mathbb{E}[C_\theta(U, V)] - 1$. We follow Genest et al. (2009) and use the copula parameter obtained by inversion of Kendall's τ. In all considered families the solution is known to be unique so this produces one parameter value under H_0 and one under H_1. To preserve space we report curves for $T = 200, 300$ and $\tau = 0.25, 0.33, 0.5, 0.75$ only.

Figure 1 shows what happens as we change the strength of dependence holding T fixed at 300. Panel (a) displays the size-power curves under H_0: Normal copula and H_1: Clayton copula, panel (b) displays the curves for H_0: Normal and H_1: Frank, panel (c) is for the test of H_0: Clayton against H_1: Normal, and panel (d) is for H_0: Clayton against H_1: Frank. We can clearly see from the figure that as the strength of dependence increases, the power of the test becomes larger. This agrees with similar observations by Genest et al. (2009) made for other copula goodness-of-fit tests. Interestingly, there are areas on the plots where the test actually has power less than its size. This happens at small enough sizes to make this observation important but the same thing occasionally happens with other "blanket" tests under weak dependence (for $\tau = 0.25$, see, e.g., Genest et al., 2009, Table 1).

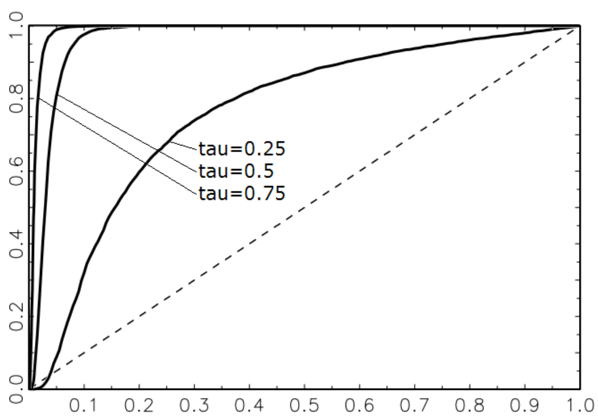
Figure 2 displays the size-power curves for different null and alternative DGPs holding



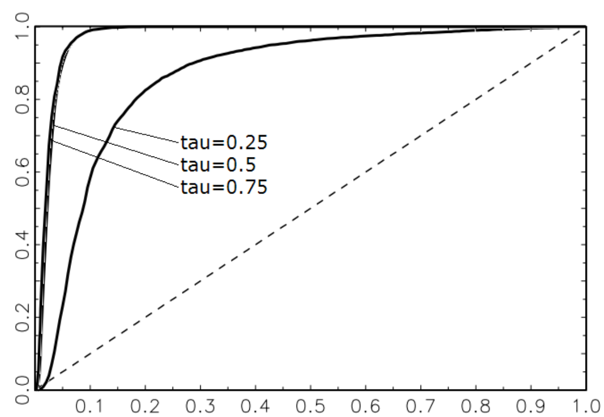
(a) H_0: Normal; H_1: Clayton



(b) H_0: Normal; H_1: Frank



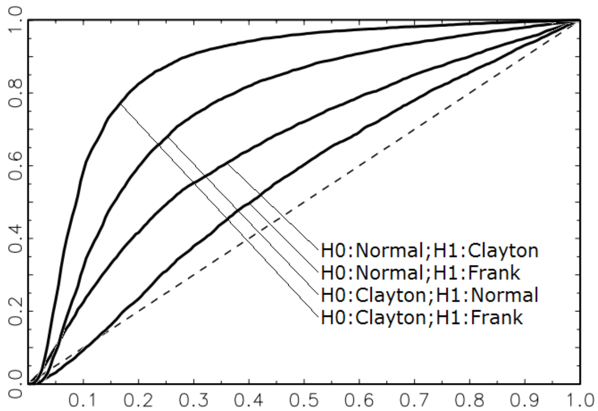
(c) H_0: Clayton; H_1: Normal



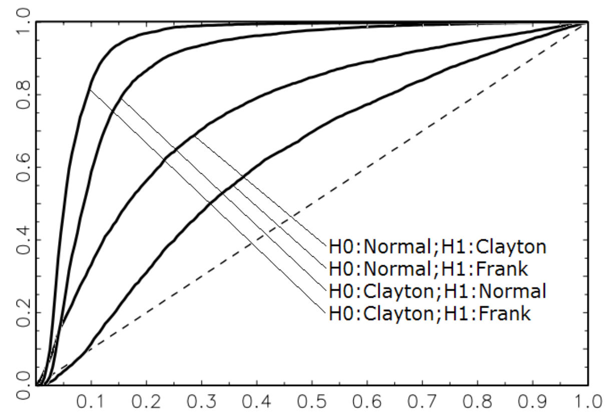
(d) H_0: Clayton; H_1: Frank

Figure 1: Size-power curves for different levels of dependence: Kendall's $\tau = 0.25$, 0.5 and 0.75. Sample size is $T = 300$.

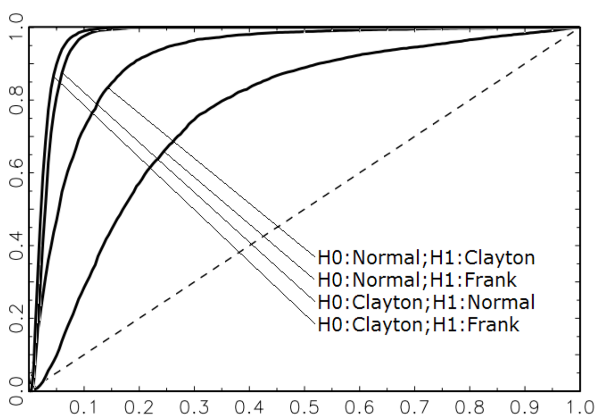
both T and τ fixed. The set of nulls and alternatives we report includes H_0: Normal vs H_1: Clayton, H_0: Normal vs H_1: Frank, H_0: Clayton vs H_1: Normal, H_0: Clayton vs H_1: Frank. An interesting observation is that the size-adjusted power of the test varies greatly for the different nulls and alternatives – something that has been noted for other tests as well. If we further allow τ to increase holding sample size fixed, the variation in power becomes much smaller. It is interesting to observe that for the tests that involve the Clayton copula under H_0, the test has much more power than for the other models we consider. Again, this interesting observation coincides with results of Genest et al. (2009) obtained for other available "blanket" tests (see their Tables 1-3). Note that the ranking of power of the various tests changes as we change strength of dependence, but the two tests involving the Clayton null remain more powerful than the others.



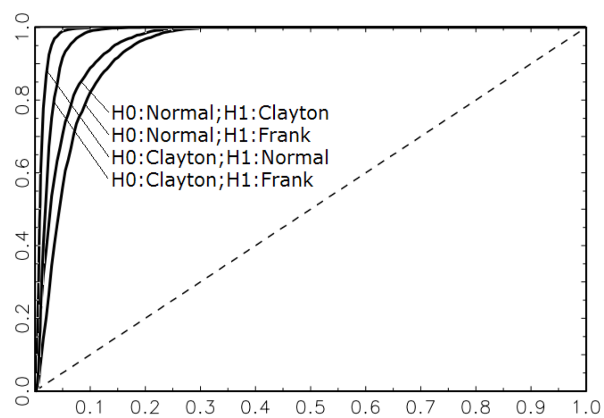
(a) $\tau = 0.25$



(b) $\tau = 0.333$



(c) $\tau = 0.5$

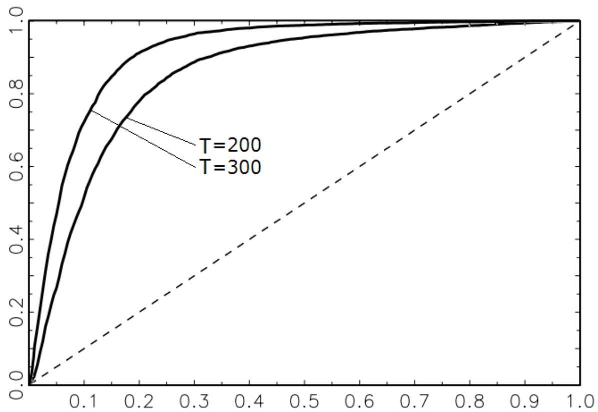


(d) $\tau = 0.75$

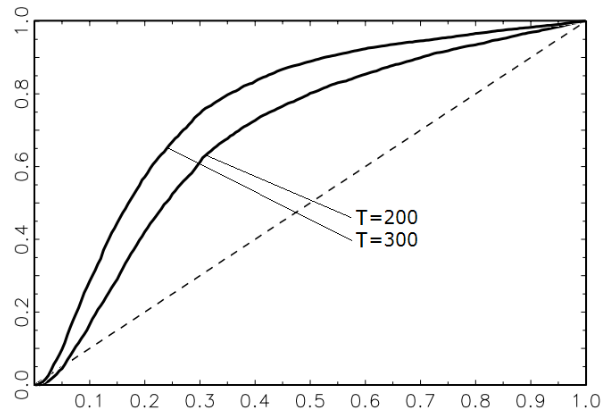
Figure 2: Size-power curves for selected copulas. Sample size is $T = 300$.

Figure 3 shows how the size-power curves shift as the sample size changes from $T = 200$ to $T = 300$. The test in each panel is the same as in Figure 1. Not surprisingly, the power increases as the sample size grows. Plots for larger samples (not reported here) illustrate that as the sample size becomes larger, H_0 is rejected with probability approaching one whenever H_1 is true, i.e. these tests are consistent.

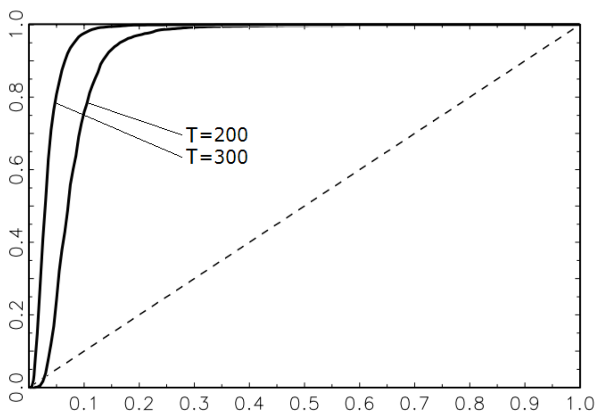
To compare our test with other "blanket" tests in more detail and also to get an idea about the extent of size distortions, we construct a size and power table similar to those reported by Genest et al. (2009). Tables 1 and 2 report size and power of our test at the 5% significance



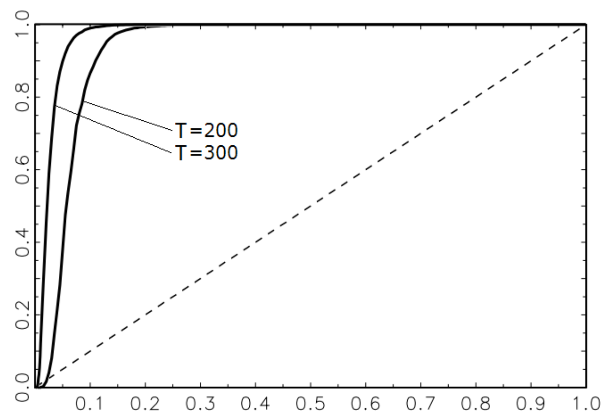
(a) H_0: Normal; H_1: Clayton



(b) H_0: Normal; H_1: Frank



(c) H_0: Clayton; H_1: Normal



(d) H_0: Clayton; H_1: Frank

Figure 3: Size-power curves for different sample sizes: $T = 200$ and $T = 300$. Kendall's $\tau = 0.5$.

level for $T = 200$ and $T = 1,000$. As before we also vary Kendall's τ from 0.25 to 0.75. In each row, we report the percentage of rejections of H_0 associated with different tests for the bootstrap test (*Simul.*) and the asymptotic test (*Asy.*). For example, when testing for the Normal copula against Clayton at $T = 200$ and $\tau = 0.75$, the chance of the bootstrap test rejecting the incorrect null is approximately 34.6%.

Similar to analogous entries for other "blanket" tests, the frequencies reported in Tables 1 and 2 show that for these sample sizes the test generally holds its nominal size. Indeed the frequencies listed in the *Simul.* columns are virtually equal to the nominal level of 5% no matter what sample size or copula family. This is hardly suprising since we are bootstrapping

Table 1: Power(Size) for T=200 at nominal size 5%

Copula under H_0	True Copula	$\tau = 0.25$ Simul.	$\tau = 0.25$ Asy.	$\tau = 0.50$ Simul.	$\tau = 0.50$ Asy.	$\tau = 0.75$ Simul.	$\tau = 0.75$ Asy.
Normal	Clayton	4.9(5)	7.7(7)	21.7(5)	34.8(8)	34.6(5)	62.9(10)
	Frank	2.5(5)	4.0(7)	3.8(5)	7.8(7)	16.5(5)	42.0(9)
	Gumbel	6.8(5)	9.6(6)	9.2(5)	18.3(8)	9.1(5)	26.7(10)
Clayton	Normal	1.3(5)	12.2(10)	29.8(5)	85.06(11)	86.1(5)	99.2(11)
	Frank	4.2(5)	26.4(10)	41.6(5)	93.2(11)	64.2(5)	94.6(11)
	Gumbel	8.6(5)	36.5(10)	60.4(5)	96.5(12)	86.4(5)	98.4(10)
Frank	Normal	6.5(5)	8.2(6)	9.2(5)	14.6(9)	3.1(5)	8.0(10)
	Clayton	4.0(5)	5.3(6)	1.5(5)	5.7(9)	2.7(5)	22.4(10)
	Gumbel	4.8(5)	5.8(6)	1.8(5)	5.4(9)	1.0(5)	8.7(10)
Gumbel	Normal	2.9(5)	5.1(8)	1.3(5)	5.2(9)	1.0(5)	9.9(10)
	Clayton	16.9(5)	30.4(8)	37.5(5)	80.0(10)	79.1(5)	97.2(10)
	Frank	3.5(5)	8.0(8)	6.3(5)	31.2(9)	32.7(5)	80.2(10)

an asymptotically pivotal statistic using as many as 10,000 replications. In this setting, the bootstrap test is very close to the exact test for a sufficiently large number of replications, regardless of the specific null or the specific sample size (see, e.g., Hall and Hart, 1990, Table 1). The same result would be expected for a sample of as few as 20 observations. On the other hand, the frequencies shown in the *Asy.* columns are often substantially higher than 5%, suggesting oversize distortions. As expected, the distortions clearly reduce as the sample size increases.

Compared to equivalent entries in Tables 1 to 3 of Genest et al. (2009), the power of our test statistic is generally lower than that of the other "blanket" tests available in the literature. However, at the sample size equal to $1,000$, our test power is usually reasonably high. Similar to other "blanket" tests, the performance of our test varies greatly with the DGPs. For some combinations of copulas under the null hypothesis and the alternative, the test's power is remarkably low. For example, if the null hypothesis is Frank and the true copula is Normal, the power of our test at $T = 1,000$ is as low as 4-6% even for $\tau = 0.75$. Interestingly, the power of other "blanket" tests is not very high for some combinations either,

Table 2: Power(Size) for T=1000 at nominal size 5%

Copula	True	$\tau = 0.25$		$\tau = 0.50$		$\tau = 0.75$	
under H_0	Copula	Simul.	Asy.	Simul.	Asy.	Simul.	Asy.
Normal	Clayton	44.0(5)	5(6)	96.9(5)	98.8(7)	93.2(5)	99.0(12)
	Frank	10.7(5)	16.2(7)	65.2(5)	80.0(8)	90.3(5)	98.1(12)
	Gumbel	58.0(5)	63.4(6)	83.4(5)	92.3(8)	78.8(5)	94.7(11)
Clayton	Normal	83.5(5)	87.8(6)	100(5)	100(7)	100(5)	100(7)
	Frank	98.6(5)	99.3(7)	100(5)	100(7)	100(5)	100(7)
	Gumbel	99.6(5)	99.8(6)	100(5)	100(7)	100(5)	100(7)
Frank	Normal	10.1(5)	10.7(5)	21.2(5)	24.0(6)	4.3(5)	5.4(6)
	Clayton	8.5(5)	9.6(6)	17.2(5)	19.9(6)	93.5(5)	95.8(6)
	Gumbel	20.2(5)	21.4(5)	14.9(5)	17.3(6)	53.8(5)	64.4(7)
Gumbel	Normal	8.3(5)	9.4(6)	20.9(5)	25.7(6)	68.3(5)	72.8(6)
	Clayton	98.2(5)	98.6(6)	100(5)	100(6)	100(5)	100(6)
	Frank	50.8(5)	53.7(6)	99.2(5)	99.5(6)	100(5)	100(6)

and for some combinations of copulas and some sample sizes, Genest et al. (2009) report even lower percentages of rejection. In such cases, the results of more than one "blanket" test should probably be considered together.

5 Application

To demonstrate how the test procedure in Section 3 can be applied in practice, in this section we test whether the bivariate Gaussian copula is appropriate for modeling dependence between an American and an European stock index. The power study demonstrated that the proposed test of the null of Normal copula has power against commonly used alternatives such as the Clayton, Frank and Gumbel copulas.

The two time series we use are FTSE100 and DJIA closing quotes from June 26, 2000 to June 23, 2008. There are 1972 pairs of returns once holidays are eliminated. Table 3 contains descriptive statistics of the returns. The statistics we use are third (m_3) and fourth (m_4) central sample moments and the Ljung-Box Q test statistics for testing autocorrelation of

Table 3: Summary statistics of returns series

	FTSE	DJIA
mean	0.0001	-.0001
st.d.	0.107	0.103
m_3	0.104	0.020
m_4	6.101	6.590
$Q(20)$ p-value	0.000	0.031
$Q^2(20)$ p-value	0.000	0.000

up to 20 lags in returns [$Q(20)$] and in squared returns [$Q^2(20)$]. Both return series display excess kurtosis and FTSE returns are a bit more skewed than DJIA.

We first apply an AR-GARCH filter to the return data. As shown in Table 4, this accounts for most of observed autocorrelation in returns and squared returns. The preferred AR-GARCH models contain up to one lag in the conditional mean equation and a GARCH (1,1) in the conditional variance with Normal innovations (allowing for Student-t innovations resulted in a relatively high estimate of the degrees of freedom (over 9) and did not improve the fit substantially). Table 4 reports the results of the AR-GARCH modeling.

The results of the test are reported in Table 5. They are based on the residuals from the AR-GARCH models. In principle, this prefiltering should affect the second step estimation and an adjustment should be required to account for that. However, Chen and Fan (2006a) show that the limiting distribution of the copula parameter is not affected by the estimation of dynamic parameters, although as before it is affected by the nonparametric estimation of marginal distributions. So, in this case, the prefiltering is innocuous.

For the bivariate Gaussian copula, the estimated parameter θ is simply the sample correlation between the margins of the bivariate normal distribution used to construct the copula. As reported in Table 5, the parameter estimate is not very large, but positive and statistically significant. Aside from the test statistic, Table 5 reports p-values obtained using both the asymptotic and the bootstrap distribution based on 10,000 replications. The test statistic is

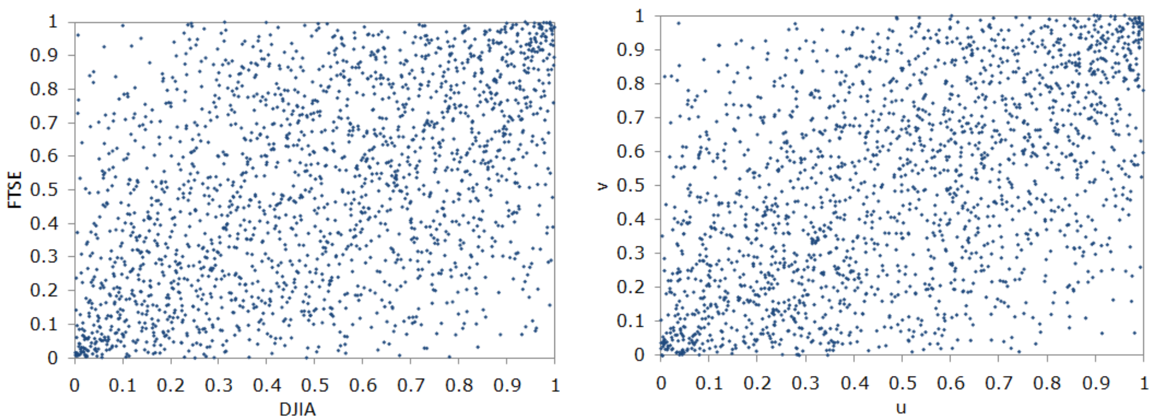
Table 4: AR-GARCH estimates and standard errors

	FTSE	DJIA
μ	-0.0006(0.0004)	-0.0007(0.0004)
AR(1)	-0.0711(0.0230)	-0.0455(0.0221)
ω	0.0000(0.0000)	0.0000(0.0000)
α	0.1158(0.0183)	0.0737(0.0167)
β	0.8742(0.0207)	0.9192(0.0196)
ll	6397.001	6438.337
m_3	-0.144	-0.096
m_4	3.349	3.724
$Q(20)$ p-value	0.320	0.372
$Q^2(20)$ p-value	0.711	0.046

Table 5: Testing the Gaussian copula

$\hat{\theta}$	0.4830(0.0188)
Asy. $p-value$ for $\mathscr{I}$	0.0489
Exact $p-value$ for $\mathscr{I}$	0.2700

quite large. Based on the asymptotic critical value, we would reject the Gaussian copula at the 5% significant level. This is a weak rejection (we would not reject at the 1% level, for example). However, we should keep in mind the reported over-rejection of this test. If we use the residual-based bootstrap critical value, we fail to reject the Gaussian copula at any conventional significance level. This is consistent with the finding of Malevergne and Sornette (2003), who report that when correlation is not very high, Gaussian copula is appropriate for financial modelling. Indeed, the rank plots for low correlations are very similar for different copulas. As a visual confirmation of this finding, we provide in Figure 4 the scatter plots of our data after transforming it into standard uniform and that of simulated data where the true copula is Gaussian with $\theta = 0.5$. The two plots look very similar.



(a) Scatter plot of data

(b) Scatter plot of Gaussian copula

Figure 4: Scatter plots of standard uniform transformed data and Gaussian copula.

6 Concluding remarks

We have proposed a new goodness-of-fit test for copulas and have shown that it has reasonable properties. The main advantage of the test is its simplicity. Basically, it is the well-studied White specification test adapted to a two-step semiparametric estimation. As such, it inherits White test's benefits and costs. The most costly feature of the test is its poor behavior in samples smaller than $1,000$. Other potential criticisms include the test's inability to detect *all* deviations from the null in finite samples, its inability to differentiate between two well-performing alternatives, and its in-sample nature.

As in-sample procedures, this and other "blanket" tests can be argued to be susceptible to overfitting and data mining. However, recent studies in the setting of predictability tests tend to question the conventional wisdom that out-of-sample tests are more credible than in-sample (see, e.g., Inoue and Kilian, 2005).

The White test is simple compared to some of the other available "'blanket" tests, which do not have such a simple asymptotic distribution and are much harder to construct. Obtaining up to three derivatives of the log-copula density is the main challenge in constructing the test statistic. However, for some families explicit formulas for the derivatives have been catalogued (see, e.g., Chen and Fan, 2006b) and, for others, symbolic algebra modules of modern software can be used to obtain them. Of course there is always the brute force method of calculating the derivatives numerically. Moreover, the test has many asymptotically equivalent forms, some of which are derived specifically to reduce the order of derivatives and to make the finite sample behavior more appealing (see, e.g., Golden et al., 2010). For example, the versions of Lancaster (1984) and Chesher (1983) do not require the third derivative while the first two derivatives of the likelihood often arise as byproducts of standard MLE optimization routines. Overall, the balance of costs and benefits speaks, we believe, in favor of this copula goodness-of-fit test, especially in large sample settings of a financial application, similar to the one we have considered.

A Proof of Proposition

We start with $N = 2$ for simplicity and later give the formulas for any N. Let $\hat{F}_{nt} = \hat{F}_n(x_{nt})$, $n = 1, 2$, $t = 1, \ldots, T$, be the empirical cdf's. Then,

$$\hat{d}_t(\theta) = vech[\nabla^2_\theta \ln c(\hat{F}_{1t}, \hat{F}_{2t}; \theta) + \nabla_\theta \ln c(\hat{F}_{1t}, \hat{F}_{2t}; \theta)\nabla'_\theta \ln c(\hat{F}_{1t}, \hat{F}_{2t}; \theta)].$$

Provided that the derivatives and expectation exist, let

$$\nabla D_\theta = E\nabla_\theta d_t(\theta)$$

and

$$\nabla \bar{D}_{\theta} = T^{-1} \sum_{t=1}^{T} \nabla_{\theta} \hat{d}_t(\theta).$$

First, expand $\sqrt{T}\bar{D}_{\hat{\theta}}$ with respect to θ:

$$\sqrt{T}\bar{D}_{\hat{\theta}} = \sqrt{T}\bar{D}_{\theta_o} + \nabla D_{\theta_o}\sqrt{T}(\hat{\theta} - \theta_o) + o_p(1).$$

Chen and Fan (2006b) show that

$$\sqrt{T}(\hat{\theta} - \theta_o) \to N(0, B^{-1}\Sigma B^{-1}),$$

where

$$B = -\mathbb{H}(\theta_0),$$
$$\Sigma = \lim_{T\to\infty} Var(\sqrt{T}A_T^*),$$
$$A_T^* = \frac{1}{T}\sum_{t=1}^{T}(\nabla_{\theta} \ln c(F_{1t}, F_{2t}; \theta_0) + W_1(F_{1t}) + W_2(F_{2t})).$$

Here terms $W_1(F_{1t})$ and $W_2(F_{2t})$ are the adjustments needed to account for the empirical distributions used in place of the true distributions. These terms are calculated as follows:

$$W_1(F_{1t}) = \int_0^1 \int_0^1 [I\{F_{1t} \leq u\} - u]\nabla^2_{\theta,u} \ln c(u, v; \theta_0)\, c(u, v; \theta_0)\mathrm{d}u\mathrm{d}v,$$
$$W_2(F_{2t}) = \int_0^1 \int_0^1 [I\{F_{2t} \leq v\} - v]\nabla^2_{\theta,v} \ln c(u, v; \theta_0)\, c(u, v; \theta_0)\mathrm{d}u\mathrm{d}v.$$

So,

$$\sqrt{T}(\hat{\theta} - \theta_o) = B^{-1}\sqrt{T}A_T^* + o_p(1).$$

Second, expand $\sqrt{T}\bar{D}_{\theta_0}$ with respect to F_{1t} and F_{1t}:

$$\sqrt{T}\bar{D}_{\theta_0} \simeq \frac{1}{\sqrt{T}}\sum_{t=1}^{T} d_t(\theta_0) + \frac{1}{T}\sum_{t=1}^{T} \nabla_{F_{1t}} d_t(\theta_0)\sqrt{T}(\hat{F}_{1t} - F_{1t}) + \frac{1}{T}\sum_{t=1}^{T} \nabla_{F_{2t}} d_t(\theta_0)\sqrt{T}(\hat{F}_{2t} - F_{2t}). \quad (2)$$

Under suitable regularity conditions,

$$\frac{1}{T}\sum_{t=1}^{T}\nabla_{F_{1t}}d_t(\theta_0)\sqrt{T}(\hat{F}_{1t}-F_{1t})$$

$$\simeq \int_0^1\int_0^1 \nabla_u vech[\nabla_\theta^2 \ln c(u,v;\theta_0)+\nabla_\theta \ln c(u,v;\theta_0)\nabla'_\theta \ln c(u,v;\theta_0)]$$

$$\sqrt{T}(\hat{F}_1(F_1^{-1}(u))-u)c(u,v;\theta_0)\mathrm{d}u\mathrm{d}v$$

$$=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\int_0^1\int_0^1 [I\{F_{1t}\leq u\}-u]$$

$$\nabla_u vech[\nabla_\theta^2 \ln c(u,v;\theta_0)+\nabla_\theta \ln c(u,v;\theta_0)\nabla'_\theta \ln c(u,v;\theta_0)]c(u,v;\theta_0)\mathrm{d}u\mathrm{d}v.$$

Denote

$$M_1(F_{1t})=\int_0^1\int_0^1 [I\{F_{1t}\leq u\}-u]$$

$$\nabla_u vech[\nabla_\theta^2 \ln c(u,v;\theta_0)+\nabla_\theta \ln c(u,v;\theta_0)\nabla'_\theta \ln c(u,v;\theta_0)]c(u,v;\theta_0)\mathrm{d}u\mathrm{d}v,$$

then

$$\frac{1}{T}\sum_{t=1}^{T}\nabla_{F_{1t}}d_t(\theta_0)\sqrt{T}(\hat{F}_{1t}-F_{1t})=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}M_1(F_{1t}).$$

Similarly, denote

$$M_2(F_{2t})=\int_0^1\int_0^1 [I\{F_{2t}\leq v\}-v]$$

$$\nabla_v vech[\nabla_\theta^2 \ln c(u,v;\theta_0)+\nabla_\theta \ln c(u,v;\theta_0)\nabla'_\theta \ln c(u,v;\theta_0)]c(u,v;\theta_0)\mathrm{d}u\mathrm{d}v,$$

then

$$\frac{1}{T}\sum_{t=1}^{T}\nabla_{F_{2t}}d_t(\theta_0)\sqrt{T}(\hat{F}_{2t}-F_{2t})=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}M_2(F_{2t}).$$

Therefore, equation (2) can be rewritten as

$$\sqrt{T}\bar{D}_{\theta_0}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}d(\theta_0)+\sqrt{T}B_T^*+o_p(1),$$

where

$$B_T^*=\frac{1}{T}\sum_{t=1}^{T}[M_1(F_{1t})+M_2(F_{2t})].$$

Finally, combining the expansions gives

$$\sqrt{T}\bar{D}_{\hat{\theta}} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} d(\theta_0) + \sqrt{T}B_T^* + \nabla D_{\theta_0} B^{-1}\sqrt{T}A_T^* + o_p(1).$$

So

$$\sqrt{T}\bar{D}_{\hat{\theta}} \to N(0, V_{\theta_0}),$$

or, equivalently,

$$T\bar{D}_{\hat{\theta}}' V_{\theta_0}^{-1}\bar{D}_{\hat{\theta}} \to \chi^2_{p(p+1)/2},$$

where

$$\begin{aligned} V_{\theta_0} = E\,\{&d_t(\theta_0) + M_1(F_{1t}) + M_2(F_{2t}) \\ &+\nabla D_{\theta_0} B^{-1}\left[\nabla_\theta \ln c(F_{1t}, F_{2t}; \theta_0) + W_1(F_{1t}) + W_2(F_{2t})\right]\big\} \\ &\times \{d_t(\theta_0) + M_1(F_{1t}) + M_2(F_{2t}) \\ &+\nabla D_{\theta_0} B^{-1}\left[\nabla_\theta \ln c(F_{1t}, F_{2t}; \theta_0) + W_1(F_{1t}) + W_2(F_{2t}))\right]\big\}'. \end{aligned}$$

Extension to $N \geq 2$ is straightforward. Now

$$d_t(\theta) = vech[\nabla_\theta^2 \ln c(F_{1t}, F_{2t}, \ldots, F_{Nt}; \theta) + \nabla_\theta \ln c(F_{1t}, F_{2t}, \ldots, F_{Nt}; \theta)\nabla_\theta' \ln c(F_{1t}, F_{2t}, \ldots, F_{Nt}; \theta)],$$

and the asymptotic variance matrix becomes

$$\begin{aligned} V_{\theta_0} = E&\left\{d_t(\theta_0) + \nabla D_{\theta_0} B^{-1}\left[\nabla_\theta \ln c(F_{1t}, F_{2t}, \ldots, F_{Nt}; \theta_0) + \sum_{n=1}^{N} W_n(F_{nt})\right] + \sum_{n=1}^{N} M_n(F_{nt})\right\} \\ &\times \left\{d_t(\theta_0) + \nabla D_{\theta_0} B^{-1}\left[\nabla_\theta \ln c(F_{1t}, F_{2t}, \ldots, F_{Nt}; \theta_0) + \sum_{n=1}^{N} W_n(F_{nt})\right] + \sum_{n=1}^{N} M_n(F_{nt})\right\}', \end{aligned} \tag{3}$$

where, for $n = 1, 2, \ldots, N$,

$$\begin{aligned} W_n(F_{nt}) = \int_0^1\int_0^1 \cdots \int_0^1 &[I\{F_{nt} \leq u_n\} - u_n]\nabla^2_{\theta, u_n} \ln c(u_1, u_2, \ldots, u_N; \theta_0) \\ &c(u_1, u_2, \ldots, u_N; \theta_0)\mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_N, \end{aligned}$$

and

$$M_n(F_{nt}) = \int_0^1 \int_0^1 \cdots \int_0^1 [I\{F_{nt} \leq u_n\} - u_n] \nabla_{u_n} vech[\nabla_\theta^2 \ln c(u_1, u_2, \ldots, u_N; \theta_0) + \nabla_\theta \ln c(u_1, u_2, \ldots, u_N; \theta_0) \nabla'_\theta \ln c(u_1, u_2, \ldots, u_N; \theta_0)] c(u_1, u_2, \ldots, u_N; \theta_0) \mathrm{d}u_1 \mathrm{d}u_2 \cdots \mathrm{d}u_N.$$

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