View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Research Papers in Economics

FEP WORKING PAPERS
FEP WORKING PAPERS
RESEARCH WORK IN PROGRESS

N. 351, DEC. 2009

COSTLY HORIZONTAL DIFFERENTIATION

JOÃO CORREIA-DA-SILVA[1,2]
JOANA PINHO[2]

[1] CEF.UP
[2] FACULDADE DE ECONOMIA, UNIVERSIDADE DO PORTO

U.PORTO
FEP FACULDADE DE ECONOMIA UNIVERSIDADE DO PORTO

Costly horizontal differentiation*

João Correia-da-Silva

CEF.UP and Faculdade de Economia. Universidade do Porto.

Joana Pinho

Faculdade de Economia. Universidade do Porto.

November 4th, 2009

Abstract. We study the effect of quadratic differentiation costs in the Hotelling model of endogenous product differentiation. The equilibrium location choices are found to depend on the magnitude of the differentiation costs (relatively to the transportation costs supported by consumers). When the differentiation costs are low, there is maximum differentiation. When they are intermediate, there is partial differentiation, with a degree of differentiation that decreases with the differentiation costs. When they are above a certain threshold, there is no equilibrium. In any case, the socially optimal degree of differentiation is always lower than the equilibrium level. We also study the case of collusion between firms. If firms can combine locations but not prices, they locate asymmetrically when differentiation costs are high and choose maximum differentiation when they are low. When collusion extends to price setting, there is partial differentiation.

Keywords: Costly product differentiation, Spatial competition, Hotelling model.

JEL Classification Numbers: D43, L13, R32.

*João Correia-da-Silva (joao@fep.up.pt) acknowledges support from CEF.UP, Fundação para a Ciência e Tecnologia and FEDER (research grant PTDC/ECO/66186/2006). Joana Pinho (080421010@fep.up.pt) acknowledges support from Fundação para a Ciência e Tecnologia (Ph.D. scholarship).

1 Introduction

In many markets, firms horizontally differentiate their products, making them more attractive to some consumers and less attractive to others. Having their own preferred variety, consumers become less sensitive to price variations. This relaxes competition, allowing firms to increase their profits.

In order to differentiate their products, firms may have to carry out costly activities, like advertising or product design. Such differentiation costs have been overlooked in most of the theoretical literature.[1] We contribute to fill this gap, by studying the effects of differentiation costs in a duopoly model with endogenous horizontal differentiation.

The standard model of horizontal differentiation is due to Hotelling (1929). In this model, there are two firms that choose their locations in a linear city, and then engage in price competition to sell their products. Consumers, who are spread along the linear city, choose which of the products to buy, taking into account, besides prices, the transportation cost that they support. Such transportation costs are most frequently assumed to be linear or quadratic (d'Aspremont, Gabszewicz and Thisse, 1979).

We study an extension of this model that considers quadratic differentiation costs (Matsushima, 2004). More precisely, it is considered that the marginal cost of production increases with the square of the distance between the firm and the center of the linear city. This additional cost may be interpreted as resulting from the process of modifying a standard product, as in the work of Eaton and Schmitt (1994), or as the cost of transporting inputs bought from suppliers that are located at the center of the city.

The model of Matsushima (2004) is more general, comprising two suppliers located arbitrarily.[2] Our setup corresponds to the particular case in which the suppliers are located at the center. Such location may be optimal for suppliers that serve many industries spread along the city.[3] On the other hand, Matsushima (2004) restricts his analysis to

[1]A notable exception is the work of Eaton and Schmitt (1994), who consider firms that produce an endogenous number of varieties which may be more or less costly to produce, according to whether their specifications are distant from or close to the specifications of a 'basic product'.

[2]Matsushima (2004) also studies the (endogenous) location of suppliers. See also the related work of Kourandi and Vettas (2009).

[3]If the firms need to buy inputs from many suppliers, uniformly dispersed along the city, then the cost of transporting these inputs also increases with the distance between the firm and the center. This provides further justification for our assumption.

the case in which the differentiation costs (supported by firms) are lower than the transportation costs (supported by consumers). Removing this assumption, we obtain several new results.

A framework that is also similar to ours was proposed by Aiura and Sato (2008). They consider that the differentiation costs are linear and that firms can locate outside the city. These two assumptions are different from ours and lead to different results. Still, they conclude, as we do, that the location of the firms depends on the relationship between the differentiation costs and the transportation costs.[4]

In the classical model (d'Aspremont, Gabszewicz and Thisse, 1979), which does not include differentiation costs, firms choose to locate at the extremes of the linear city in order to soften price competition. In our setup, the differentiation costs increase the attractiveness of the central locations, as the production cost increases with the distance to the center. This generates richer equilibrium possibilities.

We find that low differentiation costs, relatively to the transportation costs, do not affect the equilibrium location choices of the firms (maximum differentiation). However, sufficiently high differentiation costs induce firms to locate in the interior of the city (partial differentiation), increasingly closer to the center as the differentiation costs increase. We obtain, in this way, a relationship between the magnitude of differentiation costs and the degree of differentiation. We also find that extremely high differentiation costs imply the non-existence of equilibrium location choices.[5]

The socially optimal locations are found to correspond to partial differentiation, with a degree of differentiation that is decreasing with the differentiation costs. In the absence of differentiation costs, firms locate in the midpoints between the center and the extremes (a known result). As the differentiation costs increase, firms locate closer and closer

[4]The model of Gupta, Kats and Pal (1994) is farther from ours. They consider a monopolist supplier, while we assume a competitive input industry. Moreover, they allow for the existence of several downstream firms and consider that the (linear) transportation costs are supported by the firms. Another alternative setup was considered by Brekke and Straume (2004), who assumed a bilateral monopoly relationship between upstream and downstream firms.

[5]More precisely, we consider that the transportation costs supported by a consumer located at x who buys from a firm located at a are $t\,(x-a)^2$ and that the differentiation cost supported by a firm located at a is $\tau\left(\frac{1}{2}-a\right)^2$. We find that maximum differentiation is the unique equilibrium if $\frac{\tau}{t}\leq\frac{1}{2}$, that partial differentiation is the unique equilibrium if $\frac{1}{2}<\frac{\tau}{t}\leq\frac{19}{2}$, and that there is no equilibrium if $\frac{\tau}{t}>\frac{19}{2}$. We complement, thus, the results of Matsushima (2004), as he restricted his analysis to the case in which $\frac{\tau}{t}\leq 1$ and did not establish uniqueness of equilibrium.

to the center, converging to a situation of no differentiation. Comparing the optimal locations with the competitive equilibrium ones, we conclude that the socially optimal degree of differentiation is always lower than the degree of differentiation that is induced by competition.[6]

Finally, we study the case of collusive behavior between firms. In the case of partial collusion, in which firms can combine locations but not prices, we obtain a rather surprising result. If differentiation costs are high, one of the firms locates at an extreme, while the other locates between this same extreme and the center (if differentiation costs are low, firms agree on maximum differentiation). When collusion is full (extends to the price setting stage), firms locate: in the midpoints between the center and the extremes, if differentiation costs are lower than transportation costs; and increasingly closer to the center, as the differentiation costs increase above this threshold.

The remainder of the article is organized as follows. In Section 2, we setup the model and introduce notation. In Sections 3, 4 and 5, we fully characterize the relationship between the degree of differentiation and the cost of differentiation, in three economic scenarios: competition (Section 3), social planning (Section 4), and collusion (Section 5). Section 6 concludes the article with some remarks.

2 The Model

There are two firms, 1 and 2, and a continuum of consumers homogeneously distributed on the unit interval, $[0, 1]$. In a first stage, firms simultaneously choose their location (which is interpreted as their product specification) on the unit interval. The location of firm 1 is denoted a, and the location of firm 2 is denoted $1 - b$. In the second stage, firms simultaneously choose the prices to charge for their goods, p_1 and p_2.

As the model of Aiura and Sato (2008), firms located farther from the center have higher production costs. This may be interpreted as the cost of modifying a standard product, or as the cost of transporting inputs bought from suppliers that are located at the center

[6]We remark that the model at hands considers that the demand is inelastic and that prices have a neutral effect on social welfare. This should be kept in mind for an adequate interpretation of the welfare implications of the model.

of the city. The marginal cost of firm 1 is $\tau\left(\frac{1}{2}-a\right)^2$ and the marginal cost of firm 2 is $\tau\left(\frac{1}{2}-b\right)^2$, with $\tau \geq 0$.[7]

Each consumer chooses whether to buy the product of firm 1 or the product of firm 2. Consumers are heterogeneous in their tastes, preferring products located near them. Assuming that such transportation costs are also quadratic, the total cost of buying goods 1 and 2, supported by a consumer located at x, are $p_1 + t(x-a)^2$ and $p_2 + t(x-1+b)^2$, respectively, with $t > 0$.[8]

It is straightforward to determine the indifferent consumer (valid for $a+b<1$):

$$\tilde{x} = \frac{1+a-b}{2} + \frac{p_2-p_1}{2t(1-a-b)}, \tag{1}$$

which corresponds to the demand for the product sold by firm 1. The demand for the product sold by firm 2 is: $1-\tilde{x} = \frac{1-a+b}{2} + \frac{p_1-p_2}{2t(1-a-b)}$.

3 Competitive equilibrium locations

We start by studying the competitive scenario. In the first stage, firms simultaneously choose their location on the unit interval. In the second stage, firms simultaneously choose the prices to charge for their goods.

The profit of firm 1, as a function of locations and prices, is:[9]

$$\Pi_1 = \left[p_1 - \tau\left(\frac{1}{2}-a\right)^2\right]\left[\frac{1+a-b}{2} + \frac{p_2-p_1}{2t(1-a-b)}\right]. \tag{2}$$

[7] The difference with respect to the model of Aiura and Sato (2008) is that here marginal costs increase quadratically instead of linearly with the distance to the center.

[8] The model that we are considering corresponds to the model proposed by Matsushima (2004) with suppliers located at the center. But we do not restrict the analysis to the case in which $\tau \leq t$.

[9] This expression is valid for $a+b<1$. It can be shown that if firms choose the same location ($a+b=1$), then the equilibrium prices are equal to marginal cost, $p_1 = \tau\left(\frac{1}{2}-a\right)^2$ and $p_2 = \tau\left(\frac{1}{2}-b\right)^2$, and, therefore, the resulting profits are null.

Profit maximization by firm 1 implies that:

$$p_1 = \frac{1}{2}\left[t - a^2t - 2bt + b^2t + p_2 + \tau\left(\frac{1}{2} - a\right)^2\right]. \tag{3}$$

Analogously, the price set by firm 2 is:

$$p_2 = \frac{1}{2}\left[t - b^2t - 2at + a^2t + p_1 + \tau\left(\frac{1}{2} - b\right)^2\right]. \tag{4}$$

From the best response functions, (3) and (4), we obtain the prices that the firms choose, as a function of their locations:

$$p_1(a,b) = t - \frac{2}{3}at - \frac{1}{3}a^2t - \frac{4}{3}bt + \frac{1}{3}b^2t + \frac{2\tau}{3}\left(\frac{1}{2} - a\right)^2 + \frac{\tau}{3}\left(\frac{1}{2} - b\right)^2, \tag{5}$$

$$p_2(a,b) = t - \frac{2}{3}bt - \frac{1}{3}b^2t - \frac{4}{3}at + \frac{1}{3}a^2t + \frac{2\tau}{3}\left(\frac{1}{2} - b\right)^2 + \frac{\tau}{3}\left(\frac{1}{2} - a\right)^2. \tag{6}$$

Given these prices, the indifferent consumer is located at:

$$\tilde{x}(a,b) = \frac{3+a-b}{6} + \frac{\tau}{t}\,\frac{\left(\frac{1}{2} - b\right)^2 - \left(\frac{1}{2} - a\right)^2}{6\left(1-a-b\right)} = \frac{1}{2} + \frac{1+\frac{\tau}{t}}{6}\left(a-b\right).$$

And the profit obtained by firm 1 is:

$$\Pi_1(a,b) = \frac{t}{18}\left(1-a-b\right)\left[3 + (a-b)\left(1+\frac{\tau}{t}\right)\right]^2. \tag{7}$$

Observe that, given the ratio $\frac{\tau}{t}$, the value of t is irrelevant for the behavior of firms (the constant $\frac{t}{18}$ could be ignored in the maximization problem).

Definition 1.

A profile of locations, (a^*, b^*), is an equilibrium if and only if:

(i) $a^* \in \arg\max_{a\in[0,1]} \{\Pi_1(a,b^*)\}$;

(ii) $b^* \in \arg\max_{b\in[0,1]} \{\Pi_2(a^*,b)\}$.

When the differentiation costs are low relatively to the transportation costs, there is *maximum differentiation* (firms locate at the extremes of the unit interval).

Proposition 1.

Maximum differentiation, $(a^*, b^*) = (0, 0)$, is an equilibrium if and only if:[10]

$$\frac{\tau}{t} \leq \frac{1}{2}.$$

On the other hand, we find that *no differentiation* cannot be an equilibrium (firms never locate simultaneously at the center).

Proposition 2.

No differentiation, $(a^*, b^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$, is never an equilibrium.

For moderately high values of the differentiation cost (relatively to the transportation cost), there is *partial differentiation.* Firms locate in the interior of the unit interval, at the same distance from the center.

Proposition 3.

Partial differentiation, $(a^*, b^*) = (d_c, d_c)$, with $d_c \in (0, \frac{1}{2})$, is an equilibrium if and only if:

$$\frac{1}{2} < \frac{\tau}{t} \leq \frac{19}{2} \quad \textit{and} \quad d_c = \frac{1}{2} - \frac{3/4}{1 + \frac{\tau}{t}}.$$

Observe that as $\frac{\tau}{t}$ increases, the firms locate closer and closer to the center. When $\frac{\tau}{t} = \frac{19}{2}$, we have $d_c = \frac{3}{7} \simeq 0.43$. This is as close to the center as they get (locating in the center would imply null profits for both firms).

Finally, we find that *asymmetric differentiation* is never an equilibrium.

Proposition 4.

Asymmetric differentiation, $(a^*, b^*) = (d_a, d_b)$, with $d_a \neq d_b$, is never an equilibrium.

[10] All the proofs are collected in the Appendix.

We have completely characterized the equilibria of the model, which is always unique: there is *maximum differentiation* if $\frac{\tau}{t} \leq \frac{1}{2}$, and partial differentiation if $\frac{1}{2} < \frac{\tau}{t} \leq \frac{19}{2}$ (with the degree of differentiation being symmetric and increasing with $\frac{\tau}{t}$). For $\frac{\tau}{t} > \frac{19}{2}$, there is no equilibrium.

Firms face a trade-off between increasing differentiation to reduce competition in the price setting stage and decreasing differentiation to lower their production costs. If differentiation costs are low, relatively to transportation costs, the market power effect dominates. There is maximum differentiation. If differentiation costs are above a certain threshold, the trade-off becomes relevant. Firms locate in the interior of the city (partial differentiation), increasingly closer to the center as the differentiation costs increase. We are able to relate, in this way, the magnitude of differentiation costs and the degree of differentiation.

4 Socially optimal locations

Consider now that a social planner decides locations and prices with the objective of maximizing social welfare, which is defined as the sum of the consumers' surplus with the profits of the firms.

Since all consumers find it optimal to buy one unit of the good (from one of the firms or the other), and since the price paid is a transference (does not affect social welfare), maximizing social welfare is equivalent to minimizing the sum of the consumers' transportation costs with the firms' differentiation costs.

Total transportation costs, supported by consumers, are given by:

$$\begin{aligned} L_t(a,b,\tilde{x}) &= \int_0^{\tilde{x}} t(x-a)^2 dx + \int_{\tilde{x}}^1 t(x-1+b)^2 dx = \\ &= t\left[(1-a-b)\,\tilde{x}^2 + \left(a^2-b^2+2b-1\right)\tilde{x} + b^2 - b + \frac{1}{3}\right]. \end{aligned}$$

And the differentiation costs, supported by the firms, are:

$$L_\tau(a,b,\tilde{x}) = \tau\left[\left(\frac{1}{2}-a\right)^2 \tilde{x} + \left(\frac{1}{2}-b\right)^2 (1-\tilde{x})\right].$$

The social planner chooses the locations of the firms (a and b) and the prices of the goods (which, then, determine $\tilde{x}$) with the objective of maximizing:

$$\begin{aligned} W(a,b,\tilde{x}) &= -t\left[(1-a-b)\tilde{x}^2+\left(a^2-b^2+2b-1\right)\tilde{x}+b^2-b+\frac{1}{3}\right]- \\ &\quad -\tau\left[\left(\frac{1}{2}-a\right)^2\tilde{x}+\left(\frac{1}{2}-b\right)^2(1-\tilde{x})\right]. \end{aligned} \tag{8}$$

Proposition 5.

The socially optimal profile of locations is $(a,b)=(d_g,d_g)$, with $d_g=\frac{2\frac{\tau}{t}+1}{4\left(\frac{\tau}{t}+1\right)}$. Given these locations, any symmetric pair of prices, $(p_1,p_2)=(p,p)$, is socially optimal.

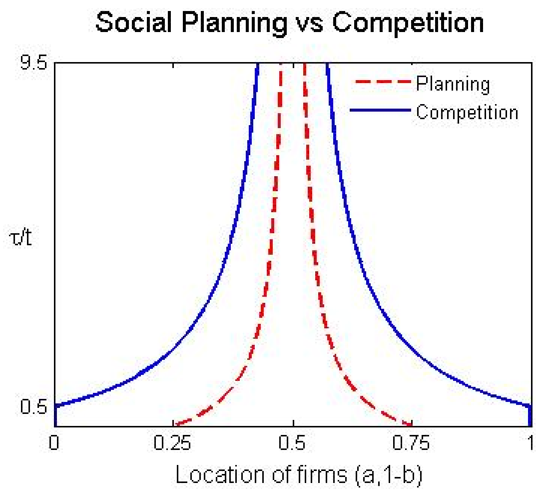

Figure 1: In a competitive equilibrium, there is an excessive degree of differentiation.

We remark that the social planner may only choose the locations, and leave price setting to the firms (the level of welfare does not change, because the resulting pair of prices remains symmetric).

Without differentiation costs, the optimal profile of locations is $(a,b)=\left(\frac{1}{4},\frac{1}{4}\right)$, which is a known result. As the differentiation cost increases (relatively to the transportation cost), the optimal locations gradually approach the center, converging to $(a,b)=\left(\frac{1}{2},\frac{1}{2}\right)$ as $\frac{\tau}{t}\to+\infty$.

Observe, in Figure 1, that the socially optimal profile of locations is always closer to the center than the equilibrium profile (for $\frac{\tau}{t}\leq\frac{19}{2}$, otherwise equilibrium does not exist).

5 Collusive locations

5.1 Partial collusion

We now study the case in which firms collude when making their (irreversible) location decisions. We start by assuming that firms can commit on their location choices, but not on the prices to set afterwards. Therefore, price-setting remains competitive.[11]

Alternatively, we can assume that firms are also able to commit on the prices to set, but that collusive price setting would be detected and punished by anti-trust authorities.

Firms jointly choose their locations, (a, b), with the objective of maximizing the sum of their profits, which is given by:[12]

$$\Pi_{pj}(a,b) = \frac{t}{9}(1-a-b)\left[9+(a-b)^2\left(1+\frac{\tau}{t}\right)^2\right]$$

We find that for relatively low differentiation costs, firms decide to locate at the extremes (maximum differentiation). For sufficiently high differentiation costs, firms decide to locate asymmetrically: one at an extreme, and the other between this same extreme and the center.

Proposition 6.

The collusive location decision (with subsequent price competition) is:[13]

(i) maximum differentiation, $(a^*, b^*) = (0, 0)$, *if* $\frac{\tau}{t} \leq 5$;

(ii) asymmetric differentiation, $(a^*, b^*) = (0, d_{pj})$ *or* $(a^*, b^*) = (d_{pj}, 0)$, *if* $\frac{\tau}{t} \geq 5$, *with* $d_{pj} = \frac{1}{3} + \frac{1}{3}\sqrt{1 - 27\left(1+\frac{\tau}{t}\right)^{-2}}$.

When the differentiation cost reaches a certain threshold, $\frac{\tau}{t} = 5$, the optimal location

[11] Friedman and Thisse (1993) study an infinitely repeated game version of the Hotelling model, in which locations are chosen (once and for all) in period 0 and prices are chosen in each period. In such a model, collusion in the price setting stage can be sustainable. In our one-shot game, collusive price setting requires the ability to make a commitment.

[12] Since we are assuming that firms compete in prices, the profit of firm 1 is still given by (7). Analogously, the expression for the profit of firm 2 is: $\Pi_2(a,b) = \frac{t}{18}(1-a-b)\left[3+(b-a)\left(1+\frac{\tau}{t}\right)\right]^2$.

[13] When $\frac{\tau}{t} = 5$, decisions (i) and (ii) are equally optimal.

jumps from $(a^*, b^*) = (0,0)$ to $(a^*, b^*) = \left(0, \frac{1}{2}\right)$. The benefit of reducing differentiation costs becomes dominant relatively to the benefit of reducing competition in the price setting stage. The optimal compromise is to have one firm in the center, minimizing the differentiation costs, and another in an extreme, providing sufficient differentiation.

As the ratio between the differentiation cost and the transportation cost parameters increases from 5 to $+\infty$, one of the firms remains located at an extreme, while the other moves from the center ($d_{pj} = \frac{1}{2}$) towards its competitor ($d_{pj} = \frac{2}{3}$). It is somewhat paradoxical that this firm becomes farther from the center, increasing differentiation costs, and closer to the competitor, increasing competition in the price setting stage. But this movement actually decreases the total differentiation costs, as the firm which is near the center increases its market share.[14]

5.2 Full collusion

To study the case of full collusion, in which collusion extends to price-setting, we need to consider a reservation utility for the consumers (otherwise firms would be able to increase prices to infinity).

Let the utility of a consumer located at x that buys the good from firm i be $u_i(x) = V - p_i - t\,(x - d_i)^2$, where V is the money equivalent of the satisfaction provided by the good, p_i is the price paid for the good and d_i is the location of firm i. If the consumer does not buy the good, her utility is null, $u_0(x) = 0$.

To guarantee full market coverage, we assume that $V \geq \frac{5t}{4} + \frac{\tau}{4}$.[15]

Lemma 1.

In the fully collusive scenario, when $V \geq \frac{5t}{4} + \frac{\tau}{4}$, firms choose locations and prices which induce all consumers to buy the good (full market coverage).

The fully collusive location choices correspond to partial differentiation. Firms locate: in the midpoints between the center and the extremes, if differentiation costs are lower than

[14]This asymmetric solution should be associated with a side-payment from the firm which locates near the center to the firm which locates at the extreme.

[15]The corresponding assumption made by Friedman and Thisse (1993), in their dynamic model without differentiation costs, was $V \geq 3t$.

transportation costs; and increasingly closer to the center, as the differentiation costs increase above this threshold (see Figure 2).

Proposition 7.

The fully collusive location choice is:[16]

(i) $(a^*, b^*) = \left(\frac{1}{4}, \frac{1}{4}\right)$, *if* $\frac{\tau}{t} \leq 1$;

(ii) $(a^*, b^*) = (d_j, d_j)$, *with* $d_j = \frac{\frac{\tau}{t}}{2\left(1+\frac{\tau}{t}\right)}$, *if* $\frac{\tau}{t} \geq 1$.

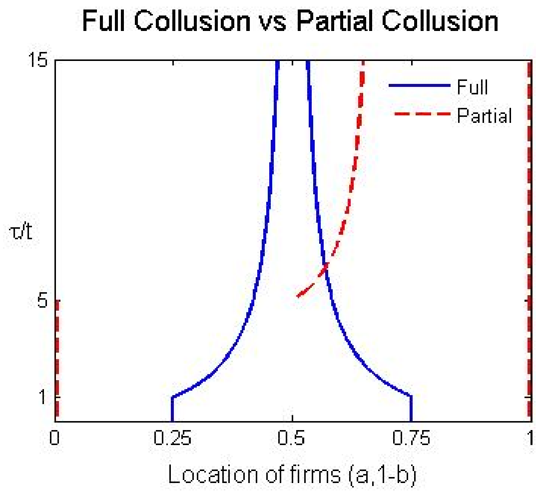

Figure 2: Partial collusion may generate asymmetric locations.

Firms could mimic a monopoly by locating in the center and setting equal prices. This turns out to be suboptimal. Partial differentiation is preferred, because it allows firms to charge a higher price.

[16]When $\frac{\tau}{t} = 1$, both (i) and (ii) are optimal.

6 Conclusion

We have shown how the existence of quadratic differentiation costs affects the degree of differentiation, in the context of the Hotelling duopoly model. According to whether the Hotelling line is interpreted as the set of possible locations in space or as the set of possible product characteristics, the differentiation cost can be interpreted as the cost of transporting raw materials from the center of a linear city or as the cost of transforming a standard product into a differentiated variety.

We have considered three different economic scenarios: competition, social planning and collusion. In any case, the location choices only depend on the ratio between the differentiation cost (incurred by firms) and the transportation cost (supported by consumers) parameters, τ/t. Firms locate symmetrically and closer to the center as τ/t increases, except in the case of partial collusion. If firms can collude on locations but not on prices and τ/t is high enough, then they choose to locate on the same side of the market.

Our model helps in understanding the choice of a costly degree of differentiation. Firms take into account not only the increase in market power that results from a unique positioning but also the cost of acquiring such a privileged situation, for example, through advertising or design. This trade-off generates a new effect of exogenous shocks: the degree of differentiation may increase or decrease as a result of variations in the transportation cost. In the standard model, maximum differentiation was robust to such perturbations.

7 Appendix

Proof of Proposition 1: From the definition of equilibrium, it follows that *maximal differentiation*, $(a^*, b^*) = (0,0)$, is an equilibrium if and only if:

(i) $0 \in \arg\max_{a\in[0,1]} \{\Pi_1(a,0)\}$;

(ii) $0 \in \arg\max_{b\in[0,1]} \{\Pi_2(0,b)\}$.

By symmetry, to prove that $(a^*, b^*) = (0,0)$ is an equilibrium, it is enough to check that (i) is true.

When $b = 0$, we have:

$$\Pi_1(a,0) = \frac{t}{18}(1-a)\left[3 + a\left(1+\frac{\tau}{t}\right)\right]^2. \tag{9}$$

The derivative of the profit function with respect to location is:

$$\frac{\partial \Pi_1(a,0)}{\partial a} = \frac{t}{18}\left(3 + a + a\frac{\tau}{t}\right)\left(-1 - 3a + 2\frac{\tau}{t} - 3a\frac{\tau}{t}\right).$$

The signal of $\frac{\partial \Pi_1(a,0)}{\partial a}$ is equal to the signal of $2\frac{\tau}{t} - 1 - 3\left(1+\frac{\tau}{t}\right)a$. It is clear that if $\frac{\partial \Pi_1(a,0)}{\partial a}$ is negative at $a = 0$, then it is negative for any $a \in [0,1)$.

Therefore, the necessary and sufficient condition for *maximal differentiation* to be an equilibrium is $\frac{\tau}{t} \leq \frac{1}{2}$. □

Proof of Proposition 2: *No differentiation*, $(a^*, b^*) = (1/2, 1/2)$, is an equilibrium if and only if:

(i) $\frac{1}{2} \in \arg\max_{a\in[0,1]} \{\Pi_1(a, 1/2)\}$;

(ii) $\frac{1}{2} \in \arg\max_{b\in[0,1]} \{\Pi_2(1/2, b)\}$.

By symmetry, to show that *no differentiation* is an equilibrium, it is only necessary to show that: $\frac{1}{2} \in \arg\max_{a\in\left[0,\frac{1}{2}\right]} \{\Pi_1(a, 1/2)\}$.

When $b = \frac{1}{2}$, the profit of firm 1 is:

$$\Pi_1(a, 1/2) = \frac{t}{18}\left(\frac{1}{2} - a\right)\left(\frac{5}{2} + a + a\frac{\tau}{t} - \frac{\tau}{2t}\right)^2.$$

Calculating the derivative with respect to location, we obtain:

$$\frac{\partial \Pi_1(a, 1/2)}{\partial a} = \frac{t}{18}\left(\frac{5}{2} + a + a\frac{\tau}{t} - \frac{\tau}{2t}\right)\left(-\frac{3}{2} - 3a - 3a\frac{\tau}{t} + \frac{3\tau}{2t}\right).$$

A necessary condition for *no differentiation* to be an equilibrium is that this derivative is positive as $a \rightarrow \frac{1}{2}^-$. But we find that it is always negative:

$$\lim_{a \rightarrow \frac{1}{2}^-} \frac{\partial \Pi_1(a, 1/2)}{\partial a} = -\frac{t}{2}.$$

$\square$

Proof of Proposition 3: *Partial differentiation*, $(a^*, b^*) = (d_c, d_c)$, with $d_c \in (0, 1/2)$, is an equilibrium if and only if:

(i) $d_c \in \arg\max_{a \in [0,1]} \{\Pi_1(a, d_c)\}$;

(ii) $d_c \in \arg\max_{b \in [0,1]} \{\Pi_2(d_c, b)\}$.

With $b = d_c$, the profit obtained by firm 1 is:

$$\Pi_1(a, d_c) = \frac{t}{18}(1 - a - d_c)\left[3 + (a - d_c)\left(1 + \frac{\tau}{t}\right)\right]^2. \quad (10)$$

The partial derivative with respect to a is:

$$\frac{\partial \Pi_1(a, d_c)}{\partial a} = \frac{t}{18}\left[3 + (a - d_c)\left(1 + \frac{\tau}{t}\right)\right]\left(-1 - 3a - d_c + 2\frac{\tau}{t} - 3a\frac{\tau}{t} - \frac{d_c \tau}{t}\right).$$

For the first-order condition of (i) to hold, we must have:

$$\left.\frac{\partial \Pi_1(a, d_c)}{\partial a}\right|_{a = d_c} = 0 \iff d_c = \frac{2\frac{\tau}{t} - 1}{4\frac{\tau}{t} + 4}.$$

Notice that $d_c \in (0, 1/2)$ if and only if $\frac{\tau}{t} > \frac{1}{2}$. Therefore, *partial differentiation* can only be an equilibrium for $\frac{\tau}{t} > \frac{1}{2}$.

To check the second-order condition, write the second derivative of the profit function with respect to location:

$$\frac{\partial^2 \Pi_1(a, d_c)}{\partial a^2} = \frac{t}{9}\left(1 + \frac{\tau}{t}\right)\left(-5 - 3a + d_c + \frac{\tau}{t} - 3a\frac{\tau}{t} + d_c\frac{\tau}{t}\right).$$

The local second order condition always holds:

$$\left.\frac{\partial^2 \Pi_1(a, d_c)}{\partial a^2}\right|_{a=d_c} = \frac{t}{9}\left(1 + \frac{\tau}{t}\right)\left(-5 - 2d_c + \frac{\tau}{t} - 2d_c\frac{\tau}{t}\right) = -\frac{t}{2}\left(1 + \frac{\tau}{t}\right).$$

The global second order condition may not hold, but there is a threshold, $a_0 > d_c$, such that $\frac{\partial^2 \Pi_1(a,d_c)}{\partial a^2}$ is negative (positive) for $a > a_0$ ($a < a_0$):

$$\frac{\partial^2 \Pi_1}{\partial a^2}(a, d_c) \leq 0 \Leftrightarrow a \geq a_0 = \frac{1}{2} - \frac{\frac{9}{4}}{1 + \frac{\tau}{t}}.$$

For $\frac{\tau}{t} \leq \frac{7}{2}$, we have $a_0 \leq 0$, which means that the global second order condition holds. Thus, for $\frac{\tau}{t} \in (1/2, 7/2]$, *partial differentiation* is an equilibrium.

For higher values of $\frac{\tau}{t}$, we must compare $\Pi_1(d_c, d_c)$ with $\Pi_1(0, d_c)$ (since Π_1 is convex for $a < a_0$, the only other candidate for a maximizer is $a = 0$). With $a = d$, the profit of firm 1 is:

$$\Pi_1(d_c, d_c) = \frac{t}{2}(1 - 2d_c) = \frac{t}{2}\left[1 - \frac{2\frac{\tau}{t} - 1}{2\left(1 + \frac{\tau}{t}\right)}\right] = \frac{3t}{4\left(1 + \frac{\tau}{t}\right)}.$$

If firm 1 decides to locate at the extreme, $a = 0$, its profit is:

$$\Pi_1(0, d_c) = \frac{3t}{4\left(1 + \frac{\tau}{t}\right)}\left[\frac{1}{864}\left(5 + 2\frac{\tau}{t}\right)\left(13 - 2\frac{\tau}{t}\right)^2\right].$$

For *partial differentiation* to be an equilibrium, we must have:

$$\begin{aligned} \Pi_1(0, d_c) &\leq \Pi_1(d_c, d_c) \Leftrightarrow \\ \frac{1}{864}\left(5+2\frac{\tau}{t}\right)\left(13-2\frac{\tau}{t}\right)^2 &\leq 1 \Leftrightarrow 8\left(\frac{\tau}{t}\right)^3 - 84\left(\frac{\tau}{t}\right)^2 + 78\frac{\tau}{t} - 19 \leq 0. \end{aligned}$$

This occurs if and only if $\frac{\tau}{t} \leq \frac{19}{2}$. □

Proof of Proposition 4: *Asymmetric differentiation*, $(a^*, b^*) = (d_a, d_b)$, with $d_a \neq d_b$, is an equilibrium if and only if:

(i) $d_a \in \arg\max_{a \in [0,1]} \{\Pi_1(a, d_b)\}$;

(ii) $d_b \in \arg\max_{b \in [0,1]} \{\Pi_2(d_a, b)\}$.

Without loss of generality, suppose that $d_a \neq 0$. From the proof of Proposition 3, we know that the location b which maximizes the profit of firm 2 is either $b = 0$ or $b = d_a$. Therefore, the only candidate for an asymmetric equilibrium is $(a^*, b^*) = (d_a, 0)$.

With $b = 0$, we have (9):

$$\Pi_1(a, 0) = \frac{t}{18}(1-a)\left[3 + a\left(1+\frac{\tau}{t}\right)\right]^2.$$

A necessary condition for $(a^*, b^*) = (d_a, 0)$ to be an equilibrium is that:

$$\left.\frac{\partial \Pi_1(a, 0)}{\partial a}\right|_{a=d_a} = 0 \Leftrightarrow \frac{t}{18}\left(3 + d_a + \frac{\tau}{t}d_a\right)\left(-1 - 3d_a + 2\frac{\tau}{t} - 3d_a\frac{\tau}{t}\right) = 0.$$

Which is equivalent to:

$$d_a = \frac{2\frac{\tau}{t} - 1}{3\left(\frac{\tau}{t} + 1\right)}. \tag{11}$$

Another necessary condition for $(a^*, b^*) = (d_a, 0)$ to be an equilibrium is that:

$$\left.\frac{\partial \Pi_2(d_a, b)}{\partial b}\right|_{b=0} = 0.$$

By symmetry, this condition is equivalent to:

$$\left.\frac{\partial\Pi_1(a,d_a)}{\partial a}\right|_{a=0}=0 \;\Leftrightarrow\; \frac{t}{18}\left[3-d_a\left(1+\frac{\tau}{t}\right)\right]\left(-1-d_a+2\frac{\tau}{t}-\frac{\tau}{t}d\right)=0.$$

Solving in order to d_a, we obtain:

$$d_a=\frac{3}{1+\frac{\tau}{t}} \quad \text{or} \quad d_a=\frac{2\frac{\tau}{t}-1}{\frac{\tau}{t}+1}.$$

The second possibility is incompatible with (11), therefore, we must have:

$$d_a=\frac{3}{1+\frac{\tau}{t}}=\frac{2\frac{\tau}{t}-1}{3\left(\frac{\tau}{t}+1\right)} \;\Leftrightarrow\; \begin{cases} d_a=\frac{1}{2}, \\ \frac{\tau}{t}=5. \end{cases}$$

With $\frac{\tau}{t}=5$, the local second order condition for $b=0$ to be a maximizer of $\Pi_2(1/2,b)$ is equivalent to:

$$\left.\frac{\partial^2\Pi_1}{\partial a^2}(a,1/2)\right|_{a=0}\leq 0 \;\Leftrightarrow\; -5+\frac{1}{2}+5+\frac{5}{2}\leq 0,$$

which does not hold. □

Proof of Proposition 5: The derivative of W with respect to $\tilde{x}$ is:

$$\frac{\partial W}{\partial \tilde{x}}=-t(\tilde{x}-a)^2+t(\tilde{x}-1+b)^2-\tau\left[\left(\frac{1}{2}-a\right)^2-\left(\frac{1}{2}-b\right)^2\right].$$

The corresponding first-order condition, $\frac{\partial W}{\partial \tilde{x}}=0$, can be written as:

$$\begin{aligned}
&2at\tilde{x}-ta^2-2t(1-b)\tilde{x}+t(1-b)^2=\tau\left[\left(\frac{1}{2}-a\right)^2-\left(\frac{1}{2}-b\right)^2\right] \Leftrightarrow \\
&\Leftrightarrow\; \tilde{x}=\frac{\frac{\tau}{t}\left[\left(\frac{1}{2}-a\right)^2-\left(\frac{1}{2}-b\right)^2\right]+a^2-(1-b)^2}{-2(1-a-b)} \Leftrightarrow \\
&\Leftrightarrow\; \tilde{x}=\frac{\frac{\tau}{t}\left[a^2-a-b^2+b\right]-(1-a-b)(a-b+1)}{-2(1-a-b)} \Leftrightarrow \\
&\Leftrightarrow\; \tilde{x}=\frac{1}{2}+\frac{a-b}{2}\left(1+\frac{\tau}{t}\right). \qquad (12)
\end{aligned}$$

Welfare is concave in $\tilde{x}$, as the second derivative is:

$$\frac{\partial^2 W}{\partial \tilde{x}^2} = -2t\left(1-a-b\right) \leq 0.$$

The first-order condition with respect to a, $\frac{\partial W}{\partial a} = 0$, is equivalent to:

$$\tilde{x}\left(\tau - 2a\tau - 2at + t\tilde{x}\right) = 0 \;\Leftrightarrow\; \tilde{x} = 0 \;\vee\; \tilde{x} = 2a + \frac{\tau}{t}\left(2a-1\right). \tag{13}$$

Similarly, the first-order condition with respect to b is:

$$\left(\tilde{x}-1\right)\left(2b\tau - t - \tau + 2bt + t\tilde{x}\right) = 0 \;\Leftrightarrow\; \tilde{x} = 1 \;\vee\; \tilde{x} = 1 - 2b + \frac{\tau}{t}\left(1-2b\right). \tag{14}$$

With $\tilde{x} = 0$ (the same would result for $\tilde{x} = 1$), social welfare is:

$$W\left(a,b,0\right) = -t\left(b^2 - b + \frac{1}{3}\right) - \tau\left(\frac{1}{2} - b\right)^2,$$

which is maximal at $b = \frac{1}{2}$, attaining the value:

$$W\left(a, \frac{1}{2}, 0\right) = -\frac{t}{12}.$$

For $\tilde{x} \neq 0$, from the first-order conditions, (12), (13) and (14), we obtain:

$$a = b = d_g = \frac{2\frac{\tau}{t} + 1}{4\left(1 + \frac{\tau}{t}\right)} \quad \text{and} \quad \tilde{x} = \frac{1}{2}.$$

To make sure that this pair of locations is optimal, we now construct the Hessian matrix:

$$H\left(a,b,\tilde{x}\right) = \begin{bmatrix} \frac{\partial^2 W}{\partial a^2} & \frac{\partial^2 W}{\partial a \partial b} & \frac{\partial^2 W}{\partial a \partial \tilde{x}} \\ \frac{\partial^2 W}{\partial b \partial a} & \frac{\partial^2 W}{\partial b^2} & \frac{\partial^2 W}{\partial b \partial \tilde{x}} \\ \frac{\partial^2 W}{\partial \tilde{x} \partial a} & \frac{\partial^2 W}{\partial \tilde{x} \partial b} & \frac{\partial^2 W}{\partial \tilde{x}^2} \end{bmatrix} =$$

$$= \begin{bmatrix} -2\tilde{x}\left(\tau+t\right) & 0 & \tau - 2a(\tau+t) + 2t\tilde{x} \\ 0 & -2\left(1-\tilde{x}\right)\left(\tau+t\right) & 2b(\tau+t) - 2t\left(1-\tilde{x}\right) - \tau \\ \tau - 2a(\tau+t) + 2t\tilde{x} & 2b(\tau+t) - 2t\left(1-\tilde{x}\right) - \tau & -2t\left(1-a-b\right) \end{bmatrix}.$$

At $(a,b,\tilde{x}) = \left(\frac{2\frac{\tau}{t}+1}{4\left(1+\frac{\tau}{t}\right)}, \frac{2\frac{\tau}{t}+1}{4\left(1+\frac{\tau}{t}\right)}, \frac{1}{2}\right) \equiv \left(d_g, d_g, \frac{1}{2}\right)$, we obtain:

$$H\left(d_g, d_g, \frac{1}{2}\right) = \begin{bmatrix} -(\tau+t) & 0 & \frac{1}{2} \\ 0 & -(\tau+t) & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{t^2}{\tau+t} \end{bmatrix}.$$

The local second-order condition for $\left(d_g, d_g, \frac{1}{2}\right)$ to be a maximum is: $|A_1| < 0$, $|A_2| > 0$ and $|A_3| < 0$. We have:

(i) $|A_1| = -(\tau+t) < 0$;

(ii) $|A_2| = \det \begin{bmatrix} -(\tau+t) & 0 \\ 0 & -(\tau+t) \end{bmatrix} = (\tau+t)^2 > 0$;

(iii) $|A_3| = \det \begin{bmatrix} -(\tau+t) & 0 & \frac{1}{2} \\ 0 & -(\tau+t) & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{t^2}{\tau+t} \end{bmatrix} = -\frac{t^2(\tau+t)}{2} < 0.$

Thus, $\left(d_g, d_g, \frac{1}{2}\right)$ is a local maximum of W. Since the domain of W, $\mathcal{D} \times [0,1]$, is a compact and W is C^∞, to show that it is a global maximum, we only need to prove that $W\left(d_g, d_g, \frac{1}{2}\right)$ is higher than W at the frontier.[17]

Calculating welfare at $\left(d_g, d_g, \frac{1}{2}\right)$, we obtain:

$$W\left(d_g, d_g, \frac{1}{2}\right) = -\frac{4\tau+t}{48\left(1+\frac{\tau}{t}\right)} = -\frac{t\left(1+4\frac{\tau}{t}\right)}{12\left(4+4\frac{\tau}{t}\right)} > -\frac{t}{12},$$

which is higher than welfare for $\tilde{x} = 0$.

By symmetry, the only remaining candidate is of the form $(0, b, \tilde{x})$.

But, at $a = 0$, $\frac{\partial W}{\partial a} > 0$, therefore, $(0, b, \tilde{x})$ cannot be a maximizer. $\square$

Proof of Proposition 6: The first-order conditions for the maximization of the joint profit are:

$$\begin{cases} \frac{\partial \Pi_{pj}}{\partial a} = 0 \\ \frac{\partial \Pi_{pj}}{\partial b} = 0 \end{cases} \Leftrightarrow \begin{cases} 9 + (a-b)^2\left(1+\frac{\tau}{t}\right)^2 = 2(1-a-b)(a-b)\left(1+\frac{\tau}{t}\right)^2, \\ 9 + (a-b)^2\left(1+\frac{\tau}{t}\right)^2 = 2(1-a-b)(b-a)\left(1+\frac{\tau}{t}\right)^2. \end{cases}$$

[17]The compact domain is $\mathcal{D} \times [0,1]$, where $\mathcal{D} = \left\{(a,b) \in [0,1]^2 :\ a+b \leq 1\right\}$.

Subtracting the second from the first, we obtain $a = b$ (which does not satisfy any of the conditions) or $a + b = 1$ (which is surely not optimal, as it implies that $\Pi_j = 0$). Therefore, there are no interior maxima.

Since $\Pi_{pj}(a, b)$ is a continuous function defined on a compact domain, it has a global maximum.[18] The only possible maximizers are the frontier points with $a = 0$ or $b = 0$.

Without loss of generality, consider $a = 0$. The joint profit becomes:

$$\Pi_{pj}(0, b) \quad = \quad \frac{t}{9}(1 - b)\left[9 + b^2\left(1 + \frac{\tau}{t}\right)^2\right].$$

The first-order condition of the maximization with respect to b can be written as:

$$2b - 3b^2 = \frac{9}{\left(1 + \frac{\tau}{t}\right)^2} \Leftrightarrow b = d_{pj} = \frac{1}{3} + \frac{1}{3}\sqrt{1 - 27\left(1 + \frac{\tau}{t}\right)^{-2}}.$$

Therefore, only $b = d_{pj}$ or the frontier point $b = 0$ can maximize $\Pi_{pj}(0, b)$.

Accordingly, firms choose $(a^*, b^*) = (0, d_{pj})$ if $\Pi_{pj}(0, d_{pj}) > \Pi_{pj}(0, 0)$, that is, if:

$$\frac{t}{9}(1 - d_{pj})\left[9 + d_{pj}^2\left(1 + \frac{\tau}{t}\right)^2\right] > t \Leftrightarrow \frac{\tau}{t} > 5.$$

And choose $(a^*, b^*) = (0, 0)$ otherwise, that is, when $\frac{\tau}{t} < 5$. $\square$

Proof of Lemma 1: Recall that $u_1(x)$ and $u_2(x)$ is the utility of the consumer located at x when she buys the good from firm 1 and from firm 2, respectively.

We start by showing that the firms are always interested in selling to the consumers who are located between them ($x \in [a, 1 - b]$). Suppose, without loss of generality, that $a < \frac{1}{2}$, and, by way of contradiction, that there exists $y \in [a, 1 - b]$ such that $u_1(y) < 0$ and $u_2(y) < 0$. There exists $y_1 \in (a, y)$ such that $u_1(y_1) = 0$. Of course that the consumers in (y_1, y) do not buy from any firm. Then, by locating at $a + \epsilon$, with $\epsilon \in (0, y - y_1)$, and keeping p_1 fixed, firm 1 increases its profit, by lowering the differentiation costs while

[18]The compact domain is $\mathcal{D} = \left\{(a, b) \in [0, 1]^2 : \; a + b \leq 1\right\}$.

maintaining (at least) its demand. The profit of firm 2 is not affected, thus we have a contradiction.

We now proceed to show that the extremes of the market are also covered. By way of contradiction, let us now suppose that there exists $\tilde{x}_1 \in (0, a)$ such that $u_1(\tilde{x}_1) = 0$ and $u_2(\tilde{x}_1) \leq 0$. Of course that the consumers located at $[\,0, \tilde{x}_1\,)$ do not buy from any firm. If $a > \frac{1}{2}$, by moving firm 1 to $a - \epsilon$, the joint profit would increase. With $a \leq \frac{1}{2}$, moving to $a - \epsilon$ increases demand (prices are kept fixed) but also increases the differentiation costs. With $\tilde{x}_2$ denoting the greatest x such that $u_2(x) \geq 0$, the joint profit is:

$$\Pi_j = \left[p_1 - \tau\left(\frac{1}{2} - a\right)^2\right](\tilde{x} - \tilde{x}_1) + \left[p_2 - \tau\left(\frac{1}{2} - b\right)^2\right](\tilde{x}_2 - \tilde{x}).$$

Since $u_1(\tilde{x}_1) = 0$, we must have $p_1 = V - t(a - \tilde{x}_1)^2$. And, as before, the indifferent consumer is given by $\tilde{x} = \frac{1+a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)}$.

The derivative of Π_j with respect to $\tilde{x}_1$ is:

$$\frac{\partial \Pi_j}{\partial \tilde{x}_1} = (\tilde{x} - \tilde{x}_1)\frac{\partial p_1}{\partial \tilde{x}_1} + \left[p_1 - \tau\left(\frac{1}{2} - a\right)^2\right]\frac{\partial(\tilde{x} - \tilde{x}_1)}{\partial \tilde{x}_1},$$

where:

$$\frac{\partial p_1}{\partial \tilde{x}_1} = 2t(a - \tilde{x}_1) \quad \text{and} \quad \frac{\partial(\tilde{x} - \tilde{x}_1)}{\partial \tilde{x}_1} = \frac{\tilde{x}_1 - a}{1 - a - b} - 1 < -1.$$

Thus:

$$\begin{aligned} \frac{\partial \Pi_j}{\partial \tilde{x}_1} &< 2t(a - \tilde{x}_1)(\tilde{x} - \tilde{x}_1) - V + t(a - \tilde{x}_1)^2 + \tau\left(\frac{1}{2} - a\right)^2 < \\ &< t - V + \frac{t}{4} + \frac{\tau}{4}. \end{aligned}$$

Therefore, $V \geq \frac{5t}{4} + \frac{\tau}{4}$ implies that $\frac{\partial \Pi_J}{\partial \tilde{x}_1} < 0$. Under this assumption, joint profit maximization implies full market coverage. $\square$

Proof of Proposition 7: Observe that if the consumers at the extremes, $x = 0$ and $x = 1$, and the indifferent consumer, $x = \tilde{x}$, buy the good, then all the other consumers also buy the good.

With a single firm producing, say firm 1, the maximum profit would be obtained by locating the firm in the center and setting $p = V - \frac{t}{4}$ (which implies $u_1(0) = u_1(1) = 0$). The resulting profit would also be equal to $V - \frac{t}{4}$. The same outcome results if both firms locate at the center and charge this same price.

If the two firms produce, there is full market coverage if and only if $u_1(0) \geq 0$, $u_2(1) \geq 0$ and $u_1(\tilde{x}) \geq 0$.

We start by maximizing the joint profit subject to $u_1(0) \geq 0$ and $u_2(1) \geq 0$, ignoring the restriction $u(\tilde{x}) \geq 0$. Then, we check whether the solution obtained satisfies this restriction. If it does, then it is the solution to our problem.

For the consumer located at $x = 0$ to buy the good from firm 1, the price charged, p_1, cannot be higher than $V - ta^2$. The corresponding margin of firm 1 is: $p_1 - V - ta^2 - \tau\left(\frac{1}{2} - a\right)^2$. The highest possible margin is obtained when (the derivative with respect to a is null):

$$-2ta - 2\tau a + \tau = 0 \;\Leftrightarrow\; a^* = \frac{\frac{\tau}{t}}{2\left(1 + \frac{\tau}{t}\right)}.$$

To maximize its margin subject to covering $x = 1$, firm 2 should locate symmetrically and set the same price. Firms are, with these decisions, maximizing their margins. If the market is fully covered (that is, the ignored restriction is satisfied), then the joint profit is being maximized. This occurs if the firm is closer to the center than to the extreme:

$$\frac{\frac{\tau}{t}}{2\left(1 + \frac{\tau}{t}\right)} \geq \frac{1}{4} \;\Leftrightarrow\; \frac{\tau}{t} \geq 1.$$

For $\frac{\tau}{t} \geq 1$, we have obtained the solution. Firms locate at $(a^*, b^*) = (d_j, d_j)$, with $d_j = \frac{\frac{\tau}{t}}{2\left(1+\frac{\tau}{t}\right)}$, set prices $p_1^* = p_2^* = V - t\frac{\left(\frac{\tau}{t}\right)^2}{4\left(1+\frac{\tau}{t}\right)^2}$, and obtain a joint profit $\Pi_j = V - \frac{t}{4}\frac{\left(\frac{\tau}{t}\right)^2 - \frac{\tau}{t}}{\left(1+\frac{\tau}{t}\right)^2}$ (higher than $V - \frac{t}{4}$).

For $\frac{\tau}{t} < 1$, the solution of the relaxed problem, obtained above, does not satisfy the ignored restriction, $u_1(\tilde{x}) \geq 0$, where $\tilde{x} \in (0, 1)$ as both firms are supposedly producing. Suppose, by way of contradiction, that the solution for this case is such that $u_1(\tilde{x}) = 0$,

$u_1(0) > 0$ and $u_2(1) \geq 0$. These conditions imply that the indifferent consumer is between both firms, $a < \tilde{x} < 1 - b$. Observe also that $u_1(0) \geq u_1(\tilde{x})$ implies $a < \frac{\tilde{x}}{2}$, and that $u_2(1) \geq u_2(\tilde{x})$ implies $b \leq \frac{1-\tilde{x}}{2}$. Therefore, $a + b \leq \frac{1}{2}$.

But, then, the joint profit is not being maximized. It can be higher if firm 1 moves to $a+\epsilon$ (towards the center) and increases p_1 for the indifferent consumer to be kept constant (revenues increase, and the differentiation costs decrease). This contradiction implies that the solution must be such that $u(\tilde{x}) = 0$, $u(0) = 0$ and $u(1) = 0$, implying that $a + b = \frac{1}{2}$.

Since $u\left(0\right) = u\left(1\right) = 0$, the charged prices must be $p_1 = V - ta^2$ and $p_2 = V - tb^2$, and the demands of firm 1 and 2 are $2a$ and $2b$, respectively. The resulting joint profit is:

$$
\begin{aligned}
\Pi_j &= 2a\left[V - ta^2 - \tau\left(\frac{1}{2} - a\right)^2\right] + 2b\left[V - tb^2 - \tau\left(\frac{1}{2} - b\right)^2\right] = \\
&= 2a\left[V - ta^2 - \tau\left(\frac{1}{2} - a\right)^2\right] + (1-2a)\left[V - t\left(\frac{1}{2} - a\right)^2 - \tau a^2\right] = \\
&= V - \frac{1}{4}t - \frac{1}{2}a\tau - 3a^2 t + a^2\tau + \frac{3}{2}at
\end{aligned}
$$

The corresponding first-order condition is:

$$
\frac{d\Pi_j}{da} = 0 \;\Leftrightarrow\; \frac{1}{2}\left(4a - 1\right)\left(\tau - 3t\right) = 0 \;\Leftrightarrow\; a = \frac{1}{4},
$$

because we are asuming that $\frac{\tau}{t} < 1$.

Consequently, if $\frac{\tau}{t} < 1$, firms choose to locate at $(a^*, b^*) = \left(\frac{1}{4}, \frac{1}{4}\right)$. The joint-profit, in this case, is $V - \frac{t+\tau}{16}$ (higher than $V - \frac{t}{4}$). $\square$

References

Aiura, H. and Sato, Y. (2008), "Welfare properties of spatial competition with location-dependent costs", *Regional Science and Urban Economics*, 38, pp. 32-48.

Brekke K.R. and Straume, O.R. (2004), "Bilateral monopolies and location choice", *Regional Science and Urban Economics*, 34, pp. 275-288.

D'Aspremont, C., Gabszewicz, J.J. and Thisse, J.-F.(1979), "On Hotelling's "Stability in competition"", *Econometrica*, 47, pp. 1145-1150.

Eaton, B.C. and Schmitt, N. (1994), "Flexible Manufacturing and Market Structure", *American Economic Review*, 84, pp. 875-88.

Friedman, J.W. and Thisse, J.-F. (1993), "Partial collusion fosters minimum product differentiation", *RAND Journal of Economics*, 24, pp. 631-645.

Gupta, B., Kats, A. and Pal, D. (1994), "Upstream monopoly, downstream competition and spatial price discrimination," *Regional Science and Urban Economics*, 24, pp. 529-542.

Hotelling, H. (1929), "Stability in competition", *Economic Journal*, 39, pp. 41-57.

Kourandi, F. and Vettas, N. (2009), "Endogenous Spatial Differentiation with Vertical Contracting", *mimeo.*

Matsushima, N. (2004), "Technology of upstream firms and equilibrium product differentiation", *International Journal of Industrial Organization*, 22, pp. 1091-1114.

Recent FEP Working Papers

Nº 350	João Correia-da-Silva and Joana Resende, "*Free daily newspapers: too many incentives to print?*", December 2009
Nº 349	Ricardo Correia and Carlos Brito, "*Análise Conjunta da Dinâmica Territorial e Industrial: O Caso da IKEA – Swedwood*", December 2009
Nº 348	Gonçalo Faria, João Correia-da-Silva and Cláudia Ribeiro, "*Dynamic Consumption and Portfolio Choice with Ambiguity about Stochastic Volatility*", December 2009
Nº 347	André Caiado, Ana Paula Africano and Aurora A.C. Teixeira, "*Firms' perceptions on the usefulness of State trade missions: an exploratory micro level empirical analysis*", December 2009
Nº 346	Luís Pinheiro and Aurora A.C. Teixeira, "*Bridging University-Firm relationships and Open Innovation literature: a critical synthesis*", November 2009
Nº 345	Cláudia Carvalho, Carlos Brito and José Sarsfield Cabral, "*Assessing the Quality of Public Services: A Conceptual Model*", November 2009
Nº 344	Margarida Catarino and Aurora A.C. Teixeira, "*International R&D cooperation: the perceptions of SMEs and Intermediaries*", November 2009
Nº 343	Nuno Torres, Óscar Afonso and Isabel Soares, "*Geographic oil concentration and economic growth – a panel data analysis*", November 2009
Nº 342	Catarina Roseira and Carlos Brito, "*Value Co-Creation with Suppliers*", November 2009
Nº 341	José Fernando Gonçalves and Paulo S. A. Sousa, "*A Genetic Algorithm for Lot Size and Scheduling under Capacity Constraints and Allowing Backorders*", November 2009
Nº 340	Nuno Gonçalves and Ana Paula Africano, "*The Immigration and Trade Link in the European Union Integration Process*", November 2009
Nº 339	Filomena Garcia and Joana Resende, "*Conformity based behavior and the dynamics of price competition: a new rational for fashion shifts*", October 2009
Nº 338	Nuno Torres, Óscar Afonso and Isabel Soares, "*Natural resources, economic growth and institutions – a panel approach*", October 2009
Nº 337	Ana Pinto Borges, João Correia-da-Silva and Didier Laussel, "*Regulating a monopolist with unknown bureaucratic tendencies*", October 2009
Nº 336	Pedro Rui Mazeda Gil, "*Animal Spirits and the Composition of Innovation in a Lab-Equipment R&D Model*", September 2009
Nº 335	Cristina Santos and Aurora A.C. Teixeira, "*The evolution of the literature on entrepreneurship. Uncovering some under researched themes*", September 2009
Nº 334	Maria das Dores B. Moura Oliveira and Aurora A.C. Teixeira, "*Policy approaches regarding technology transfer: Portugal and Switzerland compared*", September 2009
Nº 333	Ana Sofia Ferreira, Leonídio Fonseca and Lilian Santos, "*Serão os 'estudantes empreendedores' os empreendedores do futuro? O contributo das empresas juniores para o empreendedorismo*", August 2009
Nº 332	Raquel Almeida, Marina Silva and Tiago Soares, "*Coesão Territorial - As relações de fronteira entre Portugal e Espanha*", August 2009
Nº 331	Custódia Bastos, Suzi Ladeira and Sofia Silva, "*Empreendedorismo nas Artes ou Artes do Empreendedorismo? Um estudo empírico do 'Cluster' da Rua Miguel Bombarda*", August 2009
Nº 330	Filipe A. Ribeiro, Ana N. Veloso and Artur V. Vieira, "*Empreendedorismo Social: Uma análise via associativismo juvenil*", August 2009
Nº 329	Argentino Pessoa, "*Outsourcing And Public Sector Efficiency: How Effective Is Outsourcing In Dealing With Impure Public Goods?*", July 2009
Nº 328	Joana Almodovar, Aurora A.C. Teixeira, "*Conceptualizing clusters through the lens of networks: a critical synthesis*", July 2009
Nº 327	Pedro Mazeda Gil, Fernanda Figueiredo and Óscar Afonso, "*Equilibrium Price Distribution with Directed Technical Change*", July 2009
Nº 326	Armando Silva, Ana Paula Africano and Óscar Afonso, "*Which Portuguese firms are more innovative? The importance of multinationals and exporters*", June 2009
Nº 325	Sofia B. S. D. Castro, João Correia-da-Silva and Pascal Mossay, "*The core-periphery model with three regions*", June 2009

Nº 324	Marta Sofia R. Monteiro, Dalila B. M. M. Fontes and Fernando A. C. C. Fontes, "*Restructuring Facility Networks under Economy of Scales*", June 2009
Nº 323	Óscar Afonso and Maria Thompson, "*Costly Investment, Complementarities and the Skill Premium*", April 2009
Nº 322	Aurora A.C. Teixeira and Rosa Portela Forte, "*Unbounding entrepreneurial intents of university students: a multidisciplinary perspective*", April 2009
Nº 321	Paula Sarmento and António Brandão, "*Next Generation Access Networks: The Effects of Vertical Spillovers on Access and Innovation*", April 2009
Nº 320	Marco Meireles and Paula Sarmento, "*Incomplete Regulation, Asymmetric Information and Collusion-Proofness*", April 2009
Nº 319	Aurora A.C. Teixeira and José Sequeira, "*Determinants of the international influence of a R&D organisation: a bibliometric approach*", March 2009
Nº 318	José Sequeira and Aurora A.C. Teixeira, "*Assessing the influence of R&D institutions by mapping international scientific networks: the case of INESC Porto*", March 2009
Nº 317	João Loureiro, Manuel M. F. Martins and Ana Paula Ribeiro, "*Cape Verde: The Case for Euroization*", March 2009
Nº 316	Ester Gomes da Silva and Aurora A.C. Teixeira, "*Does structure influence growth? A panel data econometric assessment of 'relatively less developed' countries, 1979-2003*", March 2009
Nº 315	Mário A. P. M. Silva, "*A Model of Growth with Intertemporal Knowledge Externalities, Augmented with Contemporaneous Knowledge Externalities*", March 2009
Nº 314	Mariana Lopes and Aurora A.C. Teixeira, "*Open Innovation in firms located in an intermediate technology developed country*", March 2009
Nº 313	Ester Gomes da Silva, "*Capital services estimates in Portuguese industries, 1977-2003*", February 2009
Nº 312	Jorge M. S. Valente, Maria R. A. Moreira, Alok Singh and Rui A. F. S. Alves, "*Genetic algorithms for single machine scheduling with quadratic earliness and tardiness costs*", February 2009
Nº 311	Abel Costa Fernandes, "*Explaining Government Spending: a Cointegration Approach*", February 2009
Nº 310	João Correia-da-Silva, "*Uncertain delivery in markets for lemons*", January 2009
Nº 309	Ana Paula Ribeiro, "*Interactions between Labor Market Reforms and Monetary Policy under Slowly Changing Habits*", January 2009
Nº 308	Argentino Pessoa and Mário Rui Silva, "*Environment Based Innovation: Policy Questions*", January 2009
Nº 307	Inês Drumond and José Jorge, "*Basel II Capital Requirements, Firms' Heterogeneity, and the Business Cycle*", January 2009
Nº 306	Adelaide Maria Figueiredo, Fernanda Otília Figueiredo and Natália Pimenta Monteiro, "*Labor adjustments in privatized firms: a Statis approach*", December 2008
Nº 305	Manuela A. D. Aguiar and Sofia B. S. D. Castro, "*Chaotic and deterministic switching in a two-person game*", December 2008
Nº 304	Ana Pinto Borges and João Correia-da-Silva, "*Using Cost Observation to Regulate Bureaucratic Firms*", December 2008
Nº 303	Miguel Fonseca, "*The Investment Development Path Hypothesis: a Panel Data Approach to the Portuguese Case*", December 2008

Editor: Sandra Silva (sandras@fep.up.pt)
Download available at:
http://www.fep.up.pt/investigacao/workingpapers/
also in http://ideas.repec.org/PaperSeries.html

www.fep.up.pt

Faculdade de Economia da Universidade do Porto
Rua Dr. Roberto Frias, 4200-464 Porto | Tel. 225 571 100
Tel. 225571100 | www.fep.up.pt