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Heterogeneous discounting in consumption-investment problems. Time consistent solutions

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Abstract

In this paper we analyze a stochastic continuous time model in finite horizon in which agents discount the instantaneous utility function and the final function at constant but different instantaneous discount rates of time preference. Within this context we can model problems in which, when the time t approaches to the final time, the valuation of the final function increases compared with previous valuations in a way that cannot be explained by using a unique constant or a variable discount rate. We derive a dynamic programming equation whose solutions are time-consistent Markov equilibria. For this class of time preferences, we study the classical consumption and portfolio rules model (Merton, 1971) for CRRA and CARA utility functions for time-consistent agents, and we compare the different equilibria with the time-inconsistent solutions. The introduction of stochastic terminal time is also discussed.

Abstract

En aquest treball s'analitza un model estocàstic en temps continu en el que l'agent decisor descompta les utilitats instantànies i la funció final amb taxes de preferència temporal constants però diferents. En aquest context es poden modelitzar problemes en els quals, quan el temps t s'acosta al moment final, la valoració de la funció final incrementa en comparació amb les utilitats instantànies. Aquest tipus d' asimetria no es pot descriure ni amb un descompte estàndard ni amb un variable. Per tal d'obtenir solucions consistents temporalment es deriva l'equació de programació dinàmica estocàstica, les solucions de la qual són equilibris Markovians. Per a aquest tipus de preferències temporals, s'estudia el model clàssic de consum i inversió (Merton, 1971) per a les funcions d'utilitat del tipus CRRA i CARA, comparant els equilibris Markovians amb les solucions inconsistents temporalment. Finalment es discuteix la introducció del temps final aleatori.

JEL classification: C61; G11; C73

Keywords: heterogeneous discounting; consumption and portfolio rules; time-consistency; dynamic programming

1 Introduction

In the study of intertemporal choices it is customary in economics to consider the so-called Discounted Utility (DU) Model, introduced in Samuelson (1937). According to the Samuelson's model, time preferences can be characterized by a single parameter, the discount rate. Since the DU model assumes a constant discount rate of time preference, it can be easily shown (due to the properties of the exponential function) that constant discounting implies that agent's time preferences are time-consistent. However, empirical observations seem to show that predictions of the DU model disagree with the actual behavior of decision makers (we refer to Frederick et al (2002) for an analysis on the topic and a review of the literature up to (2002)). These anomalies can be of several types.

The best documented DU anomaly is hyperbolic discounting (or non-constant discounting, in general). Strotz (1956) studied the effects of choosing a variable rate of time preference, illustrating how for a very simple model preferences are time consistent if, and only if, the discount function is an exponential with a constant discount rate. Effects of the so-called quasi-hyperbolic (or quasi-geometric) discount functions introduced by Phelps and Pollak (1968) have been extensively studied in a discrete time context, within the field of behavioral economics. The most relevant effect of non-constant discounting is that preferences change along time. In this sense, an agent making a decision at time t has different time preferences compared with those at the initial time t_0. In a continuous time setting, a dynamic programming equation (DPE) providing a time-consistent solution was introduced in Karp (2007) in a deterministic framework. This DPE was extended to the case where the evolution of the state variables is governed by a set of stochastic differential equations in Ekeland and Pirvu (2008) and Marín-Solano and Navas (2010).

Although hyperbolic discounting relaxes the assumption of using a constant discount rate for all time periods, it does not solve all the anomalies of the DU model. As pointed out in Frederick et al (2002), the DU model assumes also that the discount rate should be the same for all types of goods and all categories of intertemporal decisions, and this is in contradiction with several empirical regularities.

In this paper we study a simple approach (giving rise, as in the case of hyperbolic discounting, to time-inconsistent preferences) which can provide a model for certain behaviors that can not be explained by the DU model or more general hyperbolic preferences. More precisely, we are interested in preferences representing a situation in which the agent discounts in a different way the utilities enjoyed along the planning horizon and that of the bequest or final function. Hence, the intertemporal utility function takes the form

$$U_t = \int_t^T d(s,t)u(x,c,s)\, ds + d(T,t)F(x(T),T) \ .$$

with $d(s,t) = e^{-\rho(s-t)}$ for $s < T$, and $d(T,t) = e^{-\bar{\rho}(T-t)}$, for $\rho \neq \bar{\rho}$, in general.

Impatient agents over-valuing the utilities $u(x(s),c(s),s)$ in comparison with the final function $F(x(T),T)$ are characterized by $\bar{\rho} > \rho$. However, when time passes, the final function increases its relative value in comparison with the instantaneous utilities $u(x(s),c(s),s)$ (usually due to consumption and hence to an immediate benefit). This asymmetric valuation cannot be described by a standard discount function or in general with non-constant discounting. There are several problems that seem to be good candidates for this description: human capital formation, where the the final function represents the utility obtained after a period of continuous effort; consumption and portfolio rule problems, where the final function represents a bequest function (the individual is more concerned with the welfare of her descendants when life is arriving to the end); or, along the same lines, retirement and pension problems. Since preferences are time-inconsistent, no optimal solutions exist, and the standard techniques in optimal control theory (the Pontryagin's Maximum Principle or the Hamilton-Jacobi-Bellman equation) give rise to time-inconsistent solutions. By reproducing the literature of non-constant discounting, we can say that an agent is naive if she does not take into account that her preferences will change in the future, so she is time-inconsistent. In order to obtain time-consistent solutions (agents are sophisticated, using the standard terminology in non-constant discounting), Markov perfect equilibria must be calculated.

This problem with heterogeneous discounting was introduced in Marín-Solano and Patxot (2011) in a deterministic setting. In that paper, a DPE providing a time-consistent solution was derived by using a variational approach, and an economic motivation was given. Such DPE is rather similar to the one first derived by Karp (2007) for the problem with non-constant discounting. An important limit in the approach introduced in that paper is that the DPE is a functional equation with a nonlocal term. As a consequence, it becomes very complicated to find solutions, not only analytically, but also numerically. In this paper we extend the results in the deterministic setting to a stochastic environment, by deriving a set of two coupled partial differential equations which are equivalent (in the deterministic setting) to the DPE derived in Marín-Solano and Patxot (2011). This approach allows us to compute (analytically or numerically) the solutions for different economic problems. In particular, we are interested in analyzing how time-inconsistent preferences with heterogeneous discounting modify the classical consumption and portfolio rules (Merton (1971)). We show that, similar to the problem with non-constant discounting, within the HARA (hyperbolic absolute risk aversion) utility functions, if the relative risk aversion is constant (logarithmic and power utility functions), the equilibrium portfolio rule does not depend on the rate of time preference. This nice property is not satisfied for more general utility functions, such as the (constant absolute risk aversion) exponential function. With respect to the consumption rules, for the case of heterogeneous discounting, they are different, not only quantitatively, but mainly qualitatively, to the equilibria derived for the case of non-constant discounting in continuous time in Marín-Solano and Navas (2010). The effects on the consumption rule of introducing heterogeneous discounting are illustrated numerically for the case of power and exponential utility functions. As a final contribution we show that, if the final time is a random variable, our problem with heterogeneous discounting transforms into a problem which is equivalent to a model introduced (in a deterministic setting) in Marín-Solano and Shevkoplyas (2011). In this case, we must search for a time-consistent equilibrium in a cooperative differential game with heterogeneous agents.

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we first derive the DPE in a discrete time setting and then, we find the formal continuous time limit. As a result, we recover the DPE in the deterministic setting as a particular case. This provides a justification to the mathematically rigorous but less intuitive procedure used in Marín-Solano and Patxot (2011). Next, we define the notion of equilibrium rule as in Marín-Solano and Patxot (2011) (which is based on the one in Ekeland and Pirvu (2008)), and the DPE is obtained by using a variational approach. In Section 4, this equation is solved for the consumption and portfolio rules problem for some particular utility functions. Section 5 analyzes the problem for the case of random time horizon. Finally, Section 6 contains the main conclusions of the paper.

2 The Model

We introduce the problem in a discrete time and deterministic setting. For each period s, $s = 0, 1, 2, \ldots, T-1$, let $x_s = (x_s^1, \ldots, x_s^n)$ be the vector of state variables and $c_s = (c_s^1, \ldots, c_s^m)$ the vector of control (or decision) variables. If $u_s(x_s, c_s, s)$ is the utility function at period s and $F(x_T, T)$ is the final (or bequest) function, in the conventional model, the intertemporal utility function of an agent taking decisions at period t takes the form

$$U_t = \sum_{s=t}^{T-1} \delta^{s-t} u_s(x_s, c_s, s) + \delta^{T-t} F(x_T, T) ,$$

where the state variables evolve according to the state equation $x_{s+1} = f(x_s, c_s, s)$, for $s = t, \ldots, T-1$. In order to maximize U_t we must solve an optimal control problem and, since the discount factor $\delta \in (0, 1]$ is always the same, the solution becomes time consistent. In general, if we consider an arbitrary discount $d(s, t)$ representing how the agent at time t discounts future utilities enjoyed at time $s \geq t$, the intertemporal utility function at period t is given by

$$U_t = \sum_{s=t}^{T-1} d(s,t) u_s(x_s, c_s, s) + d(T,t) F(x_T, T) .$$

In the standard case, $d(s, t) = \delta^{s-t}$. If time preferences are quasi-hyperbolic, $d(s, t) = \beta\delta^{s-t}$ for $s > t$, and $d(t, t) = 1$. In this paper we are interested in preferences representing a situation in which the agent discounts in a different way the utilities enjoyed along the planning horizon, and the final function. In particular, we assume that the discount rate takes the form $d(s, t) = \delta^{s-t}$ for $s < T$, and $d(T, t) = \bar{\delta}^{T-t}$. The intertemporal utility function becomes

$$U_t = \sum_{s=t}^{T-1} \delta^{s-t} u_s(x_s, c_s, s) + \bar{\delta}^{T-t} F(x_T, T) .$$

Following Marín-Solano and Patxot (2011), we call these time preferences *heterogeneous discounting*.

Next, we extend the model to a continuous time setting. Let $x = (x^1, \ldots, x^n) \in X \subseteq \mathbf{R}^n$ be the vector of state variables, $c = (c^1, \ldots, c^m) \in U \subseteq \mathbf{R}^m$ the vector of control (or decision) variables, $u(x(s), c(s), s)$ the instantaneous utility function at time s, T the planning horizon (terminal time) and $F(x(T), T)$ the final or bequest function. Then the corresponding intertemporal utility function is

$$U_t = \int_t^T e^{-\rho(s-t)} u(x, c, s)\, ds + e^{-\bar{\rho}(T-t)} F(x(T), T) . \tag{1}$$

As we present in the Introduction, impatient agents over-valuing utilities $u(x, c, s)$ in comparison with the final function $F(x(T), T)$ are characterized by $\bar{\rho} > \rho$ (or $\delta > \bar{\delta}$ in the discrete time setting). However, with these time preferences, when time passes, the final function increases its value in comparison with the utilities $u(x, c, s)$. This asymmetric valuation cannot be described by using a standard geometric discounting or, in general, with hyperbolic preferences (with a unique non-constant discount rate). Note that with (non)constant discounting the bias to the present (to *their* present) does not change from the viewpoint of the different t-agents (in the hyperbolic discounting literature, an agent taking decisions at time t is called the t-agent). With heterogeneous discounting, the bias to the present changes along time. We refer to Marín-Solano and Patxot (2011) for a discussion of this effect (in that paper heterogeneous discounting were used as an attempt to describe, e.g., the behavior of an undergraduate student who is planning on how hard to work in each of the years of her program).

Problems which can be represented by this model include consumption and portfolio rule problems or retirement and pension problems. For instance, consider a decision-maker who is planning on how much to save for her retirement. Typically, individuals are much more concerned with life quality after retirement when retirement age is approaching[1], in comparison with their concern about their post retirement life when they look at it from a long distance, for instance, when they are young. This saving effort can be viewed as a disutility during the first periods, since the agent does not spend the saved resources in consumption and hence in immediate gratification. Within this setting, let us briefly compare the type of time-inconsistency for an impatient agent (say, agent A) with hyperbolic discounting (with a non-increasing discount rate) with the effects of impatience of and agent with heterogeneous discounting with $\bar{\rho} > \rho$ (agent B).

For agent A, the willingness to increase her final year's saving effort in return for a better retirement (and higher subsequent welfare) is higher at the beginning of the planning horizon than at the end of the planning horizon, since she is always more impatient in her short-run decisions than in her long-run decisions. For this reason, this agent would like to commit herself, in the first year, to save harder in the final year, compared to her actual willingness to make the saving effort when the final year arrives. In particular, if this agent is naive (time-inconsistent), when the final year arrives, she actually ends up saving less than she planned in the first year.

Next, we look at the behavior of agent B. For a long time horizon and from the first year perspective, it is natural to assume that the agent can hardly imagine her post-retirement life, so she decides to save an small amount of money. As the prospect of retirement looms, she takes things more seriously and decides in the last year to save harder than she planned at the beginning of her planning horizon. This is the effect that we can capture by using a different instantaneous discount rates for instantaneous utilities and for the final function. In order to see this effect, consider the case $\bar{\rho} > \rho$ and rewrite the final function in (1) as $e^{-\rho(T-t)}e^{-(\bar{\rho}-\rho)(T-t)}F(x(T),T)$. In this way, the actual valuation of the final function of the agent is given by $e^{-(\bar{\rho}-\rho)(T-t)}F(x(T),T)$, which is an increasing function in t. Hence, as long as the agent approaches to the end of the planning horizon, the current final function increases, i.e., $e^{-(\bar{\rho}-\rho)(T-s_2)}F(x(T),T) > e^{-(\bar{\rho}-\rho)(T-s_1)}F(x(T),T)$ for $s_1 < s_2$, $s_i \in (t,T)$.

Summarizing, the main difference between agents A and B (or between hyperbolic and heterogeneous discounting) is the time evolution of the bias to the present. An agent taking decisions with hyperbolic preferences has always the same bias to her present, as in the case of standard (exponential) discounting. On the contrary, for agent B (with heterogeneous discounting), there is also a bias to the present, but this bias changes (decreases when $\rho < \bar{\rho}$) as long as she approaches the end of the planning horizon. If $\bar{\rho} > \rho$ the agent procrastinates (as in hyperbolic discounting), in the sense of undervaluing the final function, but this procrastination decreases along time. With a similar argument, in case that $\rho > \bar{\rho}$, the agent will have a decreasing valuation of the final function as long as she reaches the final time T.

We finish this section by introducing the problem in a stochastic setting. In the discrete

[1] Alternatively, we could think in an agent solving a consumption-portfolio rules problem where the final function represents a bequest function for her descendants. The individual is much more concerned with life quality of her descendants when she becomes older.

time case, the difference equation is now subject to random disturbances and the state equation becomes $X_{t+1} = f(X_t, c_t, t, V_{t+1})$, $X_0 = x_0$, $V_0 = v_0$. We restrict our attention to the case when V_{t+1} is a random variable taking values in a finite set $\mathcal{V}$. Each t-agent will look for maximizing in c_t the expected intertemporal utility function

$$E\left[\sum_{s=t}^{T-1} \delta^{s-t} u_s(X_s, c_s, s|x_t, v_t) + \bar{\delta}^{T-t} F(X_T, T|x_t, v_t)\right] \tag{2}$$

subject to

$$X_{s+1} = f(X_s, c_s, s, V_{s+1})\,, \;\; X_s = x_s\,, \;\; V_s = v_s\,, \;\; s = t, \ldots, T-1\,. \tag{3}$$

In continuous time, the problem becomes

$$\max E\left[\int_t^T e^{-\rho(s-t)} u(X(s), c(s), s) + e^{-\bar{\rho}(T-t)} F(X(T), T) \mid x_t\right] \tag{4}$$

subject to

$$dX(s) = f(X(s), c(s), s)ds + \sigma(X(s), c(s), s) \cdot dW(s), \;\; X(t) = x_t \;\; \text{given}\,. \tag{5}$$

3 Dynamic programming equation

The solution provided by the use of standard optimal control techniques is time-inconsistent if the intertemporal utility function takes the form (2) or (4). In Marín-Solano and Patxot (2011) a DPE for sophisticated (time-consistent) agents in a deterministic framework was derived by following a variational approach. In this section we derive first a Dynamic Programming Equation (DPE) for the stochastic problem in a discrete time setting. Next, we obtain the DPE in continuous time by discretizing first the problem and defining then the DPE as the (formal) continuous time limit. This derivation is similar to that in Karp (2007) and Marín-Solano and Navas (2010) for the case of non-constant discounting in deterministic and stochastic environments, respectively. Finally, we provide an alternative derivation of the DPE by using a variational approach.

3.1 Dynamic Programming Equation in discrete time

First, let us assume that the probability that $V_{t+1} = v \in \mathcal{V}$, $P_t(v|v_t)$, may depend on the outcome v_t at time t, as well as explicitly on time t, but it is independent on the state and control variables x_t and c_t. In addition, functions u and f are assumed to be continuous in (x, c). We search for an equilibrium rule $c_t^* = \phi_t(x_t, v_t)$, characterized by the property that no decision-maker in the sequence of decision-makers wants to deviate from it. Let T be finite. The value function for the t-agent is given by

$$W(x_t, t, v_t) = \sup_{\{c_t\}} E\left[\sum_{s=t}^{T-1} \delta^{s-t} u_s(X_s, c_s, s|x_t, v_t) + \bar{\delta}^{T-t} F(X_T, T|x_t, v_t)\right] \tag{6}$$

where $c_s = \phi_s(x_s, v_s)$, for $s = t+1, \ldots, n$. The computation of the expectation in (6) is based on conditional probabilities of the form $p^*(v_{t+1}, \ldots, v_s) = P_t(v_{t+1}|\, v_t) \cdot P_{t+1}(v_{t+2}|\, v_{t+1}) \cdots P_{s-1}(v_s|\, v_{s-1})$. We adapt the derivation of the DPE in the classical case $\delta = \bar{\delta}$ (see e.g.

Seierstad (2009)) as follows. In the final period T we define $W(x_T, T, v_T) = F(x_T, T)$ as usual. At period $T-1$,

$$W(x_{T-1}, T-1, v_{T-1}) = \sup_{\{c_{T-1}\}} \left\{ E\left[u_{T-1}(x_{T-1}, c_{T-1}, T-1) + \bar{\delta} F(X_T, T) \,|\, x_{T-1}, v_{T-1} \right] \right\},$$

where the expectation is calculated over V_T given v_{T-1}. Since $F(X_T, T)$ depends on V_T via $X_T = f(x_{T-1}, c_{T-1}, T-1, V_T)$, we can write

$$W(x_{T-1}, T-1, v_{T-1}) = u_{T-1}(x_{T-1}, \phi_{T-1}(x_{T-1}, v_{T-1}), T-1) +$$

$$+\bar{\delta} E\left[F(X_T, T) \,|\, x_{T-1}, v_{T-1} \right] = u_{T-1}(x_{T-1}, \phi_{T-1}(x_{T-1}, v_{T-1}), T-1) + \bar{\delta} L_T^{T-1} ,$$

where we define $L_T^{T-1} = E\left[F(X_T, T) \,|\, x_{T-1}, v_{T-1} \right]$. In general, if

$$L_\tau^s = E[\cdots [E[E[u(X_\tau, \phi_\tau(X_\tau, \tau), \tau) \,|\, X_{\tau-1}, V_{\tau-1}] \,|\, X_{\tau-2}, V_{\tau-2}] \cdots] \,|\, x_s, v_s] ,$$

it is clear that

$$W(x_t, t, v_t) = \sup_{\{c_t\}} \left\{ u_t(x_t, c_t, t) + \sum_{s=t+1}^{T-1} \delta^{s-t} L_s^t + \bar{\delta}^{T-t} L_T^t \right\} . \tag{7}$$

In a similar way, $W(x_{t+1}, t+1, v_{t+1}) = \sum_{s=t+1}^{T-1} \delta^{s-t-1} L_s^{t+1} + \bar{\delta}^{T-t-1} L_T^{t+1}$ and therefore

$$E\left[W(X_{t+1}, t+1, V_{t+1} \,|\, x_t, v_t) \right] = \sum_{s=t+1}^{T-1} \delta^{s-t-1} L_s^t + \bar{\delta}^{T-t-1} L_T^t . \tag{8}$$

By solving L_T^t in (8) and substituting in (7) we obtain the Dynamic Programing Equation, which proceeds backward in time:

$$W(x_T, T, v_T) = F(x_T, T) ,$$

$$\bar{\delta}^{T-t-1} W(x_t, t, v_t) = \sup_{\{c_t\}} \left\{ \bar{\delta}^{T-t-1} u_t(x_t, c_t, t) + \sum_{s=t+1}^{T-1} \left[\delta^{s-t} \bar{\delta}^{T-t-1} - \delta^{s-t-1} \bar{\delta}^{T-t} \right] L_s^t + \right.$$

$$\left. + \bar{\delta}^{T-t} E\left[W(X_{t+1}, t+1, V_{t+1} \,|\, x_t, v_t) \right] \right\} , \tag{9}$$

$$X_{s+1} = f(X_s, c_s, s, V_{s+1}) , \quad X_s = x_s, \quad V_s = v_s .$$

The decision rules solving the right hand term in equation (9) are the Markov Perfect Equilibria.

Remark 1 *Note that, if the discount rates coincide, $\delta = \bar{\delta}$, the term in the sum in (9) vanishes and we recover the standard Bellman equation.*

We can easily extend our previous results to the case when $P_t[V_{t+1} = v] = P_t(v |\, x_t, c_t, v_t)$ depends, not only on time t and the previous outcome v_t, but also on the state and control variables x_t and c_t. We present the details in the Appendix.

3.2 The continuous time case: a formal limiting procedure

Now, let us extend the DPE (9) to a continuous time setting, by following a formal limiting procedure as in Karp (2007) and Marín-Solano and Navas (2010). In the continuous time setting, the agent at time t (the t-agent) aims to solve Problem 4-5. Let us discretize the problem by following the classical Euler (or Euler-Mayurama) method. If we divide the interval $[0, T]$ into N periods of constant length ϵ, in such a way that we identify $T = N\epsilon$, and $s = j\epsilon$, for $j = 0, 1, \ldots, N$, then Equation (5) becomes $X(t+1) = X(t) + f(X(t), c(t), t) + \sigma(X(t), c(t), t)(w(t+1) - w(t))$ where $w(t)$ is a Wiener process. Denoting $X(j\epsilon) = X_j$ and $c(j\epsilon) = c_j$, for $j = 0, \ldots, N-1$, the objective of the agent in period $t = j\epsilon$ is to maximize

$$E\left[\sum_{s=j}^{N-1} e^{-\rho(s-j)\epsilon} u(X_s, c_s, s) + e^{-\bar{\rho}(N-j)\epsilon} F(X_T, T)\right] \tag{10}$$

subject to

$$X_{i+1} = X_i + f(X_i, c_i, i) + \sigma(X_i, c_i, i)(w_{i+1} - w_i) \ , \tag{11}$$

for $i = j, \ldots, T-1$, x_j given. Note that Problem 10-11 is equivalent to Problem 2-3.

Remark 2 *For a given decision rule $c(x, s)$, a condition assuring the uniform convergence (in the mean square sense) of the solution of the discretized equation (11) to the true solution to (5) is that functions f and σ satisfy uniform growth and Lipschitz conditions in x, and are Hölder continuous of order $1/2$ in the second variable.*

Definition 1 *We define the value function $V(x, t)$ for Problem (4-5) as the solution to the DPE obtained by taking the formal continuous time limit when $\epsilon \to 0$ of the DPE (9) obtained for the discrete approximation (10-11) to the problem, assuming that such a limit exists and that the solution is of class $C^{2,1}$.*

Next, let us derive the DPE for the problem with heterogeneous discounting in the spirit of the previous definition. Let $V(x, t)$ be the value function of the t-agent, with initial condition $x(t) = x_t$. Since $s = j\epsilon$ and $X(t+\epsilon) = x(t) + f(x(t), c(t), t)\epsilon + \sigma(x(t), c(t), t)(w(t+\epsilon) - w(t))$, then $W(x_j, j\epsilon, v_j) = V(x_t, t)$ and

$$V(x_{t+\epsilon}, t+\epsilon) = V(x_t, t) + \nabla_{x_t} V(x_t, t) f(x_t, c(t), t)\epsilon + \nabla_{x_t} V(x_t, t)\sigma(x, c(t), t) \cdot (w_{t+\epsilon} - w_t) +$$

$$+ \nabla_t V(x_t, t)\epsilon + \frac{1}{2} tr\left(\sigma(x_t, c(t), t) \cdot \sigma'(x_t, c(t), t) \cdot \nabla_{x_t x_t} V(x_t, t)\right)\epsilon + o(\epsilon)$$

where $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$. In addition, $e^{-\bar{\rho}(n-j)\epsilon} = e^{-\bar{\rho}(n-j-1)\epsilon}\left[1 - \bar{\rho}\epsilon + o(\epsilon)\right]$ and $e^{-\rho k\epsilon} = e^{-\rho(k-1)\epsilon}\left[1 - \rho\epsilon + o(\epsilon)\right]$. By substituting in (9) we obtain

$$V(x_t, t) = \sup_{\{c_t\}} \{u(x_t, c_t, t)\epsilon + \sum_{k=j+1}^{n-1} \left[e^{-\rho(k-j-1)\epsilon}(\bar{\rho} - \rho)\epsilon\right] L_k^j \epsilon +$$

$$+ V(x_t, t) + \nabla_{x_t} V(x_t, t) f(x_t, c_t, t)\epsilon + E\left[\nabla_{x_t} V(x_t, t)\sigma(x_t, c_t, t) \cdot (w_{t+\epsilon} - w_t)\right] + \nabla_t V(x_t, t)\epsilon +$$

$$+ \frac{1}{2} tr\left(\sigma(x_t, c_t, t) \cdot \sigma'(x_t, c_t, t) \cdot \nabla_{x_t x_t} V(x_t, t)\right)\epsilon -$$

$$- \rho\epsilon V(x_t, t) - \rho\epsilon E\left[\nabla_{x_t} V(x_t, t)\sigma(x_t, v, t)(w_{t+\epsilon} - w_t)\right] + o(\epsilon)\} \ .$$

Therefore,

$$0 = \sup_{\{c_t\}} \{ u(x_t, c_t, t)\epsilon + \sum_{k=j+1}^{n-1} \left[e^{-\rho(k-j-1)\epsilon} (\bar{\rho} - \rho)\epsilon \right] L_k^j \, \epsilon + \nabla_{x_t} V(x_t, t) f(x_t, c_t, t)\epsilon +$$

$$+ \nabla_t V(x_t, t)\epsilon + \frac{1}{2} tr \left(\sigma(x_t, c_t, t) \cdot \sigma'(x_t, c_t, t) \cdot \nabla_{x_t x_t} V(x_t, t) \right) \epsilon - \rho\epsilon V(x_t, t) + o(\epsilon) \Big\} \ . \quad (12)$$

Dividing equation (12) by ϵ and taking the limit $\epsilon \to 0$ we obtain:

Proposition 1 *Let $V(x,t)$ be a function of class $C^{2,1}$ in (x,t) satisfying the DPE*

$$\bar{\rho} V(x,t) - \nabla_t V(x,t) - K(x,t) = \quad (13)$$

$$= \sup_{\{c\}} \left\{ u(x,c,t) + \nabla_x V(x,t) f(x,c,t) + \frac{1}{2} tr \left(\sigma(x,c,t) \cdot \sigma'(x,c,t) \cdot \nabla_{xx} V(x,t) \right) \right\} \ ,$$

with

$$V(x,T) = F(x,T) \ , \quad (14)$$

and

$$K(x,t) = (\bar{\rho} - \rho) E \left[\int_t^T e^{-\rho(s-t)} u(X_s, \phi(X_s, s), s) \, ds \right] \ . \quad (15)$$

Then $V(x,t)$ is the value function for Problem 4-5. If, for each pair (x,t), there exists a decision rule $c^* = \phi(x,t)$, with corresponding state trajectory $X^*(t)$, such that c^* maximizes the right hand side term of (13), then $c^* = \phi(x,t)$ is called a Markov equilibrium rule for the problem with heterogeneous discounting.

Remark 3 *Again, if $\rho = \bar{\rho}$, the term $K(x,t)$ vanishes and we recover the standard Hamilton-Jacobi-Bellman equation.*

In the proof of the previous proposition the pass to the limit is "formal" and needs to be mathematically justified. With respect to the classical DPE, in Fleming and Soner (2006) the convergence of finite difference approximations to Hamilton-Jacobi-Bellman equations is discussed. We refer also to Kushner and Dupuis (2001) for a study of the convergence of numerical methods to the value function in the standard case.

Finally, note that we can write

$$K(x,t) = (\bar{\rho} - \rho) E \left[\int_t^T e^{-\rho(s-t)} u(X(s), \phi(X(s), s), s) \, ds \right] \quad (16)$$

and, by differentiating K in (16) with respect to t we obtain the "auxiliary dynamic programming equation"

$$\rho K(x,t) - \nabla_t K(x,t) = (\bar{\rho} - \rho) u(x, \phi(x,t), t) + \nabla_x K(x,t) \cdot f(x, \phi(x,t), t) +$$

$$+ \frac{1}{2} tr \left(\sigma(x, \phi(x,t), t) \cdot \sigma'(x, \phi(x,t), t) \cdot \nabla_{xx} K(x,t) \right) \ . \quad (17)$$

Hence we have:

Corollary 1 *Let $V(x,t)$ and $K(x,t)$ be two functions of class $C^{2,1}$ in (x,t) such that $V(x,t)$, $K(x,t)$ and the strategy $c^* = \phi(x,t)$ satisfy the set of two DPEs (13) and (17) with boundary conditions $V(x,T) = F(x,T)$, $K(x,T) = 0$. Then $V(x,t)$ is the value function for Problem (4-5), and the strategy $c^* = \phi(x,t)$ maximizing the right hand side term of Equation (13) is a Markov equilibrium rule for the problem with heterogeneous discounting.*

3.3 Dynamic programming equation in continuous time: a variational approach

Next we provide an alternative derivation of the DPE (13-15), by using a variational approach similar to that introduced, for the case of non-constant discounting, in Ekeland and Pirvu (2008). In particular, we extend to a stochastic setting the derivation of a DPE in the deterministic problem with heterogeneous discounting first derived in Marín-Solano and Patxot (2011). To do that we assume that decision rules are progressively measurable processes such that the stochastic differential equation (5) admits a unique strong solution (see e.g. Theorem 6.3 in Yong and Zhou (1999) for conditions for the existence of strong solutions). For the problem analyzed in Section 4, described by a linear SDE, the existence of strong unique solutions is guaranteed.

Equilibrium policies are defined as follows. If $c^*(s) = \phi(X(s), s)$ is the equilibrium rule, for $\epsilon > 0$ let us consider the variations

$$c_\epsilon(s) = \begin{cases} v(s) & \text{if} \quad s \in [t, t+\epsilon] \ , \\ \phi(X, s) & \text{if} \quad s > t+\epsilon \ . \end{cases}$$

If the t-agent can precommit her behavior during the period $[t, t+\epsilon]$, the value function for the perturbed control path c_ϵ is given by

$$V_\epsilon(x,t) = \max_{\{v(s),\ s\in[t,t+\epsilon]\}} E\left[\int_t^{t+\epsilon} e^{-\rho(s-t)} u(X(s), v(s), s)\, ds + \right.$$
$$\left. + \int_{t+\epsilon}^T e^{-\rho(s-t)} u(X(s), \phi(X(s), s), s)\, ds + e^{-\bar{\rho}(T-t)} F(X(T), T)\right] \ .$$

Definition 2 *Let $V_\epsilon(x,t)$ be differentiable in ϵ in a neighbourhood of $\epsilon = 0$. Then $c^*(s) = \phi(x(s), s)$ is called an equilibrium rule if*

$$\lim_{\epsilon\to 0^+} \frac{V(x,t) - V_\epsilon(x,t)}{\epsilon} \geq 0 \ .$$

The definition above can be interpreted as follows. For ϵ sufficiently small, from the continuity of V_ϵ with respect to ϵ, the maximum of V_ϵ in the limit when $\epsilon = 0$ is $V(x,t)$.

Proposition 2 *If the value function is of class $C^{2,1}$, then the solution $c = \phi(X,t)$ to the right hand term of the DPE (13-15) is an equilibrium rule, in the sense that it satisfies Definition 2.*

Proof: See the Appendix.

Remark 4 *In Marín-Solano and Shevkoplyas (2011) a DPE characterizing time-consistent solutions was derived for the general problem of mazimizing $\int_t^T d(s,t) u(x(s), c(s), s)\, ds + d(T,t) F(x(T), T)$ in a deterministic setting, where $d(s,t)$ is an arbitrary discount function. For this problem, the following DPE for time-consistent equilibria was obtained:*

$$\frac{\partial d(T,t)}{\partial t} V(x,t) + \int_t^T \left[d(T,t) \frac{\partial d(s,t)}{\partial t} - d(s,t) \frac{\partial d(T,t)}{\partial t}\right] u(x(s), \sigma(x(s), s), s) ds -$$
$$- d(T,t) \frac{\partial V(x,t)}{\partial t} = d(T,t) \max_{\{c\}} \left[u(x,c,t) + \frac{\partial V(x,t)}{\partial x} \cdot f(x,c,t)\right] \ .$$

If we extend the proof in Marín-Solano and Shevkoplyas (2011) to the stochastic case, we have just to add the expectation operator in the integral term in the equation above, and the standard second order term $\frac{1}{2} tr(\sigma(x,c,t) \cdot \sigma'(x,c,t) \cdot \nabla_{xx} V(x,t))$ in the right hand term.

4 An investment-consumption model with heterogeneous discounting

In this section, we apply the results in the previous section in order to analyze which are the effects of introducing different discount rates for utilities obtained, in an investment-consumption problem, from consumption enjoyed along time and from bequest. We obtain the equilibrium consumption and portfolio rules for this modified version of the classical Merton's model (Merton (1971)).

The financial market consists of 2 securities. One of them is risk-free (a cash account, for instance), and the price $P_0(t)$ of one unit is assumed to evolve according to the ordinary differential equation $\frac{dP_0(t)}{P_0(t)} = \mu_0\, dt$, where $P_0(0) = p_0 > 0$ and $\mu_0 > 0$ accounts for the return on the sure asset. There is also a risky security whose price $P_1(t)$ evolves according to $\frac{dP_1(t)}{P_1(t)} = \mu_1\, dt + \sigma\, dz$, where $P_1(0) = p_1 > 0$, μ_1 is the expected percentage change in price per unit time and $z(t)$ is a standard Brownian motion process. The agent can invest a proportion $w(t)$ of her wealth at time t, $W(t)$, in the risky asset and a proportion $(1 - w(t))$ in the risk free asset. In addition the agent can allocate an amount of $c(t)$ to consumption. The consumer's wealth process evolves according to

$$dW(t) = [w(t)(\mu_1 - \mu_0)W(t) + (\mu_0 W(t) - c(t))]\, dt + w(t)\sigma W(t) dz(t) \ , \tag{18}$$

with $W(0) = W_0$. The objective of the agent at time t is to choose the consumption and investment strategies, $c(s)$, $w(s)$, $s \in [t, T]$, in order to maximize

$$E\left[\int_t^T e^{-\rho(s-t)} u(c(s)) ds + e^{-\bar{\rho}(T-t)} F(W(T))\right] \tag{19}$$

subject to (18), given $W(t) = W_t$. Both the utility function $u(\cdot)$ and the bequest function $F(\cdot)$ are assumed to be strictly concave functions on their arguments[2].

If the agent can commit herself to follow in the future the "optimal" solution obtained from the viewpoint of her preferences at time $t = 0$, she will solve the classical Hamilton-Jacobi-Bellman (HJB) equation

$$\rho V^0 - \frac{\partial V^0}{\partial s} = \max_{\{c,w\}} \left\{ u(c) + [w(\mu_1 - \mu_0)W + (\mu_0 W - c)] \frac{\partial V^0}{\partial W} + \frac{1}{2} w^2 \sigma^2 W^2 \frac{\partial^2 V^0}{\partial W^2} \right\} \ , \tag{20}$$

where $V^0(W, s)$ denotes the current value function. The "optimal" controls are the solution to

$$u'(c(s)) = \frac{\partial V^0}{\partial W} \ , \quad w(s) = -\frac{(\mu_1 - \mu_0)}{\sigma^2} \left[\frac{\frac{\partial V^0}{\partial W}}{W \frac{\partial^2 V^0}{\partial W^2}} \right] \ . \tag{21}$$

Both the HJB equation (20) and the decision rules (21) do not depend explicitly on the new discount rate $\bar{\rho}$. The difference with the standard problem with a unique discount rate appears via the final condition. Note that we can write the bequest function as $e^{-\bar{\rho}T} F(W(T), T) = e^{-\rho T} e^{-(\bar{\rho}-\rho)T} F(W(T), T)$. Hence, in the current value formulation, the terminal condition to be imposed in (20) is now

$$V^0(W, T) = e^{-(\bar{\rho}-\rho)T} F(W) \ . \tag{22}$$

If $\rho = \bar{\rho}$ we recover the classical solution, which is time consistent. Otherwise, if the agent can not precommit her future actions, she will be time-inconsistent. Note that, if $V^t(W, s)$,

[2]The extension to the problem with an arbitrary number of risky assets is straightforward.

$s \in [t, T]$, denotes the current value function at time t according to the time-preferences of the t-agent, she will look for the solution to the classical HJB equation

$$\rho V^t - \frac{\partial V^t}{\partial s} = \max_{\{c,w\}} \left\{ u(c) + [w(\mu_1 - \mu_0)W + (\mu_0 W - c)] \frac{\partial V^t}{\partial W} + \frac{1}{2} w^2 \sigma^2 W^2 \frac{\partial^2 V^t}{\partial W^2} \right\} \quad (23)$$

with the boundary condition

$$V^t(W, T) = e^{-(\bar{\rho} - \rho)(T-t)} F(W) \, . \quad (24)$$

At different initial times $t \in [0, T]$ the agent has to solve the same HJB equation (23) but she applies a different terminal condition (24). In general, if the agent does not commit her decision rule at any time t, and does not take into account that her time preferences will change in the future, she will be continuously modifying her choices. This kind of extremely time-inconsistent behavior is usually referred to as the naive behavior or the naive solution in the non-constant discounting literature. In order to obtain time consistent solutions we must solve the DPE (13-15). We will do it for the family of CRRA (power and logarithmic) and CARA (exponential) utility functions.

4.1 Power utility function

Let us study the problem for the case of power utilities, $u(c) = \frac{c^\gamma}{\gamma}$ and $F(W(T)) = \frac{W(T)^\gamma}{\gamma}$, with $\gamma < 1$, $\gamma \neq 0$.

First we briefly derive the time-inconsistent (naive) solution. The "optimal solution" according to the time preferences of the t-agent can be obtained by solving the HJB equation (23) with the boundary condition (24). It is easy to prove that, in this case, the value function is given by $V^t(W, s) = \alpha^t(s) \frac{W(s)^\gamma}{\gamma}$, where

$$\alpha^t(s) = \left[\frac{1-\gamma}{\varsigma^t} + \left(e^{\frac{1}{\gamma-1}(\bar{\rho}-\rho)(T-t)} - \frac{1-\gamma}{\varsigma^t} \right) e^{\frac{\varsigma^t}{\gamma-1}(T-s)} \right]^{1-\gamma}$$

with

$$\varsigma^t = \rho - \mu_0 \gamma + \frac{1}{2} \frac{\gamma(\mu_1 - \mu_0)^2}{\sigma^2(\gamma - 1)} \, .$$

The corresponding consumption and investment rules are $c^t(s) = (\alpha^t(s))^{\frac{1}{\gamma-1}} W$, $w^t(s) = \frac{-(\mu_1 - \mu_0)}{\sigma^2(\gamma-1)}$.

In particular, if the agent can precommit her decision rule at time $t = 0$, we obtain the precommitment solution, characterized by

$$\alpha^P(s) = \left[\frac{1-\gamma}{\varsigma} + \left(e^{\frac{1}{\gamma-1}(\bar{\rho}-\rho)T} - \frac{1-\gamma}{\varsigma} \right) e^{\frac{\varsigma}{\gamma-1}(T-s)} \right]^{1-\gamma} \, . \quad (25)$$

Otherwise, if the agent is naive, since the naive t-agent follows her decision rule just at time $s = t$, her actual consumption rule can be obtained by taking $s = t$, so

$$\alpha^N(t) = \left[\frac{1-\gamma}{\varsigma^N} + \left(e^{\frac{1}{\gamma-1}(\bar{\rho}-\rho)(T-t)} - \frac{1-\gamma}{\varsigma^N} \right) e^{\frac{\varsigma^N}{\gamma-1}(T-t)} \right]^{1-\gamma} \, .$$

In order to obtain a time-consistent solution, according to Proposition 1, Markov equilibria can be obtained by solving the DPE

$$\bar{\rho} V^S(W, t) - K(W, t) - V_t^S(W, t) =$$

$$\max_{\{c,w\}}\{u(c)+[w(\mu_1-\mu_0)W+(\mu_0 W-c)]\,V_W^S(W,t)+\frac{1}{2}w^2\sigma^2W^2V_{WW}^S(W,t)\}\ , \tag{26}$$

with $K(W,t)$ given by

$$K(W,t)=E\left[\int_t^T e^{-\rho(s-t)}(\bar\rho-\rho)u(\phi(W,s))ds\right]\ , \tag{27}$$

where $c^*=\phi(W,s)$ is the equilibrium consumption rule obtained by solving the right hand term in (26). In particular, if we apply Corollary 1, we obtain the set of two coupled partial differential equations

$$\bar\rho V^S(W,t)-K(W,t)-V_t^S(W,t)=$$

$$\max_{\{c,w\}}\left\{\frac{c^\gamma}{\gamma}+[w(\mu_1-\mu_0)W+(\mu_0W-c)]\,V_W^S(W,t)+\frac{1}{2}w^2\sigma^2W^2V_{WW}^S(W,t)\right\}\ , \tag{28}$$

$$\rho K(W,t)-K_t(W,t)=$$

$$(\bar\rho-\rho)\frac{c^{*\gamma}}{\gamma}+\left[w(\mu_1-\mu_0)W+(\mu_0W-c^+)\right]K_W(W,t)+\frac{1}{2}w^2\sigma^2W^2K_{WW}(W,t)\,. \tag{29}$$

As a candidate to the value function we guess $V^S(W,t)=\alpha^S(t)\frac{W(t)^\gamma}{\gamma}$ and $K(W,t)=A(t)\frac{W(t)^\gamma}{\gamma}$. We easily obtain $c^*=(\alpha(t)^S)^{\frac{1}{\gamma-1}}W$, $w^*=\frac{-(\mu_1-\mu_0)}{\sigma^2(\gamma-1)}$.

Then by substituting in (28-29) and collecting terms in $W(t)^\gamma$, we obtain that functions $A(t)$ and $\alpha^S(t)$ are the solution to the following system of ordinary differential equations:

$$\rho\frac{1}{\gamma}A(t)-\frac{1}{\gamma}\dot A(t)=(\bar\rho-\rho)\frac{1}{\gamma}(\alpha^S(t))^{\frac{\gamma}{\gamma-1}}-A(t)\frac{1}{2}\frac{(\mu_1-\mu_0)^2}{\sigma^2(\gamma-1)}+A(t)\mu_0-(\alpha^S(t))^{\frac{1}{\gamma-1}}A(t)\ , \tag{30}$$

$$\bar\rho\frac{1}{\gamma}\alpha^S(t)-\frac{1}{\gamma}\dot\alpha^S(t)-\frac{1}{\gamma}A(t)=\frac{1}{\gamma}(\alpha^S(t))^{\frac{\gamma}{\gamma-1}}-\alpha^S(t)\frac{1}{2}\frac{(\mu_1-\mu_0)^2}{\sigma^2(\gamma-1)}+\alpha^S(t)\mu_0-(\alpha^S(t))^{\frac{\gamma}{\gamma-1}}\ . \tag{31}$$

Table 1 summarizes the results obtained for the power utility for the different behaviors of the agent: precommitment, naive or time-consistent. The results for the particular case in the limit $\gamma=0$ (logarithmic utility) are presented in Table 2.

Consumption rule	**Portfolio rule**
$c^P(t)=\frac{W_t}{\frac{1-\gamma}{\varsigma}+\left(e^{\frac{1}{\gamma-1}(\bar\rho-\rho)T}-\frac{1-\gamma}{\varsigma}\right)e^{\frac{\varsigma}{\gamma-1}(T-s)}}$	$w^P=\frac{\mu_1-\mu_0}{(1-\gamma)\sigma^2}$
$c^N(t)=\frac{W_t}{\frac{1-\gamma}{\varsigma^N}+\left(e^{\frac{1}{\gamma-1}(\bar\rho-\rho)(T-t)}-\frac{1-\gamma}{\varsigma^N}\right)e^{\frac{\varsigma^N}{\gamma-1}(T-t)}}$	$w^N=\frac{\mu_1-\mu_0}{(1-\gamma)\sigma^2}$
$c^S(t)=\left(\alpha(t)^S\right)^{\frac{1}{1-\gamma}}W_t$, $\alpha^S(t)$ given by (30-31)	$w^S=\frac{\mu_1-\mu_0}{(1-\gamma)\sigma^2}$

Table 1: Power utility function.

It is interesting to observe that, in the case of logarithmic utility functions $u(c)=\ln(c)$ and $F(W(T))=\ln(W(T))$, the naive solution is time-consistent, since it verifies the corresponding DPE. This result is similar to that described in Marín-Solano and Navas (2010) for the case of non-constant discounting (or hyperbolic preferences).

Consumption rule	Portfolio rule
$c^P(t) = \frac{W_t}{e^{-\bar{\rho}T+\rho t}+\frac{1}{\rho}\left[1-e^{-\rho(T-t)}\right]}$	$w^P = \frac{\mu_1-\mu_0}{\sigma^2}$
$c^N(t) = \frac{W_t}{e^{-\bar{\rho}(T-t)}+\frac{1}{\rho}\left[1-e^{-\rho(T-t)}\right]}$	$w^N = \frac{\mu_1-\mu_0}{\sigma^2}$
$c^S(t) = \frac{W_t}{e^{-\bar{\rho}(T-t)}+\frac{1}{\rho}\left[1-e^{-\rho(T-t)}\right]}$	$w^S = \frac{\mu_1-\mu_0}{\sigma^2}$

Table 2: Logarithmic utility function.

Next we illustrate numerically the above results. In all the figures we consider the following values for the main parameters: $T = 30$, $\gamma = -3$, $W_0 = 1000$, $\mu_0 = 0.03$, $\mu_1 = 0.09$ and $\sigma = 0.3$.

In Figure 1 we compare the consumption rules for the precommitment, naive and time-consistent (sophisticated) solutions. The discount rates are $\rho = 0.03$ (for instantaneous utilities) and $\bar{\rho} = 0.12$ (for the bequest function). Note that, for t small, the three solutions are quite similar. However, when time t approaches to the final time $T = 30$, the three solutions become different. The naive and time-consistent solutions indicate how the time preferences evolve along time, in comparison with the precommitment solution, which does not take into account the changing preferences.

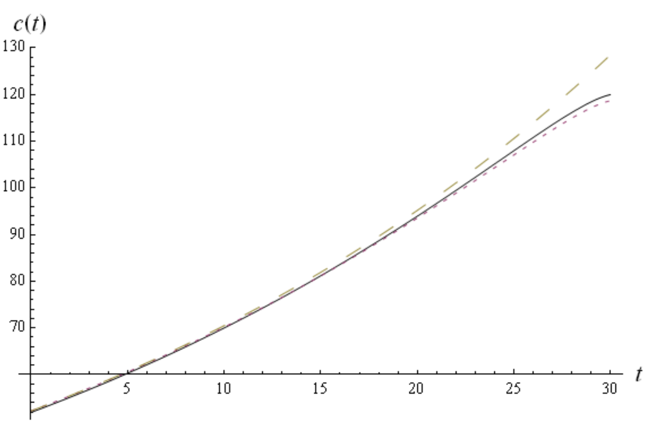

Figure 1: Precommitment (Dashed large), naive (Dashed small) and time-consistent solutions (black)

Table 3 represents the values of consumption for several values of time t. The precommitment and naive solutions coincide just at the initial time and, later on, consumption increases faster in the precommitment solution than in the other solution. Time-consistent agents consume less at the beginning and, at the middle of the time horizon, they begin to consume more than naive agents.

t	$c^P(t)$	$c^N(t)$	$c^S(t)$
0	52.2425	52.2425	51.9128
1	53.8335	53.8153	53.4908
2	55.473	55.4341	55.1163
3	57.1624	57.0998	56.7907
...	...	...	...
10	70.52	70.1727	69.9935
11	72.6676	72.2548	72.1075
12	74.8807	74.3934	74.2824
13	77.1611	76.589	76.519
14	79.511	78.8422	78.8184
15	81.9325	81.1533	81.1811
16	84.4277	83.5222	83.6076
17	86.9989	85.9484	86.0977
18	89.6484	88.431	88.6509
19	92.3786	90.9685	91.2658
20	95.192	93.5582	93.94
...	...	...	...
27	117.436	112.448	113.56
28	121.013	114.955	116.156
29	124.698	117.159	118.424
30	128.496	118.675	119.968

Table 3: Comparison of solutions.

Next, in Figure 2 we analyze the sensibility of the time-consistent solution for different values of $\bar{\rho}$. For $\rho = 0.03$ we take $\bar{\rho}_1 = 0.03$ (standard case), $\bar{\rho}_2 = 0.10$, $\bar{\rho}_3 = 0.15$ and $\bar{\rho}_4 = 0.20$.

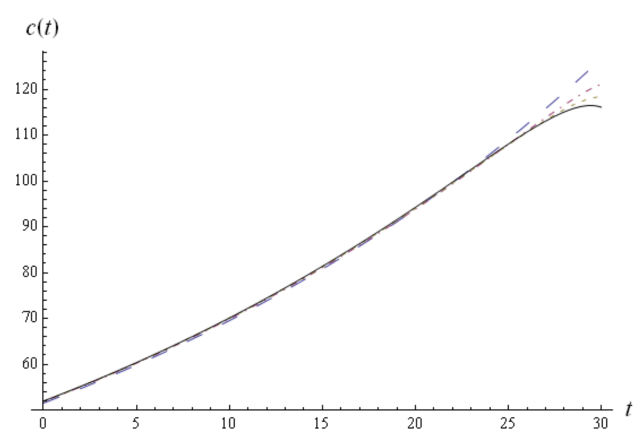

Figure 2: Standard case (Dashing large). $\bar{\rho}_2 = 0.10$ (DotDashed). $\bar{\rho}_3 = 0.15$ (Dashing small). $\bar{\rho}_4 = 0.20$ (Black).

Finally, Figure 3 illustrates the sensibility of consumption in the time-consistent solution for different values of the risk aversion γ.

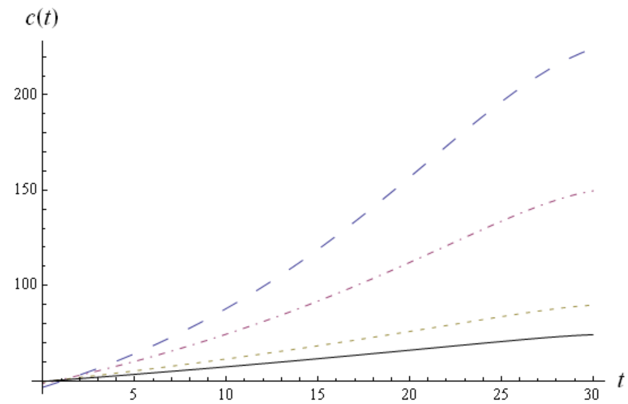

Figure 3: $\gamma = -1$ (Dashing large). $\gamma = -2$ (DotDashed). $\gamma = -5$ (Dashing small). $\gamma = -8$ (Black).

4.2 Exponential utility function

Now, let us solve the problem for the (constant absolute risk aversion) exponential utility function $u(c) = -\frac{1}{\gamma}e^{-\gamma c}$, $\gamma > 0$, with final function $F(W(T)) = -ae^{-\gamma W}$,

$$\max_{\{c,w\}} E\left[\int_t^T e^{-\rho(s-t)}\frac{-1}{\gamma}e^{-\gamma c}ds + e^{-\bar{\rho}(T-t)}(-ae^{-\gamma W(T)})\right]$$

subject to (18) with initial condition $W(t) = W_t$. Once again, we first derive the precommitment and naive solutions. A (time-inconsistent) t-agent looks for the solution to the HJB equation (23) with the utility function specified above. By guessing $V^t(W,s) = -ae^{-\gamma(\alpha^t(s)+\beta^t(s)W)}$, the consumption and portfolio rules are given by

$$c^t(s) = \alpha^t(s) + \beta^t(s)W - \frac{\ln(a\gamma\beta^t(s))}{\gamma}, \quad w^t(s) = \frac{(\mu_1 - \mu_0)}{\sigma^2\gamma\beta^t(s)W}. \tag{32}$$

We substitute (32) in (23) to obtain that $\alpha^t(s)$ and $\beta^t(s)$ must satisfy

$$\dot{\alpha}^t - \alpha^t\beta^t = \frac{\beta^t}{\gamma} - \frac{\rho}{\gamma} - \frac{1}{2}\frac{(\mu_1 - \mu_0)^2}{\sigma^2\gamma} - \frac{\beta^t}{\gamma}\ln(a\gamma\beta^t), \tag{33}$$

$$\dot{\beta}^t = (\beta^t)^2 - \mu_0\beta^t, \tag{34}$$

together with the terminal conditions $\alpha^t(T) = \frac{1}{\gamma}(\bar{\rho} - \rho)(T - t)$, $\beta^t(T) = 1$, respectively. The solution to the Bernoulli differential equation (34) is

$$\beta^t(s) = \frac{\mu_0}{1 + (\mu_0 - 1)e^{-\mu_0(T-s)}}.$$

Note that the function $\beta^t(s)$ does not depend on t. Hence, the value of $\beta(s)$ for both, the 0-agent under commitment and the naive t-agent, coincides for all $s \in [0,T]$, i.e. $\beta(s) = \beta^0(s) = \beta^N(s)$. By substituting the value of $\beta^t(s)$ in equation (33) we find that

$$\alpha^t(s) = \frac{1}{\gamma} e^{-\int_s^T \beta(\tau)d\tau} \left[(\bar{\rho} - \rho)(T-t) - \int_s^T \upsilon^e(\tau) e^{\int_\tau^T \beta(z)dz} d\tau \right],$$

where

$$\upsilon^e(\tau) = \beta(\tau) - \frac{1}{2}\frac{(\mu_1 - \mu_0)^2}{\sigma^2} - \beta(\tau)\ln(a\gamma\beta(\tau)) - \rho\,.$$

Taking $t=0$ and $s=t$ we obtain the precommitment and naive solutions, respectively,

$$\alpha^P(s) = \alpha^0(s) = \frac{1}{\gamma} e^{-\int_s^T \beta(\tau)d\tau} \left[(\bar{\rho} - \rho)T - \int_s^T \upsilon^e(\tau) e^{\int_\tau^T \beta(z)dz} d\tau \right],$$

$$\alpha^N(s) = \frac{1}{\gamma} e^{-\int_s^T \beta(\tau)d\tau} \left[(\bar{\rho} - \rho)(T-s) - \int_s^T \upsilon^e(\tau) e^{\int_\tau^T \beta(z)dz} d\tau \right].$$

Finally, let us compute the time consistent equilibrium which, according to Proposition 1, can be obtained by solving the DPE

$$\bar{\rho}V^S(W,t) - K(W,t) - V_t^S(W,t) =$$
$$\max_{\{c,w\}} \left\{ \frac{-1}{\gamma} e^{-\gamma c} + [w(\mu_1 - \mu_0)W + (\mu_0 W - c)]\, V_W^S(W,t) + \frac{1}{2} w^2 \sigma^2 W^2 V_{WW}^S(W,t) \right\},$$

with $K(W,t)$ given by

$$K(W,t) = E\left[\int_t^T e^{-\rho(s-t)} (\bar{\rho} - \rho) \frac{-1}{\gamma} e^{-\gamma c^*} ds \right].$$

Applying Collorary 1 we obtain the set of two coupled partial differential equations

$$\bar{\rho}V^S(W,t) - K(W,t) - V_t^S(W,t) =$$
$$\max_{\{c,w\}} \left\{ \frac{-1}{\gamma} e^{-\gamma c} + [w(\mu_1 - \mu_0)W + (\mu_0 W - c)]\, V_W^S(W,t) + \frac{1}{2} w^2 \sigma^2 W^2 V_{WW}^S(W,t) \right\}, \quad (35)$$
$$\rho K(W,t) - K_t(W,t) =$$
$$(\bar{\rho} - \rho)\frac{-1}{\gamma} e^{-\gamma c^*} + [w(\mu_1 - \mu_0)W + (\mu_0 W - c)]\, K_W(W,t) + \frac{1}{2} w^2 \sigma^2 W^2 K_{WW}(W,t), \quad (36)$$

where c^* is the maximizer of the right hand term in (35). As a candidate to the value function we guess $V^S(W,t) = -ae^{-\gamma(\alpha^S(t) + \beta^S(t)W)}$ and $K(W,t) = A(t)e^{-\gamma(\alpha^S(t)+\beta^S(t)W)}$. If these choices prove to be consistent the consumption and portfolio rules are

$$c^* = \alpha^S(t) + \beta^S(t)W - \frac{\ln(a\gamma\beta^S(t))}{\gamma}, \quad w^* = \frac{(\mu_1 - \mu_0)}{\sigma^2 \gamma \beta^S(t) W}. \quad (37)$$

Next, it is not difficult to check that $\beta^S(t)$ coincides with $\beta(t)$. By substituting (37) in (35-36) we obtain that functions $\alpha^S(t)$ and $A(t)$ are the solution to the following system of ordinary differential equations:

$$\rho A(t) - \dot{A}(t) + \gamma A(t)\dot{\alpha}^S(t) =$$

$$-a(\bar{\rho}-\rho)\beta^S(t) - \left[\frac{(\mu_1-\mu_0)^2}{\sigma^2\gamma\beta^S(t)} - \alpha^S(t) + \frac{\ln(a\gamma\beta^S(t))}{\gamma}\right]\gamma A(t)\beta^S(t) + \frac{1}{2}\frac{(\mu_1-\mu_0)^2}{\sigma^2}A(t)\,,$$

$$\bar{\rho}a + a\gamma\dot{\alpha}^S(t) + A(t) =$$

$$a\beta^S(t) - \left[\frac{(\mu_1-\mu_0)^2}{\sigma^2\gamma\beta^S(t)} - \alpha^S(t) + \frac{\ln(a\gamma\beta^S(t))}{\gamma}\right]\gamma a\beta^S(t) + \frac{1}{2}\frac{(\mu_1-\mu_0)^2}{\sigma^2}a\,.$$

In Figure 4 and Figure 5 we analyze the sensibility of the time-consistent consumption and portfolio rules, respectively, for different values of $\bar{\rho}$. For $\rho = 0.3$ we take $\bar{\rho}_1 = 0.03$ (standard case), $\bar{\rho}_2 = 0.10$, $\bar{\rho}_3 = 0.15$ and $\bar{\rho}_4 = 0.20$. The values for the others parameters are: $T = 30$, $\gamma = 0.1$, $W_0 = 1000$, $\mu_0 = 0.03$, $\mu_1 = 0.09$ and $\sigma = 0.3$.

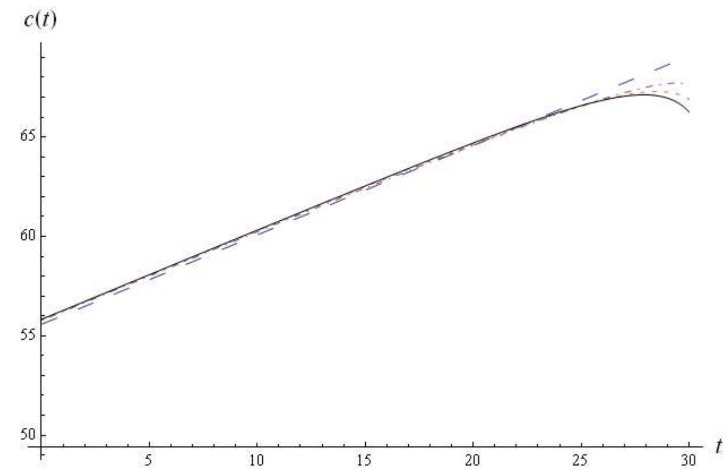

Figure 4: Standard case (Dashing large). $\bar{\rho}_2 = 0.10$ (DotDashed). $\bar{\rho}_3 = 0.15$ (Dashing small). $\bar{\rho}_4 = 0.20$ (Black).

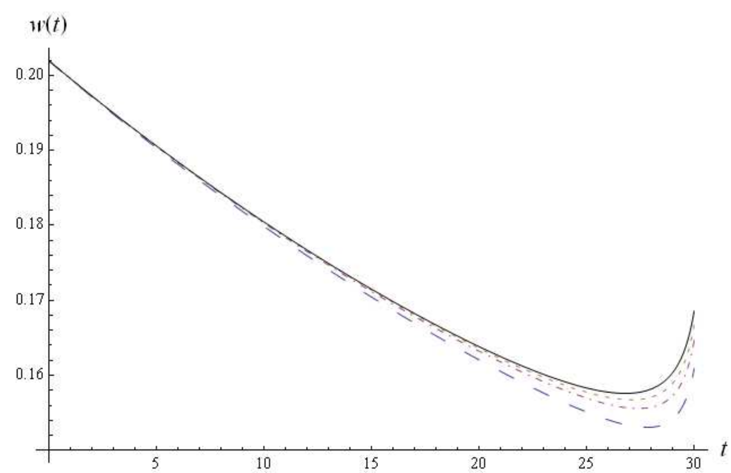

Figure 5: Standard case (Dashing large). $\bar{\rho}_2 = 0.10$ (DotDashed). $\bar{\rho}_3 = 0.15$ (Dashing small). $\bar{\rho}_4 = 0.20$ (Black).

Finally, let us briefly compare the results corresponding to the investment strategy according to the precommitment, naive and time-consistent solutions. In the case of power utilities, we have proved that the portfolio rule is always the same for these three solutions (although the consumption rule differs, as expected). For the case of exponential utilities, the investment rule is calculated according to the same formula $w^* = \frac{\mu_1 - \mu_0}{\sigma^2 \gamma \beta(t) W}$, where $\beta(t)$ coincides for all the solution concepts. However, since w^* depends on W, and W evolves in a different way for precommitment, naive and time-consistent agents, the coincidence of portfolio rules in the power utility case is lost in the case of (CARA) exponential utilities.

5 The case of stochastic terminal time

Finally, let us assume that the final time T is a random variable taking values in $[t_0, \bar{T}]$ ($\bar{T}$ can be finite or infinite) with a known (maybe subjective) distribution function $G(\tau)$ and finite expectation. For instance, in the case of uncertain lifetime presented by Yaari (1965), the distribution function $G_t(s)$ is the conditional probability that a consumer will die before time s, given that she is alive at time t, for $t < s$. Let us assume that $G(\tau)$ has density function, $G'(\tau) = g(\tau)$. The conditional distribution function satisfies $G_t(\tau) = \frac{G(\tau) - G(t)}{1 - G(t)}$ and $g_t(\tau) = \frac{dG_t(\tau)}{d\tau} = \frac{g(\tau)}{1 - G(t)}$. Under heterogeneous discounting and random duration the t-agent will look for maximizing the expected value of (4), i.e.,

$$E\left[\int_t^T e^{-\rho(s-t)} u(X(s), c(s), s) + e^{-\bar{\rho}(T-t)} F(X(T), T) \mid x_t \, , t \, ; \; T > t\right] =$$

$$E\left[\int_t^{\bar{T}} dG_t(\tau) \left[\int_t^\tau ds \, e^{-\rho(s-t)} U(X(s), c(s), s)\right] + \int_t^{\bar{T}} dG_t(\tau) e^{-\bar{\rho}(\tau-t)} F(X(\tau), \tau) \mid x_t\right] =$$

$$E\left[\int_t^{\bar{T}} \left[e^{-\rho(s-t)} (1 - G_t(s)) U(X(s), c(s), s) + e^{-\bar{\rho}(s-t)} g_t(s) F(X(s), s)\right] ds \mid x_t\right] . \tag{38}$$

For the problem of maximizing (38) subject to (5), we can easily derive the corresponding dynamic programming equation by reproducing the steps in Section 3. Let $c^*(s) = \phi(x(s), s)$ an equilibrium rule, and assume that functions $V_1(x, t)$, $V_2(x, t)$ given by

$$V_1(x, t) = E\left[\int_t^{\bar{T}} e^{-\rho(s-t)} (1 - G(s)) U(X(s), \phi(X(s), s) \, ds \mid x_t\right] ,$$

$$V_2(x, t) = E\left[\int_t^{\bar{T}} e^{-\bar{\rho}(s-t)} g(s) F(X(s), s) \, ds \mid x_t\right]$$

are of class $C^{2,1}$ in (x, t). Then the solution to the DPE

$$-\sum_{i=1}^2 \frac{\partial V_i(x, t)}{\partial t} + \rho V_1(x, t) + \bar{\rho} V_2(x, t) = \max_{\{c\}} \{(1 - G(t)) U(x, c, t) + g(t) F(x, t) +$$

$$+ \sum_{i=1}^2 \left[\nabla_x V_i(x, t) \cdot f(x, c, t) + \frac{1}{2} tr\left(\sigma(x, c, t) \cdot \sigma'(x, c, t) \cdot \nabla_{xx} V_i(x, t)\right)\right]\right\} . \tag{39}$$

is an equilibrium policy. Note that, in addition, V_1 and V_2 verify the following partial differential equations system:

$$-\frac{\partial V_1(x,t)}{\partial t} + \rho V_1(x,t) = (1-G(t))U(x,\phi(x,t),t) + \nabla_x V_1(x,t)\cdot f(x,\phi(x,t),t) + \\ +\frac{1}{2}tr\left[\sigma(x,\phi(x,t),t)\cdot\sigma'(x,\phi(x,t),t)\cdot\nabla_{xx}V_1(x,t)\right] \ , \tag{40}$$

$$-\frac{\partial V_2(x,t)}{\partial t} + \bar{\rho} V_2(x,t) = g(t)F(x,t) + \nabla_x V_2(x,t)\cdot f(x,\phi(x,t),t) + \\ +\frac{1}{2}tr\left[\sigma(x,\phi(x,t),t)\cdot\sigma'(x,\phi(x,t),t)\cdot\nabla_{xx}V_2(x,t)\right] \ . \tag{41}$$

Consider, for instance, the saving-consumption problem of maximizing (19), where T is a random variable taking values in $[0,\infty)$, subject to (18), with $\mu_1=\sigma=0$ (there is just one risk-free asset). In the log-utility case, $U(c)=\ln c$, $F(W)=a\ln W$, by maximizing the right hand term in (39) we obtain the consumption rule

$$c^*(x,t) = \frac{1-G(t)}{\nabla_W V_1(W,t)+\nabla_W V_2(W,t)} \ .$$

By substituting in (40) and (41) and by guessing $V_1(W,t)=\alpha(t)\ln W+\beta(t)$, $V_2(W,t)=\gamma(t)\ln W+\delta(t)$, we obtain that $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are the solution to the system of coupled nonlinear differential equations

$$\rho\alpha(t)-\dot{\alpha}(t) = \ln\frac{1-G(t)}{\alpha(t)+\gamma(t)} \ ,$$

$$\rho\beta(t)-\dot{\beta}(t) = (1-G(t))\ln\frac{1-G(t)}{\alpha(t)+\gamma(t)} + \alpha(t)\left(\mu_0-\frac{1-G(t)}{\alpha(t)+\gamma(t)}\right) \ ,$$

$$\bar{\rho}\gamma(t)-\dot{\gamma}(t) = ag(t) \ ,$$

$$\bar{\rho}\delta(t)-\dot{\delta}(t) = \gamma(t)\left(\mu_0-\frac{1-G(t)}{\alpha(t)+\gamma(t)}\right) \ .$$

6 Concluding remarks

In this paper, we have extended the heterogeneous discounting framework introduced in Marín-Solano and Patxot (2011) to a stochastic environment, both in a discrete and in a continuous time setting, characterizing time consistent optimal policies as the solution of a system of two coupled partial differential equations. This procedure allows us to obtain the solutions in an easier way than the equivalent but different approach there used, where time consistent policies were associated with the solution of a functional equation as in our Proposition 1.

By discounting differently the instantaneous payoffs and the final function, we can capture a changing (increasing or decreasing) relative valuation of the final function as long as the agent approaches the end of the planning horizon. In our opinion, there are several problems that seem to be good candidates for incorporating this feature as, for instance, human capital formation or consumption and portfolio rules where the final function accounts for a retirement plan or inheritance for descendants. Then, in order to illustrate the usefulness of our approach, we obtain the time consistent optimal consumption and portfolio rules for an extension of the classical Merton's model with a decision maker with heterogeneous time preferences and power and exponential utility. Finally, a discussion on the general problem with an uncertain time is briefly analyzed.

7 Appendix

DPE in discrete time: general case. Let us assume that the probabilities $P_t[V_{t+1} = v] = P_t(v|\, x_t, c_t, v_t)$ depend, not only on time t and the previous outcome v_t, but also on the state and control variables x_t and c_t. Given the policies $c_0(x_0, v_0), \ldots, c_T(x_T, v_T)$, the state X_s depends on the outcomes $V_1, \ldots, V_s$, i.e., $X_s = X_s(V_1, \ldots, V_s)$. If $p^*(v_1, \ldots, v_t)$ denotes the probability of the joint event $V_1 = v_1, \ldots, V_t = v_t$, then the expectation in (2) becomes $\sum_{s=t}^{n-1} \sum_{v_t,\ldots,v_s} \delta^{s-t} u_s(X_s, c_s(X_s, V_s), s) p^*(v_t, \ldots, v_s) + \sum_{v_t,\ldots,v_T} \bar{\delta}^{T-t} F(X_T, T) p^*(v_t, \ldots, v_T)$. Since the probabilities $p^*(v_t, \ldots, v_s)$, and hence the expected value, depend on the policies chosen, we can denote the above expectation as $E_{c_t,\ldots,c_T}$. We define the value function

$$W(x_t, t, v_t) = \sup_{\{c_t\}} E_{c_t,\ldots,c_T} \left[\sum_{s=t}^{T-1} \delta^{s-t} u_s(X_s, c_s, s) + \bar{\delta}^{T-t} F(X_T, T) \,|\, x_t, v_t \right] ,$$

where the supremum is taken over the policy $c_t = c_t(x_t, v_t)$, provided that future s-agents follow the equilibrium rule $c_s^* = \phi_s(x_s, v_s)$, for $s = t+1, \ldots, n$. In the final period T, the value function is $W(x_T, T, v_T) = F(X_T, T)$. For $s \geq \tau$, we define

$$L_s^\tau = \sum_{v_{\tau+1}} \cdots \sum_{v_s} P_\tau(v_{\tau+1}|\, x_\tau, c_\tau^*, v_\tau) \cdots P_{s-1}(v_s|\, x_{s-1}, c_{s-1}^*, v_{s-1}) u_s(X_s, \phi_s(X_s, V_s), s) \,. \tag{42}$$

For $s = T-1$ we have $W(x_{T-1}, T-1, v_{T-1}) = \sup_{\{c_{T-1}\}} \{u_{T-1}(x_{T-1}, c_{T-1}, T-1) + E_{c_{T-1}} \bar{\delta} F(X_T, T) | x_{T-1}, v_{T-1}]\}$. Let $c_{T-1}^* = \phi(x_{T-1}, v_{T-1})$ be the solution to this equation. Since $E_{c_{T-1}}[F(X_T)|x_{T-1}, v_{T-1}] = \sum_{v_T \in \mathcal{V}} P_{T-1}(v_T|x_{T-1}, c_{T-1}^*, v_{T-1}) F(X_T, T)$, then $W(x_{T-1}, T-1, v_{T-1}) = u_{T-1}(x_{T-1}, \phi(x_{T-1}, v_{T-1}), T-1) + \bar{\delta} \sum_{v_T \in \mathcal{V}} P_{T-1}(v_T|x_{T-1}, c_{T-1}^*, v_{T-1}) F(X_T, T) = u_{T-1}(x_{T-1}, \phi(x_{T-1}, v_{T-1}), T-1) + \bar{\delta} L_T^{T-1}$. In general $W(x_{t+1}, t+1, v_{t+1}) = \sum_{s=t+1}^{T-1} \delta^{s-t-1} L_s^{t+1} + \bar{\delta}^{T-t-1} L_T^{t+1}$ and

$$W(x_t, t, v_t) = \sup_{\{c_t\}} \left\{ u_t(x_t, c_t, t) + \sum_{s=t+1}^{T-1} \delta^{s-t} E_{c_t} \left[L_s^{t+1} \,|\, x_t, v_t \right] + \bar{\delta}^{T-t} E_{c_t} \left[L_n^{t+1} \,|\, x_t, v_t \right] \right\} =$$

$$= \sup_{\{c_t\}} \{ u_t(x_t, c_t, t) + \sum_{s=t+1}^{T-1} \delta^{s-t} L_s^t + \bar{\delta}^{T-t} L_T^t \} \,. \tag{43}$$

Taking the expectation of $W(x_{t+1}, t+1, V_{t+1})$ conditioned to x_t and v_t we have

$$E_{c_t} \left[W(X_{t+1}, t+1, V_{t+1}) \,|\, x_t, v_t \right] = \sum_{s=t+1}^{T-1} \delta^{s-t-1} L_s^t + \bar{\delta}^{T-t-1} L_T^t \,. \tag{44}$$

Finally, solving L_T^t in (44) and substituting in (43) we obtain the DPE (9).

□

Proof of Proposition 2. It is a rather straightforward extension of the proof in the deterministic case (see Marín-Solano and Patxot (2011)). We include a sketch of the proof.

First note that, if $\bar{x}(s)$ is the state trajectory corresponding to the decision rule $c_\epsilon(s)$, then $V(x,t) - V_\epsilon(x,t) = E \Big[\int_t^{t+\epsilon} e^{-\rho(s-t)} \left[u(X(s), \phi(X(s), s), s) - u(\bar{X}(s), v(s), s) \right] ds + \int_{t+\epsilon}^T e^{-\rho(s-t)} \left[u(X(s), \phi(X(s), s), s) - u(\bar{X}(s), \phi(\bar{X}(s), s), s) \right] ds + e^{-\bar{\rho}(T-t)} (F(X(T), T) - F(\bar{X}(T), T)) \Big]$.

Next, we can write $E\left[\int_{t+\epsilon}^{T} e^{-\rho(s-t)} u(X(s),\phi(X(s),s),s)\,ds + e^{-\rho(T-t)}F(X(T),T)\right] =$
$V(x(t+\epsilon),t+\epsilon) - E\left[\int_{t+\epsilon}^{T}\left(e^{-\rho(s-t-\epsilon)} - e^{-\rho(s-t)}\right)u(X(s),\phi(X(s),s),s)\,ds + \right.$
$\left.\left(e^{-\bar{\rho}(T-t-\epsilon)} - e^{-\bar{\rho}(T-t)}\right)F(X(T),T)\right]$

and $E\left[\int_{t+\epsilon}^{T} e^{-\rho(s-t)} u(\bar{X}(s),\phi(\bar{X}(s),s),s)\,ds + e^{-\bar{\rho}(T-t)}F(\bar{X}(T),T)\right] = V(\bar{x}(t+\epsilon),t+\epsilon) -$
$E\left[\int_{t+\epsilon}^{T}\left(e^{-\rho(s-t-\epsilon)} - e^{-\rho(s-t)}\right)u(\bar{X}(s),\phi(\bar{X}(s),s),s)\,ds + \right.$
$\left.\left(e^{-\bar{\rho}(T-t-\epsilon)} - e^{-\bar{\rho}(T-t)}\right)F(\bar{X}(T),T)\right].$

Third, note that

$$\lim_{\epsilon\to 0^+} \frac{V(x,t) - V_\epsilon(x,t)}{\epsilon} = A + B + C$$

where
$A = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon} E\left[\int_{t}^{t+\epsilon} e^{-\rho(s-t)}\left(u(X(s),\phi(X(s),s),s) - u(\bar{X}(s),v(s),s)\right)\,ds\right]$
$= u(x(t),\phi(x(t),t),t) - u(x(t),v(t),t),$

$B = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon} E\left[\int_{t+\epsilon}^{T}\left(e^{-\rho(s-t)} - e^{-\rho(s-t-\epsilon)}\right)(u(X(s),\phi(X(s),s),s)\right.$
$\left. -u(\bar{X}(s),\phi(\bar{X}(s),s),s)\right)\,ds + \left(e^{-\bar{\rho}(T-t)} - e^{-\bar{\rho}(T-t-\epsilon)}\right)\left(F(x(T),T) - F(\bar{x}(T),T)\right)\Big] = 0,$

and
$C = \lim_{\epsilon\to 0^+} \frac{V(x(t+\epsilon),t+\epsilon) - V(x(t),t)}{\epsilon} - \lim_{\epsilon\to 0^+} \frac{V(\bar{x}(t+\epsilon),t+\epsilon) - V(x(t),t)}{\epsilon}$
$= \left(\nabla_x V(x,t)\cdot f(x,\phi(x,t),t) + \frac{1}{2}tr(\sigma(x,\phi(x,t),t)\cdot\sigma'(x,\phi(x,t),t)\cdot\nabla_{xx}V(x,t))\right)$
$- \left(\nabla_x V(x,t)\cdot f(x,v(t),t) + \frac{1}{2}tr(\sigma(x,v,t)\cdot\sigma'(x,v,t)\cdot\nabla_{xx}V(x,t))\right).$

Therefore,

$$\lim_{\epsilon\to 0^+} \frac{V(x,t) - V_\epsilon(x,t)}{\epsilon} =$$

$$\left[u(x,\phi(x,t),t) + \nabla_x V(x,t)\cdot f(x,\phi(x,t),t) + \frac{1}{2}tr(\sigma(x,\phi(x,t),t)\cdot\sigma'(x,\phi(x,t),t)\cdot\nabla_{xx}V(x,t))\right] -$$

$$\left[u(x,v(t),t) + \nabla_x V(x,t)\cdot f(x,v(t),t) + \frac{1}{2}tr(\sigma(x,v,t)\cdot\sigma'(x,v,t)\cdot\nabla_{xx}V(x,t))\right] \geq 0\ ,$$

since $c^* = \phi(x,t)$ is the maximizer of the right hand term in (13).

□

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