CARF

CENTER FOR ADVANCED RESEARCH IN FINANCE

CARF Working Paper

CARF-F-258

Efficient Combinatorial Exchanges

Hitoshi Matsushima
The University of Tokyo

November 2011

CARF is presently supported by Bank of Tokyo-Mitsubishi UFJ, Ltd., Citigroup, Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Nippon Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

Efficient Combinatorial Exchanges[1]

Hitoshi Matsushima[2]

Department of Economics, University of Tokyo

First Version: November 11, 2011

This Version: November 14, 2011

Abstract

We investigate combinatorial exchanges as a generalization of combinatorial auctions and bilateral trades, where the multiple commodities to be traded are possessed by participants and a central planner as endowments. Private values, risk neutrality, and independent types are assumed. Efficiency, Bayesian Incentive Compatibility, and Interim Individual Rationality are required. We characterize the least upper bound of the central planner's expected revenue. We introduce a stability notion, namely, the marginal core, to the assumption that the central planner's endowment is unprotected. We show that the central planner has a deficit in expectation if and only if the marginal core is non-empty.

Keywords: Combinatorial Exchanges, Groves Mechanisms, Outside Opportunity, Efficient Endowment, Deficit, Marginal Core

JEL Classification Numbers: D44, D61, D82

---

[1] This study was supported by a grant-in-aid for scientific research (KAKENHI 21330043) from the Japan Society for the Promotion of Science (JSPS) and the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of the Japanese government.

[2] Department of Economics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan. E-mail: hitoshi[at]e.u-tokyo.ac.jp

1. Introduction

This paper investigates collective decision problems that have incomplete information, namely, *combinatorial exchanges*. Combinatorial exchanges are regarded to unify and generalize both cases of *bilateral trades* concerning a bargaining aspect of trading, such as those investigated by Myerson and Satterthwaite (1983),[3] and *combinatorial auctions*, which have been explored by several authors such as Rassenti, Bulfin, and Smith (1982), Kelso and Crawford (1982), and Ausubel and Milgrom (2002). In the same manner as combinatorial auctions, multiple heterogeneous commodities are traded altogether such as spectrums; these commodities are divided into multiple packages to be allocated to participants (players), according to a specified revelation mechanism with side payments, along with these participants' announcements.

Combinatorial auctions generally assume that the central planner (mediator or government) possesses all the commodities to be traded as his (or her) initial endowment. In realistic situations such as spectrum allocations, however, each participant's valuations of these commodities is crucially dependent on his valuations of those commodities that are possessed by other participants or himself as their initial endowments, which are regarded as substitutes and complements. Hence, the central planner expects to improve welfare further by exchanging their initial endowments with each other and allocating the central planner's initial endowment at the same time.

The framework of combinatorial exchanges does allow tradable commodities to be possessed not only by the central planner but also by players as their initial endowments; each participant sells his initial endowment and purchases another package of commodities at the same time.[4] However, each player has the *outside opportunity* not to participate in the collective decision problem and instead to consume his initial endowment by himself; he could thus have significant bargaining power over the central planner in this case. Consequently, in order to implement efficient allocations

---

[3] For related studies, such as double auctions, see Chatterjee and Samuelson (1983), Wilson (1985), and Matsushima (2008), for instance.

[4] For the argument about the importance of combinatorial exchanges, see Milgrom (2007). See also Chapter 1 of Milgrom (2004).

in an incentive-compatible manner, the central planner has to make considerable subsidies that fulfill their informational rents. For instance, as Myerson and Satterthwaite (1983) pointed out, in the opposing case of combinatorial exchanges such as bilateral trades, where the central planner has no initial endowment to be traded, it might be inevitable for the central planner to have a deficit in expected revenue. This contrasts with the case of combinatorial auctions, which guarantees the positivity of the central planner's expected revenue.

Based on these observations, the purpose of this paper is to clarify the degree of financial burden on the central planner when implementing efficient allocations in the context of combinatorial exchanges in a manner that is consistent not only with *Bayesian Incentive Compatibility* (BIC) but also with *Interim Individual Rationality* (IIR). IIR requires each player's interim expected payoff to be at least the same as his type-dependent outside opportunity value. In particular, the main concern of this paper is to clarify what is the necessary and sufficient condition under which the central planner has a deficit in expected revenue.

This paper permits each player's consumption to have an *externality* effect on other players' welfare. We assume quasi-linearity, risk-neutrality, private values, and an independent distribution of types. We also assume the *payoff/revenue equivalence* property in terms of a Bayesian Nash equilibrium, according to which, along with efficiency, we can focus only on *Groves mechanisms* that are consistent with IIR.

We derive the least upper bound of the central planner's expected revenue in general collective decision problems. We then show a full characterization of the case that the central planner has a deficit in expected revenue in the context of combinatorial exchanges from the viewpoint of *stability*. We introduce a new concept that is a weaker version of the core, namely the *marginal core*, which is defined as the collection of all efficient imputations that are unblocked by any coalition that consists of all players but a single player, i.e., are *marginally unblocked*.

Besides the restriction on possible blocking coalitions, there is a substantial difference from the standard definition of the core in that the imputation for the central planner is assumed to be *zero* in our definition; it was assumed in our definition that each player's initial endowment is protected by his private property right, while the central planner's initial endowment is *unprotected*. This assumption excludes an aspect of the functioning of the competition among players, making the non-emptiness of the marginal core difficult to be satisfied whenever the central planner possesses sufficient commodities.

Based on these observations, we introduce a key condition named *Efficient Endowment* (EE). This condition implies that for each player, there is a particular type with which the consumption of his initial endowment by himself is valuable to the point that the efficient allocation rule will assign it to him, irrespective of other players' types. The condition of EE makes each player's bargaining power over the central planner the *strongest*. Under EE, we show that *the marginal core is non-empty if and only if the central planner has a deficit in expected revenue*.

This characterization result unifies and generalizes the cases of combinatorial auctions and bilateral trades. In combinatorial auctions, where the central planner possesses all commodities, it is inevitable that the marginal core is empty, which automatically implies that the central planner can earn nonnegative expected revenue. By contrast, Myerson and Satterthwaite (1983) investigated bilateral trades between a single seller and a single buyer, where the seller possesses a single unit of an indivisible commodity, while the buyer and central planner have no initial endowments. They showed that no efficient mechanism satisfies BIC, IIR, or the balanced budgets across participants, implying that it is inevitable for the central planner to have a deficit in expected revenue. The model of Myerson and Satterthwaite could be regarded as an example of a special case of combinatorial exchanges in which the central planner possesses no initial endowment. Under the condition of EE, it is inevitable that the marginal core is non-empty, automatically implying that the central planner has a deficit in expected revenue in any efficient mechanism that is consistent with BIC and IIR. In this case, the central planner loses the amount of money equivalent to *the maximal net expected surplus in the entire economy*.

Based on this characterization, it is shown to be generally impossible to make stability in terms of the marginal core compatible with BIC and IIR. Whenever a player possesses a sufficient initial endowment, the exclusion of him from the collective decision problem results in a decrease in other players' welfare. By excluding this player, they consequently lose the valuable chance to win the commodities that this excluded player possessed. This makes the marginal core unlikely to be empty, but, at the same time, allows players to have significant *bargaining powers* over the central planner, making his expected revenue negative. Our characterization implies that the non-emptiness of the marginal core is equivalent to the negativity in the central planner's expected revenue.

The standard definition of the coalitional game has been intensively considered in previous studies of combinatorial auctions such as Bernheim and Whinston (1986), Ausubel and Milgrom (2002), Milgrom (2007), Day and Raghavan (2007), and Day and Milgrom (2008). Day and Raghavan (2007) and Day and Milgrom (2008) investigated so-called core-selecting mechanisms that have the advantage of stability over a Groves mechanism; a core-selecting mechanism assigns to each type profile an imputation that is included in the core associated with the standard coalitional game. These works commonly defined the stability notion as the robustness of an imputation in terms of the possibility of any coalition persuading the central planner to allow its members to consume his endowment exclusively. Consequently, the requirement of this stability might make the central planner's revenue greater than that for a Groves mechanism.

By contrast, the present paper differently defines a stability notion as the existence of an imputation that is robust to the possibility of any size $(n-1)$ coalition conspiring to *steal* the central planner's initial endowment without his allowance by removing the other player. The central planner's initial endowment is assumed to be unprotected by his property right, and he has no means of retaliating for theft, implying that his imputation can never be positive. However, in terms of possible retaliation measures, the removed player can cancel his participation by withdrawing his initial endowment from the collective decision problem, removing the opportunity of its exchange from all members of the coalition. Hence, any imputation could be regarded as being stable if any size $(n-1)$ coalition hesitates to conspire to steal the central planner's initial endowment because they are afraid of the removed player's subsequent retaliation.[5]

We further investigate a special case where a single player possesses all commodities as his initial endowment. With minor restrictions, the central planner's expected deficit is the *worst* of all possible distributions of initial endowments; the loss of the central planner's expected revenue in this single seller case, compared with in the

---

[5] We note that in this case these players cannot enjoy the positive externality effect induced by the removed player's consumption.

combinatorial auction case, could be equal to *the gross surplus induced by efficient allocations*.

We further investigate mechanisms that are not of Groves' type, and show an important result implying that *with EE, the emptiness of the marginal core is a necessary and sufficient condition under which there exists a Bayesian incentive compatible mechanism with Interim Individual Rationality that makes the central planner's ex post revenue nonnegative at all times.*

Several works in the mechanism design literature, such as Cremer and McLean (1985, 1988), Matsushima (1990a, 1990b, 2007), and Aoyagi (1999), have investigated the correlated types distribution. In particular, Matsushima (2007) showed a sufficient condition for the existence of efficient mechanisms that satisfy BIC, IIR, and the balanced budgets, implying that the central planner's expected revenue can be nonnegative. In contrast to these works, the present paper assumes the independent types distribution rather than correlated types.

Cramton, Gibbons, and Klemperer (1987) investigated the problem of dissolving partnerships as a special case of bilateral trades; players have nearly equal shares in their partnership and trade these shares with each other. They showed that the achievement of efficiency can be compatible with BIC, IIR, and the balanced budgets. Their case, however, does not satisfy EE; the efficient allocation rule always assigns the total share to the player who appreciates the value of their partnership more than does the other player, contradicting EE.

Several works, such as Jehiel and Moldovanu (1996), Jehiel, Moldovanu, and Stacchetti (1999), and Figueroa and Skreta (2009), have investigated auctions that have externality. Figueroa and Skreta (2009) investigated a single-unit auction that has externality where each player's outside opportunity depends on his type. They assumed that the central planner can make a binding commitment to make inefficient allocations as a device for threatening any player who considers not participating in this auction. In contrast to their work, the present paper does not allow any such commitment device; it is assumed that the central planner invariably implements efficient allocations for the members who actually participated, regardless of whether a particular player decided not to participate.

The organization of the remainder of this paper is as follows. Section 2 describes a basic model for general collective decision problems and demonstrates a calculation method for the least upper bound of the central planner's expected revenue. Section 3 explains combinatorial exchanges and EE, and describes a tractable characterization of the least upper bound. Section 4 describes a main theorem that under EE, the non-emptiness of the marginal core is necessary and sufficient for the central planner's deficit in expected revenue. Section 5 considers special cases such as bilateral trades, combinatorial auctions, and single seller cases. Section 6 gives several discussions about equal endowment distributions, incompatibility with stability, ex post revenue and deficit, and an issue concerning complexity and privacy. Section 7 concludes.

2. The Basic Model

Let us consider a *collective decision problem that has incomplete information* in the following manner. Let $N \equiv \{1,2,...,n\}$ denote the finite set of *players* (traders or agents), where $n \geq 2$. Each player $i \in N$ has a *type* $\omega_i \in \Omega_i$ that is unknown to either other players or the *central planner* (mediator or government), where Ω_i denotes the set of possible types for player i. Let $\Omega \equiv \prod_{i \in N} \Omega_i$ and $\Omega_{-i} \equiv \prod_{j \in N/\{i\}} \Omega_j$. The types ω_i are *independently* distributed across players according to a probability measure that have the full support of Ω. Let A denote the set of all *alternatives* that have typical element a. Each player $i's$ payoff function has a *quasi-linear* and *risk-neutral* form with *private values*, i.e., $v_i(a, \omega_i) + t_i$, where $t_i \in R$ denotes the monetary transfer from the central planner to him and $v_i : A \times \Omega_i \to A$ is his type-dependent valuation function for the alternatives.

For every $i \in N$, let $U_i^* : \Omega_i \to R$ denote player $i's$ *outside opportunity* function, where the outside opportunity for player i that has type ω_i is given by $U_i^*(\omega_i) \in R$, implying the interim expected payoff that he can receive when he does not participate in the collective decision problem. Let $U_N^* = (U_i^*)_{i \in N}$.

A *direct mechanism* is defined as (f, x), where $f : \Omega \to A$ is an *allocation rule*, $x : \Omega \to R^n$ is a *payment rule*, $x = (x_i)_{i \in N}$, and $x_i : \Omega \to R$. When each player $i \in N$ announces $\omega_i \in \Omega_i$, the central planner selects the alternative $f(\omega) \in A$ and makes the transfer payment to each player i, i.e., $x_i(\omega) \in R$, where we denote $\omega = (\omega_i)_{i \in N} \in \Omega$ and $x(\omega) = (x_i(\omega))_{i \in N} \in R^n$. We assume that the allocation rule f is *efficient* in the sense that for every $\omega \in \Omega$, the corresponding allocation $f(\omega) \in A$ maximizes the sum of players' valuations in the ex-post term, i.e.,

$$\sum_{i \in N} v_i(f(\omega), \omega_i) = \max_{a \in A} \sum_{i \in N} v_i(a, \omega_i).$$

In order to make the collective decision problem non-trivial, we assume that

(1) $$E[\sum_{i\in N} v_i(f(\omega),\omega_i)] - E[\sum_{i\in N} U_i^*(\omega_i)] > 0\,.$$[6]

This assumption implies that the efficient allocation rule f induces a *positive net expected surplus* in the ex-ante term, which is expressed by the left-hand side of (1), implying the difference between the expected aggregate value induced by the allocation rule, i.e., $E[\sum_{i\in N} v_i(f(\omega),\omega_i)]$, and the sum of players' expected outside opportunities, i.e., $E[\sum_{i\in N} U_i^*(\omega_i)]$.

BIC (Bayesian Incentive Compatibility): A direct mechanism (f,x) satisfies *BIC* if for every $i \in N$, every $\omega_i \in \Omega_i$, and every $m_i \in \Omega_i$,

$$E[v_i(f(\omega),\omega_i) + x_i(\omega) \mid \omega_i] \geq E[v_i(f(m_i,\omega_{-i}),\omega_i) + x_i(m_i,\omega_{-i}) \mid \omega_i]\,.$$

IIR (Interim Individual Rationality): A direct mechanism (f,x) and a profile of outside opportunity functions U_N^* satisfy *IIR* if for every $i \in N$ and every $\omega_i \in \Omega_i$,

$$E[v_i(f(\omega)),\omega_i) + x_i(\omega) \mid \omega_i] \geq U_i^*(\omega_i)\,.$$

BIC implies that truth-telling is a Bayesian Nash equilibrium in the collective decision problem. IIR implies that each player has the incentive to participate in the collective decision problem, irrespective of his type.

The *revenue* for the central planner is defined as the sum of the transfers from all players to the central planner, i.e., $-\sum_{i\in N} x_i(\omega)$. Given the efficient allocation rule f, the central planner's purpose is to design a payment rule x such that the associated direct mechanism (f,x) maximizes his ex-ante expected revenue $-E[\sum_{i\in N} x_i(\omega)]$ under the constraints of efficiency, BIC, and IIR. We implicitly assume that the central planner's preference has a *lexicographic order* in the sense that the achievement of

[6] $E[\cdot]$ denotes the ex-ante expectation operator in terms of $\omega \in \Omega$. $E[\cdot \mid \omega_i]$ denotes the interim expectation operator in terms of $\omega_{-i} \in \Omega_{-i}$ conditional on $\omega_i \in \Omega_i$.

efficiency is the first aim and revenue maximization is the second aim.

Let X denote the set of all payment rules. A payment rule $x \in X$ is said to be a *Groves payment rule for an efficient allocation rule* f if there exists a function $h_i : \Omega_{-i} \to R$ for each $i \in N$ such that

$$x_i(\omega) = \sum_{j \in N \setminus \{i\}} v_j(f(\omega), \omega_j) + h_i(\omega_{-i}) \text{ for all } \omega \in \Omega .$$

Let $X(f) \subset X$ denote the set of all Groves payment rules for f. In the mechanism design literature, a direct mechanism (f, x) is called a *Groves mechanism* if and only if $x \in X(f)$. It is evident that any Groves mechanism (f, x) satisfies *strategy-proofness* in the sense that for every $i \in N$ and every $\omega \in \Omega$,

$$v_i(f(\omega), \omega_i) + x_i(\omega) \geq v_i(f(m_i, \omega_{-i}), \omega_i) + x_i(m_i, \omega_{-i}) \text{ for all } m_i \in \Omega_i .$$[7]

It is evident that strategy-proofness implies BIC.

This paper implicitly assumes the *payoff equivalence* property[8] in that for every payment rule $x \in X$, if (f, x) satisfies BIC, then there exists a Groves payment rule $y \in X(f)$ that induces the same interim expected values of transfer payment, i.e., satisfies that for every $i \in N$,

$$E[x_i(\omega) | \omega_i] = E[y_i(\omega) | \omega_i] \text{ for all } \omega_i \in \Omega_i .$$

This also implies the *revenue equivalence* property in that

$$E[-\sum_{i \in N} x_i(\omega)] = E_{\omega_{-i}}[-\sum_{i \in N} y_i(\omega)].$$

Hence, we confine our attention to the subset $X(f)$.

Let us denote by $X(f, U_N^*) \subset X(f)$ the set of all Groves payment rules $x \in X(f)$ such that (f, x) and U_N^* satisfy IIR. Let us denote by $r_0 = r_0(U_N^*) \in R$ the *least upper bound* of the expected revenue for the central planner under the

[7] See Vickrey (1961), Clarke (1971), Groves (1973), Green and Laffont (1977), Holmstrom (1979), and Milgrom (2004). A Groves mechanism is sometimes called a VCG (Vickery–Clarke–Groves) mechanism.

[8] Krishna and Maenner (2001) showed mild conditions such as convexity and regular Lipschitzian that are sufficient for the payoff equivalence property in a broad class of environments that have multidimensional types. See also Krishna and Perry (2000), Milgrom and Segal (2002), and Bikhchandani et al. (2006). For works related to multidimensional types, see also Rochet and Stole (2003) and Pavan, Segal, and Toikka (2011).

constraints of $x \in X(f, U_N^*)$;

$$r_0 \equiv \max_{x \in X(f, U_N^*)} E[-\sum_{i \in N} x_i(\omega)].$$

The following theorem characterizes this least upper bound.

Theorem 1: *It holds that*

(2) $$r_0 = -(n-1)E[\sum_{i \in N} v_i(f(\omega), \omega_i)] - \sum_{i \in N} \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

Proof: Let us consider an arbitrary Groves payment rule with IIR, $x \in X(f, U_N^*)$. For every $i \in N$ and every $\omega_i \in \Omega_i$,

$$E[v_i(f(\omega)), \omega_i) + x_i(\omega) \mid \omega_i]$$

$$= E[v_i(f(\omega), \omega_j) + \sum_{j \in N \setminus \{i\}} v_j(f(\omega), \omega_j) + h_i(\omega_{-i}) \mid \omega_i]$$

$$= E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i] + E[h_i(\omega_{-i})].$$

Hence, IIR is equivalent to the inequalities given by

$$E[h_i(\omega_{-i})] \geq \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\} \text{ for all } i \in N.$$

Hence,

$$E[x_i(\omega) \mid \omega_i] \geq E[\sum_{j \in N \setminus \{i\}} v_j(f(\omega), \omega_j) \mid \omega_i]$$

$$+ \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\} \text{ for all } i \in N.$$

This implies that for every $x \in X(f, U_N^*)$,

$$E[\sum_{i \in N} x_i(\omega)] \geq (n-1)E[\sum_{i \in N} v_i(f(\omega), \omega_i)]$$

$$+ \sum_{i \in N} \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

Hence, it follows that

$$r_0 \leq -(n-1)E[\sum_{i \in N} v_i(f(\omega), \omega_i)] - \sum_{i \in N} \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

For every $i \in N$, let us specify h_i in a manner that

$$E[h_i(\omega_{-i})] = \max_{\omega_i \in \Omega_i}\{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\},$$

that is,

$$E[x_i(\omega)] = E[\sum_{j \in N \setminus \{i\}} v_j(f(\omega), \omega_j) \mid \omega_i] + \max_{\omega_i \in \Omega_i}\{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

It is clear that the specified payment rule x satisfies IIR, and

$$E[\sum_{i \in N} x_i(\omega)] = (n-1)E[\sum_{i \in N} v_i(f(\omega), \omega_i)] + \sum_{i \in N} \max_{\omega_i \in \Omega_i}\{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\},$$

which implies (2).

Q.E.D.

From Theorem 1, it follows that for every $i \in N$ and every pair of profiles of outside opportunity functions $(U_N^*, \tilde{U}_N^*)$, if

$$\tilde{U}_i^*(\omega_i) \geq U_i^*(\omega_i) \text{ for all } i \in N \text{ and all } \omega_i \in \Omega_i,$$

then it holds that $r_0(\tilde{U}_N^*) \geq r_0(U_N^*)$; the higher players' outside opportunities are, the lesser the central planner's expected revenue is.

The proof of Theorem 1 showed that whenever a Groves payment rule $x \in X(f)$ induces the least upper bound of the central planner's expected revenue, the corresponding ex-ante expected payoff for each player $i \in N$, denoted by $r_i = r_i(U_N^*)$, is given by

(3) $$r_i = E[\sum_{i \in N} v_i(f(\omega), \omega_i)] - \max_{\omega_i \in \Omega_i}\{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

3. Combinatorial Exchanges

From this section on, we focus on *combinatorial exchanges* as a special case of the collective decision problem, in which players (mobile phone companies, for instance) and the central planner (e.g., the government) possess multiple commodities (spectrums) as their initial endowments and trade these objects altogether with each other at the same time.

3.1 The Model

There exist L heterogeneous items. For each $l \in \{1,...,L\}$, the total amount of the $l-th$ item to be traded is given by a positive integer $e^l > 0$. Let $e \equiv (e^l)_{l=1}^L \in R_+^L$. We specify the set of all alternatives A as the set of all nL – dimensional vectors of nonnegative integers $a = (a_i)_{i \in N}$ satisfying that for every $l \in \{1,...,L\}$,

$$\sum_{i \in N} a_i^l \le e^l \text{, and } a_i^l \ge 0 \text{ for all } i \in N,$$

where we denote $a_i = (a_i^l)_{l=1}^L$.[9] [10] Let $a = (a_i)_{i \in N} \in A$ and $a_i = (a_i^l)_{l=1}^L$, where $a_i^l \in R$ implies the amount of the $l-th$ item that is allocated to player i. Let us denote $f(\omega) = (f_i(\omega))_{i \in N}$. For every non-empty subset $S \subset N$, we denote $a_S \equiv (a_i)_{i \in S} \in R^{|S|}$.

Let an L – dimensional vector of nonnegative integers $e_i = (e_i^l)_{l=1}^L \ge 0$ denote the *initial endowment* for player $i \in N$. Let us denote by $e_N \equiv (e_i)_{i \in N}$ the profile of their initial endowments, where we assume that

$$\sum_{i \in N} e_i^l \le e^l \text{ for all } l \in \{1,...,L\}.$$

Note that the central planner possesses $e^l - \sum_{i \in N} e_i^l \ge 0$ amount of the $l-th$ item for

---

[9] It is an irrelevant assumption that the set of alternatives is discrete. We can make the same arguments even if we replace it with a subset of multidimensional Euclidean space. For the case of divisible commodities, see Wilson (1979) and Back and Zender (1993), for instance.

[10] We can make basically the same argument even if we specify A as a non-empty proper subset of such nL – dimensional vectors.

each item $l \in \{1,...,L\}$ as his initial endowment; the initial endowments possessed by the players and central planner are traded altogether.

For every non-empty subset (coalition) $S \subset N$, let us define a subset $A(S) \subset A$ as the set of all alternatives $a \in A$ such that

$$a_i = e_i \text{ for all } i \in S,$$

implying that any player who belongs to the coalition S, $i \in S$, does not participate in the collective decision problem and consumes (or utilizes) his initial endowment e_i by himself. Let us specify a function $f^S : \Omega_{N \setminus S} \to A(S)$, which is regarded as the efficient allocation rule for the difference coalition $N \setminus S$, in a manner that for every $\omega_{N \setminus S} \in \Omega_{N \setminus S}$,

$$\sum_{i \in N \setminus S} v_i(f^S(\omega_{N \setminus S}), \omega_i) = \max_{a \in A(S)} \sum_{i \in N \setminus S} v_i(a, \omega_i),$$

where we denote $\Omega_{N \setminus S} = \underset{i \in N \setminus S}{\times} \Omega_i$ and $\omega_{N \setminus S} = (\omega_i)_{i \in N \setminus S} \in \Omega_{N \setminus S}$. According to f^S, the central planner implements efficient allocations for participants, i.e., players who belong to $N \setminus S$, provided that non-participants, i.e., players who belong to S, consume their respective initial endowments.

We permit each player's consumption to have an *externality* effect on other players' welfare; $v_i(a, \omega_i)$ depends on a_{-i}.[11] We also assume free disposal in that $v_i(a, \omega_i)$ is non-decreasing with respect to a_i.

We assume that

$$(4) \qquad U_i^*(\omega_i) = E[v_i(f^{\{i\}}(\omega_{-i}), \omega_i) \mid \omega_i] \text{ for all } i \in N \text{ and all } \omega_i \in \Omega_i.$$

Each player i has the outside opportunity not to participate in the collective decision problem and instead to consume his initial endowment e_i by himself; in this case, the central planner allocates the remaining commodities $e - e_i$ in order to maximize the sum of other players' (participants') expected payoffs and his expected revenue. Under Assumption (4), we can rewrite $r_0 = r_0(e_N)$ and $r_i = r_i(e_N)$ instead of $r_0 = r_0(U_N^*)$

[11] It is implicit in this paper to assume that the market for the players after the combinatorial exchange is well regulated so that these players' aggregate welfare is positively correlated to the total surplus including consumers' welfare. Section 7 gives further discussions.

and $r_i = r_i(U_N^*)$.

Theorem 2: *It holds that*

(5) $$r_0 \geq -(n-1)E[\sum_{i\in N} v_i(f(\omega),\omega_i)] + E[\sum_{i\in N}\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j)].$$

and for each $i \in N$,

(6) $$r_i \leq E[\sum_{i\in N} v_i(f(\omega),\omega_i)] - E[\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j)].$$

Proof: From (4) and the definition of $f^{\{i\}}$, it follows that for every $\omega_i \in \Omega_i$,

$$U_i^*(\omega_i) - E[\sum_{j\in N} v_j(f(\omega),\omega_j) \mid \omega_i]$$

$$= U_i^*(\omega_i) - E[\max_{a\in A}\sum_{j\in N} v_j(a,\omega_j) \mid \omega_i]$$

$$\leq U_i^*(\omega_i) - E[\max_{a\in A(\{i\})}\sum_{i\in N\setminus S} v_i(a,\omega_i) \mid \omega_i]$$

$$= U_i^*(\omega_i) - E[\sum_{j\in N} v_j(f^{\{i\}}(\omega_{-i}),\omega_j) \mid \omega_i]$$

$$= -E[\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j) \mid \omega_i],$$

which along with Theorem 1 and the arguments in Section 2 implies (5) and (6).

Q.E.D.

3.2. Efficient Endowment

A key condition for this paper, EE (Efficient Endowment), can be described as follows.

EE (Efficient Endowment): For every $i \in N$, there exists $\tilde{\omega}_i \in \Omega_i$ such that

$$f_i(\tilde{\omega}_i,\omega_{-i}) = e_i \text{ for all } \omega_{-i} \in \Omega_{-i}.$$

EE implies that for each player $i \in N$, there is a particular type $\tilde{\omega}_i \in \Omega_i$ with which the consumption of his initial endowment e_i by himself is valuable to the point that the efficient allocation rule f assigns it to him, irrespective of other players' types. EE excludes the case of the dissolution of partnerships investigated by Cramton, Gibbons, and Klemperer (1987); the achievement of efficiency was compatible with BIC, IIR, and the balanced budgets in their case.

Under EE, we can replace Theorem 1 and its related arguments with the following theorem, demonstrating a full characterization of the least upper bound of the central planner's expected revenue and the corresponding ex-ante expected payoff for each player.

Theorem 3: *Under EE, it holds that*

(7) $$r_0 = -(n-1)E[\sum_{i\in N} v_i(f(\omega),\omega_i)] + E[\sum_{i\in N}\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j)],$$

and for each $i \in N$,

(8) $$r_i = E[\sum_{i\in N} v_i(f(\omega),\omega_i)] - E[\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j)].$$

Proof: EE, along with the efficiency of f, implies that

$$E[\sum_{j\in N} v_j(f(\omega),\omega_j) \mid \tilde{\omega}_i] = E[\sum_{j\in N} v_j(f^{\{i\}}(\omega_{-i}),\omega_j)],$$

which along with the proof of Theorem 2 implies that

$$\max_{\omega_i\in\Omega_i}\{U_i^*(\omega_i) - E[\sum_{j\in N} v_j(f(\omega),\omega_j) \mid \omega_i]\}$$

$$= -E[\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j)].$$

This along with (2) and (3) implies (7) and (8).

Q.E.D.

The gross surplus induced by the efficient allocation in the economy without player i, i.e., the value of $\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j)$, can be regarded as player $i's$ bargaining power over the central planner; the larger his initial endowment is, the lesser the gross surplus without him is.

Without EE, it might be the case that the least upper bound r_0 is greater than the right-hand side of (5). EE, i.e., the presence of a particular type $\tilde{\omega}_i \in \Omega_i$ for each player i with which the consumption of his initial endowment e_i by himself is sufficiently valuable, makes each player's bargaining power over the central planner the *strongest*, i.e., makes the central planner's financial burden caused by the informational incompleteness on players' types the heaviest.

4. Unprotected Endowment and Marginal Core

This section demonstrates a full characterization of the case that the central planner has a deficit in expected revenue from the following viewpoint of *stability*. Let us define the *coalitional game* $\varpi : 2^N \setminus \{\phi\} \to R$ as assigning to each proper coalition $S \subset N$ the maximal expected gross surplus in the economy without all players who belong to $N \setminus S$, i.e.,

$$\varpi(S) \equiv E[\sum_{j \in S} v_j(f^{N \setminus S}(\omega_S), \omega_j)] \text{ for all } S \in 2^N \setminus \{\phi\}.$$

where we must note that

$$\varpi(N) \equiv E[\sum_{j \in N} v_j(f(\omega), \omega_j)].$$

Theorem 4: *It holds that*

$$r_0 \geq -(n-1)\varpi(N) + \sum_{i \in N} \varpi(N \setminus \{i\}),$$

and for every $i \in N$,

$$r_i \leq \varpi(N) - \varpi(N \setminus \{i\}).$$

Under EE, it holds that

$$r_0 = -(n-1)\varpi(N) + \sum_{i \in N} \varpi(N \setminus \{i\}),$$

and for every $i \in N$,

$$r_i = \varpi(N) - \varpi(N \setminus \{i\}).$$

Proof: From the definition of ϖ, Theorem 2, and Theorem 3, it is clear that this theorem is correct.

Q.E.D.

Let us call any $n-$dimensional vector $\pi = (\pi_i)_{i \in N} \in R^n$ an *imputation*, where π_i implies player $i's$ ex-ante expected payoff. It is implicitly assumed that any imputation assigns to the central planner *zero* expected revenue. We define the *marginal core* as the collection of all imputations π satisfying that

(9) $$\sum_{i \in N} \pi_i = \varpi(N),$$

and for every $i \in N$,

(10) $$\sum_{i \in S} \pi_i > \varpi(N \setminus \{i\}).$$

The marginal core implies the collection of all efficient imputations that are *marginally unblocked*, i.e., unblocked by any size $(n-1)$ coalition.

The given definition of a coalitional game is substantially different from the standard definition in related studies[12] because the imputation for the central planner is assumed to be *zero* in our definition. This assumption excludes the effect of players competing with each other over the central planner's initial endowment on the stability of allocation, threatening the non-emptiness of the marginal core.

An interpretation of the coalitional game and marginal core follows. The initial endowment of each player $i \in N$, i.e., e_i, is protected by this player's private property right, while the initial endowment of the central planner, i.e., $e - \sum_{i \in N} e_i$, is *unprotected*; any size $(n-1)$ coalition can conspire to steal the central planner's initial endowment by removing the other player. In terms of possible retaliation measures, this removed player can cancel his participation by withdrawing his initial endowment from the collective decision problem, removing the opportunity of its exchange from all members of the coalition. Hence, any imputation can be regarded as being stable if any size $(n-1)$ coalition hesitates to conspire to steal the central planner's initial endowment because they are afraid of the removed player's subsequent retaliation. The more the central planner possesses his initial endowment, the more likely the marginal core is to be empty.

It is evident that the marginal core is empty whenever the central planner possesses all commodities as his initial endowment, as shown in the next section. By contrast, the marginal core can be non-empty if the central planner's initial endowment is sufficiently small.

[12] See Bernheim and Whinston (1986), Ausubel and Milgrom (2002), and Milgrom (2007). For a definition of core-selecting mechanisms, see Day and Raghavan (2007) and Day and Milgrom (2008).

Theorem 5: *The marginal core is non-empty if and only if*

(11) $$(n-1)\varpi(N) > \sum_{i\in N}\varpi(N\setminus\{i\}) .$$

Proof: Suppose that the marginal core is non-empty. Then, there exists $\pi \in R^n$ that satisfies (9) and (10). Then,

$$\varpi(N) = \sum_{i\in N}\pi_i = \frac{1}{n-1}\sum_{i\in N}\sum_{j\in N\setminus\{i\}}\pi_j > \frac{1}{n-1}\sum_{i\in N}\varpi(N\setminus\{i\}),$$

which implies (11).

Suppose that (11) holds. Then, there exists $\tilde{\pi} = (\tilde{\pi}_i)_{i\in N} \in R^n$ satisfying that

(12) $$\sum_{j\in N\setminus\{i\}}\tilde{\pi}_j = \varpi(N\setminus\{i\}) \text{ for all } i \in N .$$

Let us specify $\pi = (\pi_i)_{i\in N} \in R^n$ by

$$\pi_i = \tilde{\pi}_i + \frac{\varpi(N) - \sum_{j\in N}\tilde{\pi}_j}{n} \text{ for all } i \in N .$$

It is evident that π satisfies (9). From (11) and (12), it follows that

$$\varpi(N) - \sum_{j\in N}\tilde{\pi}_j > 0 ,$$

and therefore,

$$\pi_i > \tilde{\pi}_i \text{ for all } i \in N ,$$

which along with (12) implies (10). Hence, the marginal core is non-empty.

Q.E.D.

The following theorem demonstrates the full characterization as the main result of this paper; *under EE, the non-emptiness of the marginal core is necessary and sufficient for the central planner to have a deficit in expected revenue.*

Theorem 6: *If* $r_0 < 0$, *then the marginal core is non-empty. Under EE, the marginal core is non-empty if and only if* $r_0 < 0$.

Proof: From Theorem 4, it is evident that $r_0 < 0$ implies (11), and that under EE,

$r_0 < 0$ is equivalent to (11). This observation along with Theorem 5 implies that this theorem is correct.

Q.E.D.

5. Special Cases

5.1. Bilateral Trades

Let us consider an example of combinatorial exchanges, namely *bilateral trades*, in which we assume that $n=2$ and that the central planner possesses no initial endowment, i.e.,

$$e_1 + e_2 = e.$$

The model of Myerson and Satterthwaite (1983) is a special case. They additionally assumed a single object with a single unit; EE automatically holds when the type spaces are the same between the seller and buyer.

In this bilateral trades case,

$$\varpi(N \setminus \{i\}) = E[U_i^*(\omega_i)] \quad \text{for each} \quad i \in \{1,2\},$$

which along with Theorem 4 implies that under EE,

$$r_0 = -E[\sum_{i\in\{1,2\}} v_i(f(\omega),\omega_i)] + E[\sum_{i\in\{1,2\}} U_i^*(\omega_i)].$$

Because of (1), under EE, it is inevitable that the central planner has a deficit in expected revenue; the central planner loses the amount of money equivalent to the *maximal net expected surplus in the entire economy*.

5.2. Combinatorial Auctions

Let us specify a profile of initial endowments $e_N = \tilde{e}_N = (\tilde{e}_i)_{i\in N}$ by

$$\tilde{e}_i = 0 \quad \text{for all} \quad i \in N,$$

which corresponds to *combinatorial auctions* in which the central planner possesses all commodities as his initial endowment. This subsection makes an assumption that restricts the positivity of the externality effect in such a weak manner that for every $i \in N$,

$$\sum_{j\in N\setminus\{i\}} v_j(f(\omega),\omega_j) < \sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j), \text{ where } e_i = 0.$$

This assumption implies that players prefer to exclude a single player and consume all commodities by themselves. We show that the central planner can earn a *positive* expected revenue as follows. Since

$$A(S) = A \quad \text{for all} \quad S \subset N,$$

it follows that

$$\varpi(N \setminus \{i\}) = \max_{\substack{a \in A: \\ a_i = 0}} \sum_{j \in N \setminus \{i\}} v_i(a, \omega_j) \quad \text{for all} \quad i \in N,$$

which implies that under EE,

$$r_0 = r_0(\tilde{e}_N) = -\sum_{i \in N} E[\sum_{j \in N \setminus \{i\}} v_j(f(\omega), \omega_j) - \max_{\substack{a \in A: \\ a_i = 0}} \sum_{j \in N \setminus \{i\}} v_i(a, \omega_j)]. \tag{13}$$

The assumption of this subsection implies that the right-hand side of (13) is positive.

5.3. Single Seller

Let us specify another profile of initial endowments $e_N = \hat{e}_N = (\hat{e}_i)_{i \in N}$ by

$$\hat{e}_1 = e, \text{ and } \hat{e}_i = 0 \quad \text{for all} \quad i \in N \setminus \{1\},$$

which implies that player 1 possesses the entire commodities as his initial endowment. This subsection makes an assumption that restricts the externality effect in such a weak manner that for every $i \in N$, every $\omega_i \in \Omega_i$, and every $a \in A$, player i's valuation of the null package equals zero at all times, i.e.,

$$v_i(a, \omega_i) = 0 \quad \text{if} \quad a_i = 0.$$

We show that under EE, it is inevitable that the central planner has a deficit in expected revenue as follows. Since

$$A(N \setminus \{1\}) = \phi, \text{ and } A(N \setminus \{i\}) = A \quad \text{for all} \quad i \in N \setminus \{1\},$$

it follows from the assumption of this subsection that

$$v_i(f^{\{1\}}(\omega_{-1}), \omega_i) = 0 \quad \text{for all} \quad i \in N \setminus \{1\}.$$

Hence,

$$\varpi(N \setminus \{1\}) = 0,$$

and for every $i \in N \setminus \{1\}$,

$$\varpi(N \setminus \{i\}) = \max_{\substack{a \in A: \\ a_i = 0}} \sum_{j \in N \setminus \{i\}} v_i(a, \omega_j),$$

implying that under EE,

(14) $$r_0 = r_0(\hat{e}_N) = -\sum_{i\in N\setminus\{1\}} E[\sum_{j\in N} v_j(f(\omega),\omega_j) - \max_{\substack{a\in A:\\ a_i=0}} \sum_{j\in N\setminus\{i\}} v_i(a,\omega_j)],$$

$$= -(n-1)\varpi(N) + \sum_{i\in N\setminus\{1\}} \varpi(N\setminus\{i\}),$$

which is negative because f is efficient. The interim expected payoff for player 1 with each type ω_1 can be given by

(15) $$E[v_1(f(\omega),\omega_1) + x_1(\omega) \mid \omega_1] = E[\sum_{j\in N} v_j(f(\omega),\omega_j) \mid \omega_1],$$

and the interim expected payoff for each player $i \in N\setminus\{1\}$ with each type ω_i can be given by

(16) $$E[v_i(f(\omega),\omega_i) + x_i(\omega) \mid \omega_i]$$

$$= E[\sum_{j\in N} v_j(f(\omega),\omega_j) - \max_{\substack{a\in A:\\ a_i=0}} \sum_{j\in N\setminus\{i\}} v_i(a,\omega_j) \mid \omega_i].$$

From (15), *the interim expected payoff for player 1 is equivalent to the maximal gross expected surplus in the entire economy*. Hence, player 1 prefers to invite potential buyers to the collective decision problem as many times as is possible. From (14) and (16), it follows that *the least upper bound of the central planner's expected revenue is equivalent to the sum of the expected payoffs for the players other than player 1*. The central planner's expected revenue does not necessarily increase as the number of players who participate in the collective decision problem increases. The central planner might not think positively about inviting new traders to the collective decision problem.

With the assumptions made in this and previous subsections, it follows from (13) and (14) that under EE,

$$r(\tilde{e}_N) - r(\hat{e}_N) = E[\max_{a\in A} \sum_{j\in N\setminus\{1\}} v_j(a,\omega_j)],$$

implying that by giving all commodities to player 1 gratis, the central planner must suffer a decrease of $E[\max_{a\in A} \sum_{j\in N\setminus\{1\}} v_j(a,\omega_j)]$ in expected revenue; the central planner loses the amount of money equivalent to the *maximal gross surplus in the combinatorial auction that does not have player 1.*

6. Discussions

6.1. Equal Endowment Distribution

This subsection assumes that the sum of the maximal *gross* surpluses in economies without any single players, given by

$$W(e_N) \equiv \sum_{i \in N} \varpi(N \setminus \{i\}) = \sum_{i \in N} E[\sum_{j \in N \setminus \{i\}} v_j(f^{\{i\}}(\omega_{-i}), \omega_j)],$$

is *convex* with respect to e_N. This assumption can be implied by the concavity of $v_i(a, \omega_i)$ with respect to a_i, provided that there is no externality effect.

It is evident from this assumption that

$$\frac{r(e_N) + r(e'_N)}{2} \leq r(\frac{e_N + e'_N}{2}),$$

indicating that in symmetric models of combinatorial exchanges, *the central planner's deficit can be suppressed when players' initial endowments are equally distributed compared with the case that the distribution of the initial endowment is divided between few people such as the single-seller case*. The central planner's expected deficit in the single seller case might be the *worst* of all possible distributions of initial endowments.

6.2. Incompatibility with Stability

We can show that irrespective of the profile of initial endowments, the central planner cannot earn nonnegative revenue in expectation in a compatible manner with stability.

Theorem 7: *Under EE, there exists no payment rule* $x \in X(f, U_N^*)$ *such that*

(17) $$E[-\sum_{i \in N} x_i(\omega)] \geq 0,$$

and

(18) $$E[\sum_{j \in N \setminus \{i\}} \{v_j(f(\omega), \omega_j) + x_j(\omega)\}] > \varpi(N \setminus \{i\}) \quad \textit{for all } i \in N.$$

Proof: From Theorem 6, it follows that under EE, whenever the marginal core is non-empty, then there is no $x \in X(f, U_N^*)$ that satisfies (17). This implies that if a payment rule $x \in X(f, U_N^*)$ satisfies (17), then it never satisfies (18), implying that Theorem 7 is correct.

Q.E.D.

Theorem 7 implies the *general impossibility* in that under EE, irrespective of the profile of initial endowments, *no marginally unblocked imputation is induced by any Groves mechanism that satisfies IIR and the nonnegativity of the central planner's expected revenue.* This supports the statement that participants cannot generally accomplish efficiency in voluntary manners.

6.3. Ex-Ante Reallocation

This subsection describes an aspect of the relationship between an arbitrary pair of profiles of initial endowments, e_N and e'_N. Let us denote by $U_i^*(\omega_i)$ and $U_i'^*(\omega_i)$ the outside opportunities for player i with type ω_i associated with his initial endowments e_N and e'_N, respectively.

Suppose that a payment scheme $x \in X(f, U_N^*)$ induces the least upper bound $r_0(e_N)$. For every $i \in N$, let us specify a real number $d_i = d_i(e_N, e'_N) \in R$ by

$$d_i \equiv \max_{\omega_i \in \Omega_i}\{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}$$

$$-\max_{\omega_i \in \Omega_i}\{U_i'^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

By using this specified vector $(d_i)_{i \in N} \in R^n$, let us specify another payment rule $x' \in X$ in a manner that for every $i \in N$ and every $\omega_i \in \Omega_i$,

$$x'_i(\omega) \equiv x_i(\omega) - d_i.$$

It is evident that x' belongs to $X(f, U_N'^*)$ and induces the least upper bound $r(e'_N)$.

We can interpret the above observations as follows. Suppose that each player $i \in N$ possesses e_i as his initial endowment. At the pre-play stage, the central planner collects all commodities possessed by players and then reallocates these collected commodities as well as the commodities that the central planner possesses to each player i by giving e_i' and the fixed amount of money d_i. After reallocating in this manner, the central planner enforces the Groves mechanism (f, x'). *The Groves mechanism* (f, x) *associated with the profiles of players' initial endowments* e_N *is the payoff/revenue equivalent to the Groves mechanism* (f, x') *that follows the replacement of* e_i *with* e_i' *accompanied with a type-independent payment* d_i *to each player* $i \in N$.

6.4. Ex Post Revenue and Deficit

We must note that even if the central planner's expected revenue is nonnegative, it might be the case that there exists a type profile at which the central planner has a deficit in the ex post term. We, however, can easily suppress this trouble by allowing the central planner to make an option contract with a risk-neutral third party in a manner that whenever players announce any type profile $\omega' \in \Omega$, then the central planner gives this third party an amount of money given by

$$E[\sum_{i \in N} x_i(\omega)] - \sum_{i \in N} x_i(\omega').$$

According to this contract, the central planner's revenue is kept constant across possible type profiles, i.e., equal to $E[-\sum_{i \in N} x_i(\omega)]$ at all times. Hence, with the availability of option contracting, Theorem 6 also implies a characterization of the case that the central planner's ex-post revenue is nonnegative at all times. The central planner can earn nonnegative revenue in the ex post term at all times if the marginal core is empty. Under EE, the central planner can earn nonnegative revenue in the ex post term at all times if and only if the marginal core is empty.

It would be more important to note that with the assumption of payoff/revenue equivalence property, we can construct a non-Groves-type mechanism that is Bayesian incentive compatible, is interim individually rational, and makes the central planner's revenue nonnegative at all times if and only if the least upper bound of the central planner's expected revenue is nonnegative.

Theorem 8: *There exists a payment rule* $x \in X$ *such that* (f, x) *satisfies BIC,* (f, x) *and* U_N^* *satisfy IIR, and*

$$-\sum_{i \in N} x_i(\omega) \geq 0 \quad \text{for all} \quad \omega \in \Omega .$$

if and only if $r_0 \geq 0$.

Proof[13]**:** It is evident from the revenue/payoff equivalence property that the "only if" part is correct. All we have to do is to prove the "if" part. Suppose that $r_0 \geq 0$. Hence, there exists a Grove payment rule with IIR, $x \in X(f, U_N^*)$, such that

$$r_0 = E[-\sum_{i \in N} x_i(\omega)] .$$

From Arrow (1979) and d'Aspremont and Gérard-Varet (1979), it is evident that there exists a payment rule $x' \in X$ such that (f, x') satisfies BIC and the balanced budgets, i.e.,

$$\sum_{i \in N} x_i'(\omega) = 0 \quad \text{for all} \quad \omega \in \Omega .$$

From the payoff/revenue equivalence property, it is evident that there exists a n-dimensional vector $(b_i)_{i \in N} \in R^n$ such that

$$E[x_i(\omega) \mid \omega_i] = E[x_i'(\omega) \mid \omega_i] + b_i \quad \text{for all} \quad i \in N \quad \text{and all} \quad \omega_i \in \Omega_i .$$

Note that

$$\sum_{i \in N} b_i = r_0 .$$

We specify another payment rule $x'' \in X$ by

$$x_i''(\omega) = x_i'(\omega) + b_i \quad \text{for all} \quad i \in N \quad \text{and all} \quad \omega \in \Omega .$$

It is evident from this specification that (f, x'') satisfies BIC, (f, x'') and U_N^*

[13] The proof of Theorem 8 is closely related to Krishna and Perry (1998). See also Chapter 5 of Krishna (2010).

satisfy IIR, and

$$-\sum_{i\in N} x_i''(\omega) = \sum_{i\in N} b_i = r_0 \geq 0 \quad \text{for all} \quad \omega \in \Omega .$$

Q.E.D.

Even without the availability of option contracting, it follows from Theorems 6 and 8 that the central planner can earn nonnegative revenue in the ex post term at all times if the marginal core is empty. Under EE, the central planner can earn nonnegative revenue in the ex post term at all times if and only if the marginal core is empty.

6.5. Complexity and Privacy

The present paper has investigated direct mechanisms in which each player reveals full information about his entire valuations. However, direct mechanisms have been criticized from the practical viewpoints concerning *complexity and privacy.*[14] In combinatorial auctions that have *no* externality, several authors have attempted to replace the standard practice of such direct revelations with less complicated and more privacy-preserved dynamical protocols such as simultaneous ascending/descending clock (Japanese) auctions.[15] In such auctions, the auctioneer continues to ask and adjust non-anonymous and non-linear price vectors to each player (buyer), and each player continues to make his demand correspondences as a price taker. Such protocols must collect sufficient information in order to achieve the allocations of the original direct mechanism while preserving players' privacy.

In combinatorial auctions that have no externality, Lahaie and Parkes (2004), Parkes (2006), and Mishra and Parkes (2007) have introduced the concept of a *universal competitive equilibrium*, which implies the competitive equilibrium properties not only in the entire economy but also in economies without a single buyer. These studies showed that a *pivot* mechanism, which is defined as a special version of a Groves

---

[14] See Rothkopf, Teisberg, and Kahn (1990), Segal (2006), Ausubel and Milgrom (2006), and Parkes (2006), for instance.

[15] See Kelso and Crawford (1982), Bikhchandani and Mamer (1997), Gul and Stacchetti (1999, 2000), Parkes and Ungar (2002), Ausubel and Milgrom (2002), Ausubel and Cramton (2004), Lahaie and Parkes (2004), Ausubel (2004, 2006), Hatfield and Milgrom (2005), Ausubel, Cramton, and Milgrom (2006), Parkes (2006), Mishra and Parkes (2007), and Matsushima (2011), for instance.

mechanism, can be implemented by an arbitrary dynamical protocol that always collects sufficient information in order to discover a universal competitive equilibrium. Subsequently, Matsushima (2011) showed a tractable method for explaining whether an arbitrary dynamical protocol can implement a pivot mechanism and clarifying its degree of privacy preservation.

This subsection briefly shows that the above arguments can be extended to combinatorial exchanges that have no externality, where $v_i(a,\omega_i)$ is assumed to be independent of a_{-i}; thus, we write $v_i(a_i,\omega_i)$ instead of $v_i(a,\omega_i)$. Let us specify a Groves payment rule $x=\bar{x}\in X(f)$ by

$$h_i(\omega_{-i})=-E[\sum_{i\in N}\max_{a\in A(\{i\})}\sum_{j\in N\setminus\{i\}}v_j(a_j,\omega_j)] \text{ for all } i\in N \text{ and } \omega_{-i}\in\Omega_{-i}.$$

The corresponding direct mechanism (f,x), which can be called a *pivot mechanism*, satisfies strategy-proofness as well as *ex-post individual rationality* in the sense that for every $i\in N$ and every $\omega\in\Omega$,

$$v_i(f_i(\omega),\omega_i)+x_i(\omega)\geq U_i^*(\omega_i).$$

This inequality holds with equality whenever $f_i(\omega)=e_i$. Let us denote $p_i:A\to R_+$ and $p=(p_i)_{i\in N}$, the latter of which is called a *price vector*. A price vector p is said to be a *universal competitive equilibrium* for $\omega\in\Omega$ if there exist $a^*\in A$, and $a^{*j}\in A(\{j\})$ for each $j\in N$, such that for every $a\in A$,

$$\sum_{i\in N}p_i(a_i^*)\geq\sum_{i\in N}p_i(a_i),$$

$$v_i(a_i^*,\omega)-p_i(a_i^*)\geq v_i(a_i,\omega)-p_i(a_i) \text{ for all } i\in N,$$

for every $j\in N$ and every $a\in A(\{j\})$,

$$\sum_{i\in N\setminus\{j\}}p_i(a_i^{*j})\geq\sum_{i\in N\setminus\{j\}}p_i(a_i),$$

and

$$v_i(a_i^{*j},\omega)-p_i(a_i^{*j})\geq v_i(a_i,\omega)-p_i(a_i) \text{ for all } i\in N\setminus\{j\}.$$

Note that

$$\sum_{i\in N}v_i(a_i^*,\omega_i)\geq\sum_{i\in N}v_i(a_i,\omega_i) \text{ for all } a\in A,$$

and for every $j \in N$,

$$\sum_{i \in N \setminus \{j\}} v_i(a_i^{*j}, \omega_i) \geq \sum_{i \in N \setminus \{j\}} v_i(a_i, \omega_i) \text{ for all } a \in A(\{i\}),$$

implying that $a^* \in A$ is an efficient allocation in the entire economy and $a^{*j} \in A$ is an efficient allocation in the economy that does not have player j. In the same manner as described by Lahaie and Parkes (2004), Parkes (2006), and Mishra and Parkes (2007), it can thus be shown that without externality, the pivot mechanism can be implemented using an arbitrary dynamical protocol if and only if this protocol always discovers a universal competitive equilibrium.

We can also extend the argument of Matsushima (2011) to combinatorial exchanges. Matsushima (2011) introduced the concept of the *representative valuation function* for each player, which assigns the minimal relative valuation to each package that has been revealed during the history of play. This representative valuation function can easily be calculated from the history of play by making minor assumptions such as revealed preference activity rules and connectedness and by describing the degree of players' privacy preservation. In the same manner as shown in Matsushima (2011), in combinatorial exchanges that do not have externality, the pivot mechanism can be implemented by an arbitrary dynamical protocol if and only if (i) there always exist efficient allocations in the entire economy and in economies that do not have single players associated with the calculated representative valuation function profile and (ii) the packages that compose these allocations have all been revealed during the history of play. We can also show that whenever the dynamical protocol implements the pivot mechanism, the resulting representative valuation function profile can be the universal competitive equilibrium for their true types.

7. Conclusion and Future Researches

The present paper investigated combinatorial exchanges that have incomplete information, in which, in contrast to standard combinatorial auctions, the multiple heterogeneous commodities to be traded are possessed not only by the central planner but also by players as their initial endowments. Each player thus has the outside opportunity not to participate in the collective decision problem and instead to consume his initial endowment by himself. According to the payoff/revenue equivalences assumption, we focused on Groves mechanisms that are compatible with IIR.

Compared with standard combinatorial auctions, players in combinatorial exchanges that possess non-negligible initial endowments can have significant bargaining powers over the central planner, making it difficult for the central planner to earn nonnegative expected revenue. We introduced the key condition of EE, which implies that for each player, there is a particular type with which the consumption of his initial endowment by himself is valuable to the point that the efficient allocation rule will assign it to him, irrespective of other players' types. Under EE, each player's bargaining power over the central planner is at its strongest. According to the standard calculation for Groves mechanisms, we characterized the least upper bound of the central planner's expected revenue. Subsequently, from the viewpoint of stability, we showed a full characterization of the case that the central planner had a deficit in expected revenue. The marginal core, which was defined as the collection of all efficient imputations across players that are marginally unblocked by any size $(n-1)$ coalition, is empty if and only if the central planner can earn nonnegative expected revenue.

Based on this characterization, it was shown to be generally impossible to make stability in terms of the marginal core compatible with BIC and IIR. Whenever a player possesses a sufficient initial endowment, the exclusion of this player from the collective decision problem results in a decrease in other players' welfare; by excluding this player, they consequently lose the valuable chance to win the commodities that this excluded player possessed. This makes the marginal core unlikely to be empty, but, at the same time, allows players to have significant bargaining powers over the central planner, making his expected revenue negative. Our characterization implies that the non-emptiness of the marginal core is equivalent to the negativity in the central planner's expected revenue.

This paper assumed that the central planner's preference followed a lexicographic order in that the achievement of efficiency is the first aim and revenue enhancement is the second aim. Future research might wish to eliminate this assumption and instead investigate the possibility that the central planner can increase his expected revenue further by employing inefficient mechanisms. This research avenue is needed in order to provide an insight into designing protocols for combinatorial exchanges that are optimal in terms of the central planner's expected revenue (see also Myerson (1981) and Riley and Samuelson (1981)). In realistic situations, the central planner should be constrained by the fact that all commodities are sold out to third parties. It might be also practically important to consider the manner in which the central planner collects the proportion of the initial endowments possessed by players in the pre-play stage in order to reduce the deficit at the expense of efficiency.

As footnote 10 pointed out, this paper has implicitly assumed that the market for players after the combinatorial exchange is well regulated so that these players' aggregate welfare is positively correlated with total surplus including consumer welfare. Without this assumption, we would need to be more cautious about the central planner's objective function concerning both the central planner's revenue and total surplus (i.e., consumer surplus as well as player welfare). This is another reason why future research should investigate inefficient allocation rules.

This paper has investigated only static models of combinatorial exchanges. Thus, a highly promising future research avenue would be to extend our model to dynamical contexts where players receive private information over time, where the population of participants could change over time, and where commodities could be resold. For related works, see Parkes and Singh (2003), Bergemann and Välimäki (2010), and Bergemann and Said (2010), for instance.

Finally, it must be noted that this paper assumed that players are fully rational and require direct mechanisms to satisfy BIC so that truth-telling is exactly the best response for any player. According to the seminal works of Parkes, Kalagnanam, and Eso (2002), Day and Milgrom (2008), and Erdil and Klemperer (2010), future research might aim to weaken this rationality assumption and instead investigate the case that the central planner attempts to design efficient mechanisms that do not satisfy BIC but keep any player’s gain from lying to a sufficiently small degree.

References

Aoyagi, M. (1999): "Correlated Types and Bayesian Incentive Compatible Mechanisms with Budget Balance," *Journal of Economic Theory* 79, 142–151.

Arrow, K. (1979): The Property Rights Doctrine and Demand Revelation under Incomplete Information," in *Economics and Human Welfare*, ed. by M. Boskin. New York, Academic Press.

Ausubel, L. (2004): "An Efficient Ascending-Bid Auction for Multiple Objects," *American Economic Review* 94, 1452–1475.

Ausubel, L. (2006): "An Efficient Dynamic Auction for Heterogeneous Commodities," *American Economic Review* 96, 602–629.

Ausubel, L. and P. Cramton (2004): "Auctioning Many Divisible Goods," *Journal of European Economic Association* 2, 480–493.

Ausubel, L., P. Cramton, and P. Milgrom (2006): "The Clock-Proxy Auction: A Practical Combinatorial Auction Design," in *Combinatorial Auctions*, ed. by P. Cramton, Y. Shoham, and R. Steinberg. MIT Press.

Ausubel, L. and P. Milgrom (2002): "Ascending Auctions with Package Bidding," *Frontier of Theoretical Economics* 1, 1–42.

Ausubel, L. and P. Milgrom (2006): "The Lovely but Lonely Vickrey Auction," in *Combinatorial Auctions*, ed. by P. Cramton, Y. Shoham, and R. Steinberg. MIT Press.

Back, K. and J. Zender (1993): "Auction of Divisible Goods: On the Rationale for the Treasury Experiments," *Review of Financial Studies* 6, 733–764.

Bergemann, D. and M. Said (2010): "Dynamic Auctions: Survey," mimeo, Yale University.

Bergemann, D. and J. Välimäki (2010): "The Dynamic Pivot Mechanism," *Econometrica* 78, 771–789.

Bernheim, D. and M. Whinston (1986): "Menu Auctions, Resource Allocation and Economic Influence," *Quarterly Journal of Economics* 101, 1–31.

Bikhchandani, S., and J. Mamer (1997): "Competitive Equilibrium in an Exchange Economy with Indivisibilities," *Journal of Economic Theory* 74, 385–413.

Bikhchandani, S., S. Chatterji, R. Lavi, A. Mu'alem, N. Nisan, and A. Sen (2006): "Weak Monotonicity Characterizes Deterministic Dominant-Strategy Implementation," *Econometrica* 74, 1109–1132.

Chatterjee, K. and W. Samuelson (1983): "Bargaining under Incomplete Information," *Operations Research* 31, 835–851.

Clarke, E. (1971): "Multipart Pricing of Public Goods," *Public Choice* 11, 17–33.

Cramton, P., R. Gibbons, and P. Klemperer (1987): "Dissolving a Partnership Efficiently," *Econometrica* 55, 615–632.

Cremer, J. and R. McLean (1985): "Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist When Demands Are Interdependent," *Econometrica* 53, 345–361.

Cremer, J. and R. McLean (1988): "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica* 56, 1247–1257.

D'Aspremont, C. and L. Gérard-Varet (1979): "Incentive and Incomplete Information," *Journal of Public Economics* 11, 25-45.

Day, R. and P. Milgrom (2008): "Core-selecting Package Auctions," *International Journal of Game Theory* 36, 393–407.

Day, R. and S. Raghavan (2007): "Fair Payments for Efficient Allocations in Public Sector Combinatorial Auctions," *Management Science* 53, 1389–1406.

Erdil, A. and P. Klemperer (2010): "A New Payment Rule for Core-Selecting Package Auctions," *Journal of European Economic Association* 8, 537–547.

Figueroa, N. and V. Skreta (2009): The Role of Outside Options in Auction Design, *Journal of Economic Theory* 144, 884–897.

Green, J. and J.-J. Laffont (1977): "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods," *Econometrica* 45, 427–438.

Groves, T. (1973): "Incentives in Teams," *Econometrica* 61, 617–631.

Gul, F. and E. Stacchetti (1999): "Walrasian Equilibrium with Gross Substitutes," *Journal of Economic Theory* 87, 95–124.

Gul, F. and E. Stacchetti (2000): "The English Auction with Differentiated Commodities," *Journal of Economic Theory* 92, 66–95.

Hatfield, J. and P. Milgrom (2005): "Matching with Contracts," *American Economic Review* 95, 913–935.

Holmstrom, B. (1979): "Groves Schemes on Restricted Domains," *Econometrica* 47, 1137–1144.

Jehiel, P. and B. Moldovanu (1996): "Strategic Nonparticipations," *RAND Journal of Economics* 27, 84–98.

Jehiel, P., B. Moldovanu, and E. Stacchetti (1999): "Multidimensional Mechanism Design for Auctions with Externality," *Journal of Economic Theory* 85, 258–293.

Kelso, A. and V. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica* 50, 1483.

Krishna, V. (2010): *Auction Theory*, Second Edition, New York, Academic Press.

Krishna, V. and E. Maenner (2001): "Convex Potentials with an Application to Mechanism Design," *Econometrica* 69, 1113–1119.

Krishna, V. and M. Perry (2000): "Efficient Mechanism Design," mimeo, Penn. State University.

Lahaie, S. and D. Parkes (2004): "Applying Learning Algorithms to Preference Elicitation in the Generalized Vickrey Auction," mimeo, Harvard University.

Matsushima, H. (1990a): "Incentive Compatible Mechanisms with Full Transferability," *Journal of Economic Theory* 54, 198–203.

Matsushima, H. (1990b): "Dominant Strategy Mechanisms with Mutually Payoff-Relevant Information and with Public Information," *Economics Letters* 34, 109–112.

Matsushima, H. (2007): "Mechanism Design with Side Payments: Individual Rationality and Iterative Dominance," *Journal of Economic Theory* 113, 1–30.

Matsushima, H. (2008): "Detail-Free Mechanism Design in Twice Iterative Dominance: Large Economies," *Journal of Economic Theory* 141, 134–151.

Matsushima, H. (2011): "Price-Based Combinatorial Auction: Connectedness and Representative Valuations," mimeo, University of Tokyo.

Milgrom, P. (2004): *Putting Auction Theory to Work*. Cambridge, UK: Cambridge University Press.

Milgrom, P. (2007): "Package Auctions and Exchanges," *Econometrica* 75, 935–965.

Milgrom, P. and I. Segal (2002): "Envelope Theorem for Arbitrary Choice Space," *Econometrica* 70, 583–601.

Mishra, D. and D. Parkes (2007): "Ascending Price Vickrey Auctions for General

Valuations," *Journal of Economic Theory* 132, 335–366.

Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research* 6, 58–73.

Myerson, R. and M. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory* 29, 265–281.

Parkes, D. (2006): "Iterative Combinatorial Auctions," in *Combinatorial Auctions*, ed. by P. Cramton, Y. Shoham, and R. Steinberg. MIT Press.

Parkes, D., J. Kalagnanam, and M. Eso (2002): "Achieving Budget-Balance with Vickrey-Based Payment Schemes in Exchanges," in *Proceedings of the Seventeenth International Joint Conference on Artificial Intelligence*. San Mateo, CA: Morgan Kaufmann, 1161–1168.

Parkes, D. and S. Singh (2003): "An MDP-Based Approach to Online Mechanism Design," in *Proceedings of the 17^{th} Annual Conference on Neural Information Processing Systems (NIPS '03)*.

Parkes, D. and L. Ungar (2002): "An Ascending-Price Generalized Vickrey Auction," mimeo, Harvard University.

Rassenti, S., R. Bulfin, and V. Smith (1982): A Combinatorial Auction Mechanism for Airport Time Slot Allocation," *Bell Journal of Economics* 13, 402–417

Pavan, A., I. Segal, and J. Toikka (2011): "Dynamic Mechanism Design: Incentive Compatibility, Profit Maximization, and Information Disclosure," Stanford University.

Riley, J. and W. Samuelson (1981): "Optimal Auction," *American Economic Review* 71, 381–392.

Rochet, J. and L. Stole (2003): "The Economics of Multidimensional Screening," in *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress*, ed. by M. Dewatripont, L. Hansen, and S. Turnovsky, Cambridge University Press: Cambridge.

Rothkopf, M., T. Teisberg, and E. Kahn (1990): "Why Are Vickrey Auctions Rare?" *Journal of Political Economy* 98, 94–109.

Segal, I. (2006): "Communication in Economic Mechanisms," in *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress, Vol. 1*, ed. by R. Blundell, W. Newey, and T. Persson, Cambridge University Press:

Cambridge.

Vickrey, W. (1961): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance* 16, 8–37.

Wilson, R. (1979): "Auction of Shares," *Quarterly Journal of Economics* 93, 675–689.

Wilson. R. (1985): "Incentive Compatibility of Double Auctions," *Econometrica* 53, 1101–1115.