Boundaries of VP and VNP

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Abstract

One fundamental question in the context of the geometric complexity theory approach to the VP vs. VNP conjecture is whether $\mathrm{VP} = \overline{\mathrm{VP}}$, where VP is the class of families of polynomials that can be computed by arithmetic circuits of polynomial degree and size, and $\overline{\mathrm{VP}}$ is the class of families of polynomials that can be approximated infinitesimally closely by arithmetic circuits of polynomial degree and size. The goal of this article is to study the conjecture in (Mulmuley, FOCS 2012) that $\overline{\mathrm{VP}}$ is not contained in VP.

Towards that end, we introduce three degenerations of VP (i.e., sets of points in $\overline{\mathrm{VP}}$), namely the stable degeneration Stable-VP, the Newton degeneration Newton-VP, and the p-definable one-parameter degeneration VP*. We also introduce analogous degenerations of VNP. We show that Stable-VP $\subseteq$ Newton-VP $\subseteq$ VP* $\subseteq$ VNP, and Stable-VNP = Newton-VNP = VNP* = VNP. The three notions of degenerations and the proof of this result shed light on the problem of separating $\overline{\mathrm{VP}}$ from VP.

Although we do not yet construct explicit candidates for the polynomial families in $\overline{\mathrm{VP}} \setminus \mathrm{VP}$, we prove results which tell us where not to look for such families. Specifically, we demonstrate that the families in Newton-VP $\setminus$VP based on semi-invariants of quivers would have to be non-generic by showing that, for many finite quivers (including some wild ones), Newton degeneration of any generic semi-invariant can be computed by a circuit of polynomial size. We also show that the Newton degenerations of perfect matching Pfaffians, monotone arithmetic circuits over the reals, and Schur polynomials have polynomial-size circuits.

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1 Introduction

One fundamental question in the context of the geometric complexity theory (GCT) approach (cf. [22, 23], [5], and [21]) to the VP vs. VNP conjecture in Valiant [27] is whether $\mathrm{VP} = \overline{\mathrm{VP}}$, where VP is the class of families of polynomials that computed by arithmetic circuits of polynomial degree and size, VNP is the class of p-definable families of polynomials, and $\overline{\mathrm{VP}}$ is the class of families of polynomials that can be approximated infinitesimally closely by arithmetic circuits of polynomial degree and size. We assume in what follows that the

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circuits are over an algebraically closed field $\mathbb{F}$. We call $\overline{\mathrm{VP}}$ the *closure* of VP, and $\overline{\mathrm{VP}} \setminus \mathrm{VP}$ the *boundary* of VP. So the question is whether this boundary is non-empty. At present, it is not even known if $\overline{\mathrm{VP}}$ is contained in VNP.

The VP vs. $\overline{\mathrm{VP}}$ question is important for two reasons. First, all known algebraic lower bounds for the exact computation of the permanent also hold for its infinitesimally close approximation. For example, the known quadratic lower bound for the permanent [20] also holds for its infinitesimally closely approximation [18], and so also the known lower bounds in the algebraic depth-three circuit models [14]; cf. App. B in [11] for a survey of the known lower bounds which emphasizes this point. These lower bounds hold because some algebraic, polynomial property that is satisfied by the coefficients of the polynomials computed by the circuits in the restricted class under consideration is not satisfied by the coefficients of the permanent. Since a polynomial property is a closed condition,[1] the same property is also satisfied by the coefficients of the polynomials that can be approximated infinitesimally closely[2] by circuits in the restricted class under consideration. This is why the same lower bound also holds for infinitesimally close approximation. We expect the same phenomenon to hold in the unrestricted algebraic circuit model as well. Hence, it is natural to expect that any realistic proof of the $\mathrm{VP} \neq \mathrm{VNP}$ conjecture will also show that $\mathrm{VNP} \not\subseteq \overline{\mathrm{VP}}$, as conjectured in [22] (note that if $\mathrm{VNP} \not\subseteq \overline{\mathrm{VP}}$ then there exists a polynomial property showing this lower bound). This is, in fact, the underlying thesis of geometric complexity theory that is implicit in [21]. But, if $\overline{\mathrm{VP}} \neq \mathrm{VP}$, as conjectured in [21], this would mean that any realistic approach to the VP vs. VNP conjecture would even have to separate the permanent from the families in $\overline{\mathrm{VP}} \setminus \mathrm{VP}$ with high circuit complexity.[3]

Second, it is shown in [21] that, assuming a stronger form of the $\mathrm{VNP} \not\subseteq \overline{\mathrm{VP}}$ conjecture, the problem NNL (short for Noether's Normalization Lemma) of computing Noether normalization of explicit varieties can be brought down from EXPSPACE, where it is currently, to P, ignoring a quasi-prefix. The existing EXPSPACE vs. P gap,[4] called the geometric complexity theory (GCT) chasm [21], in the complexity of NNL may be viewed as the common cause and measure of the difficulty of the fundamental problems in geometry (NNL) and complexity theory (Hardness). If $\overline{\mathrm{VP}} = \mathrm{VP}$, then it follows [21] that NNL is in PSPACE. Thus the conjectural inequality between $\overline{\mathrm{VP}}$ and VP is the main difficulty that needs to be overcome to bring NNL from EXPSPACE to PSPACE unconditionally, and is the main reason why the standard techniques in complexity theory may not be expected to work in the context of the $\mathrm{VP} \neq \mathrm{VNP}$ conjecture.

The goal of this article is to study the conjecture in [21] that $\overline{\mathrm{VP}}$ is not contained in VP.

1.1 Degenerations of VP and VNP

Towards that end, we introduce three notions of degenerations of VP and VNP; "degeneration" is the standard term in algebraic geometry for a limit point or infinitesimal approximation. These are subclasses of $\overline{\mathrm{VP}}$ and $\overline{\mathrm{VNP}}$, respectively; cf. Sec. 3 for formal definitions.

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[1] It is defined by the vanishing of a continuous function, namely, a (meta) polynomial.

[2] This means the polynomials are the limits of the polynomials computed by the circuits in the restricted class under consideration.

[3] Although some lower bounds techniques in the restricted models do distinguish between different polynomials with high circuit complexity (e.g., [25]), we need a better understanding of the families in $\overline{\mathrm{VP}} \setminus \mathrm{VP}$ in order to know which techniques in this spirit could even potentially be useful in the setting of the VNP versus $\overline{\mathrm{VP}}$ problem.

[4] Or, the EXPH vs. P gap, assuming the Generalized Riemann Hypothesis.

The first notion is that of a stable degeneration. Recall [24] that a polynomial f in $\mathbb{F}[x_1, \dots, x_n]$ is called *stable* with respect to the natural action of $G = \mathrm{SL}(n, \mathbb{F})$ on $\mathbb{F}[x_1, \dots, x_n]$ if the G-orbit of f is closed (in the Zariski topology). We say that a polynomial f is a *stable degeneration* of $g \in \mathbb{F}[x_1, \dots, x_n]$ if f lies in a closed G-orbit (which is unique [24]) in the closure of the G-orbit of g. The degeneration is called stable since f in this case is stable. For any class of polynomials $\mathcal{C}$, the class Stable-$\mathcal{C}$ is defined to be the class of families of polynomials that are either in $\mathcal{C}$ or are stable degenerations thereof.

The second notion is that of a Newton degeneration. We say that a polynomial f is a *Newton degeneration* of g if it is obtained from g by keeping only those terms whose associated monomial-exponents lie in some specified face of the Newton polytope of g. For any class of polynomial families $\mathcal{C}$, the class Newton-$\mathcal{C}$ is defined to be the class of families of polynomials that are Newton degenerations of the polynomials in $\mathcal{C}$, or are linear projections of such Newton degenerations.[5]

The third notion, motivated by the notion of p-definability in Valiant [27], is that of a p-definable one-parameter degeneration. We say that a family $\{f_n\}$ of polynomials is a *p-definable one-parameter degeneration* of a family $\{g_n\}$ of polynomials, if $f_n = \lim_{t \to 0} g_n(t)$, where $g_n(t)$ is obtained from g_n by transforming its variables linearly such that (1) the entries of the linear transformation matrix are Laurent polynomials in t of possibly exponential degree (in n), and (2) there exists a small circuit C_n of size polynomial in n such that any coefficient of the Laurent polynomial in any entry of the transformation matrix can be obtained by evaluating C_n at the indices of that entry and the index of the coefficient.[6] Thus a p-definable one-parameter degeneration is a one-parameter degeneration of exponential degree that can be encoded by a small circuit. For any class $\mathcal{C}$, the class $\mathcal{C}^*$ is then defined to be the class of families of polynomials that are p-definable one-parameter degenerations of the families in $\mathcal{C}$.

$\overline{\mathrm{VP}}$ and $\overline{\mathrm{VNP}}$ are closed under these three types of degenerations (cf. Propositions 6, 8, 11). Since we want to compare $\overline{\mathrm{VP}}$ with VP, and $\overline{\mathrm{VNP}}$ with VNP, we ask how VP and VNP behave under these three degenerations. This is addressed in the following result.

▶ **Theorem 1.**
(a) *Stable-VNP = Newton-VNP =* VNP^* *= VNP, and*
(b) *Stable-VP* $\subseteq$ *Newton-VP* $\subseteq$ VP^* $\subseteq$ *VNP.*

The statement of this result tells us nothing as to whether any of the inclusions in the sequence Stable-VP $\subseteq$ Newton-VP $\subseteq$ VP^* $\subseteq$ $\overline{\mathrm{VP}}$ can be expected to be strict or not. But its proof, as discussed below, does shed light on this subject.

Theorem 1 is proved by combining the Hilbert–Mumford–Kempf criterion for stability [15] with the ideas and results in Valiant [27]. The Hilbert–Mumford–Kempf criterion [15] shows that, for any polynomial f_n in the unique closed G-orbit in the G-orbit-closure of any $g_n \in \mathbb{F}[x_1, \dots, x_n]$, with $G = \mathrm{SL}_n(\mathbb{F})$, there exists a one-parameter subgroup of G that drives g_n to f_n. Furthermore, by Kempf [15], such a subgroup can be chosen in a canonical manner. As a byproduct of the proof of Theorem 1, we get a complexity-theoretic form of this criterion (cf. Theorem 18), which shows that such a one-parameter group can be chosen

[5] Taking a Newton degeneration and a linear projection need not commute, so the set of Newton degenerations alone will not in general be closed under linear projections. For example, any polynomial f is a linear projection of a sufficiently large determinant, but the Newton degenerations of the determinant only consist of polynomials of the form $\det(X')$ where X' is matrix consisting only of variables and 0s.

[6] It is assumed here that the indices are encoded as the lists of 0-1 variables.

so that the resulting one-parameter degeneration of any $\{g_n\} \in \mathrm{VP}$ to $\{f_n\} \in$ Stable-VP is p-definable. Thus the inclusion of Stable-VP in VNP ultimately depends on the existence of a very special type of one parameter degeneration of $\{g_n\}$ to $\{f_n\}$, as provided by the Hilbert–Mumford–Kempf criterion, which can be encoded by a small circuit. However, no such degeneration scheme, which can be encoded by a small circuit, is known if f_n is allowed to be any polynomial in the $\mathrm{GL}(n, \mathbb{F})$-orbit-closure of g_n.

If such a scheme exists for every f_n in the $\mathrm{GL}(n, \mathbb{F})$-orbit-closure of g_n, then it would follow that $\overline{\mathrm{VP}} \subseteq \mathrm{VP}^*$, and in conjunction with Theorem 1, that $\overline{\mathrm{VP}} \subseteq \mathrm{VNP}$. This is one plausible approach to show that $\overline{\mathrm{VP}} \subseteq \mathrm{VNP}$, if this is true. If, on the other hand, no such special scheme akin to the Hilbert–Mumford–Kempf criterion for stability exists for every f_n in the $\mathrm{GL}(n, \mathbb{F})$-orbit-closure of g_n, as the extensive research in geometric invariant theory [24] in the last century since the work of Hilbert [12] suggests, then this may be taken as an indication that $\overline{\mathrm{VP}}$ is not contained in VP^*, and hence, also not in VP.

The complexity-theoretic form of the Hilbert-Mumford criterion mentioned above (Theorem 18) also provides an exponential (in n) upper bound on the degree of the canonical Kempf-one-parameter subgroup that drives g_n to f_n, with $\{g_n\} \in \mathrm{VP}$ and $\{f_n\} \in$ Stable-VP. This canonical Kempf-one-parameter subgroup is known to be the fastest way to approach a closed orbit [16]. If one could prove a polynomial upper bound on this degree, then it would follow that Stable-VP = VP (cf. Lemma 17). On the other hand, if a worst-case superpolynomial lower bound on this degree can be proved, then it would be an indication that Stable-VP, and hence $\overline{\mathrm{VP}}$, are different from VP. In other words, this suggests a possible route to formally separate VP and $\overline{\mathrm{VP}}$.

An analogue of Theorem 1 also holds for $\mathrm{VP}_{\mathrm{ws}}$, the class of families of polynomials that can be computed by symbolic determinants of polynomial size.

Next we ask if one can construct an *explicit* family in Newton-$\mathrm{VP}_{\mathrm{ws}}$ that can reasonably be conjectured to be not in $\mathrm{VP}_{\mathrm{ws}}$ or even VP. With this mind, we first construct an explicit family $\{f_n\}$ of polynomials that can be approximated infinitesimally closely by symbolic determinants of size $\leq n$, but conjecturally cannot be computed exactly by symbolic determinants of $\Omega(n^{2+\delta})$ size, for a small enough positive constant $\delta < 1$; cf. Section 5. This construction follows a suggestion made in [22, Section 4.2]. The family $\{f_n\}$ is a Newton degeneration of the family of perfect matching Pfaffians of graphs. However, this family $\{f_n\}$ turns out to be in $\mathrm{VP}_{\mathrm{ws}}$. So we need to extend this idea much further to construct an explicit family in Newton-$\mathrm{VP}_{\mathrm{ws}}$ that can be conjectured to be not in VP.

To see how, note that perfect matching Pfaffians are derived from a semi-invariant of the symmetric quiver with two vertices and one arrow. This suggests that to upgrade the conjectural $\Omega(n^{2+\delta})$ lower bound to obtain a candidate for a superpolynomial lower bound a possible route is to replace perfect matching Pfaffians by appropriate representation-theoretic invariants. This leads to the second line of investigation, which we now discuss.

1.2 On Newton degeneration of generic semi-invariants

Our next result suggests that these invariants should be non-generic by showing that, for many finite quivers, including some wild ones, Newton degeneration of any generic semi-invariant can be computed by a symbolic determinant of polynomial size.

A quiver $Q = (Q_0, Q_1)$ [6, 8] is a directed graph (allowing multiple edges) with the set of vertices Q_0 and the set of arrows Q_1. A linear representation V of a quiver associates to each vertex $x \in Q_0$ a vector space V^x, and to each arrow $\alpha \in Q_1$ a linear map V^α from $V^{s\alpha}$ to $V^{t\alpha}$, where $s\alpha$ denotes the start (tail) of α and $t\alpha$ its target (head). The dimension vector of V is the tuple of non-negative integers that associates $\dim(V^x)$ to each vertex $x \in Q_0$. Given a dimension vector $\beta \in \mathbb{N}^{|Q_0|}$, let $\mathrm{Rep}(Q, \beta)$ denote the space of all representations of Q with the dimension vector β. We have the natural action of $\mathrm{SL}(\beta) := \prod_{x \in Q_0} \mathrm{SL}(\beta(x), \mathbb{F})$ on $\mathrm{Rep}(Q, \beta)$ by change of basis. Let $\mathrm{SI}(Q, \beta) = \mathrm{Rep}(Q, \beta)^{\mathrm{SL}(\beta)}$ denote the ring of semi-invariants. The *generic* semi-invariants in this ring (see [6]) will be recalled in Section 6.

We will be specifically interested in the following well-known types of quivers, cf. [7]. The m-Kronecker quiver is the quiver with two vertices, and m arrows between the two vertices with the same direction. It is wild if $m \geq 3$; wildness is a universality property in representation theory, analogous to NP-completeness (see, e.g., [1]). The k-subspace quiver is the quiver with $k + 1$ vertices $\{x_1, \dots, x_k, y\}$ and k arrows $(x_1, y), \dots, (x_k, y)$. It is wild if $k \geq 5$. The A-D-E Dynkin quivers are the only quivers of finite representation type – they have only finitely many indecomposable representations.

The following result tells us where *not* to look for explicit candidate families in $\overline{\mathrm{VP}} \setminus \mathrm{VP}$.

▶ **Theorem 2.** *Let Q be an m-Kronecker quiver, or a k-subspace quiver, or an A-D-E Dynkin quiver. Then any Newton degeneration of a generic semi-invariant of Q with dimension vector β and degree d can be computed by a weakly skew circuit (or equivalently a symbolic determinant) of* $\mathsf{poly}(|\beta|, d)$ *size, where $|\beta| = \sum_{x \in Q_0} \beta(x)$.*

The proof strategy for Theorem 2 is as follows. Define the coefficient complexity $\mathrm{coeff}(E)$ of a set E of integral linear equalities in $\mathbb{R}^m$ as the sum of the absolute values of the coefficients of the equalities. Define the coefficient complexity of a face of a polytope in $\mathbb{R}^m$ as the minimum of $\mathrm{coeff}(E)$, where E ranges over all integral linear equality sets that define the face, in conjunction with the description of the polytope; cf. Section 6.1.

Theorem 2 is proved by showing that the coefficient complexity of every face of the Newton polytope of a generic semi-invariant of any quiver as above is polynomial in $|\beta|$ and d, though the number of vertices on a face can be exponential.

In view of this result and its proof, to construct an explicit family in Newton-$\mathrm{VP}_{\mathrm{ws}} \setminus \mathrm{VP}_{\mathrm{ws}}$, we should look for appropriate *non-generic* invariants of representations of finitely generated algebras whose Newton polytopes have faces with *superpolynomial coefficient complexity* and *superpolynomial number of vertices.*

Of course, we do not have to confine ourselves to Newton-VP in the search of an explicit candidate family in $\overline{\mathrm{VP}} \setminus \mathrm{VP}$. We may search within VP^*, or even outside VP^*.

Organization. The rest of this article is organized as follows. In Section 2 we cover the preliminaries. In Section 3, we formally define the three degenerations of VP and VNP. In Section 4, we prove Theorem 1. In Section 5 we construct an explicit family $\{f_n\}$ that can be approximated infinitesimally closely by symbolic determinants of size $\leq n$, but conjecturally cannot be computed exactly by symbolic determinants of $\Omega(n^{2+\delta})$ size, for a small enough positive constant $\delta < 1$. In Section 6, we prove Theorem 2 for generalized Kronecker quivers. Due to page constraints, some proofs are deferred to the full version. In particular, there we give additional examples of representation-theoretic symbolic determinants whose Newton degenerations have small circuits. All these examples suggest that explicit families in Newton-$\mathrm{VP}_{\mathrm{ws}} \setminus \mathrm{VP}_{\mathrm{ws}}$ have to be rather delicate.

2 Preliminaries

For $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$. We denote by $\mathbf{x} = (x_1, \dots, x_n)$ a tuple of variables; $\mathbf{x}$ may also denote $\{x_1, \dots, x_n\}$. Let $\mathbf{e} = (e_1, \dots, e_n)$ be a tuple of nonnegative integers. We usually use $\mathbf{e}$ as the exponent vector of a monomial in $\mathbb{F}[x_1, \ldots, x_n]$. Thus, $\mathbf{x}^{\mathbf{e}}$ denotes the monomial with the exponent vector $\mathbf{e}$. Let $|\mathbf{e}| := \sum_{i=1}^{n} e_i$.

For a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F})$ denotes the characteristic of $\mathbb{F}$. Throughout this paper, we assume that $\mathbb{F}$ is algebraically closed. S_n denotes the symmetric group consisting of permutations of n objects.

We say that a polynomial $g = g(x_1, \ldots, x_n)$ is a *linear projection* of $f = f(y_1, \ldots, y_m)$ if g can be obtained from f by letting y_j's be some (possibly non-homogeneous) linear combinations of x_i's with coefficients in the base field $\mathbb{F}$.

A family of polynomials $\{f_n\}_{n\in\mathbb{N}}$ is p-bounded if f_n is a polynomial in $\mathsf{poly}(n)$ variables of $\mathsf{poly}(n)$ degree. The class VP [27] consists of p-bounded polynomial families $\{f_n\}_{n\in\mathbb{N}}$ over $\mathbb{F}$ such that f_n can be computed by an arithmetic circuit over $\mathbb{F}$ of $\mathsf{poly}(n)$ size.

Convention: We call a class $\mathcal{C}$ of families of polynomials *standard* if it contains only p-bounded families, and is closed under linear projections.

By a symbolic determinant of size m over the variables $x_1, \ldots, x_n$, we mean the determinant of an $m \times m$ matrix, whose each entry is a possibly non-homogeneous linear function of $x_1, \ldots, x_n$ with coefficients in the base field $\mathbb{F}$. The class $\mathrm{VP}_{\mathrm{ws}}$ is the class of families of polynomials that can be computed by weakly skew circuits of polynomial size, or equivalently, by symbolic determinants of polynomial size [19].

The class VNP is the class of p-definable families of polynomials [27], that is, those families (f_n) such that f_n has $\mathsf{poly}(n)$ variables and $\mathsf{poly}(n)$ degree, and there exists a family $(g_n(x, y)) \in \mathrm{VP}$ such that $f_n(x) = \sum_{e\in\{0,1\}^{\mathsf{poly}(n)}} g_n(x, e)$.

The class $\overline{\mathrm{VP}}$ is defined as follows [22, 5]. Over $\mathbb{F} = \mathbb{C}$, we say that a polynomial family $\{f_n\}_{n\in\mathbb{N}}$ is in $\overline{\mathrm{VP}}$, if there exists a family of sequences of polynomials $\{f_n^{(i)}\}_{n\in\mathbb{N}}$ in VP, $i = 1, 2, \ldots,$ s.t. for every n, the sequence of polynomials $f_n^{(i)}$, $i = 1, 2, \ldots,$ goes infinitesimally close to f_n, in the usual complex topology. Here, polynomials are viewed as points in the linear space of polynomials. There is a more general definition that works over arbitrary algebraically closed fields – including in positive characteristic – using the Zariski topology. For a direct treatment, see, e.g. [4, App. 20.6]. The operational version of this definition we use is as follows: $\{f_n(x_1, \ldots, x_m)\} \in \overline{\mathrm{VP}}$ if there exist polynomials $f_{n,t}(x_1, \ldots, x_m) \in \mathrm{VP}_{\mathbb{C}((t))}$ – $f_{n,t}$ is a polynomial in the x_i whose coefficients are Laurent series in t – such that $f_n(x)$ is the coefficient of the term in $f_{n,t}(x)$ of lowest degree in t.

The classes $\overline{\mathrm{VP}_{\mathrm{ws}}}$, $\overline{\mathrm{VNP}}$, and $\overline{\mathcal{C}}$, for any standard class $\mathcal{C}$, are defined similarly.

By the determinantal complexity $\mathrm{dc}(f)$ of a polynomial $f(x_1, \ldots, x_n)$, we mean the smallest integer m s.t. f can be expressed as a symbolic determinant of size m over $x_1, \ldots, x_n$. By the approximative determinantal complexity $\overline{\mathrm{dc}}(f)$, we mean the smallest integer m s.t. f can be approximated infinitesimally closely by symbolic determinants of size m.

Thus the $\mathrm{VP}_{\mathrm{ws}} \neq \mathrm{VNP}$ conjecture in Valiant [27] is equivalent to saying that $\mathrm{dc}(\mathrm{perm}_n)$ is not $\mathsf{poly}(n)$, where perm_n denotes the permanent of an $n \times n$ variable matrix. The $\mathrm{VNP} \not\subseteq \overline{\mathrm{VP}_{\mathrm{ws}}}$ conjecture in [22] is equivalent to saying that $\overline{\mathrm{dc}}(\mathrm{perm}_n)$ is not $\mathsf{poly}(n)$.

A priori, it is not at all obvious that dc and $\overline{\mathrm{dc}}$ are different complexity measures. The following two examples should make this clear.

▶ **Example 3** (Example 9 in [17]). Let $f = x_1^3 + x_2^2 x_3 + x_2 x_4^2$. Then $\mathrm{dc}(f) \geq 5$, but $\overline{\mathrm{dc}}(f) = 3$.

▶ **Example 4** (Proposition 3.5.1 in [18]). Let n be odd. Given an $n \times n$ complex matrix M, let M_{ss} and M_s denote its skew-symmetric and symmetric parts. Since n is odd, $\det(M_{ss})$=0. Hence, for a variable t, $\det(M_{ss} + tM_s) = tf(M) + O(t^2)$, for some polynomial function

$f(M)$. Clearly, $\overline{\mathrm{dc}}(f) = n$, since $\det(M_{ss} + tM_s)/t$ goes infinitisimally close to $f(M)$ when t goes to 0. But $\mathrm{dc}(f) > n$.

The $\mathrm{VP_{ws}} \neq \overline{\mathrm{VP_{ws}}}$ conjecture in [21] is equivalent to saying that there exists a polynomial family $\{f_n\}$ such that $\overline{\mathrm{dc}}(f_n) = \mathsf{poly}(n)$, but $\mathrm{dc}(f_n)$ is not $\mathsf{poly}(n)$. Instead of this conjecture, we will focus on the $\mathrm{VP} \neq \overline{\mathrm{VP}}$ conjecture in [21], since the considerations for the former conjecture are entirely similar.

A (convex – we will only consider convex ones here) *polytope* is the convex hull in $\mathbb{R}^n$ of a finite set of points. A *face* of a polytope P is the intersection of P with linear halfspace $H = \{v \in \mathbb{R}^n | \ell(v) \geq c\}$ for some linear function ℓ and constant c such that H contains no points of the (topological) interior of P. Equivalently, a polytope is the intersection of finitely many half-spaces, a half-space $H_{\ell,c} = \{v | \ell(v) \geq c\}$ is tight for P if $P \subseteq H_{\ell,c}$ and $P \nsubseteq H_{\ell,c'}$ for any $c' > c$, and a face of P is the intersection of P with a half-space of the form $H_{-\ell,-c}$ where $H_{\ell,c}$ is tight for P.

3 Degenerations of VP and VNP

To understand the relationship between VP, VNP, and their closures $\overline{\mathrm{VP}}$ and $\overline{\mathrm{VNP}}$, we now introduce three degenerations of VP and VNP. The considerations for $\mathrm{VP_{ws}}$ and $\overline{\mathrm{VP_{ws}}}$ are entirely similar.

3.1 Stable degeneration

First we define stable degenerations of VP and VNP.

Consider the natural action of $G = \mathrm{SL}(n, \mathbb{F})$ on $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, \ldots, x_n]$ that maps $f(\mathbf{x})$ to $f(\sigma^{-1}\mathbf{x})$ for any $\sigma \in G$. Following Mumford et al. [24], call $f = f(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$ *stable* (with respect to the G-action) if the G-orbit of f is Zariski-closed. It is known [24] that the closure of the G-orbit of any $g \in \mathbb{F}[\mathbf{x}]$ contains a unique closed G-orbit. We say that f is a *stable degeneration* of g if f lies in the unique closed G-orbit in the G-orbit-closure of g. (If the G-orbit of g is already closed then this just means that f lies in the G-orbit of g.)

We now define the class Stable-$\mathcal{C}$, the stable degeneration of any standard class $\mathcal{C}$, as follows. We say that $\{f_n\}_{n\in\mathbb{N}}$ is in Stable-$\mathcal{C}$ if (1) $\{f_n\} \in \mathcal{C}$, or (2) there exists $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that each f_n is a stable degeneration of g_n with respect to the action of $G = \mathrm{SL}(m_n, \mathbb{F})$, where $m_n = \mathsf{poly}(n)$ denotes the number of variables in f_n and g_n.

▶ **Proposition 5.** *For any class $\mathcal{C}$ of p-bounded families of polynomials, Stable-$\mathcal{C} \subseteq \overline{\mathcal{C}}$. In particular, Stable-VP $\subseteq \overline{\mathrm{VP}}$ and Stable-VNP $\subseteq \overline{\mathrm{VNP}}$.*

▶ **Proposition 6.** *Stable-$\overline{\mathcal{C}} = \overline{\mathcal{C}}$, in particular Stable-$\overline{\mathrm{VP}} = \overline{\mathrm{VP}}$, and Stable-$\overline{\mathrm{VNP}} = \overline{\mathrm{VNP}}$.*

This is a direct consequence of the definitions.

3.2 Newton degeneration

Next we define Newton degenerations of VP and VNP.

Given a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$, suppose $f = \sum_{\mathbf{e}} \alpha_{\mathbf{e}} \mathbf{x}^{\mathbf{e}}$. We collect the exponent vectors of f and form the convex hull of these exponent vectors in $\mathbb{R}^n$. The resulting polytope is called the *Newton polytope* of f, denoted $\mathrm{NPT}(f)$. Given an arbitrary face Q of $\mathrm{NPT}(f)$, the Newton degeneration of f to Q, denoted $f|_Q$, is the polynomial $\sum_{\mathbf{e}\in Q} \alpha_{\mathbf{e}} \mathbf{x}^{\mathbf{e}}$.

We now define the class Newton-$\mathcal{C}$, the Newton degeneration of any class $\mathcal{C}$, as follows: $\{f_n\}_{n\in\mathbb{N}}$ is in Newton-$\mathcal{C}$, if there exists $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that each f_n is the Newton degeneration of g_n to some face of $\mathrm{NPT}(g_n)$, or a linear projection of such a Newton degeneration.

▶ **Theorem 7.** *Let $\mathcal{C}$ be any standard class (cf. Section 2). Then Newton-$\mathcal{C} \subseteq \overline{\mathcal{C}}$. In particular, Newton-VP $\subseteq \overline{VP}$ and Newton-VNP $\subseteq \overline{VNP}$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be in Newton-$\mathcal{C}$, and suppose $f_n \in \mathbb{F}[x_1, \dots, x_{m(n)}]$.Then there exists $\{g_n\}_{n\in\mathbb{N}} \in \mathcal{C}$, such that $g_n \in \mathbb{F}[x_1, \dots, x_m]$, $m = m(n)$, and $f_n = g_n|_Q$, where Q is a face of $\mathrm{NPT}(g_n)$. Suppose the supporting hyperplane of Q is defined by $\langle \mathbf{a}, \mathbf{x}\rangle = b$, where $\mathbf{a} = (a_1, \dots, a_m)$. If necessary, by replacing $(\mathbf{a}, b)$ with $(-\mathbf{a}, -b)$, we make sure that for an arbitrary exponent vector $\mathbf{e}$ in g_n, $\langle \mathbf{a}, \mathbf{e}\rangle \geq b$. That is, among all exponent vectors, exponent vectors on Q achieve the minimum value b in the direction $\mathbf{a}$.

Now introduce a new variable t, and replace x_i with $t^{a_i}x_i$ to obtain a polynomial $g'_n(x_1, \dots, x_m, t) = g_n(t^{a_1}x_1, \dots, t^{a_m}x_m) \in \mathbb{F}[x_1, \dots, x_m, t]$. By the definition of f_n, $g'_n = t^b \cdot f_n +$ higher order terms in t. Therefore, $\{f_n\} \in \overline{\mathcal{C}}$. ◀

Noting that if $\mathcal{C}$ is closed under linear projections, then so is $\overline{\mathcal{C}}$, we have:

▶ **Corollary 8.** *For any standard class $\mathcal{C}$, Newton-$\overline{\mathcal{C}} = \overline{\mathcal{C}}$. In particular, Newton-$\overline{VP} = \overline{VP}$ and Newton-$\overline{VNP} = \overline{VNP}$.*

3.3 P-definable one-parameter degeneration

Finally, we define p-definable one-parameter degenerations of VP and VNP. We say a family $\{f_n(x_1, \dots, x_{m_n})\}$, $m_n = \mathsf{poly}(n)$, is a *one-parameter degeneration* of $\{g_n(y_1, \dots, y_{l_n})\}$, for $l_n = \mathsf{poly}(n)$, of *exponential degree*, if, for some positive integral function $K(n) = O(2^{\mathsf{poly}(n)})$, there exist $c_n(i, j, k) \in \mathbb{F}$, $1 \leq i \leq l_n$, $0 \leq j \leq m_n$, $-K(n) \leq k \leq K(n)$, such that $f_n = \lim_{t\to 0} g_n(t)$, where $g_n(t)$ is obtained from g_n by substitutions of the form

$$y_i = a^i_0 + \sum_{j=1}^{m_n} a^i_j x_j, \quad 1 \leq i \leq l_n, \text{ where } a^i_j = \sum_{k=-K(n)}^{K(n)} c_n(i, j, k)t^k, \quad 1 \leq i \leq l_n,\ 0 \leq j \leq m_n.$$

Note that by [3], $\overline{\mathrm{VP}}$ consists exactly of those one-parameter degenerations of VP of exponential degree.

We say that the family $\{f_n(x_1, \dots, x_{m_n})\}$, $m_n = \mathsf{poly}(n)$, is a *one-parameter degeneration* of $\{g_n(y_1, \dots, y_{l_n})\}$, $l_n = \mathsf{poly}(n)$, of *polynomial degree* if $K(n)$ above is $O(\mathsf{poly}(n))$ (instead of $O(2^{\mathsf{poly}(n)})$).

We say that a family $\{f_n(x_1, \dots, x_{m_n})\}$, $m_n = \mathsf{poly}(n)$, is a *p-definable one-parameter degeneration* of $\{g_n(y_1, \dots, y_{l_n})\}$, $l_n = \mathsf{poly}(n)$, if, for some $K(n) = O(2^{\mathsf{poly}(n)})$, there exists a $\mathsf{poly}(n)$-size circuit family $\{C_n\}$ over $\mathbb{F}$ such that $f_n = \lim_{t\to 0} g_n(t)$, where $g_n(t)$ is obtained from g_n by substitutions of the form

$$y_i = a^i_0 + \sum_{j=1}^{m_n} a^i_j x_j, \quad 1 \leq i \leq l_n, \text{ where } a^i_j = \sum_{k=-K(n)}^{K(n)} C_n(i, j, k)t^k, \quad 1 \leq i \leq l_n,\ 0 \leq j \leq m_n.$$

Here it is assumed that the circuit C_n takes as input $\lceil \log_2 l_n\rceil + \lceil \log_2 m_n\rceil + \lceil \log_2(K(n)+1)\rceil$ many 0-1 variables, which are intended to encode three integers (i, j, k) satisfying $1 \leq i \leq l = l_n$, $0 \leq j \leq m = m_n$, and $|k| \leq K(n)$, treating 0 and 1 as elements of $\mathbb{F}$.

Thus a p-definable one-parameter degeneration is a one-parameter degeneration of exponential degree that can be specified by a circuit of polynomial size.

For any class $\mathcal{C}$ we now define $\mathcal{C}^*$, called the p-definable one-parameter degeneration of $\mathcal{C}$, as follows. We say that $\{f_n\} \in \mathcal{C}^*$ if there exists $\{g_n\} \in \mathcal{C}$ such that $\{f_n\}$ is a p-definable one-parameter degeneration of $\{g_n\}$.

▶ **Lemma 9.** *For any standard class $\mathcal{C}$ (cf. Section 2), Newton-$\mathcal{C} \subseteq \mathcal{C}^*$. In particular, Newton-VP $\subseteq$ VP* and Newton-VNP $\subseteq$ VNP*.*

This follows from the proof of Theorem 7, noting that we may always take the coefficients of a face to have size at most $2^{\mathsf{poly}(n)}$. The following are easy consequences of the definitions:

▶ **Proposition 10.** *VP* $\subseteq \overline{\text{VP}}$, and VNP* $\subseteq \overline{\text{VNP}}$.*

▶ **Proposition 11.** *$\overline{\text{VP}}^* = \overline{\text{VP}}$, and $\overline{\text{VNP}}^* = \overline{\text{VNP}}$.*

4 Stable-VNP = Newton-VNP = VNP* = VNP

We now prove Theorem 1, by a circular sequence of inclusions.

Proof of Theorem 1. Since VNP $\subseteq$ Stable-VNP by definition, Theorem 1(a) follows from the facts that Stable-VNP $\subseteq$ Newton-VNP (cf. Theorem 12 below), Newton-VNP $\subseteq$ VNP* (Lemma 9), and VNP* $\subseteq$ VNP (cf. Theorem 15 below).

Theorem 1(b) follows from the facts that Stable-VP $\subseteq$ Newton-VP (cf. Theorem 12 below), Newton-VP $\subseteq$ VP* (Lemma 9), and VP* $\subseteq$ VNP (cf. Corollary 16 below). ◀

▶ **Theorem 12.** *For any class $\mathcal{C}$ of families of p-bounded polynomials, Stable-$\mathcal{C} \subseteq$ Newton-$\mathcal{C}$. In particular, Stable-VP $\subseteq$ Newton-VP and Stable-VNP $\subseteq$ Newton-VNP.*

Proof. Suppose $\{f_n\} \in$ Stable-$\mathcal{C}$. If $\{f_n\} \in \mathcal{C}$ then there is nothing to show. Otherwise, there exists $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ s.t. each f_n is a stable degeneration of g_n with respect to the action of $G = SL_{m_n}(\mathbb{F})$, where m_n denotes the number of variables in f_n and g_n.

It suffices to show that $f = f_n(x_1, \ldots, x_m)$, $m = m_n$, is a Newton degeneration of $g = g_n(x_1, \ldots, x_m)$. Let $\mathbf{x} = (x_1, \ldots, x_m)$.

By the Hilbert–Mumford–Kempf criterion for stability [15], there exists a one-parameter subgroup $\lambda(t) \subseteq G$ such that $\lim_{t\to 0} \lambda(t).g = f$. Let T be the canonical maximal torus in G such that the monomials in x_i's are eigenvectors for the action of T. After a linear change of coordinates (which is allowed since Newton-$\mathcal{C}$ is closed under linear transformations by definition), we can assume that $\lambda(t)$ is contained in T. Thus $\lambda(t) = \mathrm{diag}(t^{k_1}, \ldots, t^{k_m})$ (the diagonal matrix with t^{k_j}'s on the diagonal), $k_j \in \mathbb{Z}$, such that $\sum k_j = 1$.

It follows that f is the Newton degeneration of g to the face of NPT(g) where the linear function $\sum_j k_j x_j$ achieves the minimum value (which has to be zero). ◀

The following result is subsumed by Theorem 15; we include its proof here both for expository clarity (it is somewhat simpler but still gives the flavor) and brevity.

▶ **Theorem 13.** *Newton-VNP $\subseteq$ VNP.*

Proof. Suppose $\{f_n\} \in$ Newton-VNP. If $\{f_n\} \in$ VNP, then there is nothing to show. Otherwise, there exists $\{g_n\}_{n\in\mathbb{N}}$ in VNP such that each f_n is the Newton degeneration of g_n to some face of NPT(g_n), or a linear projection of such a Newton degeneration. Since VNP is closed under linear projections, we can assume, without loss of generality, that f_n is the Newton degeneration of g_n to some face of NPT(g_n).

By Valiant [27], we can assume that $g = g_n(x_1, \dots, x_m)$, $m = m_n = \mathsf{poly}(n)$, is a projection of $\mathrm{perm}(X)$,[7] where X is a $k \times k$ variable matrix, with $k = \mathsf{poly}(n)$. This means $g = \mathrm{perm}(X')$, where each entry of X' is some variable x_i or a constant from the base field F. Since $f = f_n$ is a Newton degeneration of g, it follows that there is some substitution, as in the proof of Theorem 7, $x_j \to x_j t^{k_j}$, $k_j \in \mathbb{Z}$, such that $f = \lim_{t\to 0} \mathrm{perm}(X'(t))$, where $X'(t)$ denotes the matrix obtained from X' after this substitution.

It is easy to ensure that $|k_j| \le O(2^{\mathsf{poly}(n)})$. Then, given any permutation $\sigma \in \mathrm{S}_k$, whether the corresponding monomial $\prod_i X'_{i\sigma(i)}$ contributes to f can be decided in $\mathsf{poly}(n)$ time. It follows that the coefficient of a monomial can be computed by an algebraic circuit summed over polynomially many Boolean inputs (convert the implicit $\mathsf{poly}(n)$-time Turing machine into a Boolean circuit, then convert it into an algebraic circuit (as in [27, Remark 1]) that incorporates the constants appearing in the projection). Hence $\{f_n\} \in \mathrm{VNP}$. ◀

Since VP $\subseteq$ VNP, the preceding result implies:

▶ **Corollary 14.** *Newton-VP $\subseteq$ VNP.*

The following result can proved similarly to Theorem 13; see the full version for its proof.

▶ **Theorem 15.** *$\mathrm{VNP}^* \subseteq$ VNP.*

▶ **Corollary 16.** *$\mathrm{VP}^* \subseteq$ VNP.*

In contrast, using the interpolation technique of Strassen [26] and Bini [2] we have:

▶ **Lemma 17** (cf. also [3], [5, §9.4], [10, Prop. 3.5.4])**.** *If $\{f_n\}$ is a one-parameter degeneration of $\{g_n\} \in$ VP of polynomial degree, then $\{f_n\} \in$ VP.*

A complexity-theoretic form of the Hilbert–Mumford–Kempf criterion. As a byproduct of the proof of Theorem 1, we get the following complexity-theoretic form of the Hilbert–Mumford–Kempf criterion [15] for stability with respect to the action of $G = \mathrm{SL}(m, \mathbb{F})$ on $\mathbb{F}[x_1, \dots, x_m]$. Given a one-parameter subgroup $\lambda(t) \subseteq G$, we can express it as $A \cdot \mathrm{diag}(t^{k_1}, \dots, t^{k_m}) \cdot A^{-1}$, for some $A \in G$ and $k_j \in \mathbb{Z}$, $1 \le j \le m$. We call $\sum_i |k_i|$ the total degree of $\lambda(t)$. The following theorem is implicit in the proofs of Theorems 12 and 13.

▶ **Theorem 18.** *Suppose $f = f(x_1, \dots, x_m)$ belongs to the unique closed G-orbit in the G-orbit-closure of $g = g(x_1, \dots, x_m) \in \mathbb{F}[x_1, \dots, x_m]$. Then there exists a one-parameter subgroup $\lambda(t) \subseteq G$ such that (1) $\lim_{t\to 0} \lambda(t) \cdot g = f$, and (2) the total degree of λ is $O(exp(m, \langle \deg(g) \rangle))$, where $\langle \deg(g) \rangle$ denotes the bitlength of the degree of g.*

It follows that if $\{f_n\}$ is a stable degeneration of $\{g_n\} \in$ VP, then $\{f_n\}$ is a p-definable one-parameter degeneration of $\{g_n\}$.

See the full version for an analogous result for reductive algebraic groups. We formally propose a question that has ramifications on the Stable-VP vs. VP question (cf. Section 1).

▶ **Question 19.** For some positive constant a, does there exist a stable degeneration $\{f_n\}$ of some $\{g_n\} \in$ VP, with an $\Omega(2^{n^a})$ lower bound on the degree of the canonical Kempf-one-parameter subgroup [15] λ_n driving $\{g_n\}$ to $\{f_n\}$?

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[7] To get the proof to work in characteristic 2 as well, simply use the Hamilton cycle polynomial $HC(X) = \sum_{\text{k-cycles } \sigma \in S_k} \prod_{i\in[k]} x_{i,\sigma(i)}$ instead, which is VNP-complete in any characteristic [27].

5 Newton degeneration of perfect matching Pfaffians

In this section, we construct an explicit family $\{f_n\}$ of polynomials such that f_n can be approximated infinitesimally closely by symbolic determinants of size n, but conjecturally requires size $\Omega(n^{2+\delta})$ to be computed by a symbolic determinant, for a small enough positive constant δ. However, the family $\{f_n\}$ turns out to be in $\mathrm{VP}_{\mathrm{ws}}$.

Suppose we have a simple undirected graph $G = (V, E)$ where $V = [n]$. Let $\{x_e \mid e \in E\}$ be a set of variables. The Tutte matrix of G is the $n \times n$ skew-symmetric matrix T_G such that, if $(i, j) = e \in E$, with $i < j$, then $T_G(i, j) = x_e$ and $T_G(j, i) = -x_e$; otherwise $T_G(i, j) = 0$. For a skew-symmetric matrix T, the determinant of T is a perfect square, and the square root of $\det(T)$ is called the Pfaffian of T, denoted $\mathrm{pf}(T)$. We call $\mathrm{pf}(T_G)$ the *perfect matching Pfaffian* of the graph G, and $\mathrm{pf}(T_G) = \sum_P \mathrm{sgn}(P) \prod_{e\in P} x_e$, where the sum is over all perfect matchings P of G, and $\mathrm{sgn}(P)$ takes ± 1 in a suitable manner. It is well-known that $\mathrm{pf}(T_G) \in \mathrm{VP}_{\mathrm{ws}}$.

Note that $\mathrm{NPT}(\mathrm{pf}(T_G))$ is the perfect matching polytope of G, which has the following description by Edmonds. For any $S \subseteq V$, we use $e \sim S$ to denote that e lies at the border of S. When $S = \{i\}$, we may write $e \sim i$ instead of $e \sim \{i\}$.

▶ **Theorem 20** (Edmonds, [9]). *The perfect matching polytope of a graph G is characterized by the following constraints:*

$$\begin{aligned} &(a) \quad \forall e \in E, x_e \geq 0; \\ &(b) \quad \forall i \in V, \sum_{e\in E, e\sim i} x_e = 1; \qquad\qquad (1) \\ &(c) \quad \forall C \subseteq V, |C| > 1 \text{ is odd}, \sum_{e\in E, e\sim C} x_e \geq 1. \end{aligned}$$

We shall refer to constraints of type (c) in Equation (1) as "odd-size constraints."

▶ **Theorem 21** (Kaltofen and Koiran, [13, Corollary 1]). *Given $f, g, h \in \mathbb{F}[\mathbf{x}]$, suppose $h = f/g$, and f and g are in VP_{ws}. Then $h \in VP_{ws}$.*

▶ **Theorem 22.** *For any graph G and any face Q of $\mathrm{NPT}(\mathrm{pf}(T_G))$, $\mathrm{pf}(T_G)|_Q \in VP_{ws}$.*

Proof. Thanks to Edmonds' description, any face of $\mathrm{NPT}(\mathrm{pf}(T_G))$ is obtained by setting some of the inequalities in Equation (1) to equalities. As setting $x_e = 0$ amounts to consider some graph G' with e deleted from G, the bottleneck is to deal with the odd-size constraints.

Suppose the face Q is obtained via setting the odd-size constraints corresponding to $C_1, \ldots, C_s$ to equalities, where $C_i \subseteq V$. Note that $s = \mathsf{poly}(n)$, because the dimension of $\mathrm{NPT}(\mathrm{pf}(T_G))$ is polynomially bounded, thus any face can be obtained by setting polynomially many constraints to equalities. Let y be a new variable. For any edge $e \in E$, let the number of $i \in [s]$ s.t. e lies at the border of C_i be k_e. Then transform x_e to $x_e y^{k_e}$. Let the skew-symmetric matrix after the transformation be $\widetilde{T_G}$. Since each perfect matching touches the border of every C_i at least once, y^s divides $\mathrm{pf}(\widetilde{T_G})$, so $f := \frac{\mathrm{pf}(\widetilde{T_G})}{y^s}$ is a polynomial. Furthermore, the y-free terms in f corresponds to those perfect matchings that touch each border exactly once. Thus, setting y to zero in f gives $\mathrm{pf}(T_G)|_Q$.

f is in $\mathrm{VP}_{\mathrm{ws}}$, because $\mathrm{pf}(\widetilde{T_G})$ and y^s are in $\mathrm{VP}_{\mathrm{ws}}$, and use Theorem 21. ◀

Construction of an explicit family. Now we turn to the construction of an explicit family $\{f_n\}$ mentioned in the beginning of this section. We assume that the base field $\mathbb{F} = \mathbb{C}$.

First, we give a randomized procedure for constructing f_n:

1. Fix a small enough constant $a > 0$, and let l be the nearest odd integer to n^a. Fix odd-size disjoint subsets $C_1, \dots, C_k \subseteq [n]$, $k = \lfloor n^{1-a} \rfloor$, of size l. For example, we can let $C_1 = \{1, \dots, l\}$, $C_2 = \{l+1, \dots, 2l+1\}$, etc.
2. Choose a random regular non-bipartite graph G_n on n nodes with degree (say) $\sqrt{n}$.
3. Let Q be the face of $\mathrm{NPT}(\mathrm{pf}(T_G))$ obtained by setting the odd-size constraints corresponding $C_1, \dots, C_k$ to equalities.
4. Let $f_n = \det(T_G)|_Q$.

Then, f_n can be approximated infinitesimally closely by symbolic determinants of size n; cf. the proof of Theorem 7. By Theorem 22, f_n can be expressed as a symbolic determinant of $\mathsf{poly}(n)$ size. But:

▶ **Conjecture 23.** If $a > 0$ is small enough, then, with a high probability, f_n cannot be expressed as a symbolic determinant of size $\leq n^{2+\delta}$, for a small enough positive constant δ.

This is because, with high probability, the coefficient complexity of Q is $\Omega(n^{1-a+1/2})$, and hence interpolation, which lies at the heart of the algorithm in Theorem 22, can be expected to incur $\Omega(n^{1+\delta})$ blow-up in the determinantal size, for a small enough constant $\delta > 0$. Specifically, if we unwind Strassen's proof of division gate elimination, the number of terms in the interpolation is the degree of the polynomial times the degree of the denominator. The former number gets increased by a multiplicative factor of the sum of absolute values of variable coefficients in the equations, and the latter number is the sum of absolute values of constant terms. It follows that the coefficient complexity determines the blow-up factor.

To get an explicit family $\{f_n\}$, we let G_n be a pseudo-random graph, instead of a random graph. Some suggestions can be found in the full version.

6 Newton degenerations of generic semi-invariants of quivers

In this section we prove Theorem 2 for the generalized Kronecker quivers. Due to page constraints, proofs for k-subspace quivers and A-D-E Dynkin quivers are in the full version. We assume familiarity with the basic notions of the representation theory of quivers; cf. [6, 8].

6.1 Newton degeneration to faces with small coefficient complexity

We begin by observing that the technique used to prove Theorem 22 can be generalized further. In the proof of Theorem 22, due to Edmonds' description of the perfect matching polytope, every face has a "small" description, by a set of linear equalities whose coefficients are polynomially bounded in magnitude.

For a face Q of a polytope P, we say that a set of linear equalities E characterizes Q with respect to P, if the description of P together with that of E characterizes Q. For E, let $\mathrm{coeff}(E)$ be the sum of the absolute values of the coefficients of the linear equalities in E. We define the *coefficient complexity* of Q as the minimum of $\mathrm{coeff}(E)$ over the linear equality sets E that characterize Q with respect to P. Adapting the proof of Theorem 22 we easily get the following; see the full version for a proof.

▶ **Theorem 24.** *Suppose $f \in \mathbb{F}[x_1, \dots, x_n]$ can be computed by a (weakly skew) arithmetic circuit of size s. Let Q be a face of $\mathrm{NPT}(f)$ whose coefficient complexity is $\mathsf{poly}(n)$. Then $f|_Q$ can be computed by a (weakly skew) arithmetic circuit of size $\mathsf{poly}(s, n)$.*

▶ Remark. If Q has $\mathsf{poly}(n)$ coefficient complexity, then it can be shown that $f|_Q$ is a one-parameter degeneration of f of $\mathsf{poly}(n)$ degree. Hence, Thm. 24 also follows from Lem. 17.

6.2 Generic semi-invariants of generalized Kronecker quivers

We now discuss Theorem 2 for the m-Kronecker quiver; the proof is deferred to the full version. The m-Kronecker quiver is the graph with two vertices s and t, with m arrows pointing from s to t. When $m \geq 3$, this quiver is wild.

Any tuple of m $n \times n$ matrices is a linear representation of the m-Kronecker quiver of dimension vector (n, n). Let $\mathbb{F}[x_{i,j}^{(k)}]$ denote the ring of polynomials in the variables $x_{i,j}^{(k)}$, where $i, j \in [n]$, and $k \in [m]$. For $k \in [m]$, let $X_k = (x_{i,j}^{(k)})$ denote the variable $n \times n$ matrix, whose (i, j)-th entry is $x_{i,j}^{(k)}$. Let $R(n, m)$ consist of those polynomials in $\mathbb{F}[x_{i,j}^{(k)}]$ that are invariant under the action of every $(A, C) \in \mathrm{SL}(n, \mathbb{F}) \times \mathrm{SL}(n, \mathbb{F})$, which sends $(X_1, \dots, X_m)$ to $(AX_1C^{-1}, \dots, AX_mC^{-1})$. $R(n, m)$ is the ring of semi-invariants for the m-Kronecker quiver for dimension vector (n, n) or "*matrix semi-invariants*" due to their similarity with the well-known matrix invariants. The following is proved using Theorem 24:

▶ **Theorem 25.** *The Newton degeneration of a generic semi-invariant of the m-Kronecker quiver with dimension vector (n, n) and degree dn to an arbitrary face can be computed by a weakly skew arithmetic circuit of size* $\mathsf{poly}(d, n)$.

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