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The integer cohomology of toric Weyl arrangements

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Abstract

A toric arrangement is a finite set of hypersurfaces in a complex torus, every hypersurface being the kernel of a character. In the present paper we prove that if $\mathcal{T}_{\widetilde{W}}$ is the toric arrangement defined by the *cocharacters* lattice of a Weyl group $\widetilde{W}$, then the integer cohomology of its complement is torsion free.

Keywords:
Arrangement of hyperplanes, toric arrangements, CW complexes, Salvetti complex, Weyl groups, integer cohomology

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Introduction

Let $T = (\mathbb{C}^*)^n$ be a complex torus and $X \subset Hom(T, \mathbb{C}^*)$ be a finite set of characters of T. The kernel of every $\chi \in X$ is a hypersurface of T:

$$H_\chi := \{t \in T \mid \chi(t) = 1\}.$$

Then X defines on T the *toric arrangement*:

$$\mathcal{T}_X := \{H_\chi, \chi \in X\}.$$

Let $\mathcal{R}_X$ be the *complement* of the arrangement:

$$\mathcal{R}_X := T \setminus \bigcup_{\chi \in X} H_\chi$$

The geometry and topology of $\mathcal{R}_X$ have been studied by many authors, see for instance [8], [9], [4], [7], [12] and [13]. In particular Looijenga (see [10]) and De Concini and Procesi (see [3]) computed the De Rham cohomology of $\mathcal{R}_X$ and, recently, Moci and Settepanella (see [14]) described a regular CW-complex homotopy equivalent to $\mathcal{R}_X$. This complex is similar to the one introduced by Salvetti (see [15]) for the complement of hyperplane arrangements.

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If $\mathcal{T}_{\widetilde{W}}$ is the toric arrangement associated to an affine Weyl group $\widetilde{W}$, the complex $T(\widetilde{W})$ homotopic to the complement

$$\mathcal{R}_W := T \setminus \bigcup_{H \in T_{\widetilde{W}}} H$$

admits a very nice description which generalizes a construction introduced in [16] and [6]. In their paper Moci and Settepanella conjectured that the integer cohomology of $T(\widetilde{W})$ (equivalently $\mathcal{R}_W$) is torsion free. Hence it coincides with the De Rham cohomology described in [3] and it is known since the Betti numbers can be easily computed using results in [11].

In the present paper we prove this conjecture generalizing to toric arrangements a well known result for hyperplane ones. Indeed Arnol'd proved that the integer cohomology of braid arrangement is torsion free in 1969 (see [1]).

In order to prove it we use a filtration introduced in [5] and generalized to braid arrangements in [17] (see subsection 1.2).

In Section 2 we prove that the above filtration involves complexes with torsion free cohomology. While in Section 3 we rewrite it for toric arrangements and we prove the main result of the paper:

Theorem 1 *The integer (co)-homology of the complement $\mathcal{R}_W$ is torsion free.*

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1 Notations and recalls

In this section we recall basic construction about affine and toric arrangements coming from Coxeter systems.

1.1 Salvetti's complex for Coxeter arrangements

Let (W, S) be the Coxeter system associated to the finite reflection group W and

$$\mathcal{A}_W = \{H_{ws_iw^{-1}} \mid w \in W \text{ and } s_i \in S\}$$

the arrangement in $\mathbb{C}^n$ obtained by complexifying the reflection hyperplanes of W, where, in a standard way, the hyperplane $H_{ws_iw^{-1}}$ is simply the hyperplane fixed by the reflection ws_iw^{-1}.

It is well known (see, for instance, [6] [16]) that the k-cells of Salvetti's complex $C(W)$ for arrangements $\mathcal{A}_W$ are of the form $E(w, \Gamma)$ with $\Gamma \subset S$ of cardinality k and $w \in W$.

While the integer boundary map can be expressed as follows:

$$\partial_k(E(w,\Gamma)) = \sum_{s_j\in\Gamma}\sum_{\beta\in W_\Gamma^{\Gamma\setminus\{s_j\}}}(-1)^{l(\beta)+\mu(\Gamma,s_j)}E(w\beta,\Gamma\setminus\{s_j\}) \tag{1}$$

where W_Γ is the group generated by Γ,

$$W_\Gamma^{\Gamma\setminus\{\sigma\}} = \{w\in W_\Gamma : l(ws) > l(w)\forall s\in\Gamma\setminus\{\sigma\}\}$$

and $\mu(\Gamma,s_j) = \sharp\{s_i\in\Gamma | i\leq j\}$. Here $l(w)$ stands for the length of w.

Remark 1.1 *Instead of the co-boundary operator we prefer to describe its dual, i.e. we define the boundary of a k-cell $E(w,\Gamma)$ as a linear combination of the $(k-1)$-cells which have $E(w,\Gamma)$ in theirs co-boundary, with the same coefficient of the co-boundary operator. We make this choice since the boundary operator has a nicer description than co-boundary operator in terms of the elements of W.*

This description holds also for Coxeter systems $(\widetilde{W},\widetilde{S})$ associated to Weyl groups $\widetilde{W}$.

1.2 A filtration for the complex $(C(W),\partial)$

It's known (see [2]) that for all $\Gamma\subset S$ the group W splits as

$$W = W^\Gamma W_\Gamma$$

with

$$W^\Gamma = \{w^\Gamma\in W \mid l(w^\Gamma s_i) > l(w^\Gamma) \text{ for all } s_i\in W_\Gamma\}. \tag{2}$$

If $w = w^\Gamma w_\Gamma$, $w^\Gamma\in W^\Gamma$ and $w_\Gamma\in W_\Gamma$, then $l(w\beta) = l(w^\Gamma)+l(w_\Gamma\beta)$ $\forall\beta\in W_\Gamma$ and the boundary map verifies

$$\partial(E(w,\Gamma)) = w^\Gamma.\partial(E(w_\Gamma,\Gamma)).$$

In [17] (see also [5]) author defines a map of complexes

$$i_m := i : \bigoplus_{j=1}^{m_1} C(W_{S_{m-1}}) \longrightarrow C(W)$$

as follows

$$i(j.E(w_{S_{m-1}},\Gamma)) = i(W^{S_{m-1}}(j).E(w_{S_{m-1}},\Gamma)) =$$
$$i(w^{S_{m-1}}.E(w_{S_{m-1}},\Gamma) = w^{S_{m-1}}.i(E(w_{S_{m-1}},\Gamma)) = w^{S_{m-1}}.E(w_{S_{m-1}},\Gamma) = E(w,\Gamma).$$

Where m_1 is the cardinality of $W^{S_{m-1}}$, $w^{S_{m-1}} = W^{S_{m-1}}(j)$ its j-th element in a fixed order and $S_h = \{s_1,\cdots,s_h\}\subset S = \{s_1,\cdots,s_m\}$.

The cokernel of the map i is the complex $F^1_m(W)$ having as basis all $E(w,\Gamma_1)$ for $w\in W$ and $\Gamma_1\subset S$ s.t. $s_m\in\Gamma_1$.

She iterates this construction getting maps

$$\begin{aligned}
&i_m[k] := i : \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} C(W_{S_{m-k-1}})[k] \longrightarrow F_m^k(W),\\
&i(w^{S_{m-k-1}}.(E(w_{S_{m-k-1}}, \Gamma))) = w^{S_{m-k-1}}.i(E(w_{S_{m-k-1}}, \Gamma))\\
&= E(w, \Gamma \cup \{s_m, \cdots, s_{m-k+1}\})
\end{aligned}$$

Each $i_m[k]$ gives rise to the exact sequence of complexes

$$0 \longrightarrow \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} C(W_{S_{m-k-1}})[k] \xrightarrow{i} F_m^k(W) \xrightarrow{j} F_m^{k+1}(W) \longrightarrow 0. \tag{3}$$

It is possible to filter the complex $F_m^0(W) = C(W)$ in a similar way through maps:

$$\begin{aligned}
&i^m[k] := i : \bigoplus_{j=1}^{m^1 \cdots m^{k+1}} C(W_{S^{k+1}})[k] \longrightarrow F_m^k(W),\\
&i(w^{S^{k+1}}.(E(w_{S^{k+1}}, \Gamma))) = w^{S^{k+1}}.i(E(w_{S^{k+1}}, \Gamma)) =\\
&= E(w, \{s_1, \cdots, s_k\} \cup \Gamma\})
\end{aligned} \tag{4}$$

for $0 \leq k \leq m$, $S^k = \{s_{k+1}, \cdots, s_m\}$ and m^i the cardinality of $W_{S^{i-1}}^{S^i}$.

1.3 Salvetti's complex for toric Weyl arrangements

Let Φ be a root system, $\langle \Phi^\vee \rangle$ be the lattice spanned by the coroots, and Λ be its dual lattice (which is called the *cocharacters* lattice). Then we define a torus $T = T_\Lambda$ having Λ as group of characters.

If $\widetilde{W}$ is the affine Weyl group associated to Φ, we can regard Λ as a subgroup of $\widetilde{W}$, acting by translations. It is well known that $\widetilde{W}/\Lambda \simeq W$, where W is the finite reflection group associated to $\widetilde{W}$. As a consequence, the toric Weyl arrangement can be described as:

$$\mathcal{T}_{\widetilde{W}} = \{H_{[w]s_i[w^{-1}]} \mid w \in W \text{ and } s_i \in \widetilde{S}\}$$

where two hypersurfaces $H_{[w]s_i[w^{-1}]}$ and $H_{[\overline{w}]s_i[\overline{w}^{-1}]}$ are equal if and only if there is a translation $t \in \Lambda$ such that $tws_i(tw)^{-1} = \overline{w}s_i\overline{w}^{-1}$, i.e. $\overline{w} = tw$.

In [14] authors prove that the complement

$$\mathcal{R}_W := T \setminus \bigcup_{H \in T_{\widetilde{W}}} H$$

has the same homotopy type of a CW-complex $T(\widetilde{W})$ which admits a description similar to $C(W)$.

Indeed the k-cells of $T(\widetilde{W})$ correspond to elements $E([w], \Gamma)$ where $[w] \in \widetilde{W}/\Lambda \simeq W$ is an equivalence class with one and only one representative $w \in W$ and $\Gamma = \{s_{i_1}, \ldots, s_{i_k}\}$ is a subset of cardinality k in $\widetilde{S}$.

The integer boundary operator is

$$\partial_k(E([w],\Gamma)) = \sum_{\sigma\in\Gamma}\sum_{\beta\in W_\Gamma^{\Gamma\setminus\{\sigma\}}}(-1)^{l(\beta)+\mu(\Gamma,\sigma)}E([w\beta],\Gamma\setminus\{\sigma\}). \tag{5}$$

Let $\Gamma\subset\widetilde{S}$ be a proper subset and W_Γ be the finite reflection group generated by Γ. The group

$$(\widetilde{W}/\Lambda)_\Gamma = \{[w]\in\widetilde{W}/\Lambda \mid w\in W_\Gamma\}\simeq W_\Gamma$$

is a well defined subgroup of $\widetilde{W}/\Lambda$. As in the finite case, we get

$$\widetilde{W}/\Lambda = (\widetilde{W}/\Lambda)^\Gamma(\widetilde{W}/\Lambda)_\Gamma$$

and the toric boundary map verifies

$$\partial(E([w],\Gamma)) = [w^\Gamma].\partial(E([w_\Gamma],\Gamma))$$

where $[w^\Gamma]\in(\widetilde{W}/\Lambda)^\Gamma$, $[w_\Gamma]\in(\widetilde{W}/\Lambda)_\Gamma$ and $[w]=[w^\Gamma][w_\Gamma]=[w^\Gamma w_\Gamma]$.

Let us remark that $(\widetilde{W}/\Lambda)_\Gamma$ is isomorphic to a subgroup of W which is not, in general, a parabolic one. In these cases the set $(\widetilde{W}/\Lambda)^\Gamma$ doesn't admit a description similar to the one in (2).

Our main interest in the sequel of this paper is to construct a filtration for $T(\widetilde{W})$ similar to the one in subsection 1.2. Also if it is not necessary to know an explicit description of $(\widetilde{W}/\Lambda)^\Gamma$ in order to filter the complex $T(\widetilde{W})$, nevertheless we believe that it would be useful to know a little bit more about it to have a better understanding of our construction. In particular, if $\widetilde{S}=\{s_0,\dots,s_m\}$, we are interested in the cases in which $\Gamma=\{s_k,\dots,s_m\}$ or $\Gamma=\{s_0,\dots s_h\}$.

It is a simple remark that, if $s_0\notin\Gamma$, then $(\widetilde{W}/\Lambda)_\Gamma\simeq W_\Gamma$ is a parabolic subgroup of W. While the case $s_m\notin\Gamma$ is a little bit more complicated. Since to remove s_0 or s_m is perfectly symmetric for $\widetilde{W}=\widetilde{A}_m,\widetilde{C}_m,\widetilde{D}_m,\widetilde{E}_6,\widetilde{E}_7$, then in these cases we always get that $(\widetilde{W}/\Lambda)_\Gamma\simeq W_\Gamma$ is a parabolic subgroup of W. Hence in the above situations $(\widetilde{W}/\Lambda)^\Gamma\simeq W^\Gamma$ admits a description as in (2).

Otherwise $W_{\widetilde{S}\setminus\{s_m\}}$ is still a finite reflection group but it is not of type W. For example if $\widetilde{W}=\widetilde{B}_m$ then $W_{\widetilde{S}\setminus\{s_m\}}=D_m$ which is not B_m. In these cases if $\Gamma\subset\widetilde{S}$ is a given subset with $s_m\notin\Gamma$ and $s_0\in\Gamma$, then $(\widetilde{W}/\Lambda)_\Gamma\simeq W_\Gamma$ is a parabolic subgroup of $W_{\widetilde{S}\setminus\{s_m\}}$ and, by [11], we have exactly

$$\frac{\mid W\mid}{\mid W_{\widetilde{S}\setminus\{s_m\}}\mid}$$

copies of $W_{\widetilde{S}\setminus\{s_m\}}$ in W.

Let W' be the subgroup of W such that $W'\simeq W_{\widetilde{S}\setminus\{s_m\}}\simeq(\widetilde{W}/\Lambda)_{\widetilde{S}\setminus\{s_m\}}$ then $W^{\widetilde{S}\setminus\{s_m\}}$ will denote the subset of W such that $W=W^{\widetilde{S}\setminus\{s_m\}}W'$ and we get

$$(\widetilde{W}/\Lambda)^\Gamma\simeq W^{\widetilde{S}\setminus\{s_m\}}W^\Gamma_{\widetilde{S}\setminus\{s_m\}}$$

where $W^\Gamma_{S\setminus\{s_m\}}$ is the subset of $W_{\widetilde{S}\setminus\{s_m\}}$ described in (2).

2 The cohomology of complexes $F_n^k(W)$

It is well known that the integer homology, and hence cohomology, of complexes $C(W)$ is torsion free, while the (co)-homology $H^*(F_n^k,\mathbb{Z})$ is not known. In this section we will prove that it is torsion free.

As above we will consider the boundary map instead of the (co)-boundary one.
The exact sequences (3) give rise to the corresponding long exact sequences in homology

$$\cdots \longrightarrow H_{*+1}(F_m^k(W),\mathbb{Z}) \xrightarrow{\Delta_*} \bigoplus_{j=1}^{m_1\cdots m_k} H_{*-k}(C(W_{S_{m-k}}),\mathbb{Z}) \xrightarrow{i_*} \\ \xrightarrow{i_*} H_*(F_m^{k-1}(W),\mathbb{Z}) \xrightarrow{j_*} H_*(F_m^k(W),\mathbb{Z}) \longrightarrow \cdots \tag{6}$$

where the map Δ_* is induced by the map on complexes:

$$\begin{aligned} \Delta : F_m^k(W) &\longrightarrow \bigoplus_{j=1}^{m_1\cdots m_k} C(W_{S_{m-k}}) \\ \Delta(E(w,\Gamma\cup S^{m-k})) &= \sum_{\beta\in W_{\Gamma\cup S^{m-k}}^{\Gamma\cup S^{m-k+1}}} (-1)^{l(\beta)} E(w\beta,\Gamma). \end{aligned} \tag{7}$$

To simplify notation from now on we will use

$$l = m-k-1$$

and $\bigoplus C(W_{S_{m-k}})$ instead of $\bigoplus_{j=1}^{m_1\cdots m_k} C(W_{S_{m-k}})$ since the number of copy $\prod_{i=1}^k m_i$ is completely determined by S_{m-k}.
We have the following theorem.

Theorem 2 *The integer (co)-homology of complexes $F_m^k(W)$ is torsion free for all $k\le m$.*

We need the following key Lemma.

Lemma 2.1 *Let $v\in F_m^k(W)$ be a boundary then one of the following occurs:*

i) $v\in i(\bigoplus C(W_{S_l})[k])$
ii) $v\in F_m^k(W)\setminus i(\bigoplus C(W_{S_l})[k])$
iii) v is a sum of two boundaries $v'\in i(\bigoplus C(W_{S_l})[k])$ and $v''\in F_m^k(W)\setminus i(\bigoplus C(W_{S_l})[k])$.

Proof. By construction any chain $v\in F_m^k(W)$ is a sum of two chains

$$v = v' + v''$$

the first one in $i(\bigoplus C(W_{S_l})[k])$ and the second one in $F_m^k(W)\setminus i(\bigoplus C(W_{S_l})[k])$.
Let v be a boundary. If v' (v'') is zero then $ii)$ ($i)$) follows.

Let v' and v'' both not zero. Ordering in a suitable way rows and columns of the boundary matrix, we get a block matrix as follows:

$$\left[\begin{array}{cc} \bigoplus i(\partial C(W_{S_l})[k]) & B_1 \\ 0 & B_2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] = \partial(F_m^k(W) \setminus i(\bigoplus C(W_{S_l})[k]).$$

Then we can diagonalize the matrix by row and column operations in such a way that the rows of the first (second) block are combined only with rows in the same block.
As consequence any element v which is in the boundary is written as a sum of two boundaries , one obtained by combinations of row in the first block, i.e. a combination of elements in $i(\bigoplus C(W_{S_l})[k])$, and the second one by elements in $F_m^k(W) \setminus i(\bigoplus C(W_{S_l})[k])$. $\square$

Remark 2.2 *If v' and v'' are boundaries in $F_m^k(W)$ as in the above Lemma, then $v' \in i(\bigoplus \partial C(W_{S_l})[k])$ while, obviously, v'' is a linear combination of elements in $F_m^k(W) \setminus i(\bigoplus C(W_{S_l})[k])$, but it is not in its boundary.*

Proof of Theorem 2 The integer cohomology of the complex $F_m^0(W) = C(W)$ is torsion free. By induction let us assume that $H^*(F_m^{k-1}(W), \mathbb{Z})$, and hence $H_*(F_m^{k-1}(W), \mathbb{Z})$, are torsion free.

As the sequence (6) is exact and $H_*(C(W_{S_{m-k}}), \mathbb{Z})$ and $H_*(F_m^{k-1}(W), \mathbb{Z})$ are torsion free, then $H_*(F_m^k(W), \mathbb{Z})$ (and hence $H^*(F_m^k(W), \mathbb{Z})$) is torsion free if and only if the image of i_* doesn't give rise to p-torsion for $p \in \mathbb{Z}$, i.e.

$$p[v] \in i_*(\bigoplus_{j=1}^{m_1 \cdots m_k} H_*(C(W_{S_{m-k}}), \mathbb{Z})) \Longleftrightarrow [v] \in i_*(\bigoplus_{j=1}^{m_1 \cdots m_k} H_*(C(W_{S_{m-k}}), \mathbb{Z})).$$

Let $[v]$ be a generator in the free module $H_*(F_m^{k-1}(W), \mathbb{Z})$. By construction

$$[v] = z' + z'' + \partial_*(F_m^{k-1}(W))$$

for $z' \in i(\bigoplus C(W_{S_l})[k])$ and $z'' \in F_m^{k-1}(W) \setminus i(\bigoplus C(W_{S_l})[k])$.
Let us assume

$$p[v] = pz' + pz'' + \partial_*(F_m^{k-1}(W)) \in i_*(\bigoplus H_*(C(W_{S_{m-k}}), \mathbb{Z})).$$

Then $p[v]$ has at list one representative in the image $i(\bigoplus C(W_{S_l})[k])$ and hence there is an element

$$\omega = \omega' + \omega'' \in \partial_*(F_m^{k-1}(W))$$

such that $pz' + pz'' + \omega \in i(\bigoplus C(W_{S_l})[k])$, i.e. $\omega' \in i(\bigoplus C(W_{S_l})[k])$ and $\omega'' = -pz''$.

By Lemma 2.1 we get that $-\omega'' = pz'' \in \partial_*(F_m^{k-1}(W))$ and hence $z'' \in \partial_*(F_m^{k-1}(W))$ since $H_*(F_m^{k-1}(W))$ has no torsion by inductive hypothesis. Then

$$[v] = z' + z'' + \partial_*(F_m^{k-1}(W)) = z' + \partial_*(F_m^{k-1}(W))$$

i.e. $[v] \in i_*(\bigoplus H_*(C(W_{S_{m-k}}), \mathbb{Z}))$ $\square$

Remark 2.3 *Obviously Theorem 2 holds also for complexes $F^k_m(W)$ obtained filtering with the inclusions in (4)*

An important consequence of the above theorem is that maps Δ_* are map between finitely generated free modules such that

$$p[v] \in \Delta_*(H_*(F^k_m(W),\mathbb{Z})) \Longleftrightarrow [v] \in \Delta_*(H_*(F^k_m(W),\mathbb{Z})).$$

A map between two free modules which satisfies the above condition will be called a *one-free* map and it can be diagonalized as:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where I is the identity matrix. It is a simple remark that composition of one-free maps is still a one-free map. This will be useful in the next section.

3 The integer cohomology of $\mathcal{R}_W$

In this section we prove that the (co)-homology of $T(\widetilde{W})$ (i.e. $\mathcal{R}_W$) is torsion free. In order to do it we construct a filtration of $T(\widetilde{W})$ similar to the one of $C(W)$.

3.1 A filtration for the complex $(T(\widetilde{W}),\partial)$

Let $\widetilde{S} = \{s_0, \cdots, s_m\}$ be the system of generators of $\widetilde{W}$ and W the finite group associated. We will keep the notation $S^k = \{s_{k+1},\ldots,s_m\} \subset \widetilde{S}$ while we introduce the new one $\widetilde{S}_h = \{s_0,\ldots,s_h\} \subset \widetilde{S}$.

Let us consider the natural inclusion

$$i_m := i : \bigoplus_{j=1}^{m_1} C(W_{\widetilde{S}_{m-1}}) \longrightarrow T(\widetilde{W}),$$

defined as:

$$\begin{aligned} &i(j.E(w_{\widetilde{S}_{m-1}},\Gamma)) = i(W^{\widetilde{S}_{m-1}}(j).E(w_{\widetilde{S}_{m-1}},\Gamma)) = \\ &i(w^{\widetilde{S}_{m-1}}.E(w_{\widetilde{S}_{m-1}},\Gamma)) = [w^{\widetilde{S}_{m-1}}].E([w_{\widetilde{S}_{m-1}}],\Gamma) = E([w],\Gamma) \end{aligned}$$

where m_1 is the cardinality of the set $W^{\widetilde{S}_{m-1}} = W^{\widetilde{S}\setminus\{s_m\}}$ defined in subsection 1.3 and $w^{\widetilde{S}_{m-1}} = W^{S_{m-1}}(j)$ its j-th element in a fixed order.

Let us remark that m_1 could be also equal to 1 depending on the type of $\widetilde{W}$ as seen in subsection 1.3.

The cokernel of the map i is the toric complex $F^1_m(\widetilde{W})$ having as basis all $E([w],\Gamma_1)$ for $w \in W$ and $\Gamma_1 \subset \widetilde{S}$ with $|\Gamma_1| \leq m$ s.t. $s_m \in \Gamma_1$.

We can iterate this construction getting maps

$$\begin{aligned} &i_m[k] := i : \bigoplus_{j=1}^{m_1\cdots m_{k+1}} C(W_{\widetilde{S}_l})[k] \longrightarrow F^k_m(\widetilde{W}), \\ &i(w^{\widetilde{S}_l}.E(w_{\widetilde{S}_l},\Gamma)) = [w^{\widetilde{S}_l}].E([w_{\widetilde{S}_l}],\Gamma) = E([w],\Gamma\cup S^{m-k}) \end{aligned}$$

with $l = m - k - 1$.

Each $i_m[k]$ gives rise to the exact sequence of complexes

$$0 \longrightarrow \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} C(W_{\widetilde{S}_l})[k] \xrightarrow{i} F_m^k(\widetilde{W}) \xrightarrow{j} F_m^{k+1}(\widetilde{W}) \longrightarrow 0. \tag{8}$$

In a similar way we can filter using the inclusion:

$$\begin{aligned} &i^m := i : C(W_{S^0}) \longrightarrow T(\widetilde{W}), \\ &i(E(w,\Gamma)) = E([w],\Gamma). \end{aligned}$$

Here $C(W_{S^0})$ is the classical Salvetti's complex for the finite reflection group $W_{S^0} = W$. The cokernel of the map i is the toric complex $F_m^1(\widetilde{W})$ having as basis all $E([w],\Gamma_1)$ for $w \in W$ and $\Gamma_1 \subset \widetilde{S}$ with $\mid \Gamma_1 \mid \leq m$ s.t. $s_0 \in \Gamma_1$.

We can iterate this construction getting maps

$$\begin{aligned} &i^m[k] := i : \bigoplus_{j=1}^{m^1 \cdots m^k} C(W_{S^k})[k] \longrightarrow F_m^k(\widetilde{W}), \\ &i(w^{S^k}.(E(w_{S^k},\Gamma))) = [w^{S^k}].i(E([w_{S^k}],\Gamma)) = E([w],\Gamma \cup \widetilde{S}_{k-1}). \end{aligned}$$

Each $i^m[k]$ gives rise to the exact sequence of complexes

$$0 \longrightarrow \bigoplus_{j=1}^{m^1 \cdots m^k} C(W_{S^k})[k] \xrightarrow{i} F_m^k(\widetilde{W}) \xrightarrow{j} F_m^{k+1}(\widetilde{W}) \longrightarrow 0. \tag{9}$$

3.2 Computation of integer cohomology

The exact sequences (8) give rise to the corresponding long exact sequences in homology

$$\begin{aligned} \cdots \longrightarrow H_{*+1}(F_m^{k+1}(\widetilde{W}),\mathbb{Z}) \xrightarrow{\widetilde{\Delta}_*} \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} H_{*-k}(C(W_{\widetilde{S}_l}),\mathbb{Z}) \xrightarrow{i_*} \\ \xrightarrow{i_*} H_*(F_m^k(\widetilde{W}),\mathbb{Z}) \xrightarrow{j_*} H_*(F_m^{k+1}(\widetilde{W}),\mathbb{Z}) \xrightarrow{\widetilde{\Delta}_*} \cdots . \end{aligned}$$

The map $\widetilde{\Delta}_*$ is the one induced by maps on complexes:

$$\begin{aligned} &\widetilde{\Delta} : F_m^{k+1}(\widetilde{W}) \longrightarrow \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} C(W_{\widetilde{S}_l}) \\ &\widetilde{\Delta}(E([w],\Gamma \cup S^l)) = \sum_{\beta \in W_{\Gamma \cup S^l}^{\Gamma \cup S^{l+1}}} (-1)^{l(\beta)} E([w\beta],\Gamma). \end{aligned} \tag{10}$$

If $H_*(F_m^{k+1}(\widetilde{W}),\mathbb{Z})$ are torsion free, then $H_*(F_m^k(\widetilde{W}),\mathbb{Z})$ are torsion free if and only if the maps $\widetilde{\Delta}_*$ are one-free maps, i.e. if a generator $[u] \in \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} H_{*-k}(C(W_{\widetilde{S}_l}),\mathbb{Z})$ is such that $p[u] \in Im\widetilde{\Delta}_*$ for an integer $p \in \mathbb{Z}$, then $[u] \in Im\widetilde{\Delta}_*$. We will prove it through an inductively argument.

When $k = m-1$ we get the last long exact sequence in homology

$$0 \longrightarrow \bigoplus_{j=1}^{m_1\cdots m_m} H_1(C(W_{\widetilde{S}_0}),\mathbb{Z}) \xrightarrow{i_*} H_m(F_m^{m-1}(\widetilde{W}),\mathbb{Z}) \xrightarrow{j_*} H_m(F_m^m(\widetilde{W}),\mathbb{Z}) \xrightarrow{\widetilde{\Delta}_*}$$

$$\xrightarrow{\widetilde{\Delta}_*} \bigoplus_{j=1}^{m_1\cdots m_m} H_0(C(W_{\widetilde{S}_0}),\mathbb{Z}) \xrightarrow{i_*} H_{m-1}(F_m^{m-1}(\widetilde{W}),\mathbb{Z}) \longrightarrow 0.$$

As in the affine case, we drop the indices m_i from the sum $\bigoplus$ when no misunderstanding is possible.

The integer homology for affine arrangements is torsion free and

$$H_m(F_m^m(\widetilde{W}),\mathbb{Z}) = F_m^m(\widetilde{W}) \simeq F_m^m(W) = H_m(F_m^m(W),\mathbb{Z})$$

are the free modules generated by $E([w],S^0) = E([w],S) \simeq E(w,S)$. Moreover, by definition, the map $\widetilde{\Delta}$ acts on $F_m^m(\widetilde{W})$ as Δ on $F_m^m(W)$.

Hence, if $C(W_\emptyset)$ denotes the complex generated by the 0-cell $E(1,\emptyset)$, we get the following commutative diagram in homology:

$$\begin{array}{ccc} H_m(F_m^m(W),\mathbb{Z}) & \xrightarrow{\Delta_*} & \bigoplus_{j=1}^{\sharp W} H_0(C(W_\emptyset),\mathbb{Z}) \\ \wr\,| & & \downarrow i_* \\ H_m(F_m^m(\widetilde{W}),\mathbb{Z}) & \xrightarrow{\widetilde{\Delta}_*} & \bigoplus_{j=1}^{m_1\cdots m_m} H_0(C(W_{\widetilde{S}_0}),\mathbb{Z}) \end{array} \tag{11}$$

induced by the corresponding maps on complexes. Then, if $k = m-1$, $\widetilde{\Delta}_*$ is one-free as composition of two one-free maps Δ_* and i_* and $H_{m-1}(F_m^{m-1}(\widetilde{W}),\mathbb{Z})$ is torsion free. This provide the base of induction.

We remark that $H_m(F_m^{m-1}(\widetilde{W}),\mathbb{Z})$ is torsion free since the map

$$0 \longrightarrow \bigoplus_{j=1}^{m_1\cdots m_m} H_1(C(W_{\widetilde{S}_0},\mathbb{Z}))$$

is obviously one-free.

We are interested in a slightly more general situation. For any two given subset $\widetilde{S}_h, S^k$ such that $\sharp(\widetilde{S}_h \cup S^k) \leq m$, we consider the complexes $F_m^{\widetilde{S}_h\cup S^k}(\widetilde{W})$ generated by cells $E([w],\Gamma)$ such that $\Gamma \supset \widetilde{S}_h \cup S^k$. Hence we define the inclusion maps:

$$i_m^h[l] := i : \bigoplus_{j=1}^{\widetilde{m}_k} F_{k+1}^{h+1}(W_{\widetilde{S}_k})[l] \longrightarrow F_m^{\widetilde{S}_h\cup S^{k+1}}(\widetilde{W})$$

as

$$i(j.E(w_{\widetilde{S}_k},\widetilde{S}_h\cup\Gamma)) = i(W^{\widetilde{S}_k}(j).E(w_{\widetilde{S}_k},\widetilde{S}_h\cup\Gamma)) =$$
$$i(w^{\widetilde{S}_k}.E(w_{\widetilde{S}_k},\widetilde{S}_h\cup\Gamma)) = [w^{\widetilde{S}_k}].E([w_{\widetilde{S}_k}],\widetilde{S}_h\cup\Gamma\cup S^{k+1}) = E([w],\widetilde{S}_h\cup\Gamma\cup S^{k+1})$$

where $W^{\widetilde{S}_k}$ is the subset of W isomorphic to $(\widetilde{W}/\Lambda)^{\widetilde{S}_k}$, $\widetilde{m}_k$ its cardinality and $w^{\widetilde{S}_k} = W^{\widetilde{S}_k}(j)$ its j-th element in a fixed order.

They provide short exact sequences as in (3) and (8):

$$0 \longrightarrow \bigoplus_{j=1}^{\widetilde{m}_k} F_{k+1}^{h+1}(W_{\widetilde{S}_k})[l] \longrightarrow F_m^{\widetilde{S}_h \cup S^{k+1}}(\widetilde{W}) \longrightarrow F_m^{\widetilde{S}_h \cup S^k}(\widetilde{W}) \longrightarrow 0 \qquad (12)$$

If $\sharp(\widetilde{S}_h \cup S^k) = m-1$ then $k = h+1$ and, for $l_h = m-h-1$, we get the last short exact sequence:

$$0 \longrightarrow \bigoplus_{j=1}^{\widetilde{m}_{h+1}} F_{h+2}^{h+1}(W_{\widetilde{S}_{h+1}})[l_h - 1] \xrightarrow{i} F_m^{\widetilde{S}_h \cup S^{h+2}}(\widetilde{W}) \xrightarrow{j} F_m^{\widetilde{S}_h \cup S^{h+1}}(\widetilde{W}) \longrightarrow 0.$$

It is a simple remark that

$$H_m(F_m^{\widetilde{S}_h \cup S^{h+1}}(\widetilde{W}), \mathbb{Z}) = F_m^{\widetilde{S}_h \cup S^{h+1}}(\widetilde{W}) \simeq \bigoplus_{j=1}^{\widetilde{m}_h} F_{h+1}^{h+1}(W_{\widetilde{S}_h})[l_h] = \bigoplus_{j=1}^{\widetilde{m}_h} H^{h+1}(F_{h+1}^{h+1}(W_{\widetilde{S}_h}), \mathbb{Z})$$

are the free modules generated by $E([w], \widetilde{S} \setminus \{s_{h+1}\}) = E([w], \widetilde{S}_h \cup S^{h+1}) \simeq E(w, \widetilde{S}_h) = w^{\widetilde{S}_h}.E(w_{\widetilde{S}_h}, \widetilde{S}_h)$.

Moreover the map

$$\widetilde{\Delta} : F_m^{\widetilde{S}_h \cup S^{h+1}}(\widetilde{W}) \longrightarrow \bigoplus_{j=1}^{\widetilde{m}_{h+1}} F_{h+2}^{h+1}(W_{\widetilde{S}_{h+1}})$$

splits as follows:

$$\begin{array}{ccc}
\bigoplus_{j=1}^{\widetilde{m}_h} F_{h+1}^{h+1}(W_{\widetilde{S}_h})[l_h] & \xrightarrow{\Delta} & \bigoplus_{j=1}^{\sharp W} C(W_\emptyset) \\
\wr\,| & & \downarrow i \\
F_m^{\widetilde{S}_h \cup S^{h+1}}(\widetilde{W}) & \xrightarrow{\widetilde{\Delta}} & \bigoplus_{j=1}^{\widetilde{m}_{h+1}} F_{h+2}^{h+1}(W_{\widetilde{S}_{h+1}})
\end{array}$$

and we get the commutative diagram in homology:

$$\begin{array}{ccc}
\bigoplus_{j=1}^{\widetilde{m}_h} H_{h+1}(F_{h+1}^{h+1}(W_{\widetilde{S}_h}), \mathbb{Z}) & \xrightarrow{\Delta_*} & \bigoplus_{j=1}^{\sharp W} H_0(C(W_\emptyset), \mathbb{Z}) \\
\wr\,| & & \downarrow i_* \\
H_m(F_m^{\widetilde{S}_h \cup S^{h+1}}(\widetilde{W}), \mathbb{Z}) & \xrightarrow{\widetilde{\Delta}_*} & \bigoplus_{j=1}^{\widetilde{m}_{h+1}} H_{h+1}(F_{h+2}^{h+1}(W_{\widetilde{S}_{h+1}}), \mathbb{Z}).
\end{array}$$

Hence if $\sharp(\widetilde{S}_h \cup S^k) = m-1$ the map $\widetilde{\Delta}_*$ is one-free since it is composition of one-free maps Δ_* and i_*. So far we proved the base of a more general induction.

Going backwards on homology exact sequences induced by (12) we get maps

$$\widetilde{\Delta}_* : H_{*+1}(F_m^{\widetilde{S}_h \cup S^k}(\widetilde{W}), \mathbb{Z}) \longrightarrow \bigoplus H_{*-l}(F_{k+1}^{h+1}(W_{\widetilde{S}_k}), \mathbb{Z}). \qquad (13)$$

Let us assume, by induction, that they are one-free maps for all $\widetilde{S}_h, S^k$ such that $n < \sharp(\widetilde{S}_h \cup S^k) \leq m-1$ (i.e. $H_*(F_m^{\widetilde{S}_h \cup S^k}(\widetilde{W}), \mathbb{Z})$ are free modules for $n \leq \sharp(\widetilde{S}_h \cup S^k) \leq m-1$).

Let $\sharp(\widetilde{S}_h \cup S^k)$ be equal to n.

We can also filter $F_m^{\widetilde{S}_h \cup S^k}(\widetilde{W})$ as follows:

$$i^m[h+1] := i : \bigoplus_{j=1}^{m^1 \dots m^{h+1}} F_{l_h}^{m-k}(W_{S^{h+1}})[h+1] \longrightarrow F_m^{\widetilde{S}_h \cup S^k}(\widetilde{W})$$
$$i(w^{S^{h+1}} E(w_{S^{h+1}}, \Gamma \cup S^k)) = E([w], \widetilde{S}_h \cup \Gamma \cup S^k).$$

We get the exact sequences

$$0 \longrightarrow \bigoplus_{j=1}^{m^1 \dots m^{h+1}} F_{l_h}^{m-k}(W_{S^{h+1}})[h+1] \longrightarrow F_m^{\widetilde{S}_h \cup S^k}(\widetilde{W}) \longrightarrow F_m^{\widetilde{S}_{h+1} \cup S^k}(\widetilde{W}) \longrightarrow 0.$$

This is equivalent to say that for any cell $E([w], \widetilde{S}_h \cup \Gamma \cup S^k) \in F_m^{\widetilde{S}_h \cup S^k}(\widetilde{W})$ we have only two possibilities:

$i) s_{h+1} \in \Gamma$ and hence $E([w], \widetilde{S}_h \cup \Gamma \cup S^k) = E([w], \widetilde{S}_{h+1} \cup \Gamma' \cup S^k) \in F_m^{\widetilde{S}_{h+1} \cup S^k}(\widetilde{W})$
or
$ii) s_{h+1} \notin \Gamma$ and hence $E([w], \widetilde{S}_h \cup \Gamma \cup S^k) = i(w^{S^{h+1}} E(w_{S^{h+1}}, \Gamma \cup S^k)) \in$
$$\in i\Big(\bigoplus_{j=1}^{m^1 \dots m^{h+1}} F_{l_h}^{m-k}(W_{S^{h+1}})[h+1]\Big).$$

As a consequence if $\widetilde{\Delta} : F_m^{\widetilde{S}_h \cup S^k}(\widetilde{W}) \longrightarrow \bigoplus_{j=1}^{\widetilde{m}_k} F_{k+1}^{h+1}(W_{\widetilde{S}_k})[l]$ is the map which induces the map $\widetilde{\Delta}_*$ in (13), $\widetilde{\Delta}$ splits as follows:

$$\begin{bmatrix} \widetilde{\Delta}_{|_{F_m^{\widetilde{S}_{h+1} \cup S^k}(\widetilde{W})}} & 0 \\ 0 & \Delta_{|_{\bigoplus F_{l_h}^{m-k}(W_{S^{h+1}})}} \end{bmatrix}.$$

Here $\widetilde{\Delta}_{|_{F_m^{\widetilde{S}_{h+1} \cup S^k}(\widetilde{W})}}$ is the map $\widetilde{\Delta}$ defined on $F_m^{\widetilde{S}_{h+1} \cup S^k}(\widetilde{W})$, i.e. on a complex such that $\sharp(\widetilde{S}_{h+1} \cup S^k) = n+1$ if $\sharp(\widetilde{S}_h \cup S^k) = n$.

From now on we will denote this map $\widetilde{\Delta}_{n+1}$ in order to distinguish it from $\widetilde{\Delta}_n$.

By previous consideration it follows that the diagram on complexes

$$\begin{array}{ccccccc} 0 \longrightarrow \bigoplus F_{l_h}^{m-k}(W_{S^{h+1}})[h+1] & \stackrel{i}{\to} & F_m^{\widetilde{S}_h \cup S^k}(\widetilde{W}) & \stackrel{j}{\to} & F_m^{\widetilde{S}_{h+1} \cup S^k}(\widetilde{W}) & \longrightarrow 0 \\ \Delta \downarrow & & \widetilde{\Delta}_n \downarrow & & \widetilde{\Delta}_{n+1} \downarrow & \\ 0 \longrightarrow \bigoplus \bigoplus C(W_{\widetilde{S}_k \setminus \widetilde{S}_{h+1}})[l][h+1] & \stackrel{i}{\to} & \bigoplus F_{k+1}^{h+1}(W_{\widetilde{S}_k})[l] & \stackrel{j}{\to} & \bigoplus F_{k+1}^{h+2}(W_{\widetilde{S}_k})[l] & \longrightarrow 0 \end{array} \tag{14}$$

is commutative.

Here $i : \bigoplus \bigoplus C(W_{\widetilde{S}_k \setminus \widetilde{S}_{h+1}})[l][h+1] \longrightarrow \bigoplus_{j=1}^{\widetilde{m}_k} F_{k+1}^{h+1}(W_{\widetilde{S}_k})[l]$ is the map of type (4) such that $i(w^{\widetilde{S}_k \setminus \widetilde{S}_{h+1}}.E(w_{\widetilde{S}_k \setminus \widetilde{S}_{h+1}}, \Gamma)) = w^{\widetilde{S}_k} E(w_{\widetilde{S}_k}, \widetilde{S}_h \cup \Gamma)$.

Let us remark that the sum

$$\bigoplus\bigoplus C(W_{\widetilde{S}_k\setminus\widetilde{S}_{h+1}})[l][h+1] = \bigoplus_{j=1}^{\sharp W/\sharp W_{\widetilde{S}_k\setminus\widetilde{S}_{h+1}}} C(W_{\widetilde{S}_k\setminus\widetilde{S}_{h+1}})[l][h+1]$$

splits in different ways depending if we are considering the horizontal exact sequence or the vertical map Δ.

The diagram (14) gives rise to the following commutative diagram in homology:

$$\begin{array}{ccccccc}
\longrightarrow \bigoplus H_{*-h-1}(F_{l_h}^{m-k}(W_{S^{h+1}}),\mathbb{Z}) & \xrightarrow{\widetilde{i}_*} & H_*(F_m^{\widetilde{S}_h\cup S^k}(\widetilde{W}),\mathbb{Z}) & \xrightarrow{\widetilde{j}_*} & H_*(F_m^{\widetilde{S}_{h+1}\cup S^k}(\widetilde{W}),\mathbb{Z}) & \longrightarrow \\
\Delta_* \downarrow & & \widetilde{\Delta}_{n*}\downarrow & & \widetilde{\Delta}_{n+1*}\downarrow & \\
\longrightarrow \bigoplus H_{*-l-h-1}(C(W_{\widetilde{S}_k\setminus\widetilde{S}_{h+1}}),\mathbb{Z}) & \xrightarrow{i_*} & \bigoplus H_{*-l}(F_{k+1}^{h+1}(W_{\widetilde{S}_k}),\mathbb{Z}) & \xrightarrow{j_*} & \bigoplus H_{*-l}(F_{k+1}^{h+2}(W_{\widetilde{S}_k}),\mathbb{Z}) & \longrightarrow
\end{array}$$

The maps i_*, j_* and Δ_* are one-free (see section 2).

Moreover $H_{*-h-1}(F_{l_h}^{m-k}(W_{S^{h+1}}),\mathbb{Z})$ and $H_*(F_m^{\widetilde{S}_h\cup S^k}(\widetilde{W}),\mathbb{Z})$ are free modules respectively by theorem 2 and by inductive hypothesis. Then the maps $\widetilde{i}_*$ and $\widetilde{j}_*$ in the diagram are one-free. Moreover $\widetilde{\Delta}_{n+1*}$ are one-free by induction and hence we get that maps $\widetilde{\Delta}_{n*}$ are one-free too.

So far we proved the main result of the paper:

Theorem 3 *The integer (co)-homology of the complement $\mathcal{R}_W$ is torsion free.*

As an immediate consequence of the above theorem, $H^*(\mathcal{R}_W,\mathbb{Z})$ coincides with the De Rham cohomology described in [3] and the Betti numbers can be easily computed using results in [11].

In general we have the following

Conjecture 3.1 *Let $\mathcal{T}_X$ be a thick toric arrangement in the sense of [14]. Then the integer cohomology of the complement is torsion free (and hence it coincides with the De Rham cohomology computed in [3]).*

References

[1] V. I. Arnol'd, *The cohomology ring of the colored braid group* Mat. Zametki 5:2, (1969), 227–231.

[2] N. Bourbaki, *Groupes et algebres de Lie (chap. 4,5,6)*, Masson, Paris, 1981.

[3] C. De Concini, C. Procesi, *On the geometry of toric arrangements*, Transformations Groups 10, (2005), 387-422.

[4] C. De Concini, C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*, to appear, available on www.mat.uniroma1.it/people/procesi/dida.html.

[5] C. De Concini, C. Procesi, and M. Salvetti, *Arithmetic properties of the cohomology of braid groups*, Topology 40 (2001),739–751.

[6] C. De Concini and M. Salvetti, *Cohomology of Coxeter groups and Artin groups*, Math. Res. Lett., **7** (2000), 213-232.

[7] R. Ehrenborg, M. Readdy, M. Slone, *Affine and toric hyperplane arrangements*, arXiv:0810.0295v1 (math.CO), 2008.

[8] G. I. Lehrer, *A toral configuration space and regular semisimple conjugacy classes*, Math. Proc. Cambridge Philos. Soc. 118 (1995), no. 1, 105-113.

[9] G. I. Lehrer, *The cohomology of the regular semisimple variety*, J. Algebra 199 (1998), no. 2, 666-689.

[10] E. Looijenga, *Cohomology of M3 and M1 3. Mapping class groups and moduli spaces of Riemann surfaces*, (Gottingen, 1991/Seattle, WA, 1991), Contemp. Math., 150, Amer. Math. Soc., 1993, 205-228.

[11] L. Moci, *Combinatorics and topology of toric arrangements defined by root systems*, Rend. Lincei Mat. Appl. 19 (2008), 293-308.

[12] L. Moci, *A Tutte polynomial for toric arrangements*, arXiv:0911.4823 [math.CO].

[13] L. Moci, *Wonderful models for toric arrangements*, arXiv:0912.5461 [math.AG].

[14] L. Moci, S. Settepanella, *The homotopy type of toric arrangements*, LEM, Working Paper Series, WP 2010/05 (2010), 1–16, `http://www.lem.sssup.it/WPLem/2010-13.html`.

[15] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbb{C}^n$, Inv. Math., 88 (1987), no.3, 603-618.

[16] M. Salvetti, *The homotopy type of Artin groups*, Math. Res. Lett. 1 (1994), no. 5, 565-577.

[17] S. Settepanella*A stability-like theorem for cohomology of Pure Braid Groups of the series A, B and D*, Topology Appl., 139 (2004), no.1-3, 37-47.