

## ON THE MEASUREMENT OF FINANCIAL MARKET INTEGRATION

(vector optimization/arbitrage portfolio/dual problem/pricing rule)

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### ABSTRACT

The paper presents some vector optimization problems to measure arbitrage and integration of financial markets. This new approach may be applied under static or dynamic asset pricing assumptions and leads to both, numerical and stochastic integration measures. Thus, the paper provides a new methodology in a very general setting, allowing many instruments in each market to test optimal arbitrage portfolios depending on the state of nature and the date. Markets with frictions are also analyzed, and some empirical results are presented.

### RESUMEN

El artículo aplica la optimización vectorial para introducir nuevos procedimientos que miden el nivel de arbitraje e integración de mercados financieros. Las técnicas son aplicables tanto bajo supuestos estáticos, como bajo supuestos dinámicos de valoración de activos. Por consiguiente el nivel de generalidad es alto, y se proporcionan instrumentos que permiten determinar estrategias de arbitraje óptimas de carácter dinámico y estocástico. Finalmente, también se analizan los mercados con fricciones y se presentan los resultados de algunas contrastaciones empíricas.

### 1. INTRODUCTION

The main objective of this paper is to present an unified version of some recent results concerning financial market integration, most of them appeared in Balbás and Muñoz (2), or Balbás *et al* (3). Thus, we will just present the main concepts and theorems with their economic meaning, but proofs will be omitted. We will also provide some new lines for future research.

The literature on financial integration has analyzed the problem under several points of view, but it is a very com-

mon way to study the existence of arbitrage portfolios composed by securities traded in different markets (from now on cross-market arbitrage). In such a case cointegration is the usual topic, and authors very often focus on simultaneous prices of some securities or portfolios in more than one market (see for instance Brenner and Kroner (4) or Harris *et al* (5)) or test some well known expressions frequently related to the Law of One Price, like for instance, the relationship between spot and future prices (Protapapadakis and Stoll (6) or Kempf and Korn (7)) or the put-call parity (Kleidon and Whaley (8), Lee and Nayar (9), or Kamara and Miller (10)).

However, although we could ensure that different markets were giving the same price to some specific securities, portfolios or well known replicas, this is far of being a sufficient condition to guarantee the absence of cross-market arbitrage, and therefore, this is far of being a sufficient condition to guarantee high degree of integration. The existence of discount factors (Chamberlain and Rothschild (11) or Hansen and Richard (12)) is the only way to guarantee the absence of cross-market arbitrage, but we can not compute the degree of market disintegration whenever the mentioned existence fails.

Thus, it is important to develop a formal theory in a more general setting. We first focus on a single market, deriving a general framework for arbitrage models. Attention is directed to maximum arbitrage profits assuming short selling restrictions or relative arbitrage profits otherwise. In order to focus attention on the main issues, the model assumes initially a single period. We apply linear and non-linear programming and duality theory to derive such maximum arbitrage profits. When considering several markets such an application of linear programming methods computes discrepancies among prices and among discount factors (the dual approach).

Two reasons have lead us to measure financial market integration in monetary terms. First, measures proposed here give a very useful information to agents-arbitrageurs.

They allow them to determine how much money they can win and which actions lead to this maximum arbitrage profits. Second, although our model does not initially consider the imperfect financial markets case (transaction costs, shortselling costs,...), taking proper account of these transaction costs would be possible under several assumptions. For instance if we assume that transaction costs are determined by the total price of all the interchanged assets. In this case, we compute the optimal arbitrage strategy leading to the maximum profit with respect to the total price of the interchanged assets. Then, we discount transaction costs from this maximum relative profit. Another possibility is to consider the economy with frictions defined by Jouini and Kallal (13). In this case, analogous integration measures can be introduced after minor modifications.

Duality theory of mathematical programming can be used to present a dual approach that also leads to the integration measures and extends the risk-neutral valuation methodology (Chamberlain and Rothschild (11)) even to no arbitrage free economies. The dual approach also allows to relate our measures to the one previously introduced in Chen and Knez (14), and it also leads to new open problems, since, following some ideas appeared in Hansen and Jagannathan (15), the measures could apply to test the degree of fulfillment of theoretical asset pricing models in the real world.

Finally, let us remark that the integration measures here introduced may be easily tested in practice, and we will present a brief synopsis of the results obtained for some Spanish financial markets. As we will show, some cross-market arbitrage portfolios appear in the Spanish case, and they can be well detected by applying the methodology provided by the integration measures.

**2. PRELIMINARIES**

Throughout the paper we will consider a time interval  $[0, T]$  (write  $[0, T]$  if  $T = \infty$ ) and  $n$  securities denoted by  $A_1, A_2, \dots, A_n$ . The (finite or infinite) set  $\Omega$  will denote the states of the world, and  $\Sigma$  and  $\mu$  will respectively be the usual  $\sigma$ -algebra on  $\Omega$ , and probability measure on  $\Sigma$ . The increasing family of  $\sigma$ -algebras  $(\Sigma_t)_{t \in [0, T]}$  (being  $\Sigma_0 = \{\Phi, \Omega\}$  and  $\Sigma_t \subset \Sigma$  for all  $t \in [0, T]$ ) provides the information available in the market at any instant, and the  $n$ -dimensional adapted stochastic process  $p(\omega, t) = (p_1(\omega, t), p_2(\omega, t), \dots, p_n(\omega, t))$ ,  $\omega \in \Omega$ ,  $t \in [0, T]$ , will represent the asset prices. Assuming usual conventions, for every instant  $t \in [0, T]$  and  $i = 1, 2, \dots, n$ , the  $\Sigma_t$ -measurable random variable  $p_i^t$  given by  $p_i^t(\omega) = p_i(\omega, t)$  (for any  $\omega \in \Omega$ ) will be the price of  $A_i$  at time  $t$ , and it will be assumed to be non negative and square integrable, i.e.  $p_i^t \in L^2(\Sigma_t)$ .

The first asset will be a numeraire, and therefore, there exists  $\alpha > 0$  such that for all  $t \in [0, T]$ , the inequality  $p_1(\omega, t) > \alpha$  holds almost everywhere (a.e. for short). This is

not a very restrictive assumption since latter property holds, for instance, if  $A_1$  is a riskless asset.

Given  $t, s \in [0, T]$  such that  $t < s$ , and given a random variable  $g \in L^2(\Sigma_s)$ , we denote by  $E(g|\Sigma_t) \in L^2(\Sigma_t)$  the conditional expectation of  $g$  relative to the information available at date  $t$ .

For a fixed  $t \in [0, T]$ , the feasible portfolios at time  $t$  will be represented by  $R^n$ -valued square integrable  $\Sigma_t$ -measurable random variables  $x = x^t = x^t(\omega) = x(\omega, t) = (x_1^t(\omega), x_2^t(\omega), \dots, x_n^t(\omega))$ . If the instant  $t$  varies, the corresponding adapted stochastic process will be also denoted by  $x$  or  $x(\omega, t)$ .

Throughout the paper we will adopt different criteria for the concepts of arbitrage and free lunch. At the moment, we will follow a simple extension of Ingersoll (16) chapter II.

**Definition 2.1.** *Let us consider two different instants  $t < s$ , and a  $R^n$ -valued  $\Sigma_t$ -measurable square integrable random variable  $x^t$ . Then,  $x^t$  is said to be an arbitrage portfolio between  $t$  and  $s$  if the following properties hold.*

**2.1.1.**  $\sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) \leq 0$  a.e.

**2.1.2.**  $\sum_{i=1}^n x_i^t(\omega) p_i^s(\omega) \geq 0$  a.e.

**2.1.3.**

$$\mu\{\omega \in \Omega; \sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) < 0\} + \mu\{\omega \in \Omega; \sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) > 0\} > 0$$

Furthermore, assuming that latter conditions hold,  $x^t$  is said to be an arbitrage portfolio of the second type if  $\mu\{\omega \in \Omega; \sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) < 0\} > 0$ . In other case,  $x^t$  is said to be of the first type.

**3. COMPUTING THE DISCREPANCY AMONG PRICES**

This section is devoted to measure the arbitrage opportunities in the sense of definition 2.1, and therefore, we will consider two fixed instants  $t < s$ . If there exists an arbitrage portfolio between  $t$  and  $s$ , and agents are not constrained and can sell or buy any quantity of any security, then the arbitrage earns are not limited. As a consequence, there are two possible ways to measure the level of arbitrage opportunities in monetary terms. First, we can impose short selling restrictions, and second, we can compute attainable relative arbitrage profits. We will show that both ways lead to similar measures.

**A. Assuming short selling restrictions**

Let us analyze the problem under short selling restrictions, and consider that short sales are bounded by a por-

folio  $h(\omega) = (h_1(\omega), h_2(\omega), \dots, h_n(\omega))$ ,  $\omega \in \Omega$ , such that  $h_i$  ( $i = 1, 2, \dots, n$ ) is a non negative square integrable  $\Sigma_t$ -measurable random variable. Then, the optimal arbitrage strategy is given by the following vector optimization problem

**Problem (Ph)**

Maximize

$$- \sum_{i=1}^n p_i^t(\omega) x_i^t(\omega) \begin{cases} \sum_{i=1}^n x_i^t(\omega) p_i^s(\omega) \geq 0 \text{ a.e.} \\ x_i^t(\omega) \geq -h_i(\omega) \text{ a.e., } i = 1, 2, \dots, n \end{cases}$$

Denote by  $m(t, s, h)$  the optimal objective value of latter problem. Since the objective function is not scalar, two important facts must be pointed out in order to clarify the exposition. First,  $m(t, s, h)$  is not a number, but a  $\Sigma_t$ -measurable random variable that depends on the state of nature  $\omega \in \Omega$  and that will be also denoted by  $m(t, s, h, \omega)$ . The reason is clear. At the present moment one observes some uncertainty about the arbitrage opportunities available at the future instant  $t$ . Second, something very usual in vector optimization, this random variable is not necessarily unique and may depend on the concrete optimal solution achieved. The following result shows that the second difficulty will never appear in our model, and furthermore, the first caveat has also a simple solution since portfolios that maximize random arbitrage earns are just those that maximize expected arbitrage earns. Hence, the arbitrage between  $t$  and  $s$  may be measured by the numerical value  $E(m(t, s, h, \omega))$ , expected value of  $m(t, s, h, \omega)$ .

**Theorem 3.1.** *Let  $x^*$  be a square integrable  $\Sigma_t$ -measurable random variable. The following properties hold.*

**3.1.1.**  *$x^*$  solves (Ph) if and only if  $x^*$  solves the following scalar problem denoted by (EPH)*

Maximize

$$- \sum_{i=1}^n \int_{\Omega} p_i^t(\omega) x_i^t(\omega) d\mu(\omega) \begin{cases} \sum_{i=1}^n x_i^t(\omega) p_i^s(\omega) \geq 0 \text{ a.e.} \\ x_i^t(\omega) \geq -h_i(\omega) \text{ a.e. } i = 1, 2, \dots, n \end{cases}$$

**3.1.2.** *Let  $x^*$  and  $x^{**}$  be two solutions of (Ph). Then*

$$\sum_{i=1}^n p_i^t(\omega) x_i^*(\omega) = \sum_{i=1}^n p_i^t(\omega) x_i^{**}(\omega) \text{ a.e.}$$

The above result guarantees that  $m(t, s, h, \omega)$  is well defined if problem (Ph) (or (EPH)) is solvable, but the assumptions are not sufficient to ensure this property. Anyway, let us remark that the inequality  $-\sum_{i=1}^n p_i^t(\omega) x_i^t(\omega) \leq \sum_{i=1}^n p_i^t(\omega) h_i(\omega)$  a.e. holds in  $L^1(\Sigma_t)$  for any portfolio  $x$  feasible for (Ph). Therefore, since  $L^1(\Sigma_t)$  is an order complete space (see Schaeffer (17)), there exists an integrable random variable  $m(t, s, h, \omega) \in L^1(\Sigma_t)$  supremum for the objective function. Furthermore, Theorem 3.1 can be slightly extended to show that the expected

value  $E(m(t, s, h, \omega))$  coincides with the supremum of problem (EPH).

To simplify the exposition, from now on we will assume that (Ph) is solvable if  $h$  is square integrable, although the following result provides sufficient conditions to guarantee this fact.

**Theorem 3.2.** *Let us assume the following conditions*

**3.2.1.** *There exists  $k_i > 0$  ( $i = 1, 2, \dots, n$ ) such that  $p_i^t(\omega) \geq k_i$  a.e.*

**3.2.2.**  $\sum_{i=1}^n h_i p_i^t \in L^2(\Sigma_t)$  for  $i = 1, 2, \dots, n$ .

*Then, problems (Ph) and (EPH) are solvable.*

The portfolio  $h$  has been considered the upper bound for short sales, and may be easily interpreted if we assume, for instance, that agents can not sell the assets they do not have. Thus, if we consider an investor who holds the portfolio  $h$  at time  $t$ , he/she can obtain at  $t$  the arbitrage profits given by  $m(t, s, h, \omega)$ . Moreover, it is easy to prove that  $m(t, s, h, \omega) \geq 0$  a.e. and  $m(t, s, h, \omega) = 0$  a.e. if and only if there are no arbitrage portfolios of the second type between  $t$  and  $s$ , bounded from below by  $-h$ .

We are now interested in a random measure  $m(t, s, \omega)$  without special mention of  $h$ . This measure can be obtained by computing the maximum value of  $m(t, s, h, \omega)$  among the portfolios  $h$  with price equal to one dollar in all the states of nature. Following programs and Theorem 3.3 allow to introduce  $m(t, s, \omega)$  with precision.

**Program (P)**

Maximize

$$- \sum_{i=1}^n p_i^t(\omega) x_i^t(\omega) \begin{cases} h_i(\omega) \geq 0 \text{ a.e., } i = 1, 2, \dots, n \\ \sum_{i=1}^n x_i^t(\omega) p_i^s(\omega) \geq 0 \text{ a.e.} \\ x_i^t(\omega) + h_i(\omega) \geq 0 \text{ a.e., } i = 1, 2, \dots, n \\ \sum_{i=1}^n p_i^t(\omega) h_i(\omega) = 1 \text{ a.e.} \end{cases}$$

**Program (EP)**

Maximize

$$- \sum_{i=1}^n \int_{\Omega} p_i^t(\omega) x_i^t(\omega) d\mu(\omega) \begin{cases} h_i(\omega) \geq 0 \text{ a.e., } i = 1, 2, \dots, n \\ \sum_{i=1}^n x_i^t(\omega) p_i^s(\omega) \geq 0 \text{ a.e.} \\ x_i^t(\omega) + h_i(\omega) \geq 0 \text{ a.e., } i = 1, 2, \dots, n \\ \sum_{i=1}^n p_i^t(\omega) h_i(\omega) = 1 \text{ a.e.} \end{cases}$$

**Program (II)**

Maximize

$$m(t, s, h, \omega) \begin{cases} h_i(\omega) \geq 0 \text{ a.e.}, i = 1, 2, \dots, n \\ \sum_{i=1}^n p_i^t(\omega) h_i(\omega) = 1 \text{ a.e.} \end{cases}$$

**Theorem 3.3.** *Let  $h^*$  and  $x^*$  be two  $R^n$ -valued square integrable  $\Sigma_t$ -measurable random variables. Then, we have the following properties.*

**3.3.1.** *The pair  $(x^*, h^*)$  solves (P) if and only if it solves (EP). If so, the inequality*

$$-\sum_{i=1}^n p_i^t(\omega) x_i^t(\omega) \leq -\sum_{i=1}^n p_i^t(\omega) x_i^*(\omega) \text{ a.e.}$$

*holds for any pair  $(x, h)$  feasible for (P).*

**3.3.2.** *The pair  $(x^*, h^*)$  solves (P) if and only if  $h^*$  solves (II) and  $x^*$  solves  $(Ph^*)$*

Although (P) is a vector problem, a result similar to 3.1.2 trivially follows from 3.3.1. Therefore, the random measure  $m(t, s, \omega)$  is well defined. We have already said that, in a market with short selling restrictions, the measure  $m(t, s, \omega)$  represents the maximum arbitrage profits available at  $t$ , when short sales do not exceed one dollar in total. Furthermore, if  $(x^*, h^*)$  solves (P), then  $h^*$  is the portfolio that an agent must hold in order to obtain the profits given by  $m(t, s, \omega)$  (in other case the arbitrage profits would be lower than  $m(t, s, \omega)$ , and  $x^*$  is the concrete arbitrage strategy that the agent must implement. Since  $\sum_{i=1}^n p_i^t(\omega) h^*(\omega) = 1 \text{ a.e.}$ , it is clear that  $0 \leq m(t, s, \omega) \leq 1 \text{ a.e.}$  The following result summarizes another interesting properties of  $m(t, s, \omega)$ .

**Theorem 3.4.**

**3.4.1.**  $0 \leq m(t, s, \omega) \leq 1 \text{ a.e.}$  Thus,  $m(t, s, \omega) \in L^p(\Sigma_t)$  for all  $1 \leq p \leq \infty$ .

**3.4.2.**  $0 \leq E(m(t, s, \omega)) \leq 1$ .

**3.4.3.**  $m(t, s, \omega) = 0 \text{ a.e.}$  if and only if there are no arbitrage opportunities of the second type between  $t$  and  $s$ .

**3.4.4.**  $E(m(t, s, \omega)) = 0$  if and only if there are no arbitrage opportunities of the second type between  $t$  and  $s$ .

The above theorem shows that the random measure  $m(t, s, \omega)$  or its expected value  $E(m(t, s, \omega))$ , provides the level of arbitrage opportunities of the second type. The arbitrage does not appear if the measures are zero, and the arbitrage profits almost vanish if the measures are close to zero, case in which the arbitrage would probably disappear after transaction costs or measurement errors. On the other hand, the arbitrage opportunities are more clear when the measures increase, and the limit case  $E(m(t, s, \omega)) = 1$  (which holds if and only if  $m(t, s, \omega) = 1 \text{ a.e.}$ ) corresponds to unrealistic situations that will never appear in practice. There exists an asset  $A_i$  such that  $p_i^t(\omega, t) > 0 \text{ a.e.}$  and  $p_i$

$(\omega, t) = 0 \text{ a.e.}$  It should be pointed out that  $m(t, s, \omega)$  (or its expected value) might vanish in presence of arbitrage portfolios of the first type. We will solve this difficulty in fifth section. At the moment, we are going to analyze the measures after relaxing the short selling restrictions.

**B. Models without short selling restrictions**

The random measure and the numerical measure also apply in a model without short selling restrictions, since they are optimal relative arbitrage profits. To prove this fact we need Lemma 3.5, interesting by itself, because it shows the relationship between the portfolios  $x^*$  and  $h^*$ . Moreover, the lemma may be easily interpreted. Since an investor must hold the initial portfolio  $h^*$  in order to obtain the optimal arbitrage earns, this portfolio is composed by the securities that the investor must sell.

**Lemma 3.5.** *Let the pair  $(x^*, h^*)$  be a solution of (P). Then, there exists a null set  $\Omega_0 \subset \Omega$ ,  $\Omega_0 \in \Sigma_t$ , such that the following conditions hold for  $\omega \notin \Omega_0$  and  $i = 1, 2, \dots, n$ .*

**3.5.1.** *If  $x_i^*(\omega) > 0$  then  $h_i^*(\omega) = 0$ .*

**3.5.2.** *If  $x_i^*(\omega) \leq 0$ , then  $h_i^*(\omega) = -x_i^*(\omega)$ .*

Given any ( $\Sigma_t$ -measurable and square integrable) portfolio  $x (= x(\omega, t))$  we can define the  $\Sigma_t$ -measurable random variable  $f(x)$  (or  $f(x, \omega)$ ) as follows.

If  $p_i^t(\omega) x_i^t(\omega) \geq 0 \ i = 1, 2, \dots, n$  then  $f(x, \omega) = 0$ . Otherwise,

$$f(x, \omega) = \frac{\sum_{i=1}^n p_i^t(\omega) x_i^t(\omega)}{\sum_{i \in A(\omega, x)} p_i^t(\omega) x_i^t(\omega)}$$

where  $A(\omega, x) = \{i = 1, 2, \dots, n; p_i^t(\omega) x_i^t(\omega) < 0\}$

Let us consider an investor who buys the portfolio  $x$ . If  $x$  is an arbitrage portfolio between  $t$  and  $s$  or, more generally, if the price of  $x$  at  $t$  is non positive a.e., then  $f(x, \omega)$  represents the ratio between the total income and the price of the sold assets. Thus, for arbitrage portfolios,  $f(x, \omega)$  provides relative arbitrage earns. The inequality  $f(x, \omega) \leq 1$  is clear, and  $f(x, \omega) \geq 0$  also holds if the price at  $t$  of  $x$  is non positive a.e. Hence,  $f(x)$  is always in the space  $L^p(\Sigma_t)$  ( $1 \leq p \leq \infty$ ). These facts and the above lemma allow to prove one of the most important results of this paper.

**Theorem 3.6.** *Let  $(x^*, h^*)$  be a solution of (P). Then,*

**3.6.1.**  $f(x^*, \omega) = m(t, s, \omega) \text{ a.e.}$

**3.6.2.** *If  $x$  is an arbitrage portfolio between  $t$  and  $s$ , then  $f(x, \omega) \leq m(t, s, \omega) \text{ a.e.}$*

**3.6.3.** If  $x$  is an arbitrage portfolio between  $t$  and  $s$ , and  $E(f(x)) = E(f(x, \omega))$  denotes the expected value of  $f(x, \omega)$ , then  $E(f(x)) \leq E(m(t, s, \omega)) = E(f(x^*))$ .

Now, it is obvious that the measure  $m(t, s, \omega)$  (respectively  $E(m(t, s, \omega))$ ) represents optimal relative arbitrage profits (respectively optimal expected relative arbitrage profits) in a market with, or without short selling restrictions. To be precise,  $x^*$  solves the following programs.

**Program (Q)**

$$\text{Maximize } f(x, \omega) \left\{ \begin{array}{l} \sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) \leq 0 \quad a.e. \\ \sum_{i=1}^n x_i^s(\omega) p_i^s(\omega) \geq 0 \quad a.e. \end{array} \right.$$

**Program (EQ)**

$$\text{Maximize } E(f(x, \omega)) \left\{ \begin{array}{l} \sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) \leq 0 \quad a.e. \\ \sum_{i=1}^n x_i^s(\omega) p_i^s(\omega) \geq 0 \quad a.e. \end{array} \right.$$

$x^*$  is also a strong solution of the vector problem (Q), i.e., a solution that dominates any other arbitrage strategy. Furthermore, if arbitrage portfolios of the second type between  $t$  and  $s$  do exist, then the converse also holds, since the solutions of (Q) (or (EQ)) jointly with Lemma 3.5 provide the solutions of (P) and (EP).

Problems (P), (EP), (Q), and (EQ) are equivalent but all them are interesting. First, the measures have different meaning that depends on the problem. Second, the measures do not depend on the short selling restriction assumed by the model. From the most constrained models (the agents can not sell the securities they do not have) to the most relaxed ones (there is no limit in the short positions that agents can hold) we obtain the same value for the measures. Of course, this also happens if one considers situations not so restrictive or so relaxed. Problem (Q) perhaps provides the most interesting interpretation, since it applies for both, models with or without short selling restrictions, and measures the arbitrage opportunities by means of random arbitrage earns. However, this problem is a multiobjective one and the objectives are non differentiable functions. Thus, it can not be solved in practice. On the other hand, problem (EP) perhaps yields a poor interpretation (expected but not real arbitrage earns, in a model with hard assumptions on the short selling restrictions) but it is a simple scalar linear problem that may be easily solved in practice, and for which a well known duality theory has been developed. This theory will be very important for us in future sections, since it will be the key to obtain the relationship between our measures and the risk neutral probabilities.

The above paragraph seems to be also important from a mathematical point of view. In fact, we have shown a procedure to solve a non differentiable vector optimization problem, and a duality theory for this problem has been

provided. The methodology might apply in more general situations.

Let us show the last interpretation of  $m(t, s, \omega)$  and  $E(m(t, s, \omega))$ , which will be specially useful in markets with frictions. We are interested in the ratio between the total income provided by an arbitrage portfolio, and the value of all the interchanged (bought and sold) securities. Thus, for any portfolio  $x = x^t$ , we will consider the following  $\sum_i$ -measurable random variable denoted by both,  $g(x, \omega)$  or  $g(x)$ .

If  $p_i^t(\omega) x_i^t(\omega) = 0, i = 1, 2, \dots, n$  then  $g(x) = g(x, \omega) = 0$ . Otherwise,

$$g(x) = g(x, \omega) = \frac{-\sum_{i=1}^n p_i^t(\omega) x_i^t(\omega)}{\sum_{i=1}^n p_i^s(\omega) x_i^s(\omega)}$$

It is clear that  $g(x, \omega) \leq 1$  a.e. and, assuming that  $x$  is an arbitrage strategy,  $g(x, \omega) \geq 0$  a.e. and the expression  $g(x, \omega) = \frac{f(x, \omega)}{2 - f(x, \omega)}$  may be easily proved. Thus, since the real function  $t \rightarrow \frac{t}{2 - t}$  increases for  $0 \leq t \leq 1$ , the following result holds.

**Theorem 3.7.** Let  $x^*$  be a solution of (Q). Then,  $x^*$  solves the following problems (R) and (ER)

$$\text{Maximize } g(x, \omega) \left\{ \begin{array}{l} \sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) \leq 0 \quad a.e. \\ \sum_{i=1}^n x_i^s(\omega) p_i^s(\omega) \geq 0 \quad a.e. \end{array} \right.$$

and

$$\text{Maximize } E(g(x, \omega)) \left\{ \begin{array}{l} \sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) \leq 0 \quad a.e. \\ \sum_{i=1}^n x_i^s(\omega) p_i^s(\omega) \geq 0 \quad a.e. \end{array} \right.$$

Moreover,  $x^*$  is a strong solution of (R).

The objective optimal value of latter problems will be represented by  $l(t, s, \omega)$  and  $E(l(t, s, \omega))$ . Once again, we have a random and a numerical measure respectively and they vanish if and only if there are no arbitrage portfolios of the second type.

Latter theorem shows the following relationship

$$l(t, s, \omega) = \frac{m(t, s, \omega)}{2 - m(t, s, \omega)}$$

Thus,  $0 \leq l(t, s, \omega) \leq m(t, s, \omega) \leq 1$  a.e. and  $0 \leq E(l(t, s, \omega)) \leq 1$ . Both random (or numerical) measures simultaneously achieve the extreme value 1 and yield analogous information about the level of arbitrage opportunities.

**C. Applications to static and discrete time dynamic asset pricing models**

Let us particularize the above results to static (or one period) and discrete time dynamic models. We will merely summarize the main results for these special cases. Further information may be found in Balbás and Muñoz (2) and Balbás *et al* (3).

To analyze a static model we only have to consider that  $t = 0$ . Let us also assume that  $s$  is any arbitrary future date, and suppose that 0 and  $s$  are the unique trading moments. Then, since  $\Sigma_0 = \{\Phi, \Omega\}$  and the random variables  $m(0, s, \omega)$  and  $p_i^0$  must be  $\Sigma_0$ -measurable, it is obvious that  $p_i^0$  and  $m(0, s, \omega)$  must be constant values and do not depend on  $\omega \in \Omega$ . As a consequence, the measure  $m(0, s, \omega)$  coincides with its expected value and may be denoted by  $m(0, s)$ . The inequalities  $0 \leq m(0, s) \leq 1$  are clear, and there are no arbitrage portfolios of the second type if and only if  $m(0, s) = 0$ . Analogous results hold for the measure  $l = l(0, s)$ .

Consider now a discrete time dynamic model. There are a finite number of trading dates  $t_0 = 0 < t_1 < \dots < t_N$ , and we can consider the random variables  $m(t_p, t_{p+1}, \omega)$ . Thus we measure the arbitrage opportunities by an adapted stochastic process that can be also denoted by  $m(t_p, \omega)$ ,  $i = 0, 1, \dots, N$ ,  $\omega \in \Omega$ . Since the security  $A_1$  is a numeraire, it can be easily proved that the absence of arbitrage of the second type between any pair of dates is equivalent to the absence of arbitrage of the second type between consecutive dates. Then, the absence of arbitrage holds if and only if the stochastic process  $m(t_p, \omega)$  vanishes. Once again, similar properties hold if we consider the process  $l(t_p, \omega)$ .

Discrete time dynamic models with infinite trading dates may also be considered. We will analyze them and continuous time models in fifth section. Furthermore, the theory we will develop also applies for static models and discrete time dynamic models with finite trading dates, and it will extend the results for arbitrage portfolios of the first type and free lunches in the sense of Harrison and Kreps (18).

**4. COMPUTING THE DISCREPANCY AMONG DISCOUNT FACTORS**

Random and numerical measures have been introduced by means of optimal arbitrage portfolios because we are interested in measuring the arbitrage in monetary terms. Nevertheless, there are many reasons to analyze the problem from a dual point of view. First, the dual approach provides a proxy for discount factors (or equivalent concepts, like the risk neutral probabilities, or the state prices) in no arbitrage free economies, and thus, we extend the methodology of risk-neutral valuation (see for instance

Chamberlain and Rothschild (11)) and can compute «right prices» for the assets and the errors committed by the agents. Second, the presence of arbitrage may be apparent but not real. Microstructure constraints, measurement errors, frictions, etc., might impede to implement the strategies in practice. Therefore, a theory that provides discount factors under not very ideal asset pricing assumptions makes the models more flexible and realistic. We will apply these ideas in fifth section to analyze markets with frictions, and will present some alternatives to very important results appeared in previous literature (see for instance Prisman (19) or Jouini and Kallal (13)). Third, very interesting papers have measured the integration of financial markets by means of discount factors (see for instance Chen and Knez (14)). Hence, since the integration of financial markets is the present paper main objective, we need the dual approach to relate our theory to previous ones. Finally, describing  $m(t, s, \omega)$ ,  $E(m(t, s, \omega))$ ,  $l(t, s, \omega)$  by means of discount factors we get new interpretations and properties that improve our knowledge of these measures.

Consider the following scalar or vector optimization problems for which the decision variables have been denoted by  $f \in L^2(\Sigma_s)$ ,  $\lambda \in L^2(\Sigma_t)$ , and  $\lambda_i \in L^2(\Sigma_t)$   $i = 1, 2, \dots, n$ .

**Program (P')**

$$\text{Minimize } \lambda(\omega) \begin{cases} E(fp_i^s | \Sigma_t) + \lambda_i = p_i^t & i = 1, 2, \dots, n \\ p_i^t(\omega) \lambda(\omega) - \lambda_i(\omega) \geq 0 \text{ a.e.}, & i = 1, 2, \dots, n \\ f(\omega) \geq 0, \lambda(\omega), \lambda_i(\omega) \geq 0 \text{ a.e.}, & i = 1, 2, \dots, n \end{cases}$$

**Program (EP')**

$$\text{Minimize } E(\lambda(\omega)) \begin{cases} E(fp_i^s | \Sigma_t) + \lambda_i = p_i^t & i = 1, 2, \dots, n \\ p_i^t(\omega) \lambda(\omega) - \lambda_i(\omega) \geq 0 \text{ a.e.}, & i = 1, 2, \dots, n \\ f(\omega) \geq 0, \lambda(\omega), \lambda_i(\omega) \geq 0 \text{ a.e.}, & i = 1, 2, \dots, n \end{cases}$$

Let us remark that constraints can be simplified and the decision variables  $\lambda_p$ ,  $i = 1, 2, \dots, n$  eliminated. Then, we get the alternative and equivalent set of constraints

$$0 \leq p_i^t - E(fp_i^s | \Sigma_t) \leq p_i^t \lambda, \quad i = 1, 2, \dots, n$$

$$f(\omega) \geq 0, \lambda(\omega) \geq 0 \text{ a.e.}$$

Latter problems allow an interesting economic meaning. For each feasible family of variables,  $f$  provides a possible proxy for discount factors,  $E(fp_i^s | \Sigma_t)$  becomes the «right price» for the  $i^{\text{th}}$ -security,  $p_i^t - E(fp_i^s | \Sigma_t)$  the «committed error» and  $\lambda$  the relative (per dollar) maximum (among the  $n$  securities) «committed error». So, problem (P') tries to find discount factors in order to minimize the random «relative maximum committed error», and (EP') minimizes the expected value.

It may be proved that the solution  $x^*$  of  $(P)$  and  $(EP)$  can be taken in  $L^\infty(\Sigma_t)$ . Hence, applying the results of Anderson and Nash (20) chapter III, problem  $(EP')$  becomes the dual problem of  $(EP)$ . This reference also shows several (alternative) conditions to ensure that there is no duality gap between  $(EP)$  and  $(EP')$  (i.e., primal and dual problems are solvable and the optimal objective value is common for both problems) and, in order to simplify the exposition, we will assume that the absence of duality gap holds in our context. Then, it is not difficult to obtain that the solution of  $(EP')$  becomes a strong solution for the vector optimization problem  $(P')$  as well. Moreover, manipulating primal and dual constraints, one can obtain that  $m(t, s, \omega) = - \sum_{i=1}^n x_i^*(\omega) p_i^t(\omega) = \lambda^*(\omega)$  a.e. if  $(x^*, h^*)$  solves  $(P)$  and  $(f^*, \lambda^*)$  solves  $(P')$ . The following result summarizes these ideas.

**Theorem 4.1.** *Assume that  $(f^*, \lambda^*)$  solves  $(EP')$ , and suppose that there is no duality gap between  $(EP)$  and  $(EP')$ . Then*

**4.1.1.**  $\lambda^*(\omega) = m(t, s, \omega)$  a.e. and  $E(m(t, s, \omega))$  becomes the optimal value of  $(EP')$ .

**4.2.2.**  $(f^*, \lambda^*)$  is a strong solution of  $(P')$ .

Latter result is important since it yields new interpretations for measures  $E(m(t, s, \omega))$  and  $m(t, s, \omega)$ .  $E(m(t, s, \omega))$  represents the average relative committed error when the market price the assets, while  $m(t, s, \omega)$  gives us the random relative committed error. Furthermore, if there are no arbitrage opportunities of the second type between  $t$  and  $s$ , the measures vanish and  $f^*$  gives discount factors in the sense that  $E(f^* p_i^s | \Sigma_t) = p_i^t$   $i = 1, 2, \dots, n$  (Hansen and Richard (12)). If the arbitrage appears,  $f^*$  gives a proxy for discount factors, and  $E(f^* p_i^s | \Sigma_t)$  may be understood as a «right» price for the  $i^{\text{th}}$ -security  $i = 1, 2, \dots, n$ .

Theorem 4.1 becomes also a duality theorem for vector optimization problems  $(P)$  and  $(P')$ . But let us point out that the natural dual of  $(P)$  is far more complex than  $(P')$  with a larger feasible set, and a more complex objective function (see for instance Balbás and Heras (21)). Theorem 4.1 shows that strong duality between  $(P)$  and its natural dual holds if and only if there is no duality gap between the scalar problems  $(EP)$  and  $(EP')$ . Thus, once again, our results may be interesting in Mathematical Programming Theory since this methodology might apply in more general situations.

## 5. EXTENDING THE MAIN RESULTS

### A. Markets with frictions

The literature on asset pricing has incorporated the market frictions (transaction costs, taxes, bid-ask spread,...)

under different points of view (see for instance Leland (23), Davis *et al* (24), or Toft(25)), and these frictions are specially important when one analyzes the existence of arbitrage strategies (see Brennan and Schwartz (22)). Throughout this section we will assume three alternative hypotheses on market frictions.

First, it is simple but realistic and useful to consider that transaction costs linearly depend on the total price of all the interchanged (sold and purchased) assets. If an investor trades the portfolio  $x (= x^t = x^t(\omega) = x(t, \omega))$ , the transaction costs will be given by  $C(x) = C_0 \sum_{i=1}^n p_i^t(\omega) |x_i^t(\omega)|$  where  $C_0 > 0$  is arbitrary and depends on the market. Then, since the attainable arbitrage profits are given by  $\sum_{i=1}^n p_i^t(\omega) |x_i^t(\omega)| l(t, s, \omega)$ , it is obvious that the existence of arbitrage between  $t$  and  $s$  holds after discounting transaction costs if and only if

$$\mu\{\omega \in \Omega; l(t, s, \omega) > C_0\} > 0$$

One can slightly relax the assumptions and suppose that the relationship between the transaction costs and the total value of trade is not necessarily linear, but given by a function  $C(x) = C_0 \left( \sum_{i=1}^n p_i^t(\omega) |x_i^t(\omega)| \right)$ . Then, the existence of arbitrage is equivalent to the existence of  $V > 0$  (total value of trade) such that

$$\mu\{\omega \in \Omega; l(t, s, \omega) V > C_0(V)\} > 0$$

Market frictions may also be introduced by following the approach in Jouini and Kallal (13), and this is the second possibility we are interested in. Their ideas are specially useful to incorporate the bid-ask spread because they consider that frictions imply two prices per security. Therefore, there are two different adapted stochastic processes  $v(t, \omega)$  and  $c(t, \omega)$  ( $t \in [0, T], \omega \in \Omega$ ) such that for an arbitrary  $t \in [0, T]$  the inequalities  $0 \leq v(t, \omega) \leq c(t, \omega)$  hold a.e. For a given portfolio  $x$ , the price of  $x$  is given by  $P(x, t, \omega) = \sum_{i=1}^n q_i(t, \omega) x_i(t, \omega)$  where  $q_i(t, \omega) = c_i(t, \omega)$  whenever  $x_i(t, \omega) \geq 0$  and  $q_i(t, \omega) = v_i(t, \omega)$  whenever  $x_i(t, \omega) < 0$ .

**Definition 5.1.** *Let us consider two arbitrary trading dates  $t < s$ . A  $\Sigma_t$ -measurable square integrable random variable  $x$  is said to be an arbitrage portfolio of the second type between  $t$  and  $s$  with market frictions if  $P(x, t, \omega) \leq 0$  a.e.,  $P(-x, s, \omega) \leq 0$  a.e., and  $\mu\{\omega \in \Omega; P(x, t, \omega) < 0\} > 0$ .*

Let us remark that Definitions 2.1 and 5.1 are equivalent if  $c_i(t, \omega) = v_i(t, \omega)$  and  $c_i(s, \omega) = v_i(s, \omega)$  a.e.,  $i = 1, 2, \dots, n$ .

The main results of third and fourth sections can be easily generalized, and thus, we will only summarize the

ideas. So, problems (EP) and (P) become respectively the following concave problems

$$\begin{aligned} & \text{Maximize} - \int_{\Omega} P(x, t, \omega) d\mu(\omega) \\ & \begin{cases} h_i(\omega) \geq 0 \text{ a.e.}, i = 1, 2, \dots, n \\ P(-x, s, \omega) \leq 0 \text{ a.e.} \\ x_i^t(\omega) + h_i(\omega) \geq 0 \text{ a.e.}, i = 1, 2, \dots, n \\ \sum_{i=1}^n v_i^t(\omega) h_i(\omega) = 1 \text{ a.e.} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \text{Maximize} - P(x, t, \omega) \\ & \begin{cases} h_i(\omega) \geq 0 \text{ a.e.}, i = 1, 2, \dots, n \\ P(-x, s, \omega) \leq 0 \text{ a.e.} \\ x_i^t(\omega) + h_i(\omega) \geq 0 \text{ a.e.}, i = 1, 2, \dots, n \\ \sum_{i=1}^n v_i^t(\omega) h_i(\omega) = 1 \text{ a.e.} \end{cases} \end{aligned}$$

Furthermore, the vector problem admits strong solution and both problems are solved by the same strategy  $x^*$ . The integration measures are the optimal values and are given by  $m(t, s, \omega) = -P(x^*, t, \omega)$  and  $E(m(t, s, \omega))$ .

The model without short-selling restrictions can be also extended since  $x^*$  also solves the corresponding problems (Q) and (R) after obvious changes in constraints and slightly modifying  $f(x, \omega)$  and  $g(x, \omega)$ . To be precise, denoting by  $x^+(t, \omega) = \sup\{x(t, \omega), 0\}$  and  $x^-(t, \omega) = \sup\{-x(t, \omega), 0\}$  then we can redefine

$$f(x, \omega) = \frac{P(x, t, \omega)}{P(-x^-, t, \omega)} \text{ and}$$

$$g(x, \omega) = \frac{-P(x, t, \omega)}{P(x^+, t, \omega) - P(-x^-, t, \omega)}$$

(write  $f(x, \omega) = 0$  or  $g(x, \omega) = 0$  if denominators vanish).

Above paragraphs show that four measures  $E(m(t, s, \omega))$ ,  $m(t, s, \omega)$ ,  $E(l(t, s, \omega))$ , and  $l(t, s, \omega)$  still make sense under Jouini and Kallal assumptions, and thus, the arbitrage can be also measured in monetary terms. Moreover, the economic meaning still holds.

We are now interested in the dual approach, the proxy for discount factors in no arbitrage free economies, and the interpretation of our measures in terms of prices. Since (P) and (EP) are concave, we only have to apply duality results for this kind of Mathematical Programming problems (see for instance Balbás and Guerra (26)). Denoting the dual decision variables by  $f \in L^2(\Sigma_t)$ ,  $\lambda \in L_2(\Sigma_t)$ , and  $\lambda_i \in L_2(\Sigma_t)$   $i = 1, 2, \dots, n$ , it may be proved that (EP') and (P') become now

**Program (P')**

*Minimize*  $\lambda(\omega)$

$$\begin{cases} v_i^t \leq E(fc_i^s | \Sigma_t) + \lambda_i & i = 1, 2, \dots, n \\ E(fv_i^s | \Sigma_t) + \lambda_i \leq c_i^t & i = 1, 2, \dots, n \\ v_i^t(\omega) \lambda(\omega) - \lambda_i(\omega) \geq 0 \text{ a.e.}, & i = 1, 2, \dots, n \\ f(\omega) \geq 0, \lambda(\omega), \lambda_i(\omega) \geq 0, \text{ a.e.}, & i = 1, 2, \dots, n \end{cases}$$

and

**Program (EP')**

*Minimize*  $E(\lambda(\omega))$

$$\begin{cases} v_i^t \leq E(fc_i^s | \Sigma_t) + \lambda_i & i = 1, 2, \dots, n \\ E(fv_i^s | \Sigma_t) + \lambda_i \leq c_i^t & i = 1, 2, \dots, n \\ v_i^t(\omega) \lambda(\omega) - \lambda_i(\omega) \geq 0 \text{ a.e.}, & i = 1, 2, \dots, n \\ f(\omega) \geq 0, \lambda(\omega), \lambda_i(\omega) \geq 0, \text{ a.e.}, & i = 1, 2, \dots, n \end{cases}$$

An analogous result to Theorem 4.1 may be established, and the economic meaning showed after this theorem, still holds. Furthermore, in this more general setting, the dual approach provides the main procedure to compute the measures in practice. In fact, dual problems become linear although primal ones do not verify this property, and therefore, the scalar program (EP') may be easily solved in practice. Next, the primal ones are solved by the usual primal-dual relationship (Balbás and Guerra (26)).

Let us now consider the third possibility on the transaction costs. This involves the previous ones since we assume that frictions are given by two terms. First, there are two prices per security (specially useful assumption to incorporate the bid-ask spread) and second, one must add a function that depends on the total value of trade. The second term may be interpreted as the total amount paid to brokers. Under these assumptions we have already introduced the stochastic measure  $l(t, s, \omega)$ , and the existence of arbitrage after both kind of frictions may be characterized by the  $\Sigma_t$ -measurable function

$$l_f(t, s, \omega) = \text{Max} \{0, l(t, s, \omega) - C_0\}.$$

**B. Measuring the arbitrage of the first type**

The (numerical and stochastic) measures above introduced are useful to analyze arbitrage portfolios of the second type but, as we have already said, there might be situations for which the measures vanish in presence of arbitrage of the first type. To solve this difficulty one only has to consider an optimization problem such that the objective function incorporates the arbitrage earns attainable at date  $s$ . There are several possibilities and, so for instance, we can maximize the expected value and the conditional expectation of



$$\sum_{i=1}^n p_i^s(\omega)x_i(\omega)$$

among the arbitrage portfolios providing at  $t$  the random relative profit  $m(t, s, \omega)$ . Thus, we can consider the vector optimization problem

$$\begin{aligned} & \text{Maximize } \sum_{i=1}^n E(p_i^s x_i^t | \Sigma_t) \\ & \left\{ \begin{aligned} \sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) &= -m(t, s, \omega) \text{ a.e.} \\ \sum_{i=1}^n x_i^t(\omega) p_i^s(\omega) &\geq 0 \text{ a.e.} \\ \sum_{i=1}^n p_i^t(\omega) h_i(\omega) &= 1 \text{ a.e.} \\ x_i^t(\omega) + h_i(\omega) &\geq 0 \text{ a.e., } i = 1, 2, \dots, n \\ h_i(\omega) &\geq 0 \text{ a.e., } i = 1, 2, \dots, n \end{aligned} \right. \end{aligned}$$

and the scalar problem

$$\begin{aligned} & \text{Maximize } \sum_{i=1}^n \int_{\Omega} p_i^s(\omega) x_i(\omega) d\mu(\omega) \\ & \left\{ \begin{aligned} \sum_{i=1}^n x_i^t(\omega) p_i^t(\omega) &= -m(t, s, \omega) \text{ a.e.} \\ \sum_{i=1}^n x_i^t(\omega) p_i^s(\omega) &\geq 0 \text{ a.e.} \\ \sum_{i=1}^n p_i^t(\omega) h_i(\omega) &= 1 \text{ a.e.} \\ x_i^t(\omega) + h_i(\omega) &\geq 0 \text{ a.e., } i = 1, 2, \dots, n \\ h_i(\omega) &\geq 0 \text{ a.e., } i = 1, 2, \dots, n \end{aligned} \right. \end{aligned}$$

It may be easily proved that the solution for both problems is attained at the same portfolio, and it is also important to point out that this solution is a strong one for the vector problem and also solves problem (P). Denoting by  $G(t, s, \omega)$  and  $E(G(t, s, \omega))$  the optimal values, we can consider the stochastic and numerical measures

$$\tilde{m}(t, s, \omega) = m(t, s, \omega) + G(t, s, \omega) \text{ and}$$

$$E(\tilde{m}(t, s, \omega)) = E(m(t, s, \omega)) + E(G(t, s, \omega)).$$

It trivially follows that  $\tilde{m}(t, s, \omega)$  and its expected value are never lower than zero and they vanish if and only if there are no arbitrage opportunities (of any kind) between  $t$  and  $s$ .

### C. Free lunches and continuous time models

The theory above developed seems to be suitable in order to compute the arbitrage earns between any couple of arbitrary dates  $t$  and  $s$ . Nevertheless, an implicit assumption is that trading after  $t$  and before  $s$  is not possible. The concept of arbitrage portfolio may be extended to incorporate new possibilities very usual in the literature. Hence, throughout this section we will consider finite subsets of trading dates  $\{t_0 < t_1 < \dots < t_k\}$ , included in  $[0, T]$ ,

and such that  $t_0 = t, t_k \leq s$  ( $t_k < s$  if  $s = \infty$ ). For a fixed subset, the adapted stochastic process  $x(t_j, \omega) j = 0, 1, \dots, k$  will represent feasible portfolios which will be called self-financing if

$$\sum_{i=1}^n x_j(t_{j-1}, \omega) p_i(t_j, \omega) = \sum_{i=1}^n x_i(t_j, \omega) p_i(t_j, \omega) \text{ a.e.}$$

for  $j = 1, 2, \dots, k$ . Then, one can maximize the random value  $-\sum_{i=1}^n x_i(t_0, \omega) p_i(t_0, \omega)$  among all the finite subsets of  $[0, T]$  and the self-financing portfolios  $x$  verifying

$$\left\{ \begin{aligned} \sum_{i=1}^n x_i(t_k, \omega) p_i(t_k, \omega) &\geq 0 \text{ a.e.} \\ x_i(t_0, \omega) + h_i(\omega) &\geq 0 \text{ a.e., } i = 1, 2, \dots, n \\ h_i(\omega) &\geq 0 \text{ a.e., } i = 1, 2, \dots, n \\ \sum_{i=1}^n p_i^t(\omega) h_i(\omega) &= 1 \text{ a.e.} \end{aligned} \right.$$

It may be proved that solutions of latter vector problem are strong solutions, and they also maximize the expected (not only random) arbitrage profits. Thus, we have extended the theory and obtained new measures  $M(t, s, \omega)$  and  $E(M(t, s, \omega))$  such that  $0 \leq m(t, s, \omega) \leq M(t, s, \omega) \leq 1$  a.e. and  $0 \leq E(m(t, s, \omega)) \leq E(M(t, s, \omega)) \leq 1$ .

## 6. FINANCIAL MARKET INTEGRATION

Chen and Knez (14) develop a measurement theory of market integration for two markets whenever there exist cross-market arbitrage portfolios. They work in a static setting and assume that both markets separately verify the arbitrage absence. They consider that the integration level may be measured by the distance between both sets of discount factors. Thus, their measure is just the minimum second order moment of random variables obtained by differences of discount factors.

In order to introduce the Chen and Knez measure with precision, consider that securities  $A_1, A_2, \dots, A_r$  are available in a first market, and securities  $A_{r+1}, A_{r+2}, \dots, A_n$  are available in a second one. Extending their approach in order to adapt their measure to the general context of this paper, let  $t$  and  $s$  be two arbitrary trading dates. Consider the set  $F_j$  ( $j = 1, 2$ ) of discount factors of market  $j$ . So,  $F_1$  (respectively  $F_2$ ) is the set of square integrable  $\Sigma_s$ -measurable random variables  $f_1$  (respectively  $f_2$ ) such that  $E(f_1 p_i^s | \Sigma_t) = p_i^t, i = 1, 2, \dots, r$  (respectively  $E(f_2 p_i^s | \Sigma_t) = p_i^t, i = r+1, r+2, \dots, n$ ). Then, the Chen and Knez (numerical) measure (hereafter denoted by  $g(t, s)$ ) is defined by

$$g(t, s) = \text{Minimum} \left\{ \int_{\Omega} (f_1(\omega) - f_2(\omega))^2 d\mu(\omega) : f_1 \in F_1, f_2 \in F_2 \right\}$$

To introduce a new integration measure between both markets (see also Balbás and Muñoz (2)) we can treat them as parts of a combined market where  $n$  securities are available

lable, and compute in this global market the measures  $m(t, s, \omega)$  and  $l(t, s, \omega)$  and their expected values. As a consequence, we provide both, numerical and stochastic integration measures.

There are interesting differences between the Chen and Knez measure and the ones here introduced. They provide different information about the level of market integration and, therefore, all of them must be considered in analyzing the integration between markets.

Let us summarize the above mentioned differences (further information may be found in Balbás and Muñoz (2)). First,  $g(t, s)$  measures discrepancy in discount factors, and thus, discrepancy between the criteria applied for both markets to price the securities. However,  $m(t, s, \omega)$  (and  $l(t, s, \omega)$  and their expected values) measures discrepancy between the prices of real securities. It is known that different criteria can sometimes lead to quite similar prices for some securities, and therefore,  $E(m(t, s, \omega))$  and  $g(t, s)$  can achieve quite different values, as shown in some examples presented in Balbás and Muñoz (2). To be precise,  $E(m(t, s, \omega))$  can achieve low values (cross-market arbitrage profits are small) while  $g(t, s)$  remains large.

Another interesting difference concerns the continuity respect to initial data and parameters.  $E(m(t, s, \omega))$  is continuous while  $g(t, s)$  does not verify this property.

This is an important fact that make  $g(t, s)$  very sensitive respect to measurement errors or the effect of market frictions.

$E(m(t, s, \omega))$  is continuous respect  $g(t, s)$  (i.e.  $E(m(t, s, \omega))$  achieves low values if so does  $g(t, s)$ ) but the converse may fail. This is an important fact since  $g(t, s)$  does not provide information in monetary terms. Thus, there might be situations for which  $g(t, s)$  gives low degree of integration ( $g(t, s)$  is large) and the available arbitrage profits almost vanish.

We can conclude that both measures provide different information and both measures are useful and must be considered.

The measure of Chen and Knez has been applied in Hansen and Jagannathan (15) to introduce a new way to compute the discount factors of real markets. Moreover, The Hansen and Jagannathan method was first applied in a particular case by Jackwerth and Rubinstein (27) to analyze the effect of the volatility smile on the discount factors that must be used to price some underlying assets and their derivative securities. Since the Chen and Knez measure is not continuous respect to initial data, the Hansen and Jagannathan method could present some difficulties in some specific situations. It is well known that, for imperfect markets, prices are not always defined with precision (for instance, one can choose among bid, ask and real transaction prices) and that could make their procedure very sensitive in practice.

It is an open problem to analyze the possibilities of  $m(t, s, \omega)$  (and the rest of measures here defined) as a tool to compute discount factors (the dual approach may be crucial) and the degree of fulfillment of theoretical asset pricing models in the real world. This possibility, and the differences respect to the Hansen and Jagannathan method, should be analyzed in future research.

## 7. EMPIRICAL TESTS

The integration level among several Spanish financial markets has been analyzed by means of the theory here presented. To be precise, we have considered the Spanish Markets "Sistema de interconexión bursátil español" (SIBE), "Mercado español de futuros financieros sobre renta variable" (MEFF-RV) and "Mercado de deuda anotada" (MDA), and we have computed the static measure appeared in Balbás and Muñoz (2) (summarized in the section 3.C of this paper) after the minor modifications proposed in section 5.A to incorporate the bid-ask spread and the transaction costs.

The analysis reveals that the measure  $m$  (numerical and not stochastic, since we are working in a static setting) sometimes achieves positive values, and empirical evidence seems to validate the existence of possible cross-market arbitrage riskless profits during stable periods. The profits significantly increase when facing high volatility situations, and further information about the results of the empirical test may be found in Balbás *et al* (28).

## 8. CONCLUSIONS

The integration of financial markets is an important question very often related to the existence of cross-market arbitrage portfolios.

Recent papers have developed some integration measures in order to reflect the size of the cross-market arbitrage opportunities, and they also suggest the convenience of extending the discussion to more complex models and economies.

The present paper extends the results of Balbás and Muñoz (2) and Balbás *et al* (3) and provides a new methodology in a very general setting, allowing many instruments in each market to test optimal arbitrage portfolios depending on the state of nature and the date. Thus, our theory applies for both, static and dynamic asset pricing models, and it may be also adapted in order to incorporate several assumptions on the market frictions.

The measures have interesting interpretations since they reflect relative attainable arbitrage profits. Furthermore, they do not depend on the short selling restrictions assumed and can be easily computed in practical situations. A dual approach also leads to these measures, and therefore, they may be also interpreted in terms of the errors

committed by agents when they price the different securities. The dual approach also yields discount factors, or a proxy for them in no arbitrage free economies. Thus, we extend the methodology of risk-neutral valuation.

Following some ideas appeared in recent literature, the integration measures here presented could be also appropriate as a tool to test the degree of fulfillment in practice of different asset pricing models. This is an interesting possibility that should be the main objective of some future research.

The degree of integration of some Spanish financial markets has been tested by means of the theory here developed, and some surprising results have been obtained since the cross-market arbitrage seems to appear, at least in some dates characterized by high volatilities.

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#### REFERENCES

1. Balbás, A. & M.J. Muñoz (1997) Stochastic measures of financial markets efficiency and integration. *Universidad Carlos III. Working Paper* 97-91 (19).
2. Balbás, A. & M.J. Muñoz (1998) Measuring the degree of fulfillment of the law of one price. Applications to financial market integration. *Investigaciones Económicas*, **22**, 2, 153-177.
3. Balbás, A., P.J. Guerra & M.J. Muñoz (1998) Measuring the arbitrage opportunities in an intertemporal dynamic asset pricing model. *Applied Decision Analysis*. Kluwer, Boston, 161-171.
4. Brenner, R.J. & K.F. Kroner (1995) Arbitrage, cointegration, and testing the unbiasedness hypothesis in financial markets. *Journal of Financial and Quantitative Analysis*, **30**, 1, 23-42.
5. Harris, F.H., T.H. McInish, G.L. Shosmith, & R.A. Wood (1995) Cointegration, error correction, and price discovery on informatively linked security markets. *Journal of Financial and Quantitative Analysis*, **30**, 4, 563-579.
6. Protopapadakis, A. & H.R. Stoll (1983) Spot and future prices and the law of one price. *The Journal of Finance*, **38**, 5, 1431-1455.
7. Kempf, A. & O. Korn (1996) Trading system and market integration. *Proceedings of the 6<sup>th</sup> AFIR Colloquium*. Nuremberg, October, 1-3, 1709-1728.
8. Kleidon, A.W. & R.E. Whaley (1992) One market? stocks, futures and options during October 1987. *The Journal of Finance*, **67**, 3, 851-877.
9. Lee, J.H. & N. Nayar (1993) A transactions data analysis of arbitrage between index options and index futures. *Journal of Future Markets*, **13**, 8, 889-902.
10. Kamara, A. & T.W. Miller (1995) Daily and intraday test of European put-call parity. *Journal of Financial and Quantitative Analysis*, **30**, 4, 519-541.
11. Chamberlain, G. & M. Rothschild (1983) Arbitrage, factor structure, and mean-variance analysis of large assets. *Econometrica*, **51**, 1281-1304.
12. Hansen, L.P. & S.F. Richard (1987) The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models. *Econometrica*, **55**, 3, 587-613.
13. Jouini, E. & H. Kallal (1995) Martingales and arbitrage in security markets with transaction costs. *Journal of Economic Theory*, **66**, 178-197.
14. Chen, Z. & P.J. Knez (1995) Measurement of market integration and arbitrage. *The Review of Financial Studies*, **8**, 2, 545-560.
15. Hansen, L.P. & R. Jagannathan (1997) Assessing Specification Errors in Stochastic Discount Factor Models. *The Journal of Finance*, **52**, 2, 567-590.
16. Ingersoll, J. (1989) *Theory of Financial Decision Making*. Tottawa, NJ: Rowan Littlefield.
17. Schaeffer, H.H. (1974) *Banach lattices and positive operators*. Springer Verlag. New York.
18. Harrison, J. & D. Kreps (1979) Martingales and arbitrage in multiperiods security markets. *Journal of Economic Theory*, **20**, 381-408.
19. Prisman, E.Z. (1986) Valuation of risky assets in arbitrage free economies with frictions. *The Journal of Finance*, **41**, 3, 545-556.
20. Anderson E.J., & P. Nash (1987) *Linear programming in infinite-dimensional spaces*. John Wiley & Sons. New York.
21. Balbás A., & A. Heras (1993) Duality theory for infinite dimensional multiobjective linear programming. *European Journal of Operational Research*, **68**, 379-388.
22. Brennan, M.J. & E.S. Schwartz (1990) Arbitrage in Stock Index Futures. *Journal of Business*, **69**, 1, 7-31.
23. Leland, H. (1985) Option pricing and replication with transaction costs. *The Journal of Finance*, **40**, 1283-1301.
24. Davis, M.H.A., V.G. Panas & T. Zariphopoulou (1993) European option pricing with transaction costs. *Siam Journal of Control and Optimization*, **31**, 470-493.
25. Toft, K.B. (1996) On the mean-variance trade-off in option replication with transaction costs. *Journal of Financial and Quantitative Analysis*, **31**, 2, 233-263.
26. Balbás A., & P.J. Guerra (1996) Sensitivity analysis for convex multiobjective programming in abstract spaces. *Journal of Mathematical Analysis and Applications*, **202**, 645-658.
27. Jackwerth, J.C. & M. Rubinstein (1996) Recovering Probability Distributions from Option Prices. *The Journal of Finance*, **50**, 6, 1611-1631.
28. Balbás, A., I.R. Longareda & A. Pardo (1997) Integration and Arbitrage in the Spanish Financial Markets: An Empirical Approach. *Universidad Carlos III. Working Paper*, 97-92 (20).