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We show that for many classes of symmetric two-player games, the simple decision rule "imitate-the-best" can hardly be beaten by any other decision rule. We provide necessary and sufficient conditions for imitation to be unbeatable in the sense that, even against a very clever opponent, imitation is subject to a money pump if and only if the relative payoff function of the game is of the rock-scissors-paper variety. For many interesting classes of games including examples like 2x2 games, Cournot duopoly, price competition, public goods games, common pool resource games, and minimum effort coordination games, we obtain an even stronger notion of the unbeatability of imitation.

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UNBEATABLE IMITATION*

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Abstract

We show that for many classes of symmetric two-player games, the simple decision rule "imitate-if-better" can hardly be beaten by any strategy. We provide necessary and sufficient conditions for imitation to be unbeatable in the sense that there is no strategy that can exploit imitation as a money pump. In particular, imitation is subject to a money pump if and only if the relative payoff function of the game is of the rock-scissors-paper variety. For many interesting classes of games including examples like 2x2 games, Cournot duopoly, price competition, public goods games, common pool resource games, and minimum effort coordination games, we obtain an even stronger notion of the unbeatability of imitation.

Keywords: Imitate-the-best, learning, symmetric games, relative payoffs, zerosum games, rock-paper-scissors, finite population ESS, potential games, quasisubmodular games, quasisupermodular games, quasiconcave games, aggregative games.

JEL-Classifications: C72, C73, D43.

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"Whoever wants to set a good example must add a grain of foolishness to his virtue: then others can imitate and yet at the same time surpass the one they imitate - which human beings love to do." Friedrich Nietzsche

1 Introduction

Psychologists and behavioral economists stress the role of simple heuristics or rules for human decision making under limited computational capabilities (see Gigerenzer and Selten, 2002). While such heuristics lead to successful decisions in some particular tasks, they may be suboptimal in others. It is plausible that decision makers may cease to adopt heuristics that do worse than others in relevant situations. If various heuristics are pitted against each other in a contest, then in the long run the heuristic with the highest payoff should survive.

The competing heuristics could be anything from very simple to rational, omniscient, and forward looking ones. Even if a specific rule is not currently among the contestants, there can always be a "mutation", i.e., an invention of a new rule, that enters the pool of rules. A heuristic that does very badly against other rules will not be around for long as it will not belong to the top performers. Being subject to exploitation by the opponent in strategic situations would be an evolutionary liability. Consequently, we would like to raise the following question: Is there a simple adaptive heuristic that can not be beaten by any strategy including even those of a rational, omniscient and forward looking maximizer in large classes of economically relevant situations?

The idea for this paper emerged from a prior observation in experimental data. In Duersch, Kolb, Oechssler, and Schipper (2010), subjects played against computers that were programmed according to various learning algorithms in a Cournot duopoly. On average, human subjects easily won against all of their computer opponents with one exception: the computer following the rule "imitate-if-better", the rule that simply prescribes to mimic the action of another player if and only if the other player received a higher payoff in the previous period. This suggested to us that imitation may be hard to beat by other strategies including strategies by forward–looking players.

In this paper, we prove that this holds more generally. The decision heuristic "imitateif-better" is very hard to beat by *any* strategy in large classes of symmetric two-player games that are highly relevant for economics and include games such as all symmetric 2x2 games, Cournot duopoly, Bertrand duopoly, rent seeking, public goods games, common pool resource games, minimum effort coordination games, Diamond's search, Nash demand bargaining, etc.

We shall consider two notions of being "unbeatable". We call imitation "essentially unbeatable" if in the infinitely repeated game there exists no strategy of the opponent with which she can obtain, in total, over an infinite number of periods, a payoff difference that is more than the maximal payoff difference for the one-period game. As a weaker notion we consider the concept of being "not subject to a money pump". We say imitation is not subject to a money pump if there is a bound on the sum of payoff differences any opponent can achieve in the infinitely repeated game. Or equivalently, if there is no cyclic strategy of the opponent, in which the imitator earns less than the opponent.

Since our results hold for all possible strategies of the imitator's opponent, they also apply to strategies by truly sophisticated opponents. In particular, the opponent may be infinitely patient, forward looking, and free of mistakes. More importantly, the opponent can be aware of the fact that she is matched against an imitator. That is, she may know exactly what her opponent, the imitator, would do at all times, including the imitator's starting value. Finally, the opponent may be able to commit to any strategy including any closed-loop strategy.

Our results are as follows. We present *necessary and sufficient conditions* for imitation to be subject to a money pump. The paradigmatic example for a money pump is playing repeatedly the game rock-paper-scissors, in which, obviously, an imitator can be exploited without bounds. The main result of this paper is that imitation is subject to a money pump *if and only if* the *relative payoff* game in question contains a generalized rock-paper-scissors submatrix.

Since the existence of a rock-paper-scissors submatrix may be cumbersome to check in some instances, we also provide a number of sufficient conditions for imitation not to be subject to a money pump that are based on more familiar concepts like quasiconcavity, generalized ordinal potentials, or quasisubmodularity/quasisupermodularity and aggregation of actions.

We also provide a number of sufficient conditions for imitation to be essentially unbeatable like exact potentials, increasing/decreasing differences, or additive separability. One such condition is that the game is a symmetric 2x2 game. To gain some intuition for this, consider the game of "chicken" presented in the following payoff matrix.

	swerve	straight
swerve	(3, 3)	1, 4
straight	$\begin{pmatrix} 4,1 \end{pmatrix}$	0,0

Suppose that initially the imitator starts out with playing "swerve". What should a

forward looking opponent do? If she decides to play "straight", she will earn more than the imitator today but will be copied by the imitator tomorrow. From then on, the imitator will stay with "straight" forever. If she decides to play "swerve" today, then she will earn the same as the imitator and the imitator will stay with "swerve" as long as the opponent stays with "swerve". Suppose the opponent is a dynamic relative payoff maximizer. In that case, the dynamic relative payoff maximizer can beat the imitator at most by the maximal one-period payoff differential of 3. Now suppose the opponent maximizes the sum of her *absolute* payoffs. The best an absolute payoff maximizer can do is to play swerve forever. In this case the imitator cannot be beaten at all as he receives the same payoff as his opponent. In either case, imitation comes very close to the top–performing heuristics and there is no evolutionary pressure against such an heuristic.

The behavior of learning heuristics has previously been studied mostly for the case when all players use the same heuristic. For the case of imitate-the-best,¹ Vega-Redondo (1997) showed that in a symmetric Cournot oligopoly with imitators, the long run outcome converges to the competitive output if small mistakes are allowed. This result has been generalized to aggregative quasisubmodular games by Schipper (2003) and Alós-Ferrer and Ania (2005). Huck, Normann, and Oechssler (1999), Offerman, Potters, and Sonnemans (1997), and Apesteguia et al. (2007, 2010) provide some experimental evidence in favor of imitative behavior. In contrast to the above cited literature, the current paper deals with the interaction of an imitator and a forward looking, very rational and patient player. Apart from experimental evidence in Duersch, Kolb, Oechssler, and Schipper (2010) we are not aware of any work that deals with this issue. For a Cournot oligopoly with imitators and myopic best reply players, Schipper (2009) showed that the imitators' long run average payoffs are strictly higher than the best reply players' average payoffs.

A recent paper by Feldman, Kalai, and Tennenholtz (2010) has a similar but complementary objective to ours. They study whether a strategy which they call "copycat" can be beaten in a symmetric two-player game by an arbitrary opponent who may have full knowledge of the game and may play any history dependent strategy. The copycat strategy, on the other hand, can only observe past actions of both players. Remarkably, the copycat strategy can nearly match the average payoff of the opponent. The strategy

¹For the two-player case, imitate-the-best and imitate-if-better are almost equivalent, the difference being that the latter specifically prescribes a tie-breaking rule (for the case of both players having equal payoffs in the previous round). Since we use imitate-if-better only in the two-player case, we do not need to specify what happens if more than one other player is observed.

used by copycat is to equalize the occurrences of action profiles (x, y) and (y, x) for any $x, y \in X$. To achieve this, an auxiliary two-player zero-sum game is introduced whose payoff at (x, y) is the difference of frequencies of (x, y) and (y, x) played so far. Thus, similar to our approach, the authors use the idea of imitation and auxiliary zero-sum games. Yet, their copycat rule is far more sophisticated than our imitation rule as it entails finding a (possibly mixed) minmax strategy in the auxiliary zero-sum game in each round. On the other hand, the possible opponents are less omniscient than the opponents in our setting. In particular, the opponents in their setting cannot perfectly predict the imitator's action in the next round, which explains why Feldman et al.'s result apply even to rock-paper-scissors.

The article is organized as follows. In the next section, we present the model and provide formal definitions for being unbeatable. Our main result, which provides a necessary and sufficient condition for a money pump, is contained in Section 3. Sufficient conditions for imitation to be essentially unbeatable are given in Section 4. Section 5 provides sufficient conditions for imitation not being subject to a money pump. We finish with Section 6, where we summarize and discuss the results.

2 Model

We consider a symmetric two-player game (X, π) , in which both players are endowed with the same (finite or infinite) set of pure actions X. For each player, the bounded payoff function is denoted by $\pi : X \times X \longrightarrow \mathbb{R}$, where $\pi(x, y)$ denotes the payoff to the player choosing the first argument when his opponent chooses the second argument. We will frequently make use of the following definition.

Definition 1 (Relative payoff game) Given a symmetric two-player game (X, π) , the relative payoff game is (X, Δ) , where the relative payoff function $\Delta : X \times X \longrightarrow \mathbb{R}$ is defined by

$$\Delta(x, y) = \pi(x, y) - \pi(y, x).$$

Note that, by construction, every relative payoff game is a symmetric zero-sum game since $\Delta(x, y) = -\Delta(y, x)$.

The *imitator* follows the simple rule "imitate-if-better". To be precise, the imitator adopts the opponent's action if and only if in the previous round the opponent's payoff was strictly higher than that of the imitator. Formally, the action of the imitator y_t in period t given the action of the other player from the previous period x_{t-1} is

$$y_{t} = \begin{cases} x_{t-1} & \text{if } \Delta(x_{t-1}, y_{t-1}) > 0\\ y_{t-1} & \text{else} \end{cases}$$
(1)

for some initial action $y_0 \in X$.

Our aim is to determine whether there exists a strategy of the imitator's opponent that obtains substantially higher payoffs than the imitator. We allow for any strategy of the opponent, including very sophisticated ones. In particular, the opponent may be infinitely patient and forward looking, and may never make mistakes. More importantly, she may know exactly what her opponent, the imitator, will do at all times, including the imitator's starting value. She may also commit to any closed loop strategy.

We now present two definitions of what we mean by "unbeatable". Consider first a situation in which an imitator starts out with a very unfavorable initial action. A clever opponent who knows this initial action can take advantage off it. Yet, from then on the opponent has no strategy that makes her better off than the imitator. Arguably, the disadvantage in the initial period should not play a role in the long run. This motivates the first definition.

Definition 2 (Essentially unbeatable) We say that imitation is essentially unbeatable if for any initial action of the imitator and any strategy of the opponent, the imitator can be beaten in total by at most the maximal one-period payoff differential, i.e., if for any y_0 and any sequence $\{x_t\}$,

$$\sum_{t=0}^{T} \Delta(x_t, y_t) \le \max_{x, y} \Delta(x, y), \text{ for all } T \ge 0,$$
(2)

where y_t is given by equation (1).

In the chicken game discussed in the Introduction, imitation was essentially unbeatable since the maximal payoff difference was 3.

Essentially unbeatable is a demanding property. The following is a weaker notion of being "unbeatable".

Definition 3 (No money pump) We say that imitation is not subject to a money pump if there exists a finite bound M such that for any initial action of the imitator y_0 and any sequence $\{x_t\}$ of actions of the opponent

$$\sum_{t=0}^{T} \Delta(x_t, y_t) \le M, \quad \text{for all } T \ge 0,$$
(3)

where y_t is given by equation (1).

Clearly, no money pump reduces to essentially unbeatable if $M = \max_{x,y} \Delta(x, y)$.

Again, one can argue that the finite disadvantage should not play a role in the long run as time goes to infinity.

The name of the latter condition is motivated by the observation that in a finite game, imitation is not subject to a money pump if the opponent cannot create a cycle of actions that strictly improve her relative payoff at every step. This is reminiscent of "no money pumps" in economics. The following definitions make this precise.

Given a symmetric two-player game (X, π) , a path in the action space $X \times X$ is a sequence of action profiles $(x_0, y_0), (x_1, y_1), \dots$ A path is constant if $(x_t, y_t) = (x_{t+1}, y_{t+1})$ for all $t = 0, 1, \dots$ Otherwise, the path is called non-constant. A non-constant finite path $(x_0, y_0), \dots, (x_n, y_n)$ is a cycle if $(x_0, y_0) = (x_n, y_n)$ for some n > 1. Let us call a cycle an *imitation cycle* if for all (x_t, y_t) and (x_{t+1}, y_{t+1}) on the path of the cycle $\Delta(x_t, y_t) > 0$ and $y_{t+1} = x_t$. An imitation cycle is thus a particular cycle along which one player always obtains a strictly positive relative payoff and the other player mimics the action of the first player in the previous round. Thus, an imitation cycle never contains an action profile on the diagonal of the payoff matrix.

Lemma 1 For any finite symmetric game (X, π) , imitation is subject to a money pump if and only if there exists an imitation cycle.

PROOF. Consider a finite symmetric game (X, π) and its relative payoff game (X, Δ) . We show that if imitation is subject to a money pump, then there is a imitation cycle. The converse is trivial.

Since the game is finite, there can not be infinitely many strictly positive relative payoff improvements unless there is a cycle. To show that such a cycle implies an imitation cycle, suppose by contradiction that there exists a period t such that $\Delta(x_t, y_t) \leq 0$. W.l.o.g. assume that $\Delta(x_{t+1}, y_{t+1}) > 0$. This is w.l.o.g. because we assumed a money pump. By equation (1) the imitator will not imitate in t + 1 the previous period's action of the opponent, i.e., $y_{t+1} = y_t$. But then, there must be a cycle with $x_t = x_{t+1}$, $x_{t+1} = x_{t+2}$, ... By applying this argument to any period t for which $\Delta(x_t, y_t) \leq 0$, we can construct a cycle with $\Delta(x_t, y_t) > 0$ for all t. The decision rule of the imitator then requires that $y_{t+1} = x_t$ for all t, which proves that such a cycle is an imitation cycle. \Box As in previous studies of imitation (see e.g. Alós-Ferrer and Ania, 2005; Schipper, 2003; Vega-Redondo, 1997), the concept of a finite population evolutionary stable strategy (Schaffer, 1988, 1989) plays a prominent role in our analysis.

Definition 4 (fESS) An action $x^* \in X$ is a finite population evolutionary stable strategy (fESS) of the game (X, π) if

$$\pi(x^*, x) \ge \pi(x, x^*) \text{ for all } x \in X.$$
(4)

In terms of the relative payoff game, inequality (4) is equivalent to

$$\Delta(x^*, x) \ge 0 \text{ for all } x \in X.$$

Already Schaffer (1988, 1989) observed that the fESS of the game (X, π) and the symmetric pure Nash equilibria of the relative payoff game (X, Δ) coincide.

3 A Necessary and Sufficient Condition for a Money Pump

The game rock–paper-scissors is the paradigmatic example for how an imitator can be exploited without bounds by a clever opponent. In our terminology, imitation is subject to a money pump.

Example 1 (Rock-Paper-Scissors) Consider the well known rock-paper-scissors game.²

$$\begin{array}{cccc}
R & P & S \\
R & \left(\begin{array}{ccc}
0 & -1 & 1 \\
1 & 0 & -1 \\
S & \left(\begin{array}{ccc}
-1 & 1 & 0
\end{array}\right)
\end{array}$$

If the imitator starts for instance with R, then the opponent can play the cycle P-S-R... In this way, the opponent could win in every period and the imitator would lose in every period. Over time, the payoff difference would grow without bound in favor of the opponent.

 $^{^2\}mathrm{In}$ the following, we will often represent symmetric payoff matrices by the matrix of the row player's payoffs only.

We can generalize Example 1 by noting that the crucial feature of the example is that a money pump is created by the fact that for each action of the imitator there is an action of the opponent which yields her a strictly positive relative payoff and which yields the imitator a strictly negative relative payoff.

Definition 5 (gRPS Matrix) A symmetric zero-sum game (X, π) is called a generalized rock-paper-scissors (gRPS) matrix if for each column there exists a row with a strictly positive payoff to the row player, i.e. if for all $y \in X$ there exists a $x \in X$ such that $\pi(x, y) > 0$.

It should be fairly obvious that if a zero–sum game contains somewhere a submatrix that is a generalized rock-paper-scissors matrix, then this is sufficient for a money pump as the opponent can make sure that the process cycles forever in this submatrix. What is probably less obvious is that the existence of such a submatrix is also necessary for a money pump.

Definition 6 (gRPS Game) A symmetric zero-sum game (X, π) is called a generalized rock-paper-scissors (gRPS) game if it contains a submatrix $(\bar{X}, \bar{\pi})$ with $\bar{X} \subseteq X$ and $\bar{\pi}(x, y) = \pi(x, y)$ for all $x, y \in \bar{X}$, and $(\bar{X}, \bar{\pi})$ is a gRPS matrix.

This leads us to our main result.

Theorem 1 Imitation is subject to a money pump in the finite symmetric game (X, π) if and only if its relative payoff game (X, Δ) is a gRPS game.

The proof follows from Lemma 1 and the following lemma.

Lemma 2 Consider a finite symmetric game (X, π) with its relative payoff game (X, Δ) . (X, Δ) is a gRPS game if and only if there exists an imitation cycle.

PROOF. " \Leftarrow ": If there exists an imitation cycle in (X, Δ) , let \bar{X} be the orbit of the cycle, i.e., all actions of X that are played along the imitation cycle. For each action (i.e., column) $y \in \bar{X}$, there exists an action (i.e., row) $x \in \bar{X}$ such that $\Delta(x, y) > 0$. Hence, $(\bar{X}, \bar{\Delta})$, where $\bar{\Delta}$ is defined by $\bar{\Delta}(x, y) = \Delta(x, y)$ for all $x, y \in \bar{X}$, is a gRPS submatrix. Thus, (X, Δ) is a gRPS game.

" \Rightarrow ": If the relative payoff game (X, Δ) is a gRPS game, then it contains a gRPS submatrix $(\bar{X}, \bar{\Delta})$. That is, for each column of the matrix game $(\bar{X}, \bar{\Delta})$ there exists a row

with a strictly positive relative payoff to player 1. Let the initial action of the imitator y be contained in \overline{X} . If the opponent selects such a row $x \in \overline{X}$ for which she earns a strict positive relative payoff, i.e., $\Delta(x, y) > 0$, then she will be imitated by the imitator in the next period. Yet, at the next period, when the imitator plays x, the opponent has another action $x' \in \overline{X}$ with a strictly positive relative payoff, i.e., $\Delta(x', x) > 0$. Thus the imitator will imitate her in the following period. More generally, for each action $y \in \overline{X}$ of the imitator, there is another action $x \in \overline{X}$, $x \neq y$ of the opponent that earns the latter a strictly positive relative payoff. Since \overline{X} is finite, such a sequence of actions must contain a cycle. Moreover, we just argued that $\Delta(x_t, y_t) > 0$ and $y_{t+1} = x_t$ for all t. Thus, it is an imitation cycle.

Theorem 1 is used to obtain an interesting necessary condition for imitation being not subject to a money pump.

Proposition 1 Let (X, π) be a finite symmetric game with its relative payoff game (X, Δ) . If (X, Δ) has no pure equilibrium, then imitation is subject to a money pump.

PROOF. By Theorem 1 in Duersch, Oechssler, and Schipper (2011), (X, Δ) has no symmetric pure equilibrium if and only if it is a gRPS matrix. Thus, if (X, Δ) has no symmetric pure equilibrium, then it is a gRPS game. Hence, by Theorem 1 imitation is subject to a money pump.

Corollary 1 If the finite symmetric game (X, π) has no fESS, then imitation is subject to a money pump.

In other words, the existence of a fESS is a necessary condition for imitation not being subject to a money pump. The reason for the existence of a fESS not being sufficient is that there could be a gRPS submatrix of the game ("disjoint" from the fESS profile) that gives rise to an imitation cycle.

Since the relative payoff game of a symmetric zero-sum game is a gRPS game if and only if the underlying symmetric zero-sum game is a gRPS game, we obtain from Theorem 1 the following corollary.

Corollary 2 Imitation is subject to a money pump in the finite symmetric zero-sum game (X, π) if and only if (X, π) is a gRPS game.

4 Sufficient Conditions for Essentially Unbeatable

In this section we present two classes of games for which imitation is essentially unbeatable. The first class is the class of 2x2 games. The second class is the class of games with an exact potential.

4.1 Symmetric 2x2 games

In this section, we extend the "chicken" example of the introduction to all symmetric 2x2 games. Note that the relative payoff game of any symmetric 2x2 game cannot be a generalized rock-paper-scissors matrix since the latter must be a symmetric zero-sum game. If one of the row player's off-diagonal relative payoffs is a > 0, then the other must be -a violating the definition of a gRPS matrix. Thus Theorem 1 implies that for any symmetric 2x2 game imitation is not subject to a money pump. We can strengthen the result to imitation being essentially unbeatable.

Proposition 2 In any symmetric 2x2 game, imitation is essentially unbeatable.

PROOF. Let $X = \{x, x'\}$. Consider a period t in which the opponent achieves a strictly positive relative payoff, $\Delta(x, x') > 0$. (If no such period t in which the opponent achieves a strictly positive relative payoff exists, then trivially imitation is essentially unbeatable.) Obviously, $\Delta(x, x') \leq \max_{x,y} \Delta(x, y)$. Since $\Delta(x, x') > 0$, the imitator imitates x in period t + 1. For there to be another period in which the opponent achieves a strictly positive relative payoff, it must hold that $\Delta(x', x) > 0$. This yields a contradiction since the relative payoff game is symmetric zero-sum and hence $\Delta(x', x) = -\Delta(x, x')$. Thus there can be at most one period in which the opponent achieves a strictly positive relative payoff.

Note that "Matching pennies" is not a counter-example since it is not symmetric.

4.2 Exact Potential Games

Next, we consider games that possess an exact potential function. The following notion is due to Monderer and Shapley (1996).

Definition 7 (Exact potential games) The symmetric game (X, π) is an exact potential game if there exists an exact potential function $P: X \times X \longrightarrow \mathbb{R}$ such that for all $y \in X$ and all $x, x' \in X$,³

$$\pi(x, y) - \pi(x', y) = P(x, y) - P(x', y),$$

$$\pi(x, y) - \pi(x', y) = P(y, x) - P(y, x').$$

The following definition may appear to be restrictive. However, we will show below that there is a fairly large number of important examples that fall into this class.

Definition 8 (Additively Separable) A relative payoff function Δ is additively separable if $\Delta(x, y) = f(x) + g(y)$ for some functions $f, g: X \longrightarrow \mathbb{R}$.

Properties such as increasing or decreasing differences are often useful for proving the existence of pure equilibria and convergence of learning processes.

Definition 9 Let X be a totally ordered set. A (relative) payoff function Δ has decreasing (resp. increasing) differences on $X \times X$ if for all $x'', x', y'', y' \in X$ with x'' > x' and y'' > y',

$$\Delta(x'', y'') - \Delta(x', y'') \le (\ge) \Delta(x'', y') - \Delta(x', y').$$
(5)

 Δ is a valuation if it has both decreasing and increasing differences.

Our original intent was to study the consequences of $\Delta(x, y)$ having either increasing or decreasing differences. However, it turns out that all of the above properties are equivalent in our context.

Proposition 3 Let (X, π) be a symmetric two-player game. Suppose that X is a compact and totally ordered set and π is continuous. Then imitation is essentially unbeatable if any of the following conditions holds:

- (i) (X,π) is an exact potential game
- (ii) (X, Δ) is an exact potential game
- (iii) Δ has increasing differences

³Given the symmetry of (X, π) , the second equation plays the role usually played by the quantifier "for all players" in the definition of potential games.

(iv) Δ has decreasing differences

(v) Δ is additively separable.

PROOF. We first note that all five conditions are equivalent in our context. Duersch, Oechssler, and Schipper (2011, Theorem 3) show that (i) and (ii) are equivalent. There, we also show that (iii) and (iv) are equivalent for all symmetric two-player zero-sum games Duersch, Oechssler, and Schipper (2011, Proposition 1). Hence, (iii) or (iv) imply that Δ is a valuation. Brânzei, Mallozzi, and Tijs (2003, Theorem 1) show that (ii) is equivalent to Δ being a valuation for zero-sum games. Finally, Topkis (1998, Theorem 2.6.4.) shows equivalence of (v) and Δ being a valuation for zero-sum games. Thus, it suffices to prove the claim for condition (v).

Let Δ be additively separable, i.e. $\Delta(x, y) = f(x) + g(y)$ for some functions $f, g : X \longrightarrow \mathbb{R}$. Thus we have for all $x'', x', x \in X$,

$$\Delta(x'', x) - \Delta(x', x) = \Delta(x'', x') - \Delta(x', x'),$$

which is equivalent to

$$\Delta(x'', x) = \Delta(x'', x') + \Delta(x', x) \tag{6}$$

because $\Delta(x', x') = 0$ since the relative payoff game is a symmetric zero-sum game.

Let $(x_0, x_1, ...)$ be a sequence of opponent's actions generated by an opponent's strategy, and let $\{\Delta(x_t, y_t)\}_{t=0,1,...}$ be her associated sequence of relative payoffs when the imitator follows his imitation rule in equation (1) with an initial action y_0 . Now consider the subsequence of strictly positive relative payoffs of the opponent, $\{\Delta(x_t, y_t)|t = 0, 1, ...; \Delta(x_t, y_t) > 0\}$. Assume the case that $\{\Delta(x_t, y_t)|t = 0, 1, ...; \Delta(x_t, y_t) > 0\}$ is not a singleton. (Otherwise the Proposition follows trivially.) Observe that for any adjacent elements of the subsequence, say $\Delta(x_k, y_k)$ and $\Delta(x_{k+\ell}, y_{k+\ell})$ (for some $\ell > 0$), we must have $\Delta(x_{k+\ell}, y_{k+\ell}) = \Delta(x_{k+\ell}, x_k)$. This is because an imitator mimics the opponent if the opponent obtained a strictly positive relative payoff and stays with his own action if the opponent's relative payoff was less than or equal to zero. Note that $\sum_{t=k}^{k+\ell} \Delta(x_t, y_t) \leq \Delta(x_{k+\ell}, x_k) + \Delta(x_k, y_k) = \Delta(x_{k+\ell}, y_k)$, where the inequality follows from the fact that all elements of the sequence strictly between k and $k + \ell$ are non-positive and the equality follows from equation (6) above. Applying this argument inductively yields that for any y_0 and T > 0 for which $\Delta(x_T, y_0) > 0$, we have that

$$\sum_{t=0}^{T} \Delta(x_t, y_t) \le \Delta(x_T, y_0) \le \max_{x, y} \Delta(x, y),$$

where $\max_{x,y} \Delta(x, y)$ exists because π is continuous and X is compact.

As sufficient condition for the additive separability of relative payoffs is provided in the next result.

Corollary 3 Consider a game (X, π) with a compact action set X and a payoff function that can be written as $\pi(x, y) = f(x) + g(y) + a(x, y)$ for some continuous functions $f, g: X \longrightarrow \mathbb{R}$ and a symmetric function $a: X \times X \longrightarrow \mathbb{R}$ (i.e., a(x, y) = a(y, x) for all $x, y \in X$). Then imitation is essentially unbeatable.

The following examples demonstrate that the assumption of additively separable relative payoffs is not as restrictive as may be thought at first glance. All of those games are also exact potential games. However, often the conditions on the relative payoffs are easier to verify than finding an exact potential function.

Example 2 (Cournot Duopoly with Linear Demand) Consider a (quasi) Cournot duopoly given by the symmetric payoff function $\pi(x, y) = x(b - x - y) - c(x)$ with b > 0. Since $\pi(x, y)$ can be written as $\pi(x, y) = bx - x^2 - c(x) - xy$, Corollary 3 applies, and imitation is essentially unbeatable.

Example 3 (Bertrand Duopoly with Product Differentiation) Consider a differentiated duopoly with constant marginal costs, in which firms 1 and 2 set prices x and y, respectively. Firm 1's profit function is given by $\pi(x, y) = (x - c)(a + by - \frac{1}{2}x)$, for a > 0, $b \in [0, 1/2)$. Since $\pi(x, y)$ can be written as $\pi(x, y) = ax - ac + \frac{1}{2}cx - \frac{1}{2}x^2 - bcy + bxy$, Corollary 3 applies, and imitation is essentially unbeatable. This example with strategic complementarities also shows that the result is not restricted to strategic substitutes.

Example 4 (Public Goods) Consider the class of symmetric public good games defined by $\pi(x, y) = g(x, y) - c(x)$ where g(x, y) is some symmetric monotone increasing benefit function and c(x) is an increasing cost function. Usually, it is assumed that g is an increasing function of the sum of provisions, x + y. Various assumptions on g have been studied in the literature such as increasing or decreasing returns. In any case, Corollary 3 applies, and imitation is essentially unbeatable.

Example 5 (Common Pool Resources) Consider a common pool resource game with two appropriators. Each appropriator has an endowment e > 0 that can be invested in an outside activity with marginal payoff c > 0 or into the common pool resource. Let $x \in X \subseteq [0, e]$ denote the opponent's investment into the common pool

resource (likewise y denotes the imitator's investment). The return from investment into the common pool resource is $\frac{x}{x+y}(a(x+y) - b(x+y)^2)$, with a, b > 0. So the symmetric payoff function is given by $\pi(x,y) = c(e-x) + \frac{x}{x+y}(a(x+y) - b(x+y)^2)$ if x, y > 0 and *ce* otherwise (see Walker, Gardner, and Ostrom, 1990). Since $\Delta(x,y) =$ $(c(e-x) + ax - bx^2) - (c(e-y) + ay - by^2)$, Proposition 3 implies that imitation is essentially unbeatable.

Example 6 (Minimum Effort Coordination) Consider the class of minimum effort games given by the symmetric payoff function $\pi(x, y) = \min\{x, y\} - c(x)$ for some cost function $c(\cdot)$ (see Bryant, 1983, and Van Huyck, Battalio, and Beil, 1990). Corollary 3 implies that imitation is essentially unbeatable.

Example 7 (Synergistic Relationship) Consider a synergistic relationship among two individuals. If both devote more effort to the relationship, then they are both better off, but for any given effort of the opponent, the return of the player's effort first increases and then decreases. The symmetric payoff function is given by $\pi(x, y) = x(c + y - x)$ with c > 0 and $x, y \in X \subset \mathbb{R}_+$ with X compact (see Osborne, 2004, p.39). Corollary 3 implies that imitation is essentially unbeatable.

Example 8 (Diamond's Search) Consider two players who exert effort searching for a trading partner. Any trader's probability of finding another particular trader is proportional to his own effort and the effort by the other. The payoff function is given by $\pi(x, y) = \alpha xy - c(x)$ for $\alpha > 0$ and c increasing (see Milgrom and Roberts, 1990, p. 1270). The relative payoff game of this two-player game is additively separable. By Proposition 3 imitation is essentially unbeatable.

Finally, a natural question is whether additive separability of relative payoffs (or equivalently the existence of an exact potential function for the underlying game) are also necessary conditions for imitation to be essentially unbeatable. The following counterexample shows that this is not the case.

Example 9 (Coordination game with outside option) Consider the following coordination game with an outside option (C) for both players of not participating (left matrix).

$$\pi = \begin{array}{ccc} A & B & C \\ A & -1 & 0 \\ 2 & 3 & 0 \\ C & C & 0 & 0 \end{array} \right) \qquad \Delta = \begin{array}{ccc} A & B & C \\ A & -1 & 0 \\ B & B & C & 0 \\ 0 & 0 & 0 \end{array} \right)$$

Note that the relative payoff game Δ (right matrix) does not have constant differences. E.g., $\Delta(A, B) - \Delta(B, B) = -3 \neq \Delta(A, C) - \Delta(B, C) = 0$. Thus, by Topkis (1998, Theorem 2.6.4.) it is not additively separable, and by Duersch, Oechssler, and Schipper (2011, Theorem 3) (X, π) is not an exact potential game. Yet, imitation is essentially unbeatable. If the imitator's initial action is A, the opponent can earn at most a relative payoff differential of 3 after which the imitator adjusts and both earn zero from there on. For other initial actions of the imitator, the maximal payoff difference is at most 0.

5 Sufficient Conditions for No Money Pump

The existence of a gRPS submatrix may be cumbersome to check in some instances. Therefore, we provide below a number of sufficient conditions for imitation not to be subject to a money pump that are based on more familiar concepts like quasiconcavity, generalized ordinal potentials, or quasisubmodularity/quasisupermodularity and aggregation of actions. Yet, quite differently to what is usually done in the literature we impose these properties on the relative payoff games rather than on the underlying games.

5.1 Relative Payoff Games with Generalized Ordinal Potentials

Potential functions are often useful for obtaining results on convergence of learning algorithms to equilibrium, existence of pure equilibrium, and equilibrium selection.⁴ In the previous section, we have shown in Proposition 3 that if the relative payoff game is an exact potential game, then imitation is essentially unbeatable. It is natural to explore the implications of more general notions of potentials. Besides exact potential games (see Definition 7), the following notion was introduced by Monderer and Shapley (1996).

Definition 10 (Generalized ordinal potential games) The symmetric game (X, π) is a generalized ordinal potential game if there exists a generalized ordinal potential function $P: X \times X \longrightarrow \mathbb{R}$ such that for all $y \in X$ and all $x, x' \in X$,

$$\pi(x, y) - \pi(x', y) > 0 \quad implies \quad P(x, y) - P(x', y) > 0,$$

$$\pi(x, y) - \pi(x', y) > 0 \quad implies \quad P(y, x) - P(y, x') > 0.$$

⁴For some of the classes of games considered here there exist convergence results for various learning processes although convergence results for imitation are rare (see Alós-Ferrer and Ania, 2005, Schipper, 2003, and Vega-Redondo, 1997). Note, however, that our results do not follow from any results in the literature since we do not consider a pair of imitators but rather one imitator against an arbitrary decision rule of the opponent.

Note that every exact potential game is a weighted potential game, every weighted potential game is an ordinal potential game, and every ordinal potential game is a generalized ordinal potential game. Monderer and Shapley (1996, Lemma 2.5 and the first paragraph on p. 129) show that any finite strategic game admitting a generalized ordinal potential possesses a pure Nash equilibrium. Thus, if (X, π) is a finite symmetric game with relative payoff game (X, Δ) and the latter is a generalized ordinal potential game, then (X, π) possesses a fESS.

A sequential path in the action space $X \times X$ is a sequence $(x_0, y_0), (x_1, y_1), ...$ of profiles $(x_t, y_t) \in X \times X$ such that for all t = 0, 1, ..., the action profiles (x_t, y_t) and (x_{t+1}, y_{t+1}) differ in exactly one player's action. A sequential path is a strict improvement path if for each t = 0, 1, ..., the player who switches her action at t strictly improves her payoff. A finite sequential path $(x_0, y_0), ..., (x_m, y_m)$ is a *strict improvement cycle* if it is a strict improvement path and $(x_0, y_0) = (x_m, y_m)$.

Lemma 3 If (X, Δ) does not contain a strict improvement cycle, then it does not contain an imitation cycle.⁵

PROOF. We prove the contrapositive. I.e., if (X, Δ) contains an imitation cycle, then it contains a strict improvement cycle. Let $(x_0, y_0), ..., (x_m, y_m)$ be an imitation cycle. From this imitation cycle, we construct a strict improvement cycle as follows: For t = 0, ..., m - 1, we add the element (x_t, y_{t+1}) as successor to (x_t, y_t) and predecessor to (x_{t+1}, y_{t+1}) . That is, instead of simultaneous adjustments of actions at each period as in an imitation cycle, we let players adjust actions sequentially by taking turns. The imitator adjusts from (x_t, y_t) to (x_t, y_{t+1}) and the opponent from (x_t, y_{t+1}) to (x_{t+1}, y_{t+1}) for t = 0, ..., m - 1. This construction yields a sequential path.

We now show that it is a strict improvement cycle. First, for the imitator, whenever he adjusts in t = 0, ..., m - 1, we claim $\Delta(y_t, x_t) < \Delta(y_{t+1}, x_t) = 0$. Note that by symmetric zero-sum, $\Delta(y_t, x_t) = -\Delta(x_t, y_t) < 0$ because (x_t, y_t) is an element of an imitation cycle, i.e., $\Delta(x_t, y_t) > 0$. $\Delta(y_{t+1}, x_t) = 0$ because the imitator mimics the action of the opponent, $y_{t+1} = x_t$. Thus $\Delta(y_{t+1}, x_t) = \Delta(x_t, x_t) = 0$ by symmetric zero-sum.

⁵Ania (2008, Proposition 3) presents a similar result according to which if all players are imitators and imitation is payoff improving, then the fESS is a Nash equilibrium action. This is different from Lemma 3 as we consider an imitator against an opponent who herself may not imitate and focus on the relationship between relative payoff games that possess a generalized ordinal potential and imitation cycles.

Second, for the opponent, whenever she adjusts in $t = 1, ..., m, \Delta(x_t, y_t) > \Delta(x_{t-1}, y_t) = 0$ because (x_t, y_t) is an element of an imitation cycle, so $\Delta(x_t, y_t) > 0$. Moreover, the imitator mimics the action of the opponent, i.e., $y_t = x_{t-1}$, and thus $\Delta(x_{t-1}, y_t) = \Delta(x_{t-1}, x_{t-1}) = 0$. Hence $(x_0, y_0), (x_0, y_1), (x_1, y_1), ..., (x_{m-1}, y_m), (x_m, y_m)$ is indeed a strict improvement cycle.

The converse is not true as the following counter-example shows.

Example 10 Consider the following relative payoff game.⁶

$$\Delta = \begin{array}{ccc} a & b & c \\ b & \left(\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 1 \\ c & \left(\begin{array}{ccc} 1 & -1 & 0 \end{array} \right) \end{array} \right)$$

Clearly, this game is not a gRPS game. Thus, by Lemma 1 it does not possess an imitation cycle. However, we can construct a strict improvement cycle (b, a), (c, a), (c, c), (b, c) and (b, a).

Proposition 4 Let (X, π) be a finite symmetric game with its relative payoff game (X, Δ) . If (X, Δ) is a generalized ordinal potential game, then imitation is not subject to a money pump.

PROOF. Monderer and Shapley (1996, Lemma 2.5) show that a finite strategic game has no strict improvement cycle (what they call the finite improvement property) if and only if it is a generalized ordinal potential game. Since this result holds for any finite strategic game, it holds also for any finite symmetric zero-sum game (X, Δ) .

Lemma 3 shows that if (X, Δ) does not contain a strict improvement cycle, then it does not contain an imitation cycle. Thus Lemma 1 implies that imitation is not subject to a money pump.

If the converse were true, then the class of generalized ordinal potential relative payoff games and relative payoff games that are not gRPS games would coincide. Yet, the converse is not true. This follows again from Example 10. It is not a gRPS game but due

⁶This example appears also in Ania (2008, Example 2), where it is used to demonstrate that the class of games where imitation is payoff improving (when all players are imitators) is not a subclass of generalized ordinal potential games.

to the existence of a strict improvement cycle by Monderer and Shapley (1996, Lemma 2.5) it does not possess a generalized ordinal potential.

For an example of a game whose relative payoff game is a generalized ordinal potential game see again the coordination game with an outside option presented in Example 9. A generalized ordinal potential function is given by

$$P = \begin{array}{ccc} A & B & C \\ B & \begin{pmatrix} -2 & -1 & -2 \\ -1 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

5.2 Quasiconcave Relative Payoff Games

Here we show that imitation is essentially unbeatable if the relative payoff game is quasiconcave.

Definition 11 (Quasiconcave) A symmetric two-player game (X, π) is quasiconcave (or single-peaked) if there exists a total order < on X such that for each $x, x', x'', y \in X$ and x' < x < x'', we have that $\pi(x, y) \ge \min \{\pi(x', y), \pi(x'', y)\}$.

For a matrix game this definition implies that the row player's payoff has in each column a single peak. In our companion paper, Duersch, Oechssler, and Schipper (2011, Theorem 2), we show that if X is finite and Δ is quasiconcave, then an equilibrium of (X, Δ) and therefore a fESS of (X, π) exists.

Proposition 5 Let (X, π) be a finite symmetric game with relative payoff game (X, Δ) . If (X, Δ) is quasiconcave, then imitation is not subject to a money pump.

PROOF. Suppose (X, Δ) is a finite quasiconcave game. Consider a symmetric submatrix (X', Δ') where $X' \subset X$ and Δ' is the restriction of Δ to X'. It follows directly from Definition 11 that (X', Δ') is also a finite quasiconcave game. Lemma 1 in Duersch, Oechssler, and Schipper (2011),⁷ then implies that (X', Δ') is not a gRPS matrix. Since we picked an arbitrary $X' \subset X$, (X, Δ) is not a gRPS game. Thus, by Theorem 1, imitation is not subject to a money pump.

⁷The lemma is reproduced in the Appendix for the reader's convenience.

The following corollary may be useful for applications. Let $X \subset \mathbb{R}^m$ be a finite subset of a finite dimensional Euclidean space. A function $f: X \longrightarrow \mathbb{R}$ is convex (resp. concave) if for any $x, x' \in X$ and for any $\lambda \in [0,1]$ such that $\lambda x + (1-\lambda)x' \in X$, $f(\lambda x + (1-\lambda)x') \leq (\geq)\lambda f(x) + (1-\lambda)f(x')$.

Corollary 4 Let (\mathbb{R}^m, π) be a symmetric two-player game for which $\pi(\cdot, \cdot)$ is concave in its first argument and convex in its second argument. If the players' actions are restricted to a finite subset X of the finite dimensional Euclidian space \mathbb{R}^m , then imitation is not subject to a money pump.

Bargaining is an economically relevant situation involving two players. Our results imply that imitation is not subject to a money pump in bargaining as modeled in the Nash Demand game.

Example 11 (Nash Demand Game) Consider the following version of the Nash Demand game (see Nash, 1953). Two players simultaneously demand an amount in \mathbb{R}_+ . If the sum is within a feasible set, i.e., $x + y \leq s$ for s > 0, then player 1 receives the payoff $\pi(x, y) = x$. Otherwise $\pi(x, y) = 0$ (analogously for player 2). The relative payoff function is quasiconcave. If the players' demands are restricted to a finite set, then Proposition 5 implies that imitation is not subject to a money pump.

Example 12 Consider a symmetric two-player game with the payoff function given by $\pi(x, y) = \frac{x}{y}$ with $x, y \in X \subset [1, 2]$ with X being finite. This game's relative payoff function is quasiconcave. Thus our result implies that imitation is not subject to a money pump. Moreover, the example demonstrates that not every quasiconcave relative payoff function is additively separable.

Finally, we would like to remark that Example 10 is an instance of a quasiconcave relative payoff game but due to the strict improvement cycle it does not posses a generalized ordinal potential. Moreover, in Duersch, Oechssler, and Schipper (2011, Example 1) we show that there are relative payoff games that are neither gRPS games nor quasiconcave.

5.3 Aggregative Games

Many games relevant to economics possess a natural aggregate of all players' actions. For instance, in Cournot games the total market quantity or the price is an aggregate. But also other games like rent-seeking games, common pool resource games, public good games etc. can be viewed as games with an aggregate. The aggregation property has been useful for the study of imitation and fESS in the literature (see Schipper, 2003, and Alós-Ferrer and Ania, 2005). In this section, we will derive results for aggregative games whose absolute payoff functions satisfy some second-order properties.⁸

We say that (X, Π) is an *aggregative game* if it satisfies the following properties.

- (i) X is a totally ordered set of actions and Z is a totally ordered set.
- (ii) There exists an aggregator $a: X \times X \longrightarrow Z$ that is
 - monotone increasing in its arguments, i.e. if (x'', y'') > (x', y'), then a(x'', y'') > a(x', y'),⁹ and
 - symmetric, i.e., a(x, y) = a(y, x) for all $x, y \in X$.

(iii) π is extendable to $\Pi: X \times Z \longrightarrow \mathbb{R}$ with $\Pi(x, a(x, y)) = \pi(x, y)$ for all $x, y \in X$.

We say that an aggregative game (X, Π) is quasisubmodular (resp. quasisupermodular) if Π is quasisubmodular (resp. quasisupermodular) in (x, y) on $X \times Z$, i.e., for all z'' > z', x'' > x',

$$\Pi(x'', z'') - \Pi(x', z'') \ge 0 \quad \Rightarrow (\Leftarrow) \quad \Pi(x'', z') - \Pi(x', z') \ge 0, \tag{7}$$

$$\Pi(x'', z'') - \Pi(x', z'') > 0 \quad \Rightarrow (\Leftarrow) \quad \Pi(x'', z') - \Pi(x', z') > 0.$$
(8)

Quasisupermodularity (resp. quasisubmodularity) is sometimes also called the (dual) single crossing property (e.g. Milgrom and Shannon, 1994).¹⁰

⁸At a first glance, the aggregation property may be less compelling in the context of two-player games. However, the results we obtain in this section allow us to cover important examples that are not covered by any of our other results.

⁹The partial order > on $X \times X$ is defined as (x'', y'') > (x', y') if and only if $x'' \ge x'$ and $y'' \ge y'$ with one of these inequalities being strict.

¹⁰It is important to realize that quasisubmodularity in (x, z) where z is the aggregate of all players' actions is different from quasisubmodularity in (x, y) where y is the aggregate of all opponents' actions. For instance, Schipper (2009, Lemma 1) shows that quasisubmodularity in (x, z) where z is the aggregate of all players' actions is satisfied in a Cournot oligopoly if the inverse demand function is decreasing. No assumptions on costs are required. It is known from Amir (1996, Theorem 2.1) that further assumptions on costs are required if the Cournot oligopoly should be quasisubmodular in (x, y) where y is the aggregate of all opponents' actions.

A finite aggregative game is *quasiconcave* (or single-peaked) if for any $x, x', x'' \in X$ with x < x' < x'' and $z \in Z$,

$$\Pi(x', z) \ge \min\{\Pi(x, z), \Pi(x'', z)\}.$$

A finite aggregative game is *quasiconvex* if for any $x, x', x'' \in X$ with x < x' < x'' and $z \in Z$,

$$\Pi(x', z) \le \max\{\Pi(x, z), \Pi(x'', z)\}.$$

It is strictly quasiconvex if the inequality holds strictly. An action $x^* \in X$ is a *fESS* of the aggregative game (X, Π) if

$$\Pi(x^*, a(x^*, x)) \ge \Pi(x, a(x^*, x)) \text{ for all } x \in X.$$

The following lemma is the key insight for our result on quasiconcave quasisubmodular aggregative games.

Lemma 4 Suppose (X, Π) is a quasiconcave quasisubmodular aggregative game. If x is between some x' and a fESS x^* , then

$$\Pi(x, a(x, x')) \ge \Pi(x', a(x, x')).$$

PROOF. Suppose that $x' \le x \le x^*$. The case $x' \ge x \ge x^*$ can be dealt with analogously. For x = x' or $x = x^*$ the proposition is trivial or follows from the definition of fESS, respectively. Thus, assume that $x' < x < x^*$.

By the definition of a fESS

$$\Pi(x^*, a(x^*, x')) - \Pi(x', a(x^*, x')) \ge 0.$$

By quasiconcavity,

$$\Pi(x, a(x^*, x')) - \Pi(x', a(x^*, x')) \ge 0.$$

The result follows then by quasisubmodularity,

$$\Pi(x, a(x, x')) - \Pi(x', a(x, x')) \ge 0,$$

since $(x^*, x') > (x, x')$ and hence $a(x^*, x') > a(x, x')$.

Proposition 6 If (X, Π) is a finite quasiconcave quasisubmodular aggregative game for which a fESS exists, then imitation is not subject to a money pump.

PROOF. We will show that from any initial action of the imitator different from a fESS, any opponent's strategy which yields a sequence of actions with strictly positive relative payoffs at each step reaches a fESS in a finite number of steps. Once reached, there are no further strictly relative payoff gains feasible for the opponent by the definition of a fESS. Hence, there does not exist an imitation cycle. It follows then from Lemma 1 that imitation is not subject to a money pump.

Note that since the game is quasiconcave, if x^* and x^{**} are fESS, then so is any $x \in X$ with $x^* < x < x^{**}$ or $x^{**} < x < x^*$. We write E for the set of fESS.

Step 1: Let $y_0 \in X$ be the starting action of the imitator. Assume that $y_0 < x^* = \min E$ (the proof for $y_0 > x^{**} = \max E$ works analogously). We claim that when the imitator switches to a different action $y_1 \neq y_0$, we must have that $y_1 > y_0$. Suppose by contradiction that $y_1 < y_0$. By equation (1), the imitator would only choose y_1 if in the previous period the opponent chose $x = y_1$ and received a strictly higher payoff than the imitator,

$$\Delta(y_1, y_0) = \Pi(y_1, a(y_1, y_0)) - \Pi(y_0, a(y_1, y_0)) > 0.$$
(9)

But this contradicts Lemma 4 as $y_1 < y_0 < x^*$. Thus, $y_1 > y_0$.

- If $y_1 \in E$, we are done.
- If $y_0 < y_1 < x^*$, then take y_1 as the new starting action and repeat Step 1.
- Else, go to Step 2.

Step 2: We have that $y_1 > x^{**}$. We claim that when the imitators switches to a new action $y_2 \neq y_1$, we must have that $y_2 < y_1$. Suppose by contradiction that $y_2 > y_1$. By equation (1), the imitator would only choose y_2 if in the previous period the opponent chose $x = y_2$ and received a higher payoff, $\Delta(y_2, y_1) > 0$. But this contradicts Lemma 4 as $y_2 > y_1 > x^{**}$. Thus $y_2 < y_1$.

- If $y_2 \in E$, we are done.
- If $y_0 < y_2 < x^*$, then take y_2 as the new starting action and repeat Step 1.
- If $x^{**} < y_2 < y_1$, then take y_2 as the new starting action and repeat Step 2.

We claim that $y_2 \leq y_0$ can be ruled out. Since X is finite, the algorithm then stops after finite periods. To verify this claim, suppose to the contrary that $y_2 \leq y_0$. By

equation (1), the imitator would only choose y_2 if in the previous period the opponent chose $x = y_2$ and received a strictly higher payoff than the imitator,

$$\Delta(y_2, y_1) = \Pi(y_2, a(y_2, y_1)) - \Pi(y_1, a(y_2, y_1)) > 0$$

By quasiconcavity, we have

$$\Pi(y_0, a(y_2, y_1)) - \Pi(y_1, a(y_2, y_1)) \ge 0.$$

Since $a(y_0, y_1) > a(y_2, y_1)$ for $y_0 > y_2$ and $a(y_0, y_1) = a(y_2, y_1)$ for $y_0 = y_2$, we have by quasisubmodularity

$$\Pi(y_0, a(y_0, y_1)) - \Pi(y_1, a(y_0, y_1)) \ge 0.$$

But this contradicts inequality (9) and proves the claim.

The following examples present applications of the previous result. The first example extends the linear Cournot oligopoly of Example 2 to general symmetric Cournot oligopoly.

Example 13 (Cournot Duopoly) Let the symmetric payoff function be $\pi(x, y) = xp(x+y) - c(x)$ and assume that $\pi(x, y)$ is quasiconcave in x. Schipper (2009, Lemma 1) shows that a symmetric Cournot duopoly with an arbitrary decreasing inverse demand function p and arbitrary increasing cost function c is an aggregative quasisubmodular game. Thus, Proposition 6 implies that imitation is not subject to a money pump in Cournot duopoly.

Example 14 (Rent Seeking) Two contestants compete for a rent v > 0 by bidding $x, y \in X \subseteq \mathbb{R}_+$. A player's probability of winning is proportional to her bid, $\frac{x}{x+y}$ and zero if both players bid zero. The cost of bidding equals the bid. The symmetric payoff function is given by $\pi(x, y) = \frac{x}{x+y}v - x$ (see Tullock, 1980, and Hehenkamp, Leininger, and Possajennikov, 2004). This game is an aggregative quasisubmodular game (see Schipper, 2003, Example 6, and Alós-Ferrer and Ania, 2005, Example 2) and $\pi(x, y)$ is concave in x. Thus Proposition 6 implies that imitation is not subject to a money pump.

For quasiconvex quasisupermodular aggregative games we can prove an analogous result. We first observe that in a strictly quasiconvex quasisubmodular game a fESS must be a "corner" solution if it exists. It follows that there can be at most two fESS. **Lemma 5** Let (X, Π) be a finite strictly quasiconvex quasisupermodular aggregative game. If x^* is a fESS, then $x^* = \max X$ or $x^* = \min X$.

PROOF. Let x^* be a fESS and suppose to the contrary that there exist $x', x'' \in X$ such that $x'^* < x''$. We distinguish four cases:

Case 1: If

$$\Pi(x'', a(x^*, x'')) \ge \Pi(x', a(x^*, x'')),$$

then by strict quasiconvexity

$$\Pi(x^*, a(x^*, x'')) < \Pi(x'', a(x^*, x'')),$$

a contradiction to x^* being a fESS.

Case 2: The case $\Pi(x', a(x^*, x')) \ge \Pi(x'', a(x^*, x'))$ is analogous to Case 1.

Case 3: If

$$\Pi(x', a(x^*, x'')) \ge \Pi(x'', a(x^*, x'')),$$

then by strict quasiconvexity

$$\Pi(x^*, a(x^*, x'')) < \Pi(x', a(x^*, x'')).$$

By quasisupermodularity,

$$\Pi(x^*, a(x^*, x')) < \Pi(x', a(x^*, x')).$$

a contradiction to x^* being a fESS.

Case 4: The case $\Pi(x''^*, x') \ge \Pi(x'^*, x')$ is analogous to Case 3.

Thus, if x^* is a fESS, then $x^* = \max X$ or $x^* = \min X$.

Proposition 7 If (X, Π) is a finite strictly quasiconvex quasisupermodular aggregative game for which a fESS exists, then imitation is not subject to a money pump.

PROOF. Again, we will show that from any initial action by the imitator different from a fESS, any opponent's strategy which yields a sequence of actions with strictly positive relative payoffs at each step reaches a fESS in a finite number of steps. Once reached, there are no further strict relative payoff gains possible for the opponent by the definition of a fESS. Hence, there does not exist an imitation cycle. It follows then from Lemma 1 that imitation is not subject to a money pump.

Consider a sequence of nontrivial actions x_1, x_2, x_3 the opponent may take. Suppose that $x_2 < x_1$ (the case $x_2 > x_1$ is dealt with analogously). By equation (1), the imitator will mimic the opponent only if her relative payoffs are strictly positive, i.e.

$$\Pi(x_2, a(x_2, x_1)) > \Pi(x_1, a(x_2, x_1)).$$
(10)

To show that the sequence of actions moves to one of the corners, we need to show that either $x_3 > x_1$ or $x_3 < x_2$. Suppose to the contrary that $x_2 < x_3 \le x_1$.¹¹ By equation (1), the imitator will mimic the opponent only if her relative payoffs are strictly positive, i.e.

$$\Pi(x_3, a(x_3, x_2)) > \Pi(x_2, a(x_3, x_2)).$$

Thus, by quasisupermodularity

$$\Pi(x_3, a(x_1, x_2)) > \Pi(x_2, a(x_1, x_2)).$$
(11)

If $x_2 < x_3 < x_1$, then from inequality (10) and strict quasiconvexity follows

$$\Pi(x_2, a(x_2, x_1)) > \Pi(x_3, a(x_2, x_1)).$$
(12)

If $x_3 = x_1$, then inequality (10) is equivalent to inequality (12). But inequality (12) contradicts inequality (11). Thus we have shown that with every nontrivial step, the opponent gets closer to a corner. Since there are only finitely many actions, a corner must be reached in finitely many steps. If the corner is a fESS, then no further changes of actions occur. Otherwise, the other corner may be reached in one additional step. This must be a fESS by Lemma 5 since a fESS is assumed to exist. Once it is reached, no further changes of actions occur.

6 Discussion

We have shown in this paper that imitation is a behavioral rule that is surprisingly robust to exploitation by any strategy. This includes strategies by truly sophisticated opponents. In Table 1 we summarize our results.¹² The only class of symmetric games in which imitation can really be beaten is the class of games whose relative payoff function is a generalized rock-paper-scissors game. According to Lemma 2 this is also the class

¹¹The case of $x_2 \neq x_3$ is already excluded by the requirement of non-trivial steps.

¹²More results on the classes of games and their relationships are contained in our companion paper, Duersch, Oechssler, and Schipper (2011).

of games in which there is an imitation cycle, i.e. a cycle in which the opponent always jumps to a new action which in turn is imitated by the imitator in the next round. Given the large number of examples and sufficient conditions we specified, it seems fair to say that imitation is very hard to beat in large and generic classes of economically relevant games.

Class	Result	Reference	Examples
Symmetric 2x2 games	essentially unbeatable	Prop. 2	Chicken, Prisoners' Dilemma, Stag Hunt
Additively separable relative payoff function or Relative payoff functions with increasing or decreasing differences	essentially unbeatable essentially unbeatable	Prop. 3	Linear Cournot duopoly Heterogeneous Bertrand duopoly Public goods Common pool resources Minimum effort coordination Supergistic relationship
or (Relative payoff) games with exact potential	essentially unbeatable	table	Diamond's search
Relative payoff games with generalized ordinal potential	no money pump	Prop. 4	Example 9
Quasiconcave relative payoff games	no money pump	Prop. 5	Nash demand game Example 10 Example 12
Quasiconcave quasisub- modular aggregative games	no money pump	Prop. 6	Cournot games Rent seeking
Quasiconvex quasisuper- modular aggregative games	no money pump	Prop. 7	
No generalized Rock-Paper-Scissors games	no money pump	Thm. 1	all of the above

Table 1: Summary of results

The property that imitate-if-better is unbeatable in such a large class of games seems to be unique among commonly used learning rules. We are not aware of any rule that shares this property with imitate-if-better.¹³ For example, there are important differences between imitate-if-better and *unconditional* imitation, when behavior is imitated regardless of its success. A well known example of the latter is tit-for-tat. To see the difference, consider the following game.

$$\pi = \begin{array}{ccc} A & B & C \\ \pi = \begin{array}{ccc} A & 0 & 0 & 0 & -1 & -1 & 0 \\ B & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 0 & -1 & 10 & 0 & 0 & 0 \end{array} \end{array} \qquad \Delta = \begin{array}{ccc} A & B & C \\ \Delta = \begin{array}{ccc} A & 0 & 1 & -1 \\ -1 & 0 & -10 \\ 1 & 10 & 0 \end{array} \right)$$

Obviously, Δ is not a generalized gRPS game. In fact, it is easy to see that imitate-ifbetter is essentially unbeatable for this game. However, tit-for-tat would be subject to a money pump by following a cycle $(A \rightarrow B \rightarrow C \rightarrow A \dots)$. The reason for this difference is that an imitate-if-better player would never leave action C whereas a tit-for-tat player can be induced to follow the opponent from C to A.

There are other modifications that may cause the imitate-if-better rule to lose the property of being unbeatable. For instance, we assumed that an imitator sticks to his action in case of a tie in payoffs. To see what goes wrong with an alternative tie-braking rule consider a homogenous Bertrand duopoly with constant marginal costs. Suppose the imitator starts with a price equal to marginal cost. If the opponent chooses a price strictly above marginal cost, her profit is also zero. If nevertheless, the opponent were imitated, she could start the money pump by undercutting the imitator until they reach again price equal to marginal cost and then start the cycle again.

Similarly, many commonly used belief learning rules, for example, best response learning or fictitious play, can easily be exploited in all games in which a Stackelberg leader achieves a higher payoff than the follower (as e.g. in Cournot games). Against such rules, the opponent can simply stubbornly choose the Stackelberg leader action knowing that the belief learning player will eventually converge to the Stackelberg follower action. Thus, belief learning rules can be beaten without bounds in such games. Yet, it remains an open question for future research whether there are other behavioral rules that perform equally well as imitate-if-better.

The restriction of our analysis to two-player games is certainly a limitation. While a full treatment of the *n*-player case is beyond the scope of the current paper, we provide here an example that shows how imitation can be beaten in a standard Cournot game when there are three players. Let the inverse demand function be p(Q) = 100 - Q and

¹³Apart from close variants of imitate-if-better like rules that imitate only with a certain probability, see e.g. Schlag's (1998) proportional imitation rule.

the cost function be $c(q_i) = 10q_i$. Now consider the case of two relative payoff maximizers and one imitator. Writing a vector of quantities as (q_I, q_M, q_M) , it is easy to check that the following sequence of action profiles (0, 22.5, 22.5), (22.5, 0, 68), (0, 22.5, 22.5), (22.5, 68, 0), (0, 22.5, 22.5) ... is an imitation cycle. The two maximizers take turns in inducing the imitator to reduce his quantity to zero by increasing quantity so much that price is below marginal cost. Since the other maximizer has zero losses, she is imitated in the next period, which yields half of the monopoly profit for both maximizers. Clearly, this requires coordination among the two maximizers but this can be achieved in an infinitely repeated game by the use of a trigger strategy. Thus, imitation is subject to a money pump. Recall, however, that we pitted imitation against truly sophisticated opponents in a particular game. Whether imitation can be beaten also by less sophisticated (e.g. human) opponents in a wider class of games remains to be seen in future experiments and in theoretical work on *n*-player games.

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Appendix (not for publication)

The following result appears in Duersch, Oechssler and Schipper (2011, Lemma 8). The entire paper can be found at http://www.econ.ucdavis.edu/faculty/schipper/zerosum.pdf

Lemma 6 A finite quasiconcave symmetric two-player zero-sum game is not a gRPS matrix.

PROOF. Suppose by contradiction that the finite quasiconcave symmetric zero-sum game (X,π) is a gRPS matrix. Note first that if $\pi(\cdot, y)$ is quasiconcave in the first argument, i.e., if x' < x < x'' implies that $\pi(x,y) \ge \min \{\pi(x',y), \pi(x'',y)\}$, then by symmetry, $\pi(y,x) \le \max \{\pi(y,x'), \pi(y,x'')\}$, i.e. $\pi(x,\cdot)$ is quasiconvex in the second argument.

Let (x_k, x_ℓ) be the left-most cell with a strictly positive entry that is above the main diagonal, i.e. $\pi(x_k, x_\ell) > 0$, where $x_\ell := \arg \min_{x''} \{\pi(x', x'') > 0 \text{ and } x'' > x'\}$ and $x_k := \arg \min_{x'} \{\pi(x', x_\ell)\}$. If there are several such entries in column x_ℓ , we choose without loss of generality the lowest one. Such an entry exists since (X, π) is a gRPS and finite (i.e., the last column must have a strictly positive entry above the main diagonal).

By symmetry, (x_{ℓ}, x_k) is below the main diagonal and $\pi(x_{\ell}, x_k) < 0$. By quasiconcavity, all entries in the column x_k below x_{ℓ} are also negative, $\pi(x, x_k) < 0$, for all $x > x_{\ell}$.

Since rows are quasiconvex, it follows that $\pi(x_{\ell}, x) \leq 0$ for all x such that $x_k < x < x_{\ell}$. The same holds for all lower rows, $\pi(x', x) \leq 0$, for all $x' > x_{\ell}, x_k < x < x'$. This defines a trapezoid \prod_{neg} of payoff entries below the diagonal that does not contain any strictly positive entries.

Now, look specifically at column $x_{\ell-1}$. Π_{neg} contains all entries in this column that are below the diagonal. However, this column must have a positive entry since the game is a gRPS matrix. Therefore, the column has to have a positive entry above the diagonal. But this is a contradiction to the fact that (x_k, x_ℓ) is the left-most cell with a positive entry above the main diagonal.