Capital Deepening and Non-Balanced Economic Growth*

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Abstract

This paper constructs a model of non-balanced economic growth. The main economic force is the combination of differences in factor proportions and capital deepening. Capital deepening tends to increase the relative output of the sector with a greater capital share (despite the equilibrium reallocation of capital and labor away from that sector). We first illustrate this force using a general two-sector model. We then investigate it further using a class of models with constant elasticity of substitution between two sectors and Cobb-Douglas production functions in each sector. In this class of models, non-balanced growth is shown to be consistent with an asymptotic equilibrium with constant interest rate and capital share in national income. Finally, we construct and analyze a model of "non-balanced endogenous growth," which extends these results to an economy with endogenous and directed technical change, and demonstrates that non-balanced technological progress can be an equilibrium phenomenon.

Keywords: capital deepening, endogenous growth, multi-sector growth, non-balanced economic growth.

JEL Classification: O40, O41, O30.

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1 Introduction

Most models of economic growth strive to be consistent with the Kaldor facts, i.e., the relative constancy of the growth rate, the capital-output ratio, the share of capital income in GDP and the real interest rate (see Kaldor, 1963, and also Denison, 1974, Barro and Sala-i-Martin, 2004). Beneath this balanced picture, however, are the patterns that Kongsamut, Rebelo and Xie (2001) refer to as the Kuznets facts, which concern the systematic change in the relative importance of various sectors, in particular, agriculture, manufacturing and services (see Kuznets, 1957, 1973, Chenery, 1960, Kongsamut, Rebelo and Xie, 2001). While the Kaldor facts emphasize the balanced nature of economic growth, the Kuznets facts highlight its non-balanced nature.

The Kuznets facts have motivated a small literature, which typically starts by positing non-homothetic preferences consistent with Engel's law. With these preferences, the marginal rate of substitution in consumption changes as an economy grows, directly leading to a pattern of non-balanced growth (e.g., Murphy, Shleifer and Vishny, 1989, Matsuyama, 1992, 2005, Echevarria, 1997, Laitner, 2000, Kongsamut, Rebelo and Xie, 2001, Caselli and Coleman, 2001, Gollin, Parente and Rogerson, 2002). An alternative perspective, proposed by Baumol (1967), emphasizes the potential non-balanced nature of economic growth resulting from differential productivity growth across sectors, but has received less attention in the literature.[1]

This paper has two aims. First, it shows that there is another, and very natural, reason to expect economic growth to be non-balanced. Differences in factor proportions across sectors (i.e., different shares of capital) combined with capital deepening will lead to non-balanced growth. The reason is simple: an increase in capital-labor ratio will raise output more in the sector with greater capital intensity. More specifically, we prove that with balanced technological progress (in the sense of equal rates of Hicks-neutral technological progress across sectors), capital deepening and differences in factor proportions necessarily cause non-balanced growth. This result holds irrespective of the exact source of economic growth or the process of accumulation.

The second objective of the paper is to present and analyze a tractable two-sector growth

[1] An exception is the recent independent paper by Ngai and Pissarides (2005), which constructs a model of multi-sector economic growth inspired by Baumol. In Ngai and Pissarides's model, there are exogenous TFP differences across sectors, but all sectors have identical Cobb-Douglas production functions. Consequently, although their model is potentially consistent with the Kuznets and Kaldor facts, it does not contain the main contribution of our paper, non-balanced growth resulting from factor proportion differences and capital deepening.

model featuring non-balance growth. We do this by constructing a class of economies with constant elasticity of substitution between two sectors and Cobb-Douglas production functions within each sector. We investigate the equilibrium of such an economy with exogenous and endogenous technological change. We show that equilibrium takes a simple form, with constant growth rate in all sectors, constant interest rate and constant share of capital in national income in the limiting (asymptotic) equilibrium.

The form of the limiting equilibrium of this class of economies depends on whether the products of the two sectors are gross substitutes or complements (meaning whether the elasticity of substitution between these products is greater than or less than one). When they are gross substitutes, the sector that is more "capital intensive" (in the sense of having a greater capital share) dominates the economy. The form of the equilibrium is more interesting when the elasticity of substitution between these products is less than one. In this case, the growth rate of the economy is determined by the more slowly growing (less capital-intensive sector). Despite the change in the terms of trade against the faster growing sector, in equilibrium sufficient amounts of capital and labor (and technological progress when this is endogenous) are deployed in this sector to ensure a faster rate of growth.

One interesting feature is that, especially when the elasticity of substitution is less than one,[2] the resulting pattern of economic growth is consistent with the Kuznets facts, without substantially deviating from the Kaldor facts. In particular, even in the limiting equilibrium both sectors grow with positive (and unequal) rates, and more importantly, we show that convergence to this limiting equilibrium may be slow, and along the transition path, growth is non-balanced, while capital share and interest rate vary only by relatively small amounts. Therefore, the equilibrium with an elasticity of substitution less than one may be able to rationalize both the Kuznets and the Kaldor facts. Naturally, whether or not this is the

[2] As we will see below, the elasticity of substitution between products will be less than one if and only if the (short-run) elasticity of substitution between labor and capital is less than one. In view of the time-series and cross-industry evidence, a short-run elasticity of substitution between labor and capital less than one appears reasonable.

For example, Hamermesh (1993), Nadiri (1970) and Nerlove (1967) survey a range of early estimates of the elasticity of substitution, which are generally between 0.3 and 0.7. David and Van de Klundert (1965) similarly estimate this elasticity to be in the neighborhood of 0.3. Using the translog production function, Griffin and Gregory (1976) estimate elasticities of substitution for nine OECD economies between 0.06 and 0.52. Berndt (1976), on the other hand, estimates an elasticity of substitution equal to 1, but does not control for a time trend, creating a strong bias towards 1. Using more recent data, and various different specifications, Krusell, Ohanian, Rios-Rull, and Violante (2000) and Antras (2001) also find estimates of the elasticity significantly less than 1. Estimates implied by the response of investment to the user cost of capital also typically imply an elasticity of substitution between capital and labor significantly less than 1 (see, e.g., Chirinko, 1993, Chirinko, Fazzari and Mayer, 1999, or Mairesse, Hall and Mulkay, 1999).

empirically correct explanation is not answered by this theoretical result.

Finally, we present and analyze a model of "non-balanced endogenous growth," which shows the robustness of our results to endogenous technological progress, and demonstrates how, in the presence of factor proportion differences, the pattern of technological progress itself will be non-balanced. To the best of our knowledge, despite the large literature on endogenous growth, there are no previous studies that combine endogenous technological progress and non-balanced growth.[3]

The rest of the paper is organized as follows. Section 2 shows how the combination of factor proportions differences and capital deepening lead to non-balanced growth using a general two-sector growth model. Section 3 constructs a more specific model with a constant elasticity of substitution between two sectors and Cobb-Douglas production functions, but exogenous technological progress. It characterizes the full dynamic equilibrium of this economy, and shows how with an elasticity of substitution less than one, the model may generate an equilibrium path that is consistent both with the Kuznets and the Kaldor facts. Section 4 introduces endogenous technological progress and shows that the results are robust to differential rates of technological progress across sectors. Section 5 concludes, and the Appendix contains proofs that are not presented in the text.

2 Capital Deepening and Non-Balanced Growth

We first illustrate how differences in factor proportions across sectors combined with capital deepening lead to non-balanced economic growth. To do this, we use a standard two-sector competitive model with constant returns to scale in both sectors, and two factors of production, capital, K, and labor, L. To highlight that the exact nature of the accumulation process is not essential for the results, in this section we take the sequence (process) of capital and labor supplies, $[K(t), L(t)]_{t=0}^{\infty}$, as given (and assume that labor is supplied inelastically). In addition, we omit explicit time dependence when this will cause no confusion.

Final output, Y, is produced as an aggregate of the output of two sectors, Y_1 and Y_2,

$$Y = F(Y_1, Y_2),$$

[3] See, among others, Romer (1986, 1990), Lucas (1988), Rebelo (1991), Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991a,b), Aghion and Howitt (1992), Jones (1995), Young (1993). Aghion and Howitt (1998) and Barro and Sala-i-Martin (2004) provide excellent introductions to endogenous growth theory. See also Acemoglu (2002) on models of directed technical change that feature endogenous, but balanced technological progress in different sectors. Acemoglu (2003) presents a model with non-balanced technological progress between two sectors, but in the limiting equilibrium both sectors grow at the same rate.

and we assume that F exhibits constant returns to scale and is continuously differentiable. Output in both sectors is produced with the production functions

$$Y_1 = A_1 G_1 (K_1, L_1) \tag{1}$$

and

$$Y_2 = A_2 G_2 (K_2, L_2). \tag{2}$$

G_1 and G_2 also exhibit constant returns to scale and are twice differentiable. A_1 and A_2 denote Hicks-neutral technology terms.[4] We also assume that the functions F, G_1 and G_2 satisfy Inada-type conditions, e.g., $\lim_{Y_1 \to 0} \partial F (Y_1, Y_2) / \partial Y_1 = \infty$ for all $Y_2 > 0$, etc. These assumptions ensure interior solutions and simplify the exposition, though they are not necessary for the results presented in this section.

Market clearing implies

$$\begin{aligned} K_1 + K_2 &= K, \\ L_1 + L_2 &= L, \end{aligned} \tag{3}$$

where K and L are the (potentially time-varying) supplies of capital and labor, given by the exogenous sequence $[K(t), L(t)]_{t=0}^{\infty}$, which we take to be continuosly differentiable functions of time. Without loss of any generality, we also ignore capital depreciation.

We normalize the price of the final good to 1 in every period, and denote the prices of Y_1 and Y_2 by p_1 and p_2, and wage and rental rate of capital (interest rate) by w and r. We assume that product and factor markets are competitive, so product prices satisfy

$$\frac{p_1}{p_2} = \frac{\partial F (Y_1, Y_2) / \partial Y_1}{\partial F (Y_1, Y_2) / \partial Y_2}, \tag{4}$$

and the wage and the interest rate satisfy[5]

$$\begin{aligned} w &= \frac{\partial A_1 G_1 (K_1, L_1)}{\partial L_1} = \frac{\partial A_2 G_2 (K_2, L_2)}{\partial L_2} \\ r &= \frac{\partial A_1 G_1 (K_1, L_1)}{\partial K_1} = \frac{\partial A_2 G_2 (K_2, L_2)}{\partial K_2}. \end{aligned} \tag{5}$$

[4] Allowing more general forms of technological progress would complicate the analysis since we would have to rule out a change in the marginal rates of substitution between capital and labor in G_1 and G_2 with changes in A_1 and A_2 that would "by chance" satisfy the conditions necessary for balanced growth.

[5] Without the Inada-type assumptions, these would have to be written as

$$w \geq \partial A_1 G_1 (K_1, L_1) / \partial L_1 \text{ and } L_1 \geq 0,$$

with complementary slackness, etc.

An equilibrium, given factor supply sequences, $[K(t), L(t)]_{t=0}^{\infty}$, is a sequence of product and factor prices, $[p_1(t), p_2(t), w(t), r(t)]_{t=0}^{\infty}$ and factor allocations, $[K_1(t), K_2(t), L_1(t), L_2(t)]_{t=0}^{\infty}$, such that (3), (4) and (5) are satisfied.

Also define the share of capital in the two sectors

$$s_1 \equiv \frac{rK_1}{p_1 Y_1} \text{ and } s_2 \equiv \frac{rK_2}{p_2 Y_2},$$

the capital to labor ratio in the two sectors,

$$k_1 \equiv \frac{K_1}{L_1} \text{ and } k_2 \equiv \frac{K_2}{L_2}$$

and the "per capita production functions" (without the Hicks-neutral technology term),

$$g_1(k_1) \equiv \frac{G_1(K_1, L_1)}{L_1} \text{ and } g_2(k_2) \equiv \frac{G_2(K_2, L_2)}{L_2}.$$

We say that there is *capital deepening* in the economy if $\dot{K}/K > \dot{L}/L$ and there is *factor proportion differences* if $s_1 \neq s_2$.[6] The next theorem shows that if there is capital deepening and factor proportion differences, then balanced technological progress is not consistent with balanced growth.

Theorem 1 Suppose that at time t, there are factor proportion differences between the two sectors, i.e., $s_1(t) \neq s_2(t)$, technological progress is balanced, i.e., $\dot{A}_1(t)/A_1(t) = \dot{A}_2(t)/A_2(t)$ and there is capital deepening, i.e., $\dot{K}(t)/K(t) > \dot{L}(t)/L(t)$, then growth is not balanced, that is, $\dot{Y}_1(t)/Y_1(t) \neq \dot{Y}_2(t)/Y_2(t)$.

Proof. Differentiating the production function for the two sectors,

$$\frac{\dot{Y}_1}{Y_1} = \frac{\dot{A}_1}{A_1} + s_1 \frac{\dot{K}_1}{K_1} + (1 - s_1) \frac{L_1}{L_1}$$

and

$$\frac{\dot{Y}_2}{Y_2} = \frac{\dot{A}_2}{A_2} + s_2 \frac{\dot{K}_2}{K_2} + (1 - s_2) \frac{L_2}{L_2}.$$

Suppose, to obtain a contradiction, that $\dot{Y}_1/Y_1 = \dot{Y}_2/Y_2$. Since $\dot{A}_1/A_1 = \dot{A}_2/A_2$ and $s_1 \neq s_2$, this implies $\dot{k}_1/k_1 \neq \dot{k}_2/k_2$ (otherwise, $\dot{k}_1/k_1 = \dot{k}_2/k_2 > 0$ because of capital deepening and if, for example, $s_1 < s_2$, then $\dot{Y}_1/Y_1 < \dot{Y}_2/Y_2$).

[6] Here $s_1(t) \neq s_2(t)$ refers to the equilibrium factor proportions in the two sectors at time t. It does not necessarily mean that these will not be equal at some future date.

Since F exhibits constant returns to scale, (4) implies

$$\frac{\dot{p}_1}{p_1} = \frac{\dot{p}_2}{p_2} = 0. \tag{6}$$

Equation (5) yields the following interest rate and wage conditions

$$\begin{aligned} r &= p_1 A_1 g_1'(k_1) \\ &= p_2 A_2 g_2'(k_2), \end{aligned} \tag{7}$$

and

$$\begin{aligned} w &= p_1 A_1 \left(g_1(k_1) - g_1'(k_1) k_1\right) \\ &= p_2 A_2 \left(g_2(k_2) - g_2'(k_2) k_2\right). \end{aligned} \tag{8}$$

Differentiating the interest rate condition, (7), with respect to time and using (6), we have

$$\frac{\dot{A}_1}{A_1} + \varepsilon_{g_1'} \frac{\dot{k}_1}{k_1} = \frac{\dot{A}_2}{A_2} + \varepsilon_{g_2'} \frac{\dot{k}_2}{k_2}$$

where

$$\varepsilon_{g_1'} \equiv \frac{g_1''(k_1) k_1}{g_1'(k_1)} \text{ and } \varepsilon_{g_2'} \equiv \frac{g_2''(k_2) k_2}{g_2'(k_2)}.$$

Since $\dot{A}_1/A_1 = \dot{A}_2/A_2$, we must have

$$\varepsilon_{g_1'} \frac{\dot{k}_1}{k_1} = \varepsilon_{g_2'} \frac{\dot{k}_2}{k_2}. \tag{9}$$

Differentiating the wage condition, (8), with respect to time, using (6) and some algebra gives

$$\frac{\dot{A}_1}{A_1} - \frac{s_1}{1-s_1} \varepsilon_{g_1'} \frac{\dot{k}_1}{k_1} = \frac{\dot{A}_2}{A_2} - \frac{s_2}{1-s_2} \varepsilon_{g_2'} \frac{\dot{k}_2}{k_2}.$$

Since $\dot{A}_1/A_1 = \dot{A}_2/A_2$ and $s_1 \neq s_2$, this equation is inconsistent with (9), yielding a contradiction and proving the claim. ■

The intuition for this result can be obtained as follows. Suppose there is capital deepening and that both capital and labor are allocated to the two sectors with constant proportions. Because factor proportions differ between the two sectors, say $s_1 < s_2$, such an allocation will generate faster growth in sector 2 than in sector 1 and induce a non-balanced pattern of growth (since there is capital deepening). In equilibrium, the faster growth in sector 2 will naturally change equilibrium prices, and the decline in the relative price of sector 2 will cause some of the labor and capital to be reallocated to sector 1. However, this reallocation cannot entirely offset the greater increase in the output of sector 2, since, if it did, the relative price change that stimulated the reallocation would not take place. Consequently, equilibrium growth will be non-balanced.

The proof of Theorem 1 makes it clear that the two-sector structure is not necessary for this result. In light of this, we also state a generalization for $N \geq 2$ sectors, where aggregate output is given by the constant returns to scale production function

$$Y = F\left(Y_1, Y_2, ..., Y_N\right).$$

The definitions for s, k and g and the other assumptions above naturally generalize to this setting. We have:

Theorem 2 Suppose that at time t, there are factor proportion differences among the N sectors in the sense that there exists i and $j \leq N$ such that $s_i(t) \neq s_j(t)$, technological progress is balanced, i.e., $\dot{A}_i(t)/A_i(t) = \dot{A}_j(t)/A_j(t)$ for all i and $j \leq N$, and there is capital deepening, i.e., $\dot{K}(t)/K(t) > \dot{L}(t)/L(t)$, then growth is not balanced, that is, there exists i and $j \leq N$ such that $\dot{Y}_i(t)/Y_i(t) \neq \dot{Y}_j(t)/Y_j(t)$.

The proof of this theorem parallels that of Theorem 1 and is omitted.

3 Two-Sector Growth with Exogenous Technology

The previous section demonstrated that differences in factor proportions across sectors and capital deepening will lead to non-balanced growth. This result was proved for a given (arbitrary) sequence of capital and labor supplies, $[K(t), L(t)]_{t=0}^{\infty}$, but this level of generality does not allow us to fully characterize the equilibrium path and its limiting properties. We now wish to analyze the equilibrium behavior of such an economy fully, which necessitates at least the sequence of capital stocks to be endogenized, and capital deepening to emerge as an equilibrium phenomenon. We will accomplish this by imposing more structure both in terms of specifying preferences and in terms of the production functions. Capital deepening will result from exogenous technological progress, which will in turn be endogenized in Section 4.

3.1 Demographics, Preferences and Technology

The economy consists of $L(t)$ workers at time t, supplying their labor inelastically. There is exponential population growth,

$$L(t) = \exp(nt)\, L(0). \tag{10}$$

We assume that all households have constant relative risk aversion (CRRA) preferences over total household consumption (rather than per capita consumption), and all population growth takes place within existing households (thus there is no growth in the number of households).[7] This implies that the economy admits a representative agent with CRRA preferences (see, for example, Caselli and Ventura, 2000):

$$\int_0^\infty \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt,$$

where $C(t)$ is aggregate consumption at time t, ρ is the rate of time preferences and $\theta \geq 0$ is the inverse of the intertemporal elasticity of substitution (or the coefficient of relative risk aversion). We again drop time arguments to simplify the notation whenever this causes no confusion, and continue to assume that there is no depreciation of capital. The flow budget constraint for the representative consumer is:

$$\dot{K} = rK + wL + \Pi - C, \tag{11}$$

where K and L denote the total capital stock and the total labor force in the economy, Π is total net corporate profits received by the consumers, w is the equilibrium wage rate and r is the equilibrium interest rate.

The unique final good is produced by combining the output of two sectors with elasticity of substitution $\varepsilon \in [0, \infty)$:

$$Y = \left[\gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\, Y_2^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{12}$$

where γ is a distribution parameter which determines the relative importance of the two goods in the aggregate production.

The resource constraint of the economy, in turn, requires that consumption and investment are less than total output, $Y = rK + wL + \Pi$, thus

$$\dot{K} + C \leq Y. \tag{13}$$

[7] The alternative would be to specify population growth taking place at the extensive margin, in which case the discount rate of the representative agent would be $\rho - n$ rather than ρ, without any substantive changes in the analysis.

The two goods Y_1 and Y_2 are produced competitively using constant elasticity of substitution (CES) production functions with elasticity of substitution between intermediates equal to $\nu > 1$:

$$Y_1 = \left(\int_0^{M_1} y_1(i)^{\frac{\nu-1}{\nu}} di \right)^{\frac{\nu}{\nu-1}} \text{ and } Y_2 = \left(\int_0^{M_2} y_2(i)^{\frac{\nu-1}{\nu}} di \right)^{\frac{\nu}{\nu-1}}, \quad (14)$$

where $y_1(i)$'s and $y_2(i)$'s denote the intermediates in the sectors that have different capital/labor ratios, and M_1 and M_2 represent the technology terms. In particular M_1 denotes the number of intermediates in sector 1 and M_2 the number of intermediate goods in sector 2.

Intermediate goods are supplied by monopolists that hold the relevant patent,[8] and are produced with the following Cobb-Douglas technologies

$$y_1(i) = l_1(i)^{\alpha_1} k_1(i)^{1-\alpha_1} \text{ and } y_2(i) = l_2(i)^{\alpha_2} k_2(i)^{1-\alpha_2}, \quad (15)$$

where $l_1(i)$ and $k_1(i)$ are labor and capital used in the production of good i of sector 1 and $l_2(i)$ and $k_2(i)$ are labor and capital used in the production of good i of sector 2.[9]

The parameters α_1 and α_2 determine which sector is more "capital intensive".[10] When $\alpha_1 > \alpha_2$, sector 1 is less capital intensive, while the converse applies when $\alpha_1 < \alpha_2$. In the rest of the analysis, we assume that

$$\alpha_1 > \alpha_2, \quad \text{(A1)}$$

which only rules out the case where $\alpha_1 = \alpha_2$, since the two sectors are otherwise identical and the labeling the sector with the greater capital share is without loss of any generality.

All factor markets are competitive, and market clearing for the two factors imply

$$\int_0^{M_1} l_1(i)di + \int_0^{M_2} l_2(i)\,di \equiv L_1 + L_2 = L, \quad (16)$$

and

$$\int_0^{M_1} k_1(i)di + \int_0^{M_2} k_2(i)\,di \equiv K_1 + K_2 = K, \quad (17)$$

[8] Monopoly power over intermediates is introduced to create continuity with the next section, where monopoly profits will motivate the creation of new intermediates. Since equilibrium markups will be constant, this monopoly power does not have any substantive effect on the form of equilibrium.

[9] Strictly speaking, we should have two indices, $i_1 \in [0, M_1]$ and $i_2 \in [0, M_2]$, but we simplify the notation by using a generic i to denote both indices, and let the context determine which index is being referred to.

[10] We use the term "capital intensive" as corresponding to a greater share of capital in value added, i.e., meaning for example that $s_1 > s_2$ in terms of the notation of the previous section. While this share is constant because of the Cobb-Douglas technologies, the equilibrium ratios of capital to labor in the two sectors depend on prices.

where the first set of equalities in these equations define K_1, L_1 and K_2, L_2 as the levels of capital and labor used in the two sectors, and the second set of equalities impose market clearing.

The number of intermediate goods in the two sectors evolve at the exogenous rates

$$\frac{\dot{M}_1}{M_1} = m_1 \text{ and } \frac{\dot{M}_2}{M_2} = m_2, \tag{18}$$

and each new intermediate is assigned to a monopolist, so that all intermediate goods are owned and produced by monopolists throughout. Since M_1 and M_2 determine productivity in their respective sectors, we will refer to them as "technology".

3.2 Equilibrium

Recall that w and r denote the wage and the capital rental rate, and p_1 and p_2 denote the prices of the Y_1 and Y_2 goods, with the price of the final good normalized to one. Let $[q_1(i)]_{i=1}^{M_1}$ and $[q_2(i)]_{i=1}^{M_2}$ be the prices for labor-intensive and capital-intensive intermediates.

An equilibrium in this economy is given by paths for factor, intermediate and final goods prices r, w, $[q_1(i)]_{i=1}^{M_1}$, $[q_2(i)]_{i=1}^{M_2}$, p_1 and p_2, employment and capital allocation $[l_1(i)]_{i=1}^{M_1}$, $[l_2(i)]_{i=1}^{M_2}$, $[k_1(i)]_{i=1}^{M_1}$, $[k_2(i)]_{i=1}^{M_2}$ such that firms maximize profits and markets clear, and consumption and savings decisions, C and $\dot{K}$, maximize consumer utility.

It is useful to break the characterization of equilibrium into two pieces: static and dynamic. The static part takes the state variables of the economy, which are the capital stock, the labor supply and the technology, K, L, M_1 and M_2, as given, and determines the allocation of capital and labor across sectors and factor and good prices. The dynamic part of the equilibrium determines the evolution of the endogenous state variable, K (the dynamic behavior of L is given by (10) and the one of M_1 and M_2 by (18)).

First, our choice of numeraire implies that the price of the final good, P, satisfies:

$$1 \equiv P = \left[\gamma^{\varepsilon} p_1^{1-\varepsilon} + (1-\gamma)^{\varepsilon} p_2^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

Next, since Y_1 and Y_2 are supplied competitively, their prices are equal to the value of their marginal product, thus

$$p_1 = \gamma \left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}} \quad \text{and} \quad p_2 = (1-\gamma)\left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}}, \tag{19}$$

and the demands for intermediates, $y_1(i)$ and $y_2(i)$, are given by the familiar isoelastic demand curves:

$$\frac{q_1(i)}{p_1} = \left(\frac{y_1(i)}{Y_1}\right)^{-\frac{1}{\nu}} \quad \text{and} \quad \frac{q_2(i)}{p_2} = \left(\frac{y_2(i)}{Y_2}\right)^{-\frac{1}{\nu}}. \tag{20}$$

The value of the monopolist for intermediate i in the s-intensive sector is given by

$$V_s(i,t) = \int_t^\infty \exp\left[-\int_t^v r(z)dz\right] \pi_s(i,v)dv, \tag{21}$$

for $s = 1, 2$, where $\pi_s(i,t) = (q_s(i,t) - mc_s(i,t))\, y_s(i,t)$ is the flow profits for firm i at time t, with q_s given by the demand curves in (20), and mc_s is the marginal cost of production in this sector. Given the production functions in (15), the cost functions take the familiar Cobb-Douglas form, $mc_1(i) = \alpha_1^{-\alpha_1}(1-\alpha_1)^{\alpha_1 - 1} r^{1-\alpha_1} w^{\alpha_1}$, and $mc_2(i) = \alpha_2^{-\alpha_2}(1-\alpha_2)^{\alpha_2 - 1} r^{1-\alpha_2} w^{\alpha_2}$. In equilibrium, all firms in the same sector will make the same profits, so we have $V_s(i,t) = V_s(t)$, and we use $V_1(t)$ and $V_2(t)$ to denote the value firms in the two sectors at time t. In Section 4, these value functions will be used to determine the equilibrium growth rate of the number of intermediate goods, M_1 and M_2.

Each monopolist chooses its price to maximize (21). Since prices at time t only influence revenues and costs at that point, profit-maximizing prices will be given by a constant mark-up over marginal cost:

$$q_1(i) = \left(\frac{\nu}{\nu - 1}\right) \alpha_1 (1-\alpha_1)^{\alpha_1 - 1} r^{1-\alpha_1} w^{\alpha_1}, \tag{22}$$

$$q_2(i) = \left(\frac{\nu}{\nu - 1}\right) \alpha_2 (1-\alpha_2)^{\alpha_2 - 1} r^{1-\alpha_2} w^{\alpha_2}. \tag{23}$$

Equations (22) and (23) imply that all intermediates in each sector sell at the same price $q_1 = q_1(i)$ for all $i \leq M_1$ and $q_2 = q_2(i)$ for all $i \leq M_2$. This combined with (20) implies that the demand for, and the production of, the same type of intermediate will be the same. Thus:

$$\begin{aligned} y_1(i) &= l_1(i)^{\alpha_1} k_1(i)^{1-\alpha_1} = y_1 = l_1^{\alpha_1} k_1^{1-\alpha_1} \qquad \forall i \leq M_1 \\ y_2(i) &= l_2(i)^{\alpha_2} k_2(i)^{1-\alpha_2} = y_2 = l_2^{\alpha_2} k_2^{1-\alpha_2} \qquad \forall i \leq M_2, \end{aligned}$$

where l_1 is the level of employment in all intermediates of sector 1, etc.

Market clearing conditions, (16) and (17), then imply that $l_1 = L_1/M_1$, $k_1 = K_1/M_1$, $l_2 = L_2/M_2$ and $k_2 = K_2/M_2$, so we have the output of each intermediate in the two sectors as

$$y_1 = \frac{L_1^{\alpha_1} K_1^{1-\alpha_1}}{M_1} \quad \text{and} \quad y_2 = \frac{L_2^{\alpha_2} K_2^{1-\alpha_2}}{M_2}. \tag{24}$$

Substituting (24) into (14), we obtain the total supply of labor- and capital-intensive goods as

$$Y_1 = M_1^{\frac{1}{\nu-1}} L_1^{\alpha_1} K_1^{1-\alpha_1} \text{ and } Y_2 = M_2^{\frac{1}{\nu-1}} L_2^{\alpha_2} K_2^{1-\alpha_2}. \tag{25}$$

Comparing these (derived) production functions to (1) and (2) highlights that in this economy, the production functions G_1 and G_2 from the previous section take Cobb-Douglas forms, with one sector always having a higher share of capital than the other sector, and also that $A_1 = M_1^{\frac{1}{\nu-1}}$ and $A_2 = M_2^{\frac{1}{\nu-1}}$.

In addition, combining (25) with (12) implies that the aggregate output of the economy is:

$$Y = \left[\gamma \left(M_1^{\frac{1}{\nu-1}} L_1^{\alpha_1} K_1^{1-\alpha_1} \right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) \left(M_2^{\frac{1}{\nu-1}} L_2^{\alpha_2} K_2^{1-\alpha_2} \right)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}. \tag{26}$$

Using (24) and (25) we can rewrite the prices for the labor- and capital-intensive intermediates as $q_1 = \gamma M_1^{\frac{1}{\nu-1}} \left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}}$ and $q_2 = (1-\gamma) M_2^{\frac{1}{\nu-1}} \left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}}$, and the flow profits from the sale of the labor- and capital-intensive intermediates are:

$$\pi_1 = \frac{\gamma}{\nu} \left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}} \frac{Y_1}{M_1} \text{ and } \pi_2 = \frac{1-\gamma}{\nu} \left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}} \frac{Y_2}{M_2}. \tag{27}$$

Finally, factor prices and the allocation of capital between the two sectors are determined by:[11]

$$w = \left(\frac{\nu-1}{\nu}\right) \gamma \alpha_1 \left(\frac{Y}{Y_1}\right)^{\frac{1}{\varepsilon}} \frac{Y_1}{L_1} \tag{28}$$

$$w = \left(\frac{\nu-1}{\nu}\right) (1-\gamma) \alpha_2 \left(\frac{Y}{Y_2}\right)^{\frac{1}{\varepsilon}} \frac{Y_2}{L_2} \tag{29}$$

$$r = \left(\frac{\nu-1}{\nu}\right) \gamma (1-\alpha_1) \left(\frac{Y}{Y_1}\right)^{\frac{1}{\varepsilon}} \frac{Y_1}{K_1} \tag{30}$$

$$r = \left(\frac{\nu-1}{\nu}\right) (1-\gamma)(1-\alpha_2) \left(\frac{Y}{Y_2}\right)^{\frac{1}{\varepsilon}} \frac{Y_2}{K_2}. \tag{31}$$

These factor prices take the familiar form, equal to the marginal product of a factor from (26) with a discount, $(\nu-1)/\nu$, due to the the monopoly markup in the intermediate goods.

[11] To obtain these equations, start with the cost functions above, and derive the demand for factors by using Shepherd's Lemma. For example, for the sector 1, these are $l_1 = \left(\frac{\alpha_1}{1-\alpha_1}\frac{r}{w}\right)^{1-\alpha_1} y_1$ and $k_1 = \left(\frac{\alpha_1}{1-\alpha_1}\frac{r}{w}\right)^{-\alpha_1} y_1$. Combine these two equations to derive the equilibrium relationship between r and w. Then using equation (22), eliminate r to obtain a relationship between w and q_1. Now combining with the demand curves in (20), the market clearing conditions, (16) and (17), and using (25) yields (28). The other equations are obtained similarly.

3.3 Static Equilibrium: Comparative Statics

Let us now analyze how changes in the state variables, L, K, M_1 and M_2, impact on equilibrium factor prices and factor shares. As noted in the Introduction, the case with $\varepsilon < 1$ is of greater interest (and empirically more relevant as pointed out in footnote 2), so throughout, we focus on this case (though we give the result for the case in which $\varepsilon > 1$, and we only omit the case with $\varepsilon = 1$, which is standard).

Let us denote the fraction of capital and labor employed in the capital-intensive sector respectively by $\kappa \equiv K_1/K$ and $\lambda \equiv L_1/L$ (clearly $1-\kappa \equiv K_2/K$ and $1-\lambda \equiv L_2/L$). Then Equations (28), (29), (30) and (31) imply:

$$\kappa = \left[1 + \left(\frac{1-\alpha_2}{1-\alpha_1}\right)\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{Y_1}{Y_2}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{-1} \tag{32}$$

and

$$\lambda = \left[\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{\alpha_2}{\alpha_1}\right)\left(\frac{1-\kappa}{\kappa}\right)+1\right]^{-1}. \tag{33}$$

Equation (33) makes it clear that the share of labor in the sector 1, λ, is monotonically increasing in the share of capital in the sector 1, κ. We next determine how these two shares change with capital accumulation and technological change.

Proposition 1 In equilibrium,

1.

$$\frac{d\ln\kappa}{d\ln K} = -\frac{d\ln\kappa}{d\ln L} = \frac{(1-\varepsilon)(\alpha_1-\alpha_2)(1-\kappa)}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-\lambda)} > 0 \Leftrightarrow (\alpha_1-\alpha_2)(1-\varepsilon) > 0. \tag{34}$$

2.

$$\frac{d\ln\kappa}{d\ln M_2} = -\frac{d\ln\kappa}{d\ln M_1} = \frac{(1-\varepsilon)(1-\kappa)/(\nu-1)}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-\lambda)} > 0 \Leftrightarrow \varepsilon < 1. \tag{35}$$

The proof of this proposition is straightforward and is omitted.

Equation (34), part 1 of the proposition, states that when the elasticity of substitution between sectors, ε, is less than 1, the fraction of capital allocated to the capital-intensive sector declines in the stock of capital (and conversely, when $\varepsilon > 1$, this fraction is increasing in the stock of capital). To obtain the intuition for this comparative static, which is useful for understanding many of the results that will follow, note that if K increases and κ remains constant, then the capital-intensive sector grows by more than the other sector. Equilibrium prices given in (19) imply that when $\varepsilon < 1$, the relative price of the capital-intensive sector falls more than proportionately, inducing a greater fraction of capital to be allocated to the sector that is less intensive in capital. The intuition for the converse result when $\varepsilon > 1$ is straightforward.

Moreover, equation (35) implies that when the elasticity of substitution, ε, is less than one, an improvement in the technology of a sector causes the share of capital going to that sector to fall. The intuition is again the same: increased production in a sector causes a more than proportional decline in its relative price, inducing a reallocation of capital away from it towards the other sector (again the converse results and intuition apply when $\varepsilon > 1$).

Proposition 1 gives only the comparative statics for κ. Equation (33) implies that the same comparative statics applies to λ.

Next, combining (28) and (30), we also obtain relative factor prices as

$$\frac{w}{r} = \frac{\alpha_1}{1-\alpha_1}\left(\frac{\kappa K}{\lambda L}\right), \tag{36}$$

and the capital share in the economy as[12]

$$s_K \equiv 1 - \frac{wL}{Y} = 1 - (1-\gamma)\,\alpha_1 \left(\frac{Y_1}{Y}\right)^{\frac{\varepsilon-1}{\varepsilon}} \frac{1}{\lambda}. \tag{37}$$

Proposition 2 In equilibrium,

1.

$$\frac{d\ln(w/r)}{d\ln K} = -\frac{d\ln(w/r)}{d\ln L} = \frac{1}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-\lambda)} > 0.$$

2.

$$\frac{d\ln(w/r)}{d\ln M_2} = -\frac{d\ln(w/r)}{d\ln M_1} = -\frac{(1-\varepsilon)(\kappa-\lambda)/(\nu-1)}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-\lambda)} < 0 \Leftrightarrow (\alpha_1-\alpha_2)(1-\varepsilon) > 0.$$

3.

$$\frac{d\ln s_K}{d\ln K} < 0 \Leftrightarrow \varepsilon < 1.$$

4.

$$\frac{d\ln s_K}{d\ln M_2} = -\frac{d\ln s_K}{d\ln M_1} < 0 \Leftrightarrow (\alpha_1-\alpha_2)(1-\varepsilon) > 0.$$

[12] Notice that we define the capital share as one minus the labor share, which makes sure that monopoly profits are included in the capital share. Also s_K refers to the share of capital in national income, and is thus different from the capital shares in the previous section, which were sector specific. Sector-specific capital shares are constant here because of the Cobb-Douglas production functions.

The proof of this proposition is provided in the Appendix.

The most important result in this proposition is part 3, which links the equilibrium relationship between the capital share in national income and the capital stock to the elasticity of substitution. Since a negative relationship between the share of capital in national income and the capital stock is equivalent to capital and labor being gross complements in the aggregate, this result also implies that, as claimed in footnote 2, the elasticity of substitution between capital and labor is less than one if and only if ε is less than one. Intuitively, as in Theorem 1, an increase in the capital stock of the economy causes the output of the more capital-intensive sector to increase relative to the output in the less capital-intensive sector (despite the fact that the share of capital allocated to the less-capital intensive sector increases as shown in equation (34)). This then increases the production of the more capital-intensive sector, and when $\varepsilon < 1$, it reduces the relative reward to capital (and the share of capital in national income). The converse result applies when $\varepsilon > 1$.

Moreover, when $\varepsilon < 1$, part 4 implies that an increase in M_2 is "capital biased" and an increase in M_1 is "labor biased". The intuition for why an increase in the productivity of the sector that is intensive in capital is biased toward labor (and vice versa) is once again similar: when the elasticity of substitution between the two sectors, ε, is less than one, an increase in the output of a sector (this time driven by a change in technology) decreases its price more than proportionately, thus reducing the relative compensation of the factor used more intensively in that sector. When $\varepsilon > 1$, we have the converse pattern, and M_1 is "capital biased," while an increase in M_2 is "labor biased"

3.4 Dynamic Equilibrium

We now turn to the characterization of the dynamic equilibrium path of this economy. We start with the Euler equation for consumers, which takes the familiar form[13]

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r - \rho). \tag{38}$$

To write the transversality condition, note that the financial wealth of the representative consumer comes from payments to capital and profits, and is given by $W(t) = K(t) + M_1(t) V_1(t) + M_2(t) V_2(t)$, where recall that $V_1(t)$ is the present discounted value of the profits of a firm in the sector 1 at time t and there are $M_1(t)$ such firms, and similarly for

[13] Throughout, we assume that in equilibrium consumption, capital and factor prices are differentiable functions of time, and work with time derivatives, e.g., $\dot{C}$, etc.

$V_2(t)$ and $M_2(t)$. The transversality condition is then:

$$\lim_{t\to\infty} W(t)\exp\left(-\int_0^t r(\tau)\,d\tau\right)=0, \tag{39}$$

which together with the resource constraint given in (13) determines the dynamic behavior of consumption and capital stock, C and K. Equations (10) and (18) give the behavior of L, M_1 and M_2.

We can therefore summarize a dynamic equilibrium as paths of interest rates, labor and capital allocation decisions, r, λ and κ, satisfying (28), (29), (30) and (31), and of consumption, capital stock, technology, values of innovation satisfying (13), (21), (38), and (39).

Let us also introduce the following notation for growth rates of the key objects in this economy:

$$\frac{\dot{L}_1}{L_1}\equiv n_1,\ \frac{\dot{L}_2}{L_2}\equiv n_2,\ \frac{\dot{L}}{L}\equiv n$$
$$\frac{\dot{K}_1}{K_1}\equiv z_1,\ \frac{\dot{K}_2}{K_2}\equiv z_2,\ \frac{\dot{K}}{K}\equiv z$$
$$\frac{\dot{Y}_1}{Y_1}\equiv g_1,\ \frac{\dot{Y}_2}{Y_2}\equiv g_2,\ \frac{\dot{Y}}{Y}\equiv g,$$

so that n_s and z_s denote the growth rate of labor and capital stock, m_s denotes the growth rate of technology, and g_s denotes the growth rate of output in sector s. Moreover, whenever they exist, we denote the corresponding asymptotic growth rates by asterisks, i.e.,

$$n_s^*=\lim_{t\to\infty} n_s\ ,\ z_s^*=\lim_{t\to\infty} z_s \text{ and } g_s^*=\lim_{t\to\infty} g_s.$$

Similarly denote the asymptotic capital and labor allocation decisions by asterisks

$$\kappa^*=\lim_{t\to\infty}\kappa \text{ and } \lambda^*=\lim_{t\to\infty}\lambda.$$

We now state and prove two lemmas that will be useful both in this and the next section.

Lemma 1 If $\varepsilon<1$, then $n_1 \gtreqless n_2 \Leftrightarrow z_1 \gtreqless z_2 \Leftrightarrow g_1 \lesseqgtr g_2$. If $\varepsilon>1$, then $n_1 \gtreqless n_2 \Leftrightarrow z_1 \gtreqless z_2 \Leftrightarrow g_1 \gtreqless g_2$.

Proof. Differentiating (28) and (29) with respect to time yields

$$\frac{\dot{w}}{w}+n_1=\frac{1}{\varepsilon}g+\frac{\varepsilon-1}{\varepsilon}g_1 \text{ and } \frac{\dot{w}}{w}+n_2=\frac{1}{\varepsilon}g+\frac{\varepsilon-1}{\varepsilon}g_2. \tag{40}$$

Subtracting the second from the first gives $n_1 - n_2 = (\varepsilon - 1)(g_1 - g_2)/\varepsilon$, and immediately implies the first part of the desired result. Similarly differentiating (30) and (31) yields

$$\frac{\dot{r}}{r} + z_1 = \frac{1}{\varepsilon} g + \frac{\varepsilon - 1}{\varepsilon} g_1 \text{ and } \frac{\dot{r}}{r} + z_2 = \frac{1}{\varepsilon} g + \frac{\varepsilon - 1}{\varepsilon} g_2. \tag{41}$$

Again, subtracting the second from the first gives the second part of the result. ■

This lemma establishes the straightforward, but at first counter-intuitive, result that when the elasticity of substitution between the two sectors is less than one, then the equilibrium growth rate of the capital stock and labor force in the sector that is growing faster must be less than in the other sector. When the elasticity of substitution is greater than one, the converse result obtains. To see the intuition, note that terms of trade (relative prices) shift in favor of the more slowly growing sector. When the elasticity of substitution is less than one, this change in relative prices is more than proportional with the change in quantities, and this encourages more of the factors to be allocated towards the more slowly growing sector.

Lemma 2 Suppose the asymptotic growth rates g_1^* and g_2^* exist. If $\varepsilon < 1$, then $g^* = \min\{g_1^*, g_2^*\}$. If $\varepsilon > 1$, then $g^* = \max\{g_1^*, g_2^*\}$.

Proof. Differentiating the production function for the final good (26) we obtain:

$$g = \frac{\left[\gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} g_1 + (1-\gamma) Y_2^{\frac{\varepsilon-1}{\varepsilon}} g_2\right]}{\left[\gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2^{\frac{\varepsilon-1}{\varepsilon}}\right]} \tag{42}$$

which, combined with $\varepsilon < 1$ implies that as $t \to \infty$, $g^* = \min\{g_1^*, g_2^*\}$. Similarly, combined with $\varepsilon > 1$, implies that as $t \to \infty$, $g^* = \max\{g_1^*, g_2^*\}$. ■

Consequently, when the elasticity of substitution is less than 1, the asymptotic growth rate of aggregate output will be determined by the sector that is growing more slowly, and the converse applies when $\varepsilon > 1$.

3.5 Constant Growth Paths

We first focus on *asymptotic equilibrium paths*, which are equilibrium paths that the economy tends to as $t \to \infty$. A *constant growth path* (CGP) is defined as an equilibrium path where the asymptotic growth rate of consumption exists and is constant, i.e.,

$$\lim_{t\to\infty}\frac{\dot{C}}{C}=g_C^*.$$

From the Euler equation (38), this also implies that the interest rate must be asymptotically constant, i.e., $\lim_{t\to\infty}\dot{r}=0$.

To establish the existence of a CGP, we impose the following parameter restriction:

$$\max\left\{\frac{m_1}{\alpha_1},\frac{m_2}{\alpha_2}\right\}\leq(\nu-1)\left(\frac{\rho}{1-\theta}-n\right). \tag{A2}$$

This assumption ensures that the transversality condition (39) holds. Terms of the form m_1/α_1 or m_2/α_2 appear naturally in equilibrium, since they capture the "augmented" rate of technological progress. In particular, recall that associated with the technological progress, there will also be equilibrium capital deepening in each sector. The overall effect on labor productivity (and output growth) will depend on the rate of technological progress augmented with the rate of capital deepening. The terms m_1/α_1 or m_2/α_2 capture this, since a lower α_1 or α_2 corresponds to a greater share of capital in the relevant sector, and thus a higher rate of augmented technological progress for a given rate of Hicks-neutral technological change. In this light, Assumption (A2) can be understood as implying that the augmented rate of technological progress should be low enough to satisfy the transversality condition (39).

The next theorem is the main result of this part of the paper and characterizes the relatively simple form of the constant growth path (CGP) in the presence of non-balanced growth. Although we characterize a CGP, in the sense that aggregate output grows at a constant rate, it is noteworthy that growth is non-balanced since output, capital and employment in the two sectors grow at *different rates.*

Theorem 3 Suppose Assumptions A1 and A2 hold. Define s and $\sim s$ such that $\frac{m_s}{\alpha_s}=\min\left\{\frac{m_1}{\alpha_1},\frac{m_2}{\alpha_2}\right\}$ and $\frac{m_{\sim s}}{\alpha_{\sim s}}=\max\left\{\frac{m_1}{\alpha_1},\frac{m_2}{\alpha_2}\right\}$ when $\varepsilon<1$, and $\frac{m_s}{\alpha_s}=\max\left\{\frac{m_1}{\alpha_1},\frac{m_2}{\alpha_2}\right\}$ and $\frac{m_{\sim s}}{\alpha_{\sim s}}=\min\left\{\frac{m_1}{\alpha_1},\frac{m_2}{\alpha_2}\right\}$ when $\varepsilon>1$. Then there exists a unique CGP such that

$$g^*=g_C^*=g_s^*=z_s^*=n+\frac{1}{\alpha_s(\nu-1)}m_s \tag{43}$$

$$z_{\sim s}^*=n-\frac{(1-\varepsilon)\,m_{\sim s}}{(\nu-1)}+\frac{[1+\alpha_{\sim s}(1-\varepsilon)]\,m_s}{\alpha_s(\nu-1)}<g^* \tag{44}$$

$$g_{\sim s}^*=n+\frac{\varepsilon m_{\sim s}}{(\nu-1)}+\frac{[1-\alpha_{\sim s}(1-\varepsilon\alpha_{\sim s}(1-\varepsilon))]\,m_s}{\alpha_s(\nu-1)[1-\alpha_{\sim s}(1-\varepsilon)]}>g^* \tag{45}$$

$$n_s^* = n \text{ and } n_{\sim s}^* = n - \frac{(1-\varepsilon)(\alpha_s m_{\sim s} - \alpha_{\sim s} m_s)}{\alpha_s (\nu - 1)}. \tag{46}$$

Proof. We prove this proposition in three steps.

Step 1: Suppose that $\varepsilon < 1$. Provided that $g_{\sim s}^* \geq g_s^* > 0$, then there exists a unique CGP defined by equations (43), (44), (45) and (46) satisfying $g_{\sim s}^* > g_s^* > 0$, where $\frac{m_s}{\alpha_s} = \min\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$ and $\frac{m_{\sim s}}{\alpha_{\sim s}} = \max\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$.

Step 2: Suppose that $\varepsilon > 1$. Provided that $g_{\sim s}^* \leq g_s^* < 0$, then there exists a unique CGP defined by equations (43), (44), (45) and (46) satisfying $g_{\sim s}^* < g_s^* < 0$, where $\frac{m_s}{\alpha_s} = \max\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$ and $\frac{m_{\sim s}}{\alpha_{\sim s}} = \min\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$.

Step 3: Any CGP must satisfy $g_{\sim s}^* \geq g_s^* > 0$, when $\varepsilon < 1$ and $g_s^* \geq g_{\sim s}^* > 0$, when $\varepsilon > 1$ with $\frac{m_s}{\alpha_s}$ defined as in the theorem.

The third step then implies that the growth rates characterized in steps 1 and 2 are indeed equilibria and there cannot be any other CGP equilibria, completing the proof.

Proof of Step 1. Let us assume without any loss of generality that $s = 1$, i.e., $\frac{m_s}{\alpha_s} = \frac{m_1}{\alpha_1}$. Given $g_2^* \geq g_1^* > 0$, equations (32) and (33) imply condition that $\lambda^* = \kappa^* = 1$ (in the case where $s = 2$, we would have $\lambda^* = \kappa^* = 0$) and Lemma 2 implies that we must also have $g^* = g_1^*$. This condition together with our system of equations, (25) , (40) and (41), solves uniquely for n_1^*, n_2^*, z_1^*, z_2^*, g_1^* and g_2^* as given in equations (43), (44), (45) and (46). Note that this solution is consistent with $g_2^* > g_1^* > 0$, since Assumptions A1 and A2 imply that $g_2^* > g_1^*$ and $g_1^* > 0$. Finally, $C \leq Y$, (11) and (39) imply that the consumption growth rate, g_C^*, is equal to the growth rate of output, g^* (suppose not, then since $C/Y \to 0$ as $t \to \infty$, the budget constraint (11) implies that asymptotically $K(t) = Y(t)$, and integrating the budget constraint gives $K(t) \to \int_0^t Y(s)\,ds$, implying that the capital stock grows more than exponentially, since Y is growing exponentially; violating the transversality condition (39)).

Finally, we can verify that an equilibrium with z_1^*, z_2^*, m_1^*, m_2^*, g_1^* and g_2^* satisfies the transversality condition (39). Note that the transversality condition (39) will be satisfied if

$$\lim_{t\to\infty} \frac{\dot{W}(t)}{W(t)} < r^*, \tag{47}$$

where r^* is the constant asymptotic interest rate. Since from the Euler equation (38) $r^* = \theta g^* + \rho$, (47) will be satisfied when $g^*(1-\theta) < \rho$. Assumption A2 ensures that this is the case with $g^* = n + \frac{1}{\alpha_1(\nu-1)} m_1$. A similar argument applies for the case where $\frac{m_s}{\alpha_s} = \frac{m_2}{\alpha_2}$.

Proof of Step 2. The proof of this step is similar to the previous one, and is thus omitted.

Proof of Step 3. We now prove that along all CGPs $g^*_{\sim s} \geq g^*_s > 0$, when $\varepsilon < 1$ and $g^*_s \geq g^*_{\sim s} > 0$, when $\varepsilon > 1$ with $\frac{m_s}{\alpha_s}$ defined as in the theorem. Without any loss of generality, suppose that $\frac{m_s}{\alpha_s} = \frac{m_1}{\alpha_1}$. We now separately derive a contradiction for two configurations, (1) $g^*_1 \geq g^*_2$, or (2) $g^*_2 \geq g^*_1$ but $g^*_1 \leq 0$.

1. Suppose $g^*_1 \geq g^*_2$ and $\varepsilon < 1$. Then, following the same reasoning as in Step 1, the unique solution to the equilibrium conditions (25), (40) and (41), when $\varepsilon < 1$ is:

$$g^* = g^*_C = g^*_2 = z^*_2 = n + \frac{1}{\alpha_2 (\nu - 1)} m_2 \tag{48}$$

$$z^*_1 = n - \frac{(1-\varepsilon) m_1}{(\nu - 1)} + \frac{[1 + \alpha_1 (1-\varepsilon)] m_2}{\alpha_2 (\nu - 1)} \tag{49}$$

$$g^*_1 = n + \frac{\varepsilon m_1}{(\nu - 1)} + \frac{[1 - \alpha_1 (1 - \varepsilon\alpha_1 (1-\varepsilon))] m_2}{\alpha_2 (\nu - 1) [1 - \alpha_1 (1-\varepsilon)]} \tag{50}$$

$$n^*_1 = n + \frac{(1-\varepsilon) [\alpha_1 m_2 - \alpha_2 m_1]}{\alpha_2 (\nu - 1)} \tag{51}$$

   But combining these equations implies that $g^*_1 < g^*_2$, which contradicts the hypothesis $g^*_1 \geq g^*_2 > 0$. The argument for $\varepsilon > 1$ is analogous.

2. Suppose $g^*_2 \geq g^*_1$ and $\varepsilon < 1$, then the same steps as above imply that there is a unique solution to equilibrium conditions (25), (40) and (41), which are given by equations (43), (44), (45) and (46). But now (43) directly contradicts $g^*_1 \leq 0$. Finally suppose $g^*_2 \geq g^*_1$ and $\varepsilon > 1$, then the unique solution is given by (48), (49), (50) and (51), then (50) directly contradicts $g^*_1 \leq 0$, and this completes the proof.

■

There are a number of important implications of this theorem. First, as long as $m_1/\alpha_1 \neq m_2/\alpha_2$, growth is non-balanced. The intuition for this result is the same as Theorem 1 in the previous section. Differential capital intensities in the two sectors combined with capital deepening in the economy (which itself results from technological progress) ensure faster growth in the more capital-intensive sector. Intuitively, if capital were allocated proportionately to the two sectors, the more capital-intensive sector would grow faster. Because of the changes in prices, capital and labor are reallocated in favor of the less capital-intensive sector, but not enough to fully offset the faster growth in the more capital-intensive sector. This result also highlights that the assumption of balanced technological progress in the previous section (in this context, $m_1 = m_2$) was not necessary for the theorem, but we simply needed to rule out the precise relative rate of technological progress between the two sectors that would ensure balanced growth (in this context, $m_1/\alpha_1 = m_2/\alpha_2$).

Second, while the CGP growth rates looks somewhat complicated because they are written in the general case, they are relatively simple once we restrict attention to parts of the parameter space. For example, when $m_1/\alpha_1 < m_2/\alpha_2$, the capital-intensive sector (sector 2) always grows faster than the labor-intensive one, i.e., $g_1^* < g_2^*$. In addition if $\varepsilon < 1$, the more slowly-growing labor-intensive sector dominates the asymptotic behavior of the economy, and the CGP growth rates are

$$\begin{aligned} g^* &= g_C^* = g_1^* = z_1^* = n + \frac{1}{\alpha_1(\nu-1)} m_1, \\ g_2^* &= n + \frac{\varepsilon m_2}{(\nu-1)} + \frac{[1-\alpha_2(1-\varepsilon\alpha_2(1-\varepsilon))]\,m_1}{\alpha_1(\nu-1)[1-\alpha_2(1-\varepsilon)]} > g^*. \end{aligned}$$

In contrast, when $\varepsilon > 1$, the more rapidly-growing capital-intensive sector dominates the asymptotic behavior and

$$\begin{aligned} g^* &= g_C^* = g_2^* = z_2^* = n + \frac{1}{\alpha_2(\nu-1)} m_2, \\ g_1^* &= n + \frac{\varepsilon m_1}{(\nu-1)} + \frac{[1-\alpha_1(1-\varepsilon\alpha_1(1-\varepsilon))]\,m_2}{\alpha_2(\nu-1)[1-\alpha_1(1-\varepsilon)]} < g^*. \end{aligned}$$

Third, as the proof makes it clear, in the limit, the share of capital and labor allocated to one of the sector tends to one (e.g., when sector 1 is the asymptotically dominant sector, $\lambda^* = \kappa^* = 1$). Nevertheless, at all points in time both sectors produce positive amounts, so this limit point is never reached. In fact, at all times both sectors grow at rates greater than the rate of population growth in the economy. Moreover, when $\varepsilon < 1$, the sector that is shrinking grows faster than the rest of the economy at all point in time, even asymptotically. Therefore, the rate at which capital and labor are allocated away from this sector is determined in equilibrium to be *exactly* such that this sector still grows faster than the rest of the economy. This is the sense in which non-balanced growth is not a trivial outcome in this economy (with one of the sectors shutting down), but results from the positive but differential growth of the two sectors.

Finally, it can be verified that the share of capital in national income and the interest rate are constant in the CGP. For example, when $m_1/\alpha_1 < m_2/\alpha_2$, $rK/Y = 1 - \alpha_1(\nu-1)/\nu$ and when $m_1/\alpha_1 > m_2/\alpha_2$, $rK/Y = 1-\alpha_2(\nu-1)/\nu$. The interest rate, on the other hand, is equal to $r = (1-\alpha_1)(\nu-1)\gamma^{\frac{\varepsilon}{\varepsilon-1}}(\chi^*)^{-\alpha_1}/\nu$ in the first case and $r = (1-\alpha_2)(\nu-1)\gamma^{\frac{\varepsilon}{\varepsilon-1}}(\chi^*)^{-\alpha_2}/\nu$ in the second case, where χ^* is defined below. These results are the basis of the claim in the Introduction that this equilibrium may account for non-balanced growth at the sectoral level, without substantially deviating from the Kaldor facts. In particular, the constant growth path equilibrium matches both the Kaldor facts and generates unequal growth between the two sectors. However, in this constant growth path equilibrium, one of the sectors has already become very small relative to the other. Therefore, this theorem does not answer whether along the equilibrium path (but away from the asymptotic equilibrium), we can have a situation in which both sectors have non-trivial employment levels and the equilibrium capital share in national income and the interest rate are approximately constant. This question and the stability of the constant growth path equilibrium are investigated in the next section.

3.6 Dynamics and Stability

The previous section characterized the asymptotic equilibrium, and established the existence of a unique constant growth path. This growth path exhibits non-balanced growth, though asymptotically the economy grows at a constant rate and the share of capital in national income is constant. We now study the equilibrium behavior of this economy away from this asymptotic equilibrium.

The equilibrium behavior away from the asymptotic equilibrium path can be represented by a dynamical system characterizing the behavior of a control variable C and four state variables K, L, M_1 and M_2. The dynamics of aggregate consumption, C, and aggregate capital stock, K, are given by the Euler equation (38) and the resource constraint (13). Furthemore, the dynamic behavior of L is given by (10) and the one of M_1 and M_2 by (18).

As noted above, when $\varepsilon > 1$, the sector which grows faster dominates the economy, while when $\varepsilon < 1$, conversely, the slower sector dominates. We want to show that, in both cases, the unique CGP of the previous section is locally stable. Since the case with $\varepsilon < 1$ is more interesting, we emphasize this case in our analysis. Moreover, without loss of generality, we restrict the discussion to the case in which asymptotically the economy is dominated by the labor-intensive sector, sector 1, so that

$$g^* = g_1^* = z_1^* = n + \frac{1}{\alpha_1(\nu-1)}m_1.$$

This means that when we assume $\varepsilon < 1$, the relevant part of the parameter space is where

$m_1/\alpha_1 < m_2/\alpha_2$, and, when $\varepsilon > 1$, we must have $m_1/\alpha_1 > m_2/\alpha_2$ (for the rest of the parameter space, it would be sector 2 that dominates the asymptotic behavior).

The equilibrium behavior of this economy can be represented by a system of autonomous non-linear differential equations in three variables,

$$c \equiv \frac{C}{LM_1^{\frac{1}{\alpha_1(\nu-1)}}}, \ \chi \equiv \frac{K}{LM_1^{\frac{1}{\alpha_1(\nu-1)}}} \text{ and } \kappa.$$

Here c is the level of consumption normalized by population and technology (of the sector which will dominate the asymptotic behavior), and is the only control variable; χ is the capital stock normalized by the same denominator, and κ determines the allocation of capital between the two sectors. These two are state variables with given initial conditions χ_0 and κ_0.[14]

The dynamic equilibrium conditions than translate into the following equations:

$$\begin{aligned}
\frac{\dot{c}}{c} &= \frac{1}{\theta}\left[(1-\alpha_1)\left(\frac{\nu-1}{\nu}\right)\gamma\left(\frac{Y}{Y_1}\right)^{\frac{1}{\varepsilon}}\lambda^{\alpha_1}(\kappa\chi)^{-\alpha_1}-\rho\right]-n-\frac{1}{\alpha_1(\nu-1)}m_1, \qquad (52)\\
\frac{\dot{\chi}}{\chi} &= \lambda^{\alpha_1}\kappa^{1-\alpha_1}\chi^{-\alpha_1}\left(\frac{Y}{Y_1}\right)-\chi^{-1}c-n-\frac{1}{\alpha_1(\nu-1)}m_1,\\
\frac{\dot{\kappa}}{\kappa} &= \frac{(1-\kappa)\left[(\alpha_1-\alpha_2)\frac{\dot{\chi}}{\chi}+\left(\frac{1}{\nu-1}\right)\left(m_2-\frac{\alpha_2}{\alpha_1}m_1\right)\right]}{\left(\frac{\varepsilon}{1-\varepsilon}\right)+\alpha_2+(1-\alpha_1)(1-\kappa)+(1-\alpha_2)\kappa+(\alpha_1-\alpha_2)(1-\lambda)},
\end{aligned}$$

where

$$\left(\frac{Y}{Y_1}\right)=\gamma^{\frac{\varepsilon}{\varepsilon-1}}\left[1+\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1}{\kappa}-1\right)\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

and

$$\lambda=\left[\left(\frac{1-\alpha_1}{\alpha_1}\right)\left(\frac{\alpha_2}{1-\alpha_2}\right)\left(\frac{1-\kappa}{\kappa}\right)+1\right]^{-1}.$$

Clearly, the constant growth path equilibrium characterized above corresponds to a steady-state equilibrium in terms of these three variables, denoted by c^*, χ^* and κ^* (i.e., in the CGP equilibrium, c , χ and κ will be constant). These steady-state values are given by $\kappa^* = 1$,

$$\chi^*=\left[\frac{\nu\left(\theta\left[n+\frac{1}{\alpha_1(\nu-1)}m_1\right]+\rho\right)}{\gamma^{\frac{\varepsilon}{\varepsilon-1}}(1-\alpha_1)(\nu-1)}\right]^{-\frac{1}{\eta}},$$

and

$$c^*=\gamma^{\frac{\varepsilon}{\varepsilon-1}}{\chi^*}^{1-\eta}-\chi^*\left(n+\frac{1}{\alpha_1(\nu-1)}m_1\right).$$

[14] χ_0 is given by definition, and κ_0 is uniquely pinned down by the static equilibrium allocation of capital at time $t = 0$, given by (32).

Since there are two state and one control variable, local (saddle-path) stability requires the existence of a (unique) two-dimensional manifold of solutions in the neighborhood of the steady state that converge to c^*, χ^* and κ^*. The next theorem states that this is the case.

Theorem 4 The non-linear system (52) is locally (saddle-path) stable, in the sense that in the neighborhood of c^*, χ^* and κ^*, there is a unique two-dimensional manifold of solutions that converge to c^*, χ^* and κ^*.

Proof. Let us rewrite the system (52) in a more compact form as

$$\dot{x} = f(x), \tag{53}$$

where $x \equiv \begin{pmatrix} c & \chi & \kappa \end{pmatrix}'$. To investigate the dynamics of the system (53) in the neighborhood of the steady state, consider the linear system

$$\dot{z} = J(x^*)\, z,$$

where $z \equiv x - x^*$ and x^* such that $f(x^*) = 0$, where $J(x^*)$ is the Jacobian of $f(x)$ evaluated at x^*. Differentiation and some algebra enable us to write this Jacobian matrix as

$$J(x^*) = \begin{bmatrix} a_{cc} & a_{c\chi} & a_{c\kappa} \\ a_{\chi c} & a_{\chi\chi} & a_{\chi\kappa} \\ a_{\kappa c} & a_{\kappa\chi} & a_{\kappa\kappa} \end{bmatrix},$$

where

$$\begin{aligned}
a_{cc} &= a_{\kappa c} = a_{\kappa\chi} = 0 \\
a_{c\chi} &= -\gamma^{\frac{\varepsilon}{\varepsilon-1}} (\chi^*)^{-\alpha_1-1} \left(\frac{\alpha_1(1-\alpha_1)}{\theta}\right)\left(\frac{\nu-1}{\nu}\right) \\
a_{c\kappa} &= \gamma^{\frac{\varepsilon}{\varepsilon-1}} (\chi^*)^{-\alpha_1} \left(\frac{1-\alpha_1}{\theta}\right)\left(\frac{\nu-1}{\nu}\right)\left[\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1+\alpha_2(1-\varepsilon)}{1-\varepsilon}\right) - \alpha_1\right] \\
a_{\chi c} &= -(\chi^*)^{-1} \\
a_{\chi\chi} &= \gamma^{\frac{\varepsilon}{\varepsilon-1}} (\chi^*)^{-\alpha_1-1} (1-\alpha_1)\left[1 - \frac{1}{\theta}\left(\frac{\nu-1}{\nu}\right)\right] + \frac{(\chi^*)^{-1}\rho}{\theta} \\
a_{\chi\kappa} &= \gamma^{\frac{\varepsilon}{\varepsilon-1}} (\chi^*)^{-\alpha_1}\left[(1-\alpha_1) + \left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1+\alpha_2(1-\varepsilon)}{1-\varepsilon}\right)\right] \\
a_{\kappa\kappa} &= -\left(\frac{1-\varepsilon}{\nu-1}\right)\left(m_2 - \frac{\alpha_2}{\alpha_1} m_1\right).
\end{aligned}$$

The determinant of the Jacobian is $\det(J(x^*)) = -a_{\kappa\kappa}a_{c\chi}a_{\chi c}$. First, $a_{c\chi}$ and $a_{\chi c}$ are clearly negative. Next, it can be seen that $a_{\kappa\kappa}$ is always negative since we are in the case with

$\varepsilon \lesseqgtr 1 \Leftrightarrow m_2/\alpha_2 \gtreqless m_1/\alpha_1$.[15] This implies that $\det\left(J\left(x^*\right)\right) > 0$, so the steady state is hyperbolic. Moreover, either all the eigenvalues are positive or two of them are negative and one positive. To determine which is the case, we look at the characteristic equation given by $\det\left(J\left(x^*\right) - vI\right) = 0$, where v denote the eigenvalues. This equation is

$$\left(a_{\kappa\kappa} - v\right)\left[v\left(a_{\chi\chi} - v\right) + a_{\chi c}a_{c\chi}\right] = 0,$$

and shows that one of the eigenvalue is equal to $a_{\kappa\kappa}$ and thus negative, so there must be two negative eigenvalues. Consequently, there exists a unique two-dimensional manifold of solutions in the neighborhood of this steady state, converging to it. This proves local (saddle-path) stability. ■

This result shows that the constant growth path equilibrium is locally stable, and when the initial values of capital, labor and technology are not too far from the constant growth path, the economy will indeed converge to this equilibrium, with non-balanced growth at the sectoral level and constant capital share and interest rate at the aggregate.

We next investigate the dynamics of some parameterized economies numerically. In particular, we wish to study the speed of convergence to the asymptotic equilibrium, and how the economy behaves along the path of convergence, for example, whether the interest rate and the share of capital change by large amounts along the transition path. Since the economy with $\varepsilon < 1$ is our main focus, we only report simulations for this case.

Recall that our economy is fully characterized by ten parameters, γ, ε, ν, α_1, α_2, ρ, θ, n, m_1, and m_2. We choose a period to correspond to a year, and take a baseline economy with a 1% annual population growth ($n = 0.01$), $\rho = 0.02$ and $\theta = 2$ as in the baseline calibration of the neoclassical growth model in Barro and Sala-i-Martin (2004). We choose the rest of the parameters to be consistent with 3% total growth rate of output ($g = 0.03$), a capital share in national income of approximately 35% ($s_K = 0.35$), and an interest rate of approximately 12% ($r = 0.12$).[16] These choices imply that $\alpha_1 \simeq 0.8$ (essentially to match the share of capital asymptotically), $\varepsilon = 0.5$ and $\nu \simeq 5$ (to match the interest rate) and $m_1 \simeq 0.125$ (to match the growth rate). Finally, in the benchmark simulation, we choose a capital intensity in sector 2

[15] As noted above, this is not a parameter restriction. When we have $\varepsilon > 1$ and $m_2/\alpha_2 > m_1/\alpha_1$, for example, then it will be sector 2 that grows more slowly in the limit, and stability will again obtain.

[16] Since there is no depreciation in our model, the interest rate also corresponds to the rental rate of capital. We choose 12% as the benchmark value to approximate a reasonable rental rate (e.g., an interest rate of 7% plus 5% depreciation).

close to that in sector 1, $\alpha_2 = 0.75$ and $m_2 = m_1 \simeq 0.125$ to highlight the non-balanced growth resulting from differential capital intensities in the two sectors. In addition, we choose a low level of γ, $\gamma = 0.15$, to generate a fraction of employment in sector 1 of approximately 40%, and the following initial values: $L(0) = 1$, $K(0) = 1$, $M_L(0) = 1$ and $M_H(0) = 0.1$.

Figure 1 shows the results of the simulations.[17] The four panels depict λ, κ, the interest rate (r) and the capital share (s_K). The solid line is for the benchmark. The first remarkable feature in the simulation is the rate of convergence. The units on the horizontal axis are years, and range from zero to 3000. This shows that it takes a very very long time for the fraction of capital and labor allocated to sector 1 to approach their asymptotic equilibrium value of 1. For example, initially, about 40% of employment is in sector 1 and even after 500 years, less than half of employment remains there. This illustrates that even though in the limit one of the sectors employs all of the factors, it takes a very long time for the economy to approach this limit point. Second, in the baseline simulation, the interest rate is essentially constant, and varies only between 0.112 and 0.115 throughout the convergence process. The share of capital in national income is declining visibly, but its range of movement is small (between 0.35 and 0.375). Moreover, in the first 500 years, the capital share essentially moves between 0.37 and 0.375). This is the basis of our claim that this type of model may lead to a pattern of non-balanced sectoral growth (as shown by λ and κ in the top two panels), while generating only small movements in the interest rate and the capital share, thus remaining approximately consistent with the Kaldor facts.

[17] To perform the simulations, we first represent the equilibrium as a two-dimensional non-autonomous system in c and χ (rather than the three-dimensional autonomous system analyzed above) since κ can be represented as a function of time only. This two dimensional system has one state and one control variable. Following Judd (1998, ch.10), we then discretize these differential equations using the Euler method, and turn them into a system of first-order difference equations in c_t and χ_t, which can itself be transformed into a second-order non-autonomous system only in χ_t. This second-order equation can be analyzed numerically by reversing time, perturbing χ away from its steady-state value, and then integrating it backward to (χ_0, κ_0) (given the exogenously given sequence of state variables, $\{L_t, M_{1,t}, M_{2,t}\}_{t=0}^{T}$ and boundary conditions $\chi_{T+1} = \chi_T = \chi^*$).

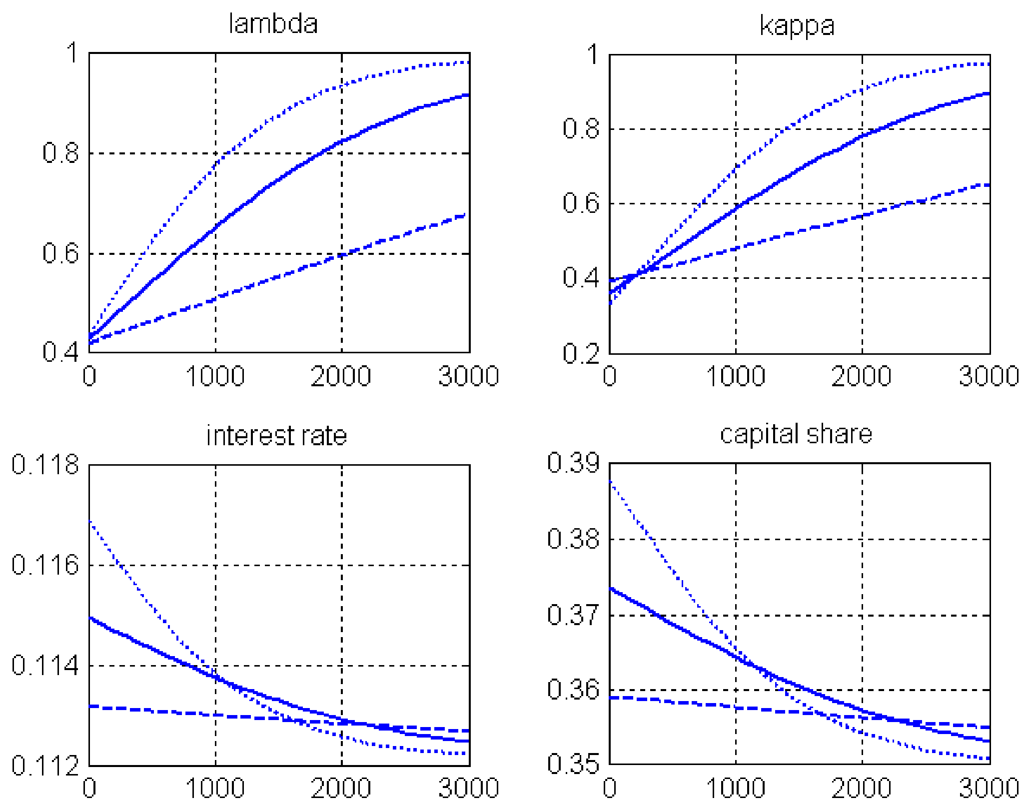

Figure 1. Solid line: $\alpha_2 = .75$. Dotted line: $\alpha_2 = .72$. Dashed line: $\alpha_2 = .78$.

The dashed and dotted lines in Figure 1 show variations with $\alpha_2 = 0.78$ and $\alpha_2 = 0.72$. The dashed line shows that when α_2 is increased even further, convergence is even slower, and now after 3000 years only less than 70% of employment is in sector 1. Moreover, the interest rate and the share of capital in national income are now essentially constant. When α_2 is reduced so that the gap between the capital intensity of the two sectors becomes larger, the speed of convergence is a little faster, but is still very very slow. The range of change of the capital share also becomes larger (between 0.35 and 0.39).

Figure 2 investigates the consequences of varying m_2. The solid line is again for the benchmark simulation, with $m_2 = m_1 \simeq 0.125$, while the dashed and the dotted lines show simulations with $m_2 = 0.12$ and $m_2 = 0.13$. They show that when the TFP growth rate in the capital-intensive sector is reduced, convergence takes much longer. For example, after 3000 years, a little above 60% of employment is in sector 1. The capital share also changes by very little over this time period. In contrast, when the TFP growth rate of the capital-intensive sector is increased, convergence is faster than the benchmark case but still very slow. For example, it takes 2000 years for the share of employment in sector 1 to reach 90%. These simulations therefore show that this class of economies may be able to generate significant non-balanced sectoral growth, without substantially deviating from the Kaldor facts.

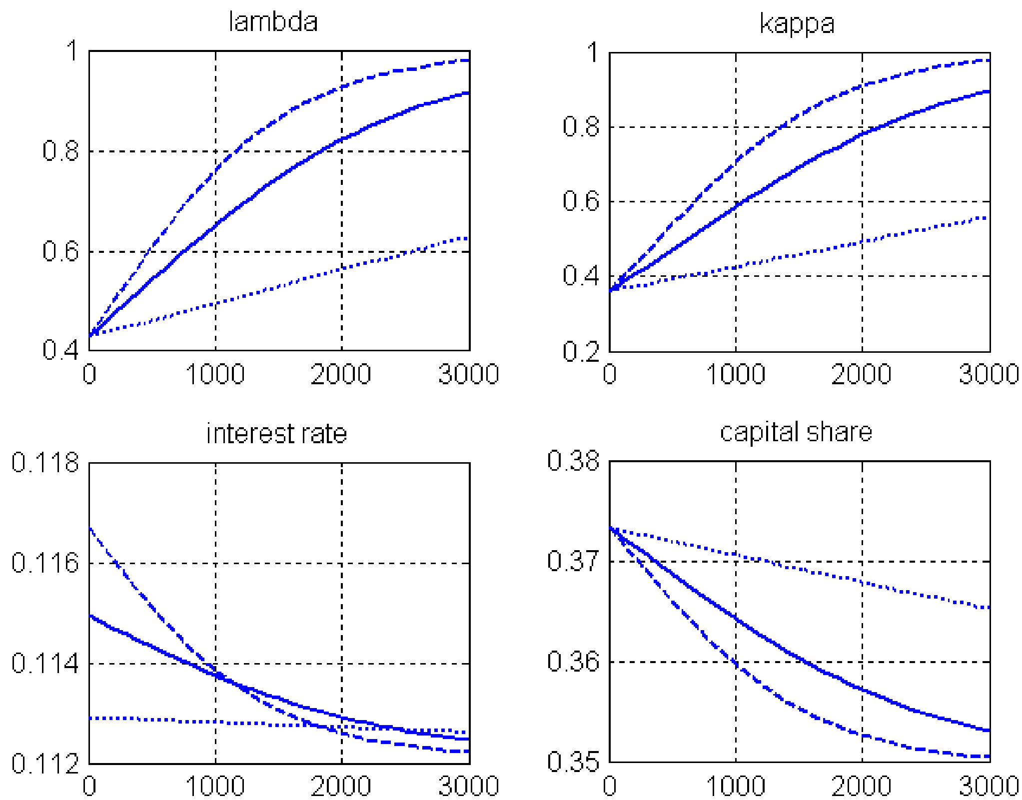

Figure 2. Solid line: $m_2 = .125$. Dotted line: $m_2 = .12$. Dashed line: $m_2 = .13$.

4 Non-Balanced Endogenous Growth

In this section we introduce endogenous technological progress. This investigation is motivated by two questions. As already emphasized, non-balanced growth results from capital deepening, and so far, capital deepening was a consequence of exogenous technological change. The first question is whether similar results obtain when technological change itself is endogenous. Second and more important, one may wonder whether endogenous technological change will take place in such a form as to restore balanced growth. The analysis in this section will explicitly show that this is not the case. Finally, endogenizing technological change in this context enables us to derive a model of non-balanced endogenous technological change, which is an important direction for models of endogenous technology given the non-balanced nature of growth in the data.

Demographics, preferences and technology are the same as described in the previous section, except that instead of the exogenous processes for technology given in (18), we now need to specify an innovation possibilities frontier, i.e., the technology to transform resources into blueprints for new varieties in the two sectors. In particular, we assume that

$$\dot{M}_1 = b_1 M_1^{-\varphi} X_1 \text{ and } \dot{M}_2 = b_2 M_2^{-\varphi} X_2, \tag{54}$$

where $X_1 \geq 0$ and $X_2 \geq 0$ are research expenditures in terms of the final good, b_1 and b_2 are strictly positive constants measuring the technical difficulty of creating new blueprints in the two sectors, and $\varphi \in (-1, \infty)$ measures the degree of spillovers in technology creation.[18]

Finally, we assume that there is free entry into research, and a firm that invents a new intermediate of either sector becomes the monopolist producer with a perpetually enforced patent. Given the value for being the monopolist for intermediate in (21), we have two free entry conditions, which will pin down equilibrium technological change:

$$V_1 \leq \frac{M_1^{\varphi}}{b_1} \text{ and } V_2 \leq \frac{M_2^{\varphi}}{b_2}, \tag{55}$$

with each condition holding as equality when there is positive R&D expenditure for that sector, i.e., when $X_1 > 0$ or $X_2 > 0$.

The resource constraint of the economy is also modified to incorporate the R&D expenditures,

$$\dot{K} + C + X_1 + X_2 \leq Y. \tag{56}$$

Now a dynamic equilibrium is represented by paths of interest wages and rates, w and r, labor and capital allocation decisions, λ and κ, satisfying (30) and (31), and of consumption,

[18] When $\varphi = 0$, there are no spillovers from the current stock of knowledge to future innovations. With $\varphi < 0$, there are positive spillovers and the stock of knowledge in a particular sector makes further innovation in that sector easier. With $\varphi > 0$, there are negative spillovers ("fishing out") and further innovations are more difficult in sectors that are more advanced (see, for example, Jones, 1995, Kortum, 1997). Similar to the results in Jones (1995), Young (1999) and Howitt (1999), there will be endogenous growth for a range of values of φ because of population growth. In the remainder, we will typically think of $\varphi > 0$, so that there are negative spillovers, though this is not important for any of the asymptotic results.

Also, this innovation possibilities frontier assumes that only the final good is used to generate new technologies. The alternative is to have a scarce factor, such as labor or scientists, in which case some amount of positive spillovers would be necessary.

capital stock, technology, values of innovation and research expenditures satisfying (21), (38), (39), (54), (55) and (56). It is also useful to define a path that satisfies all of these equations, possibly except the transversality condition, (39), as a *quasi-equilibrium.*

Since the case with $\varepsilon < 1$ is both more interesting, and in view of the discussion in footnote 2, also more realistic, in this section we focus on this case exclusively.

We first note that Propositions 1 and 2 from Section 3 still apply and characterize the comparative static responses, and Lemmas 1 and 2 from there determine the behavior of the growth rate of sectoral output, capital and employment. For the analysis of the economy with endogenous technology, we also need an additional result in the next lemma. It shows that provided that (i) $\varepsilon < 1$, (ii) there exists a constant asymptotic interest rate r^* (i.e., $\lim_{t\to\infty} \dot{r} = 0$), and (iii) there is positive population growth, in the asymptotic equilibrium the free entry conditions in (55) will both hold as equality:

Lemma 3 Suppose that $\varepsilon < 1$, $n > 0$, and $\lim_{t\to\infty} \dot{r} = 0$, then $\lim_{t\to\infty} (V_1 - M_1^{\varphi}/b_1) = 0$ and $\lim_{t\to\infty} (V_2 - M_2^{\varphi}/b_2) = 0$.

The proof of this lemma is rather long and is provided in the Appendix.

This lemma is an important result for the analysis of non-balanced endogenous growth. It enables us to solve for the asymptotic growth rates from the free entry conditions and obtain a relatively simple characterization of the constant growth path equilibrium.

The economic intuition for the lemma comes from population growth; with population growth, it is always optimal to allocate more capital to each sector, which increases the profitability of intermediate producers in that sector. Consequently, the value of a new blueprint increases asymptotically. This rules out asymptotic equilibrium paths with slack free-entry conditions, because along such paths, the cost of creating a new blueprint would remain constant, ultimately violating the free-entry condition.

To establish the existence of a CGP, we now impose the following parameter restriction, which replaces (A2) in the exogenous technology case:

$$\zeta > \max\left\{\frac{1}{\alpha_1}, \frac{1}{\alpha_1}\left[1 - \frac{(1-\theta)}{\rho}n\right]^{-1}\right\} \tag{A3}$$

where $\zeta \equiv (\nu - 1)(1 + \varphi)$. This assumption ensures that the transversality condition (39) holds. Notice that in contrast to Assumption A2 only α_1 features in this assumption, since, given Assumption A1, $\alpha_1 < \alpha_2$ and with endogenous technology, this will make sure that sector

1 is the more slowly growing sector in the asymptotic equilibrium. Assumption A3 also rules out quasi-equilibrium paths where output and consumption grow more than exponentially.

Lemma 4 Suppose Assumption A3 holds and $\varepsilon < 1$, then there exists no quasi-equilibria with $\lim_{t\to\infty} \dot{C}/C = \infty$.

This lemma is proved in the Appendix, where, for completeness, we also show that Assumption A3 is "tight" in the sense that, if first inequality in this assumption, $\zeta > 1/\alpha_1$, did not hold, there always exist quasi-equilibria with more than exponential growth.

Combined Lemmas 3 and 4 imply that the asymptotic equilibrium has to converge either to a limit with constant growth of consumption, or to a limit cycle. Using these results, the next theorem establishes the existence of a unique constant growth path, with non-balanced sectoral growth, but constant share of capital and constant interest rate in the aggregate. It is therefore the analogue of Theorem 3 of the previous section, and is the main result of this section.

Theorem 5 Suppose Assumptions A1 and A3 hold, $\varepsilon < 1$, and $n > 0$, then there exists a unique CGP where

$$g^* = g_C^* = g_1^* = z_1^* = \frac{\alpha_1\zeta}{\alpha_1\zeta - 1}n, \tag{57}$$

$$g_2^* = \frac{\alpha_1\zeta\,(1-\varepsilon+\zeta) + \varepsilon\zeta\,(\alpha_1-\alpha_2)}{(\alpha_1\zeta-1)\,(1-\varepsilon+\zeta)}n > g^*, \tag{58}$$

$$z_2^* = \frac{\alpha_1\zeta\,(1-\varepsilon+\zeta) - \zeta\,(\alpha_1-\alpha_2)\,(1-\varepsilon)}{(\alpha_1\zeta-1)\,(1-\varepsilon+\zeta)}n < g^*, \tag{59}$$

$$n_1^* = n \quad \text{and} \quad n_2^* = \left[1 - \frac{\zeta\,(1-\varepsilon)\,(\alpha_1-\alpha_2)}{(\alpha_1\zeta-1)\,[1-\varepsilon+\zeta]}\right] n, \tag{60}$$

$$m_1^* = \frac{1}{1+\varphi}z_1^* \quad \text{and} \quad m_2^* = \frac{1}{1+\varphi}z_2^*. \tag{61}$$

The proof of this theorem is also given in the Appendix.

There a number of important features worth noting. First, this theorem shows that the equilibrium of this non-balanced endogenous growth economy takes a relatively simple form. Second, given the equilibrium rates of technological change in the two sectors, m_1^* and m_2^*, the asymptotic growth rates are identical to those in Theorem 3 in the previous section, so that the economy with endogenous technological change gives the same insights as the one with exogenous technology. In particular, there is non-balanced sectoral growth, but in the aggregate, the economy has a limiting equilibrium with constant share of capital in national income and a constant interest rate. Finally, technology is also endogenously non-balanced, and in fact, tries to counteract the non-balanced nature of economic growth. Specifically, there is more technological change towards the sector that is growing more slowly (recall we are focusing on the case where $\varepsilon < 1$), so that the sector with a lower share of capital has an increasing share of capital, employment and technology in the economy. Nevertheless, this sector still grows more slowly than the more capital-intensive sector, and the non-balanced nature of the growth process remains. Therefore, endogenous growth does not restore balance between the two sectors as long as capital intensity (factor proportion) differences between the two sectors remain.[19]

5 Conclusion

This paper shows how differences in factor proportions across sectors combined with capital deepening lead to a non-balanced pattern of economic growth. We first illustrated this point using a general two-sector growth model, and then characterized the equilibrium fully using a class of economies with constant elasticity of substitution between sectors and Cobb-Douglas production technologies. This class of economies shows how the pattern of equilibrium may be simultaneously consistent with non-balanced sectoral growth (the so-called Kuznets facts) while also generating asymptotically constant share of capital in national income and interest rate in the aggregate (the Kaldor facts). We also constructed and analyzed a model with endogenous technology featuring non-balanced economic growth.

The main contribution of the paper is theoretical, but it also raises a number of empirical questions. In particular, the analysis suggests that differences in factor proportions across sectors will introduce a powerful force towards non-balanced growth, which could be important in accounting for the cross-sectoral patterns of output and employment growth. Whether this is so or not, especially in the context of the differential growth of agriculture, manufacturing and services, appears to be a fruitful area for future research.

[19] Since there are two more endogenous state variables and two more control variables now, it is not possible to show local stability analytically. In particular, in addition to c, χ and κ, we need to keep track of the endogenous evolution of M_1, M_2, X_1 and X_2 (or their stationary transformations). Given the size of this system, we are unable to prove local (saddle-path) stability.

6 Appendix

6.1 Proof of Proposition 2

Parts 1 and 2 follow from differentiating equation (36) and Proposition 1. Here we prove parts 3 and 4. Recall the expression for the equilibrium capital share

$$s_K \equiv \frac{rK}{Y} = 1 - \gamma\alpha_1 \left(\frac{Y_1}{Y}\right)^{\frac{\varepsilon-1}{\varepsilon}} \lambda^{-1}$$

with

$$\lambda = \left[\left(\frac{1-\eta}{1-\alpha}\right)\left(\frac{\alpha}{\eta}\right)\left(\frac{1}{\kappa}-1\right)+1\right]^{-1}$$

where combining the production function and the equilibrium capital allocation we get

$$\begin{aligned}\left(\frac{Y_1}{Y}\right)^{\frac{\varepsilon-1}{\varepsilon}} &= \left[\gamma + (1-\gamma)\left(\frac{Y_1}{Y_2}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{-1} \\ &= \gamma^{-1}\left(1+\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1}{\kappa}-1\right)\right)^{-1}\end{aligned}$$

Then, using the results of Proposition 1, we have

$$\frac{d\ln s_K}{d\ln K} = -S\left(\frac{1-s_K}{s_K}\right)\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\frac{(1-\varepsilon)(\alpha_1-\alpha_2)(1-\kappa)/\kappa}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-x)} \tag{62}$$

and

$$\frac{d\ln s_K}{d\ln M_2} = -\frac{d\ln s_K}{d\ln M_1} = S\left(\frac{1-s_K}{s_K}\right)\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\frac{(1-\varepsilon)(1-\kappa)/\kappa(\nu-1)}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-x)} \tag{63}$$

where

$$S \equiv \left[\left(1+\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1}{\kappa}-1\right)\right)^{-1} - \left(\left(\frac{\alpha_1}{\alpha_2}\right)+\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1}{\kappa}-1\right)\right)^{-1}\right]$$

with

$$S < 0 \Leftrightarrow \alpha_1 > \alpha_2.$$

This establishes the desired result that

$$\frac{d\ln s_K}{d\ln K} < 0 \Leftrightarrow \varepsilon < 1$$

and

$$\frac{d\ln s_K}{d\ln M_1} = -\frac{d\ln s_K}{d\ln M_2} > 0 \Leftrightarrow (\alpha_1-\alpha_2)(1-\varepsilon) > 0.$$

■

6.2 Proof of Lemma 3

We will prove this lemma in four steps.

Step 1: $m_1^* = m_2^* = 0$ imply $g_2^* = g_1^* = n$.

Step 2: $\limsup_{t\to\infty}(V_1 - M_1^\varphi/b_1) \geq 0$ or $\limsup_{t\to\infty}(V_2 - M_2^\varphi/b_2) \geq 0$.

Step 3: $\limsup_{t\to\infty}(V_1 - M_1^\varphi/b_1) \geq 0$ and $\limsup_{t\to\infty}(V_2 - M_2^\varphi/b_2) \geq 0$.

Step 4: $\lim_{t\to\infty}(V_1 - M_1^\varphi/b_1) = 0$ and $\lim_{t\to\infty}(V_2 - M_2^\varphi/b_2) = 0$.

Proof of Step 1: We first prove that $m_1^* = m_2^* = 0$ imply $g_2^* = g_1^* = n$. To see this, combine equations (30) and (31) to obtain

$$\left(\frac{Y_2}{Y_1}\right)^{\frac{1-\varepsilon}{\varepsilon}} = \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{1-\alpha_2}{1-\alpha_1}\right)\frac{K_1}{K_2}.$$

Differentiating gives

$$\frac{1-\varepsilon}{\varepsilon}(\alpha_2 n_2 - \alpha_1 n_1) + \frac{1-\varepsilon}{\varepsilon}(1-\alpha_2) z_2 - \frac{1-\varepsilon}{\varepsilon}(1-\alpha_1) z_1 = z_1 - z_2. \tag{64}$$

To derive a contradiction, suppose that $g_2^* > g_1^*$. Then, from Lemma 1 $n_2^* < n_1^*$ and $z_2^* < z_1^*$ and from Lemma 2, $g^* = g_1^*$. Next, differentiating (25), we obtain

$$g_1 = \alpha_1 n_1 + (1-\alpha_1) z_1 + \frac{1}{\nu - 1} m_1 \text{ and } g_2 = \alpha_2 n_2 + (1-\alpha_2) z_2 + \frac{1}{\nu-1} m_2. \tag{65}$$

Moreover, since $\lim_{t\to\infty}\dot{r} = 0$, equation (41) and the fact that $g^* = g_1^*$ imply that $g_1^* = z_1^*$. This combined with equation (65) and $m_1^* = 0$ implies that $z_1^* = n_1^* = n$, which together with (64) implies

$$0 > \frac{1-\varepsilon}{\varepsilon}(1-\alpha_2)(z_2^* - z_1^*) > z_1^* - z_2^*,$$

yielding a contradiction. The argument for the case in which $g_2^* < g_1^*$ is analogous. Since $g_2^* = g_1^* = g^*$ and $m_1^* = m_2^* = 0$, it must also be the case that $g_2^* = g_1^* = n$, completing the proof of step 1.

Proof of Step 2: First note that $\limsup_{t\to\infty}(V_1 - M_1^\varphi/b_1) < 0$ and $\limsup_{t\to\infty}(V_2 - M_2^\varphi/b_2) < 0$ imply that the free entry conditions, (55), are asymptotically slack, so m_1^* and m_2^* exist, and $\lim_{t\to\infty} m_1(t) = m_1^* = 0$ and $\lim_{t\to\infty} m_2(t) = m_2^* = 0$ (since they cannot be negative). In particular, note that if $\limsup_{t\to\infty}(V_1 - M_1^\varphi/b_1) < 0$, this implies that there exists no "infinitely-recurring" interval in the limit where the free entry condition holds for sector 1.

Now to derive a contradiction, suppose that $m_2^* = m_1^* = 0$, which, as shown above, implies $g_1^* = g_2^* = g^* = n > 0$. Then, differentiating the second equation in (27), we obtain:

$$\lim_{t\to\infty} \frac{\dot{\pi}_2}{\pi_2} = g^* > 0. \tag{66}$$

Combining this with $\lim_{t\to\infty} \dot{r} = 0$ and the value function in (21) yields: $\lim_{t\to\infty} V_2 = \infty$. Since $m_2^* = 0$ by hypothesis, $M_2^{-\varphi}$ is constant, and we have $\lim_{t\to\infty} V_2 = \infty > \lim_{t\to\infty} M_2^{\varphi}/b_2$, violating the free entry condition (55). This proves that m_1^* and m_2^* cannot both equal to 0, and thus $\limsup_{t\to\infty} (V_2 - M_2^{\varphi}/b_2) < 0$ and $\limsup_{t\to\infty} (V_1 - M_1^{\varphi}/b_1) < 0$ is not possible.

Proof of Step 3: We next prove that $\limsup_{t\to\infty} (V_1 - M_1^{\varphi}/b_1) \geq 0$, $\limsup_{t\to\infty} (V_2 - M_2^{\varphi}/b_2) \geq 0$, $\limsup_{t\to\infty} m_1 > 0$ and $\limsup_{t\to\infty} m_2 > 0$.

Suppose, to derive a contradiction, $\limsup_{t\to\infty} m_2 = m_2^* = 0$ and $\limsup_{t\to\infty} (V_2 - M_2^{\varphi}/b_2) < 0$ (the other case is proved analogously). Since, as shown above, $m_1^* = m_2^* = 0$ is not possible, we must have $\limsup_{t\to\infty} (V_1 - M_1^{\varphi}/b_1) \geq 0$ and $\limsup_{t\to\infty} m_1 > 0$.

We start by noting from (27), that asymptotically

$$\lim_{t\to\infty} \frac{\dot{\pi}_2}{\pi_2} = \frac{1}{\varepsilon}(g - g_2) + g_2, \tag{67}$$

because $m_2^* = 0$. Since

$$\lim_{t\to\infty} \frac{\dot{r}}{r} = \frac{1}{\varepsilon}(g - g_2) + g_2 - z_2 = 0$$

from (31), $z_2 > 0$ implies that $\lim_{t\to\infty} \dot{\pi}_2/\pi_2 > 0$. But by the same argument as in Step 2, we have $\lim_{t\to\infty} V_2 = \infty > \lim_{t\to\infty} M_2^{\varphi}/b_2$, violating the free entry condition (55). We therefore only have to show that $z_2 > 0$ (i.e., $\limsup_{t\to\infty} z_2 > 0$). Suppose, to obtain a contradiction, that $z_2^* = 0$. Using (40) and (41) from the proof of Lemma 1, we have $z_2 - n_2 = z_1 - n_1$, which implies that $n_2^* = 0$ (recall that either $n_1 \geq n$ or $n_2 \geq n$, and since $\lim_{t\to\infty} g \geq n$, either $z_1 \geq n$ or $z_2 \geq n$). But then with $n_2^* = z_2^* = m_2^* = 0$, we have $g_2^* = 0 < g_1^*$, which contradicts $n_2^* < n_1^* = n$ from Lemma 1. A similar argument for the other sector completes the proof that $\limsup_{t\to\infty} (V_1 - M_1^{\varphi}/b_1) \geq 0$ and $\limsup_{t\to\infty} (V_2 - M_2^{\varphi}/b_2) \geq 0$.

Proof of Step 4. From the free entry conditions in (55), we have that $V_1 - M_1^{\varphi}/b_1 \leq 0$ and $V_2 - M_2^{\varphi}/b_2 \leq 0$, thus $\limsup_{t\to\infty} (V_1 - M_1^{\varphi}/b_1) \leq 0$ and $\limsup_{t\to\infty} (V_2 - M_2^{\varphi}/b_2) \leq 0$. Combined with Step 3, this implies $\limsup_{t\to\infty} (V_1 - M_1^{\varphi}/b_1) = 0$ and $\limsup_{t\to\infty} (V_2 - M_2^{\varphi}/b_2) = 0$. Hence, we only have to prove that $\liminf_{t\to\infty} (V_1 - M_1^{\varphi}/b_1) \geq 0$

and $\liminf_{t\to\infty}\left(V_2 - M_2^{\varphi}/b_2\right) \geq 0$. We prove the first inequality (the proof of the second is similar).

Suppose, to derive a contradiction, that $\liminf_{t\to\infty}\left(V_1 - M_1^{\varphi}/b_1\right) < 0$. This implies that there exists a (recurring) interval (t_0', t_2') such that $V_1(t) - M_1^{\varphi}(t)/b_1 < 0$ for all $t \in (t_0', t_2')$. Suppose that (t_0', t_2') is unbounded; this would imply that $\limsup_{t\to\infty} m_1 = m_1^* = 0$, yielding a contradiction with Step 1. Thus (t_0', t_2') must be bounded, so there exists $(t_0, t_2) \supset (t_0', t_2')$ such that for $t \in (t_0, t_2) \setminus (t_0', t_2')$, we have $V_1(t) - M_1^{\varphi}(t)/b_1 = 0$. Moreover, since $\limsup_{t\to\infty} m_1 > 0$, there also exists an interval $(t_0'', t_2'') \supset (t_0, t_2)$ such that for all $t \in (t_0, t_2) \setminus (t_0'', t_2'')$, $m_1 > 0$.

Next, since $m_1 = 0$ for all $t \in (t_0', t_2')$, we also have $M_1(t_0') = M_1(t_2')$. This implies

$$V_1\left(t_2'\right) = \frac{M_1^{\varphi}(t_0')}{b_1} = V_1\left(t_0'\right). \tag{68}$$

Figure A1 shows this diagrammatically.

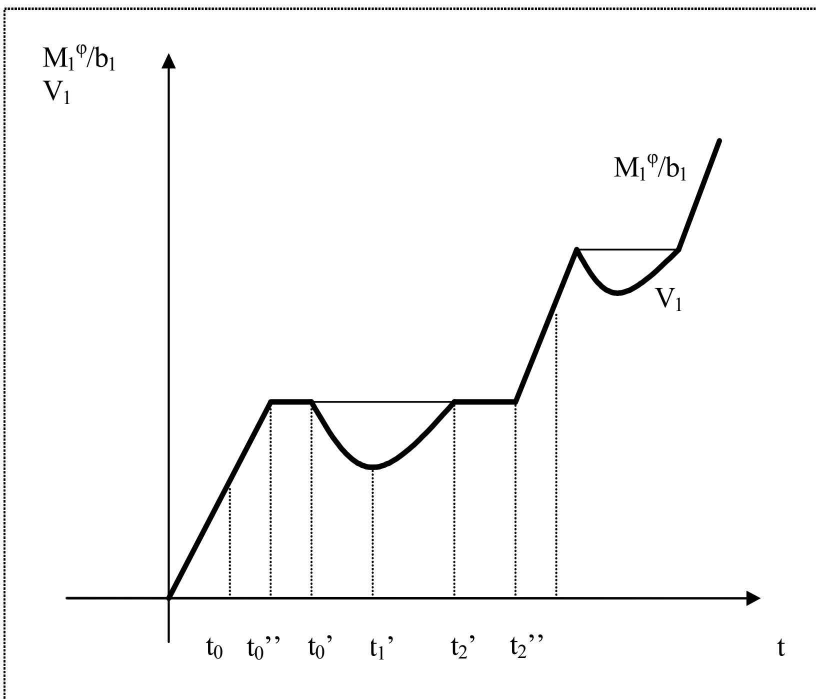

Figure A1: The solid line represents M_1^{φ}/b_1 and the thick line represents V_1.

Let us rewrite (21) in the Bellman equation form

$$\frac{\dot{V}_1(t)}{r} = V_1(t) - \frac{\pi_1(t)}{r} \quad \forall t. \tag{69}$$

Equation (69) also shows that $\dot{V}_1(t)$ is well-defined, so $V_1(t)$ is continuously differetiable in t. Equation (68) and the fact that $V_1(t) - M_1^{\varphi}(t'_0)/b_1 < 0$ for all $t \in (t'_0, t'_2)$ imply that $V_1(t)$ reaches a minimum over (t'_0, t'_2) with $\dot{V}_1(t) = 0$. Let $t'_1 < t'_2$ be such that $V_1(t'_1)$ is the first local minimum after t'_0, which naturally satisfies $V_1(t'_1) < V_1(t'_0)$. Moreover, using (27) and (31), we have that for all t

$$\frac{\pi_1(t)}{r^*} = \frac{1}{(\nu-1)(1-\alpha_1)} \frac{K_1(t)}{M_1(t)},$$

where $r^* = \lim_{t\to\infty} r(t)$ is the asymptotic equilibrium interest rate, which exists by hypothesis that $\lim_{t\to\infty} \dot{r}(t) = 0$. Also, using the fact that $\lim_{t\to\infty} \dot{r}(t) = 0$ and the interest rate condition, (30), we obtain that since $m_1 = 0$ and $n > 0$, $K_1(t'_1) > K_1(t'_0)$. In addition, since $M_1(t'_1) = M_1(t'_0)$, we can use (69) to write

$$\frac{\dot{V}_1(t'_1)}{r^*} = V_1(t'_1) - \frac{1}{(\nu-1)(1-\alpha_1)} \frac{K_1(t'_1)}{M_1(t'_1)} < V_1(t'_0) - \frac{1}{(\nu-1)(1-\alpha_1)} \frac{K_1(t'_0)}{M_1(t'_0)} = \frac{\dot{V}_1(t'_0)}{r^*} < 0$$

which contradicts the fact that $V_1(t'_1)$ is a local minimum, completing the proof of the lemma.■

6.3 Proof of Lemma 4 and The Converse Result

Proof of Lemma 4: First, recall that $C \leq Y$, (11). Hence it is enough to prove that $\lim_{t\to\infty} g = \infty$ will violate the resource constraint. We will prove this separately in two cases, when $g_2^* \geq g_1^*$ and when $g_2^* < g_1^*$

Suppose $g_2^* \geq g_1^*$ and $g^* = \infty$. Then, Lemma 2 implies $g_1^* = g^* = \infty$, and equation (41) together with (40) and (65) yields

$$g = n - \left(\frac{1-\alpha_1}{\alpha_1}\right)\frac{\dot{r}}{r} + \frac{1}{\alpha_1(\nu-1)} m_1.$$

Given $n < \infty$ and $\lim_{t\to\infty} \dot{r}/r > 0$, it must be that asymptotically

$$g^* = \frac{1}{\alpha_1(\nu-1)} m_1^*. \tag{70}$$

Combining the technology possibility frontier (54) and (70) we have

$$\lim_{t\to\infty} \frac{\dot{X}_1}{X_1} = \lim_{t\to\infty} \frac{\dot{m}_1}{m_1} + \alpha_1(1+\varphi)(\nu-1) g^*.$$

Then, the first inequality in Assumption A3, $\zeta > 1/\alpha_1$, implies

$$\lim_{t\to\infty} \frac{\dot{X}_1}{X_1} > g^*,$$

which violates the resource constraint (13).

Next suppose that $g_1^* > g_2^*$and $g^* = \infty$. Then, following the steps of above, Lemma 2 implies $g_2^* = g^* = \infty$, and equation (41) together with (40) and (65) yields

$$\left(\frac{1-\alpha_1(1-\varepsilon)}{\varepsilon}\right) g_1 = \alpha_1 n_1 + \left(\frac{1-\alpha_1}{\varepsilon}\right) g - (1-\alpha_1)\frac{\dot{r}}{r} + \left(\frac{1}{\nu-1}\right) m_1.$$

Since $g^* = \infty$, a fortiori $g_1^* = \infty$, and, given $n < \infty$ and $\lim_{t\to\infty} \dot{r}/r > 0$, we have that asymptotically

$$m_1^* = (\nu-1)\left[\left(\frac{1-\alpha_1(1-\varepsilon)}{\varepsilon}\right) g_1^* - \left(\frac{1-\alpha_1}{\varepsilon}\right) g^*\right]. \tag{71}$$

Once again the innovation possibilities frontier (18) implies

$$\lim_{t\to\infty}\frac{\dot{X}_1}{X_1} = \lim_{t\to\infty}\frac{\dot{m}_1}{m_1} + \alpha_1(1-\varphi)\, m_1^*.$$

Then equation (70) together with the first inequality in Assumption A3, $\zeta > 1/\alpha_1$, implies

$$\lim_{t\to\infty}\frac{\dot{X}_1}{X_1} > \alpha_1(1-\varphi)(\nu-1)\, g^* > g^*,$$

which violates the resource constraint (13), completing the proof that when Assumption A3 holds any quasi-equilibrium with more than exponential growth violates the resource constraints.■

For completeness, we also prove the converse of Lemma 4, which shows that the use of the first inequality in Assumption A3, $\zeta > 1/\alpha_1$, in this lemma is "tight" in the sense that, if it were relaxed, the converse result would obtain.

Lemma 4': Suppose A1 holds, but $\zeta \equiv (\nu-1)(1-\varphi) \leq \frac{1}{\alpha_1}$, then there exists quasi-equilibria with $\lim_{t\to\infty} g = \infty$.

Proof. This lemma will be proved by showing that in this case

$$g_2^* = g_1^* = g^* = \infty \quad \text{and} \quad z_2^* = z_1^* = z^* \tag{72}$$

$$z^* = g^* - \frac{\dot{r}}{r} \tag{73}$$

$$\frac{\dot{m}_1}{m_1} = \frac{\dot{m}_2}{m_2} = [1-\alpha_1\zeta]\, g \tag{74}$$

is a quasi-equilibrium.

From the interest rate conditions in the two sectors (30) and (31), and (72) we obtain

$$z^* = g^* - \lim_{t\to\infty}\frac{\dot{r}}{r}$$

which is exactly condition (73). By substituting into (65), we obtain

$$g^* = n_1 - \left(\frac{1-\alpha_1}{\alpha_1}\right) \lim_{t\to\infty} \frac{\dot{r}}{r} + \frac{1}{\alpha_1(\nu-1)} m_1^*$$

$$g^* = n_2 - \left(\frac{1-\alpha_2}{\alpha_2}\right) \lim_{t\to\infty} \frac{\dot{r}}{r} + \frac{1}{\alpha_2(\nu-1)} m_2^*$$

which gives $g^* = \frac{1}{\alpha_1(\nu-1)} m_1^*$ and $g^* = \frac{1}{\alpha_2(\nu-1)} m_2^*$, and hence

$$m_1^* = \frac{\alpha_1}{\alpha_2} m_2^*.$$

Differentiating this condition gives equation (74).

Finally, we need to check feasibility, i.e., that the R&D expenditures do not grow faster than output. From the technology possibilities frontiers, (18), this requires

$$\lim_{t\to\infty} \frac{\dot{X}_1}{X_1} = \frac{\dot{m}_1^*}{m_1^*} + \alpha_1 \zeta g^* \leq g^*$$

$$\lim_{t\to\infty} \frac{\dot{X}_2}{X_2} = \frac{\dot{m}_2^*}{m_2^*} + \alpha_2 \zeta g^* \leq g^*$$

and both these conditions are satisfied given (74) and $\alpha_1 > \alpha_2$. ■

6.4 Proof of Theorem 5

We prove this proposition in two steps.

Step 1: Provided that $g_2^* \geq g_1^* > 0$, then there exists a unique CGP defined by equations (57), (58), (59), (60) and (61), satisfying $g_2^* > g_1^* > 0$.

Step 2: All CGP must satisfy $g_2^* \geq g_1^* > 0$.

Proof of Step 1. Lemma 3 establishes that as $t \to \infty$ the free-entry conditions (55) must asymptotically hold as equality. Combining (55) as equality with (69) (and the equivalent for sector 2), we obtain the following conditions that must hold as $t \to \infty$:

$$\frac{\frac{\gamma}{\nu}\left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}} Y_1}{r - \varphi m_1} = \frac{M_1^{1+\varphi}}{b_1} \text{ and } \frac{\frac{\gamma}{\nu}\left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}} Y_2}{r - \varphi m_2} = \frac{M_2^{1+\varphi}}{b_2} \tag{75}$$

Differentiating (75) yields:

$$g_1 - \frac{1}{\varepsilon}(g_1 - g) - (1+\varphi) m_1 = 0 \text{ and } g_2 - \frac{1}{\varepsilon}(g_2 - g) - (1+\varphi) m_2 = 0. \tag{76}$$

Then $g_2^* > g_1^* > 0$ and Lemma 2 imply that we must also have $g^* = g_1^*$. This condition together our system of equations, (40), (41), (65), (76), solves uniquely for n_1^*, n_2^*,z_1^*, z_2^*, m_1^*, m_2^*, g_1^* and g_2^* (and $\kappa^* = \lambda^* = 1$) as given in equations (57), (58), (59), (60) and (61). Note that this solution is consistent with $g_2^* > g_1^* > 0$, since Assumptions A1 and A3 immediately imply that $g_2^* > g_1^*$ and $g_1^* > 0$ (which is also consistent with Lemma 3). Finally, $C \leq Y$, (11) and (39) imply that the consumption growth rate, g_C^*, is equal to the growth rate of output, g^*.

Finally, we can verify that an equilibrium with κ^*, λ^*, n_1^*, n_2^*, z_1^*, z_2^*, m_1^*, m_2^*, g_1^* and g_2^* satisfies the transversality condition (39) with a similar argument to the one spelled in the first step of the proof of Theorem 3.

Proof of Step 2. The proof that along all CGPs $g_2^* > g_1^* > 0$ must be true, is analogous to the one of the second step in the proof of Theorem 3.■

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