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THE ROLE OF INFORMATION IN REPEATED GAMES WITH FREQUENT ACTIONS

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THE ROLE OF INFORMATION IN REPEATED GAMES WITH FREQUENT ACTIONS

BY YULIY SANNIKOV AND ANDRZEJ SKRZYPACZ[1]

We show that in repeated interactions the avenues for effective provision of incentives depend crucially on the type of information players observe. We establish this conclusion for general repeated two-player games in which information arrives via a continuous-time stationary process that has a continuous multidimensional Brownian component and a Poisson component, and in which the players act frequently. The Poisson jumps can be used to effectively provide incentives both with transfers and value burning, while continuous Brownian information can be used to provide incentives only with transfers.

KEYWORDS: Repeated games, imperfect monitoring, frequent actions, Brownian motion, Poisson process, Levy decomposition.

1. INTRODUCTION

CONSIDER A DYNAMIC INTERACTION in which players learn information continually over time. According to the Lévy decomposition theorem, if the information process has independent and identically distributed increments (conditional on current actions), it can be decomposed into a continuous Brownian component and a discontinuous Poisson component. Figure 1 illustrates continuous and discontinuous information processes (specifically, log likelihood ratios for a statistical test of cooperative behavior by one of the players).

The arrival of information can be classified into these two categories not just on an abstract mathematical level, but also in practice. For example, members of a team inside a firm may see continuously how close they are to completion of a project. They may also learn information from breakdowns and accidents that arrive discontinuously. Firms colluding in a market for chemicals with secret price discounts can trace market prices of futures on their product relatively continuously (with each change containing little information about strategies) and can monitor infrequent (and informative) purchasing decisions of large clients.

This paper shows that the effective use of information to provide incentives in repeated two-player games with frequent actions depends crucially on whether information arrives continuously or via sudden, informative events. Motivated by the Lévy decomposition theorem, we assume that players learn information through a mix of Brownian and Poisson processes. We assume that the players' actions affect only the drifts but not the volatilities of the Brownian processes, to ensure that players learn information gradually (as illustrated

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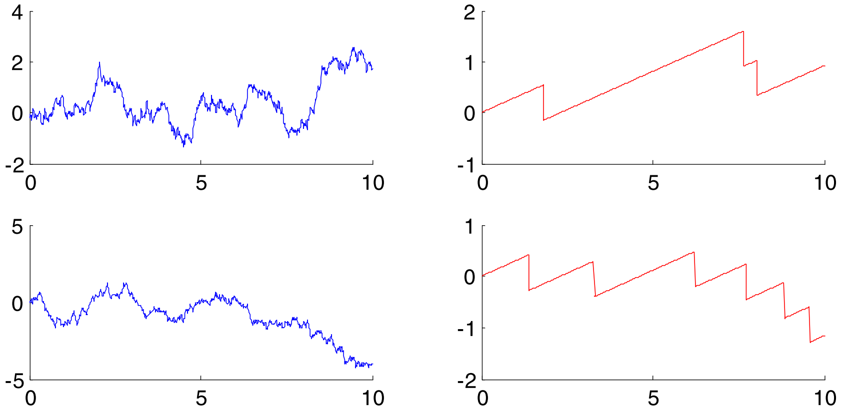

FIGURE 1.—The evolution of log likelihood ratios if the null is true (cooperative behavior; top panels) and if the alternative is true (bottom panels).

in the left panel of Figure 1), since volatility can be precisely estimated from even small sample paths of the process.[2] We require information flows (and payoffs) to be invariant to changes in the frequency of moves. The information about current actions is observed without delay; that is, the current actions completely determine the distribution of current signals, independently of past actions.

Existing theory describes two main ways to provide incentives in repeated games. The first way is by burning value, as in the oligopoly of Green and Porter (1984), where incentives to collude are created by the threat of triggering a price war. Strongly symmetric equilibria of Abreu, Pearce, and Stacchetti (1986) (hereafter APS86), in which all players choose the same action in each period, use only this way of creating incentives. The second way is by transferring continuation payoffs among players along tangent hyperplanes. Fudenberg, Levine, and Maskin (1994) (hereafter FLM) proved the folk theorem for a class of games that satisfy identifiability conditions by constructing equilibria that rely on this second way of creating incentives.[3]

We relate these two ways of providing incentives to the types of information players observe in games with frequent actions. We establish that in the limit

[2]See Fudenberg and Levine (2007) for a model in which actions affect volatility, but players observe only cumulative signals rather than their sample paths.

[3]Mailath and Samuelson (2006) provided an excellent exposition of the current theory of discrete-time repeated games. In Proposition 8.2.1, Mailath and Samuelson (2006) showed that the folk theorem typically fails for strongly symmetric equilibria, in which incentives are created via value burning. In asymmetric equilibria, even though the folk theorem holds under appropriate conditions, for discount factors less than 1 payoffs are bounded away from efficiency. The reason is that transfers of continuation values along tangent hyperplanes necessarily require value burning when the set of equilibrium payoffs is strictly convex.

as the period between the players' actions, Δ, shrinks to 0, the set of payoffs that players can attain equilibrium is bounded by

(a) using the Brownian information only to transfer value along tangent hyperplanes (tangent to the set of achievable equilibrium payoffs),

(b) using the Poisson signals separately from the Brownian information,

(c) ignoring multiple Poisson arrivals, and

(d) using the Brownian information *linearly* (i.e., making continuation payoffs a linear function of the Brownian information).

Poisson signals can be used both to transfer value along tangent hyperplanes and to destroy value (by moving orthogonally to the tangent hyperplane).

To prove our result, we observe that restrictions (a)–(d) on the use of information (even without additional restrictions on the size of transfers and value destruction) bound the set of attainable payoffs to a possibly smaller set, which we call M. In particular, we show that for any discount rate, the set M bounds the set of attainable payoffs in pure-strategy sequential equilibria (SEp) when the players move sufficiently frequently (Theorem 1). Restrictions (a)–(d) matter when players act frequently: for large Δ, the set of attainable payoffs can be significantly larger than M.

For small Δ, impatient players cannot attain all of M, since they can only transfer and destroy bounded amounts of continuation payoffs, and can only do so inefficiently. To complement our main result, in Section 5.1 we show that for generic games, players can attain any payoff profile inside M if the discount rate r and the period between actions Δ are sufficiently close to 0. Moreover, payoff profiles inside M can be attained while respecting restrictions (a)–(d) on the use of information more and more strictly as $\Delta \to 0$.[4] Thus, for high frequency of moves and low discount rates, any other way of using information contributes very little to satisfying the incentive compatibility constraints and cannot significantly expand the set of payoffs attainable in equilibrium. These "ineffective" ways of using information include

(a′) conditioning on Poisson and Brownian information jointly,

(b′) conditioning on multiple Poisson signals,

(c′) triggering value burning using Brownian information, and

(d′) using Brownian information nonlinearly.

What if players are impatient? The bound M on the set of attainable payoffs still applies, and in Section 6 we also explore the interaction between restrictions (a)–(d) and incentive compatibility constraints/SEp payoff sets in games with impatient players. While the results are weaker due to technical difficulties, particularly with respect to restriction (d), Section 6 suggests that informational restrictions are important in general, and not just for patient players.

Several papers have studied games with frequent actions but focused on the creation of incentives via value burning. Abreu, Milgrom, and Pearce (1991)

[4]Some restrictions, like linearity, cannot be satisfied exactly for $\Delta > 0$ because payoffs are bounded and signals are not.

(hereafter AMP) used a repeated prisoners' dilemma with Poisson signals to show the distinction between increasing the frequency of actions while keeping the flow of information constant, and increasing the discount factor toward 1. They have also shown that under frequent actions, incentives can be provided effectively by triggering punishments after single (but not multiple) arrivals of Poisson signals. Sannikov and Skrzypacz (2007) showed that it is impossible to provide incentives by using Brownian information to trigger punishments in a repeated Cournot duopoly.[5] Unlike Poisson signals, Brownian information leads to much higher costs of type I errors, that is, triggering a punishment when no deviation has occurred. Fudenberg and Levine (2007) studied a repeated game between one long-run player and a sequence of short-run players, and assumed that players observe only cumulative signals at the end of each period, instead of continuously. In this setup, incentives can be created only by value burning. They showed that attainable payoffs depend on whether signals are Brownian or Poisson, and whether actions affect the variance or only the mean of Brownian signals. If a deviation increases the variance of a Brownian signal, the cost of type I errors drops dramatically. Fudenberg and Levine (2009) focused on the ways that a continuous-time information process can arise from a sequence of processes in discrete time. In contrast, we focus on observations of the time path of a fixed continuous-time process at discrete points of time. We build upon the intuition about the costs of type I errors that appear in AMP, Sannikov and Skrzypacz (2007), and Fudenberg and Levine (2007). The novelty of this paper is that it is the first to study repeated games with general action spaces and payoff functions, the first to allow for both transfers and value burning, and the first to consider information arrival through a mix of continuous and discontinuous processes.

The intuition for our results is as follows. First, deviations yield per period benefits on the order of Δ, the length of a period. Since punishments are bounded by the range of continuation payoffs, events with probability less than $O(\Delta)$ are negligible for incentives. Therefore, conditioning on multiple Poisson arrivals is ineffective, as these events happen with probabilities on the order of Δ^2 per period. Second, treating the Poisson and Brownian parts independently does not influence incentives much because the rare Poisson arrivals are much more informative than the Brownian information (as shown by the changes in the log likelihood ratios in Figure 1). Third, with Brownian information it is too costly to provide incentives via value burning. This is because with normally distributed log likelihood ratios, the optimal test for detecting a deviation has a disproportionately large type I error ($\approx O(\Delta^{1/2})$ per period) if a deviation increases the probability of punishment by $O(\Delta)$. With Poisson signals, a similar

[5]For the Cournot game they studied, they showed that collusion is impossible even in asymmetric equilibria. To prove this result, they assumed that goods are *homogenous*, so that deviations of different players cannot be statistically identified by looking at the common market price.

test would have a type I error $O(\Delta)$. Therefore, burning value upon Poisson arrivals can be a part of an optimal incentive scheme if it is not possible to provide incentives solely with transfers. Fourth, the linear use of Brownian information to implement transfers is hardest to explain in words. Fundamentally, the result has to do with the curvature of the set of available continuation payoffs. If the set is smooth, its curvature defines a locally quadratic cost of transferring value between players, and as $\Delta \to 0$, less and less information arrives per period and transfers need to be small.[6] A constrained maximization problem with a quadratic cost function has a linear solution, hence the result. Of course, since the set M is typically nonsmooth, we approximate it with smooth sets from both outside and inside. Using outer approximations, Section 5 shows that nonlinear use of Brownian information cannot significantly improve the set of attainable payoffs over M. At the same time, Section 5.1 shows that inner approximations can be generically attained with approximately linear transfers for small enough discount rates and high enough frequency of moves.[7]

Our construction of the set M uses the method of decomposing payoffs on half-spaces (Fudenberg and Levine (1994), Kandori and Matsushima (1998)) with restrictions (a)–(d). These informational restrictions are closely connected with the theory of repeated games in continuous time. In fact, in continuous time, restrictions (a)–(d) above are the only ways of using information. For example, in the continuous-time games of Sannikov (2007), which involve only Brownian information, continuation values in optimal equilibria move tangentially along the boundary of the set of equilibrium payoffs.[8] In the games of Faingold and Sannikov (2007) between a large player and a population of small players, the only way to provide incentives is by burning value (and hence the set of equilibrium payoffs in those games collapses to the static Nash payoffs when players can act continuously because, as we show here, no dynamic incentives can be provided in the limit).

This paper is organized as follows. Section 2 presents the model. Section 3 describes the construction of set M and relates it to intuition from continuous

[6]One may wonder why we ignore extreme realizations of Brownian signals even though they are very informative about the players' actions (see Mirrlees (1974), Holmström and Milgrom (1987), Müller (2000), Hellwig and Schmidt (2002), and Fudenberg and Levine (2007)). There are two reasons. First, transfers in a repeated game are bounded by the set of feasible payoffs (unlike in the standard principal-agent models where it is assumed that agent's utility is unbounded from below), and extreme realizations of Brownian signals are very unlikely. As a result, incentives created by conditioning on those extreme realizations are negligible. Second, if the set of continuation payoffs is strictly convex, large transfers are too costly since they need to be accompanied by value burning (and small transfers for small probability events have negligible impact on incentives).

[7]The folk theorem of FLM also uses a smooth inner approximation for the set of feasible and individually rational payoffs.

[8]The characterization of Sannikov (2007) requires additional assumptions, such as *pairwise identifiability* of action profiles. These assumptions are not needed to characterize the set M in our paper.

time. Section 4 provides examples. Section 5 presents the main result formally for games with small discount rates and Section 6 extends our main insights to general discount rates. Section 7 concludes by discussing the issues of efficiency, games with more than two players, and the connection between Brownian signals and Poisson signals that arrive frequently. The Appendix contains main proofs and the Supplemental Material (Sannikov and Skrzypacz (2010)) contains additional, more technical proofs.

2. THE MODEL

Consider a repeated game with frequent moves. Two players choose actions a_1 and a_2 from finite sets A_1 and A_2, respectively. We denote an action profile by $a = (a_1, a_2)$. Players can change their actions only at discrete time points $t \in \{0, \Delta, 2\Delta, \dots\}$,[9] but they observe signals that carry imperfect information about their actions continuously.

The flow of information is independent of Δ. For any fixed action profile, public signals arrive via a continuous-time process with independent and identically distributed (i.i.d.) increments. Motivated by the Lévy decomposition theorem, we divide public signals into their continuous and discontinuous components.[10] The continuous component is given by the k-dimensional process

$$dX_t = \mu(a_1, a_2)\, dt + dZ_t^k,$$

where Z_t^k is a k-dimensional standard Brownian motion and μ is a function from action profiles to $\mathbb{R}^k$.[11] The discontinuous Poisson component of the public signal takes values y from a finite set Y and has a value-specific intensity $\lambda(y|a)$, conditional on the actions taken by players. We make the following assumptions.

ASSUMPTION 1: *$\lambda(y|a)$ is positive for all (y, a) (full support).*

ASSUMPTION 2: *The dimension of the Brownian signal is $k \geq 1$.*[12]

ASSUMPTION 3: *Actions do not affect the volatility of Brownian signals.*

[9] Throughout the paper, we suppress the dependence of actions on time to simplify notation.

[10] Continuous-time processes with i.i.d. increments are called *Lévy processes* (see Barndorff-Nielsen, Mikosch, and Resnick (2001, Theorems 1.1–1.3) or Sato (1999)). By the Lévy decomposition theorem (see Protter (2005, Theorem 42)), any Lévy process in $\mathbb{R}^k$ can be represented as a sum of a k-dimensional Brownian motion (a continuous component) and a compounded Poisson process (a jump process that can take many values).

[11] As we mentioned in footnote 2, we restrict the actions to affect only the drift (and not the volatility). See Fudenberg and Levine (2007) for discussion of games in which players affect volatility. We also assume that the volatility is 1, but this is without loss of generality.

[12] It is possible for the Brownian signal to be completely uninformative, as in Examples 2 and 3 in Section 4. In this case, the Brownian signal is just a public randomization device.

The last assumption is required to ensure that Brownian signals carry information continuously (as illustrated in the left panel of Figure 1), because volatility is observed *instantaneously* and perfectly.

The flow of payoffs that the players receive is also defined independently of Δ. At any moment of time t, player i's incremental payoff is given by

$$dG_t(a_i, a_j) = b_i(a_i)\, dX_t + c_i(a_i)\, dt + \sum_{y\in Y} h_i(a_i, y)\, dJ_{y,t}$$

for some functions $b_i : A_i \to \mathbb{R}^k$, $c_i : A_i \to \mathbb{R}$, and $h_i : A_i \times Y \to \mathbb{R}$, where $dJ_{y,t}$ is the indicator function for a realization of a jump with value y at time t. The interpretation is that b is the sensitivity of player i's payoff to the Brownian component of the signal, c_i is the private benefit or cost of action a_i, and h_i is the realization of player i's payoff for the jump y (these can be asymmetric across the players). This definition presents the most general way a player's payoff can depend on his *current* action and the *current* public signal.[13]

The expected flow of payoffs is

$$g_i(a_i, a_j) = b_i(a_i)\mu(a) + c_i(a_i) + \sum_{y\in Y} h_i(a_i, y)\lambda(y|a).$$

We denote $g(a) = (g_1(a), g_2(a))$.

Players discount payoff flows at a common discount rate r and maximize the sum of (normalized) expected discounted payoffs

$$r\int_0^\infty e^{-rt} g_i(a_i, a_j)\, dt.$$

A *public* strategy is a mapping from history of the public signals X_t and J_{yt} into actions. Without loss of generality, we consider public strategies that are not functions of the entire paths of signals X_t and J_{yt}, but only of the history of sufficient statistics about the players' actions, $(x, (j_y))$, where

(1) $$x = X_t - X_{t-\Delta} \sim N(\Delta\mu(a), \Delta I)$$

and j_y is the number of arrivals of Poisson signals of type y in a given period.[14]

[13]A reader unfamiliar with imperfect public monitoring games may be surprised that dG_i does not depend directly on a_j. If it did, player i could infer something about a_j from his payoffs and not only from the public signal. Note that the expected payoffs do depend on both actions (because actions affect the distribution of signals).

[14]There is no need to use the paths of signals for public randomization because the signal space is continuous (since $k \geq 1$) and so by Abreu, Pearce, and Stacchetti (1990), public randomization is not required to convexify the set of equilibrium payoffs.

A public perfect equilibrium (PPE) is a profile of public strategies that induces a Nash equilibrium after any public history of the repeated game. Assumption 1 (full support) implies that the set of pure-strategy PPE is equivalent to the set of pure-strategy sequential equilibria (SEp). We focus on the set $V(\Delta, r)$ of SEp payoff profiles for a game with period length Δ and discount rate r. Abreu, Pearce, and Stacchetti (1990) (APS90) together with Assumption 2 (that the signal space is continuous) imply that the set $V(\Delta, r)$ is convex. To guarantee existence, we assume that the stage game has at least one Nash equilibrium in pure strategies.

Denote by V the set of all feasible payoff profiles (i.e., the convex hull of $g(a)$), and denote by V^* the set of feasible and individually rational payoff profiles (V less the profiles with a payoff smaller than the pure-strategy minimax payoff of one of the players). Let $\bar{V}$ denote the maximal distance between any two points of V. Then $\bar{V}$ is a bound on transfers and value burning in any equilibrium. We call any unit vector in the payoff space a *direction*.

3. THE USE OF INFORMATION WHEN Δ IS SMALL

Our goal is to show that the set of payoff profiles, attainable in equilibria when the time period between actions is small, is bounded by what can be achieved by using information in a limited number of ways. We find informational restrictions that apply differently to two standard ways of creating incentives in repeated games: (i) transfers of continuation values between players along tangent hyperplanes (as in the folk theorem of FLM) and (ii) value burning, that is, choosing continuation payoffs orthogonally to the tangent hyperplane, as in Green–Porter equilibria (or APS86) that involve jumps to a price war on the equilibrium path. Specifically, we show that for any r, as $\Delta \to 0$, the set of equilibrium payoffs is bounded by a set M constructed under the following informational restrictions:

(a) using the Brownian signals only to transfer value along tangent hyperplanes (tangent to the set of achievable equilibrium payoffs),

(b) using the Poisson signals separately from the Brownian signals,

(c) ignoring multiple arrivals of Poisson signals, and

(d) using the Brownian signals *linearly* (i.e., making continuation payoffs a linear function of the Brownian signals).

Poisson information can be used both to transfer value tangentially and to destroy value by moving orthogonally to the tangent hyperplane. The results suggests that these are the only effective ways of using information when Δ is small. As a complement, we show that for generic games, any point in the interior of M is attainable in equilibrium uniformly for all small Δ when players are sufficiently patient (both results are presented formally in Section 5). When players are impatient, for small Δ, the set of attainable payoffs, $V(\Delta, r)$, is strictly smaller than M. Section 6 discusses the relationship between $V(\Delta, r)$

and the set of payoffs that can be achieved in equilibria with a restricted use of information.[15]

The set M is defined using linear programs, with embedded informational restrictions, that bound the weighted sum of the players' payoffs in all directions. For a given direction (set of weights) $N = (N_1, N_2)$, the bound is given by the program

$$D(N) = \max_{a,\beta,d(y)} \left(g(a) + \sum_{y \in Y} d(y)\lambda(y|a) \right) \cdot N \quad \text{s.t.} \quad d(y) \cdot N \le 0 \tag{2}$$

and

$$g_i(a) - g_i(a') + \beta(\mu(a) - \mu(a'))T_i + \sum_{y \in Y} d_i(y)(\lambda(y|a) - \lambda(y|a')) \ge 0 \tag{IC}$$

for all alternative action profiles $a' = (a_i', a_j)$ in which one of the players i deviates while his opponent j follows the action profile a (i.e., (IC) is a set of incentive compatibility constraints that look at all single-player deviations). In this program, $T = (T_1, T_2)$ denotes the unit tangent vector obtained by rotating N in the clockwise direction, $\beta \in \mathbb{R}^k$ denotes the *linear* impact of the Brownian signal x on incentives, and $d(y) = (d_1(y), d_2(y))$ refers to the jump in the players' continuation payoffs following a *single* arrival of a Poisson signal of type y. Thus $\sum_{y \in Y} d(y) \cdot N \lambda(y|a)$ represents the value burning necessary to enforce the profile a. Note that value burning is defined relative to the direction N in which program (2) maximizes expected payoffs and does not necessarily mean a reduction in expected continuation payoffs for both players. Using a continuous-time limit game (as $\Delta \to 0$), we provide heuristic justification for these constraints in Section 3.1.

With the bounds $D(N)$ on the weighted sums of players payoffs for all N, the set M is defined as an intersection of half-spaces

$$M = \bigcap_N H(N), \quad \text{where} \quad H(N) = \{v \cdot N \le D(N)\}.$$

Figure 2 shows how the maximal half-space $H(N)$ in the direction N is generated using program (2). The idea is to minimize the expected "value burning" in equilibrium, that is, how much below the boundary of this half-space the players have to move to provide incentives.

REMARK: Mechanically, our construction of M is similar to that of Fudenberg and Levine (1994) (hereafter FL),[16] who found the set of payoffs that

[15]Section 6 uses the same restrictions as the definition of the set M with the exception of linearity with respect to Brownian signals.

[16]See also Kandori and Matsushima (1998).

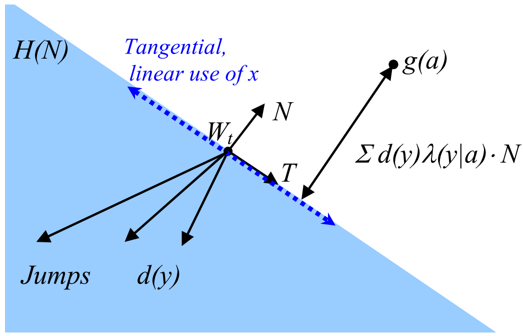

FIGURE 2.—Vectors N, T and a half-space $H(N)$ generated by destroying value $\sum_{y\in Y} d(y)\lambda(y|a)\cdot N$.

patient players can attain when the folk theorem fails. The difference is that we place restrictions on the use of information to characterize attainable payoffs as $\Delta \to 0$, rather than as $r \to 0$ analyzed by FL. If we carried out FL's construction directly without restrictions on the use of information, we would typically find that the entire set V^* of all feasible and individually rational payoffs is attainable in equilibrium as $r \to 0$. This difference implies that for many games,

$$\lim_{r\to 0}\lim_{\Delta\to 0} V(\Delta, r) \subsetneq \lim_{\Delta\to 0}\lim_{r\to 0} V(\Delta, r).$$

It is the case in Example 1 in Section 4: in this case, point $(2, 2)$ can be achieved for any Δ in the limit as $r \to 0$ by burning value conditional on arrival of extreme realizations of x, but it cannot be achieved for *any* r in the limit as $\Delta \to 0$.

3.1. *Heuristic Interpretation of* $D(N)$

Heuristically, in continuous time when the current pair of promised utilities W_t is on the boundary of the equilibrium value set, the motion of continuation values is locally described by

$$(3) \qquad dW_t = r(W_t - g(a))\,dt + r\beta(dX_t - \mu(a)\,dt)T + r\sum_{y\in Y} d(y)(dJ_{y,t} - \lambda(y|a)\,dt),$$

where a is the current action profile, T is the direction tangent to the boundary at W_t, β is a vector controlling the sensitivity of the tangential motion of payoffs to the Brownian component of the signal, $dJ_{y,t}$ is a counting process equal to 1 whenever Poisson shock y arrives, and $d(y)$ is the jump in continuation values that occurs when Poisson shock y arrives (such that $W_t + d(y)$ lies in the equilibrium payoff set).

The term $r(W_t - g(a))\,dt$ stands for promise keeping: the expected change in continuation payoffs is equal to the difference in promised and expected realized payoffs. The expected value of all the other elements, which represent provision of incentives via continuation values, is equal to zero. The expression $r\beta(dX_t - \mu(a)\,dt)T$ represents incentives provided by monitoring the Brownian component of the signal and moving continuation payoffs only along the tangent to the boundary of the equilibrium value set with sensitivity β. It disallows the normal component of the motion of W_t to depend on the Brownian signal to prevent continuation values from escaping the equilibrium value set. The expression

$$r\sum_{y\in Y} d(y)(dJ_{y,t} - \lambda(y|a)\,dt)$$

represents incentives provided by monitoring the Poisson process.

Expression (3) confirms our claims about the use of information directly in continuous time and explains the constraints (IC). In (3), the Brownian component of the signal is used to provide incentives *only* through *tangential* transfers that are *linear* in dX_t.[17] This is the only option in continuous time, but the result of this paper is that it is close to optimal in discrete time as $\Delta \to 0$ as well, because all other ways of using Brownian information have a negligible effect on incentives as $\Delta \to 0$. At the same time, the Poisson component can be used to provide incentives both via transfers and via value burning. Finally, the two signals are not used jointly.

With such restrictions on the use of information to provide incentives, it is easy to see that (IC) follow from (3). Indeed, the left hand side of (IC) represents the joint effect of a deviation on the current payoff and continuation value, where the effect on continuation value can be read from (3).

4. EXAMPLES

This section illustrates the construction of M on a simple partnership game. Two players choose effort $a_i = 0$ or 1, and the expected stage-game payoffs are given by

$$g_i(a_1, a_2) = 4a_1 + 4a_2 - a_1a_2 - 5a_i.$$

Each partner gets her share of the expected revenue $4a_1 + 4a_2 - a_1a_2$ but pays the cost of effort $5a_i$. The static Nash equilibrium of this game is $(0, 0)$, and

[17]Sannikov (2007) has shown that in a class of continuous-time games with a Brownian noise only, in optimal equilibria, continuation values move tangentially along the boundary of the equilibrium set.

the matrix of expected stage-game payoffs is

	0	1
0	0, 0	4, −1
1	−1, 4	2, 2

We will analyze three monitoring/production technologies with these expected payoffs:

EXAMPLE 1—Continuous Monitoring: The first technology has a flow cost of 2 and yields a stochastic stream of revenue $2\,dX_t$, where $dX_t = \mu(a_1, a_2)\,dt + dZ_t$ and $\mu(a_1, a_2) = 4a_1 + 4a_2 - a_1a_2 + 1$.

EXAMPLE 2—Discontinuous Monitoring With Good News: With the second technology, revenue arrives in amounts of 2 with a Poisson intensity $\lambda^G(a_1, a_2) = 4a_1 + 4a_2 - a_1a_2 + 1$ and there is also a fixed cost flow of 2.[18]

EXAMPLE 3—Discontinuous Monitoring With Bad News: The third technology brings a continuous revenue flow of 14.5, except for occasional sudden losses. These losses cost 2 each and arrive with a Poisson intensity of $\lambda^B(a_1, a_2) = 7.25 - (4a_1 + 4a_2 - a_1a_2)$.

The three technologies correspond to the following "business models":

Model 1. The partners manage a large number of small accounts; their efforts are the management of their salespeople. The revenues (net of costs) come from a large number of small customers with i.i.d. decisions.

Model 2. The partners run a business with a small number of large accounts and spend time jointly preparing proposals for clients. Since the partners have different areas of expertise, they are not able to value each other's input to the proposals. Thus the only way to judge their efforts is by clients' decisions.

Model 3. The partners manage a production technology with a long-term contract providing a steady stream of revenue, but occasional large repairs or customer complaints (caused by production mistakes) draw large one-time expenses.

Next, we find the set M for each monitoring technology.

Set 1. With continuous monitoring, instruments $(\beta^1, \beta^2) = (\beta T_1, \beta T_2)$ enforce action profiles in which each player maximizes

$$g_i(a_1, a_2) + \beta^i \mu(a_1, a_2) = (1 + \beta^i)(4a_1 + 4a_2 - a_1a_2) + \beta^i - 5a_i.$$

Therefore, player i chooses action 1 if $a_j = 0$ and $\beta^i \geq 1/4$ or if $a_j = 1$ and $\beta^i \geq 2/3$. Various pairs (β^1, β^2) give rise to stage-game payoffs illustrated in the left panel of Figure 3. From this figure we can read which payoff pairs are

[18]To be consistent with Assumption 2, in Examples 2 and 3 the players also observe a noninformative Brownian signal. However, this signal plays no role in the definition of M.

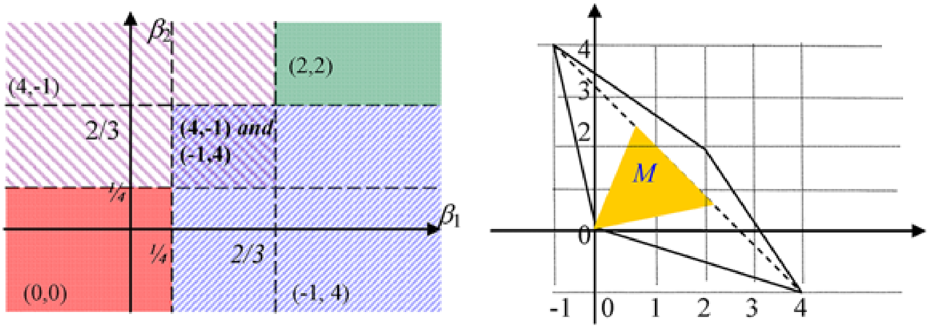

FIGURE 3.—Payoffs achievable by pairs (β^1, β^2) and M in Example 1.

enforceable on each tangent hyperplane. For example, all payoffs *except* for $(2, 2)$ are enforceable on the negative 45-degree tangent (since in the direction that maximizes the sum of payoffs $T_1 = -T_2$ so that $\beta^1 = -\beta^2$). Ironically, $(2, 2)$ is the most efficient payoff profile, so the maximal hyperplane in the negative 45-degree direction passes through points $(4, -1)$ and $(-1, 4)$. The right panel of Figure 3 shows the set M constructed with the help of the left panel.

Set 2. In this case, $\lambda^G(a_1, a_2)$ is the same as $\mu(a_1, a_2)$ from the previous example, and the mapping from instruments (d_1, d_2) to payoffs in Figure 4 looks exactly the same as in Figure 3. However, now it is possible to burn value when a Poisson signal arrives (so we do not need to have $(d_1, d_2) = (dT_1, dT_2)$). From Figure 4, we see that the set M becomes larger due to value burning to enforce payoffs $(4, -1)$ and $(-1, 4)$. We omit the detailed derivation of this set. Interestingly, even though good news signals are not useful for providing incentives in high-payoff strongly symmetric equilibria (as shown by AMP and Fudenberg and Levine (2007)), they can be useful in providing incentives in asymmetric equilibria (so the set M contains higher average payoffs than the best equilibrium in AMP).

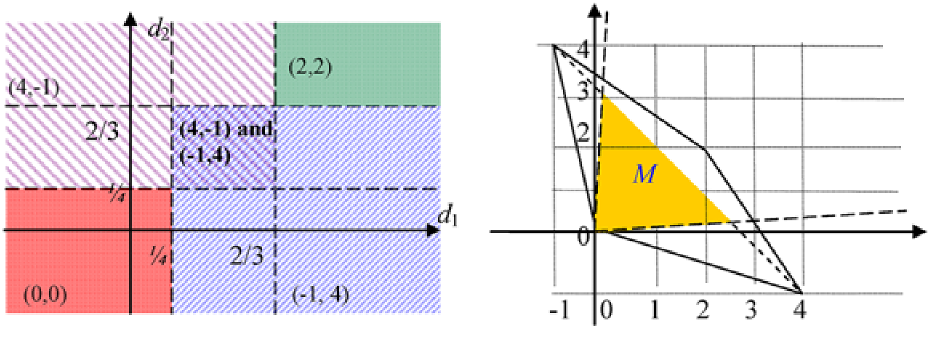

FIGURE 4.—Payoffs achievable by pairs (d^1, d^2) and M in Example 2.

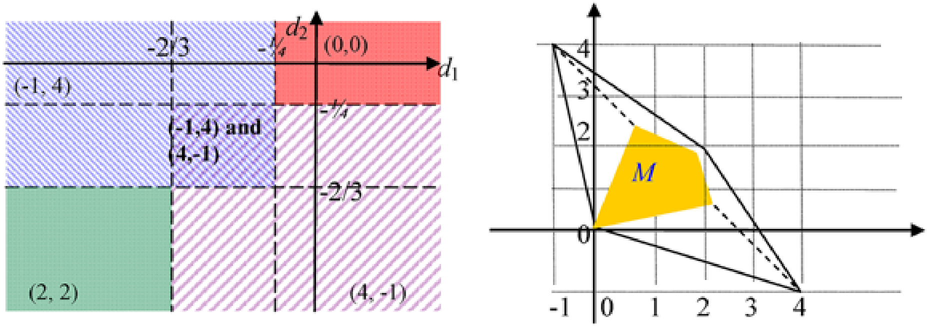

FIGURE 5.—Enforceable actions and M in Example 3.

Set 3. Figure 5 illustrates the construction of the set M for this case. In this case it is possible to enforce the payoff pair $(2, 2)$ by burning $2/3$ units of payoff for each player (i.e., $d = (-2/3, -2/3)$) when a bad news jump arrives (which happens with intensity 0.25).

5. PROVING THE BOUND IN GAMES WITH FREQUENT ACTIONS

In this section we prove the main theorem, which shows that the set M bounds payoffs attainable as $\Delta \to 0$ for any $r > 0$.

THEOREM 1: *For any $\varepsilon > 0$ and $r > 0$, there exists Δ^* such that for any $\Delta < \Delta^*$, there is no SEp achieving a payoff vector that is at distance at least ε from the set M.*

The rest of this section sketches the proof of Theorem 1 and the Appendices (both text and online) fill in the details. We start with the following definition:

DEFINITION 1: A payoff profile w is generated by the set W if there is a current-period action profile a and a map $\omega(x, (j_y))$ from signals to continuation-value transitions that satisfy the feasibility constraint $w + \omega(x, (j_y)) \in W$, the promise-keeping constraint

$$w = (1 - e^{-r\Delta})g(a) + e^{-r\Delta}E\big[w + \omega(x, (j_y))|a\big] \tag{4}$$

$$\Rightarrow \quad w = g(a) + \frac{e^{-r\Delta}}{1 - e^{-r\Delta}}E\big[\omega(x, (j_y))|a\big],$$

and the IC constraints

$$(g_i(a) - g_i(a')) + \frac{e^{-r\Delta}}{1 - e^{-r\Delta}}\big(E\big[\omega_i(x, (j_y))|a\big] - E\big[\omega_i(x, (j_y))|a'\big]\big) \geq 0 \tag{5}$$

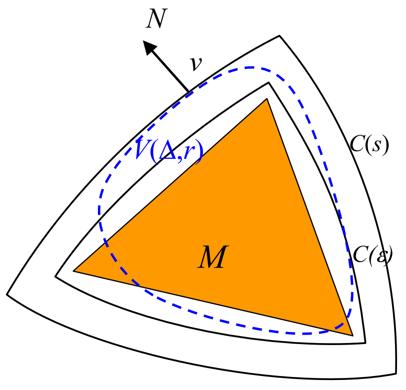

FIGURE 6.—The set M and the family of curves $C(s)$.

for any a' (such that $a'_j = a_j$ and $a'_i \in A_i$).[19]

The first important step of the proof is a construction of a family of continuously expanding closed convex curves $C(s)$, $s \in [\varepsilon, \bar{s}]$ containing M such that the following conditions hold:

(i) The distance between any point of $C(\varepsilon)$ and M is at most ε.

(ii) The set V is contained inside $C(\bar{s})$.

(iii) For sufficiently small $\Delta > 0$ and any $s \in [\varepsilon, \bar{s}]$, not a single point of $C(s)$ can be generated by the convex hull of $C(s)$.

If we have a family of curves around M that satisfies conditions (i), (ii), and (iii), it follows immediately that the set of SEp payoffs, $V(\Delta, r)$, lies inside $C(\varepsilon)$ for sufficiently small Δ. Otherwise, there are arbitrarily small Δ such that $V(\Delta, r)$ sticks outside the set $C(\varepsilon)$, as shown in Figure 6. If so, let us find the smallest set $C(s)$ that contains $V(\Delta, r)$ for a given Δ. Because sets $C(s)$ expand continuously, there exists an extreme point v of $V(\Delta, r)$ that touches the boundary of $C(s)$. Then, following APS90, since v is generated by $V(\Delta, r)$, it must also be generated by the convex hull of $C(s)$, leading to a contradiction.

The family of curves $C(s)$ required for our proof is constructed in Appendix A. However, instead of making sure that property (iii) holds directly, in Appendix A we ensure that no curve has a single point generated by the curve directly in continuous time, even if the constraints are relaxed by ε'. The family of curves $C(s)$ constructed in Appendix A satisfies properties (i), (ii), (iii′), and a technical uniform curvature condition (iii″):

(iii′) There exists $\varepsilon' > 0$ such that no point on any curve $C(s)$ is ε'-generated using continuous-time instruments by the curve $C(s)$.

[19]Throughout the paper we write continuation payoffs as $w + \omega(x, (j_y))$ where w is the equilibrium expected payoff vector in the current period.

(iii″) There is some $\kappa > 0$ such that the curvature of any curve $C(s)$ at any point is at least κ.[20]

The following definition of ε'-generation (used in condition (iii′)), motivated by the continuous-time intuition of Section 3.1, works for our argument:

DEFINITION 2: A point w on a curve of curvature κ is ε'-generated using continuous-time instruments $\{a, \beta, d\}$ by the curve if $|d(y)| \leq \bar{V}$, $d(y) \cdot N \leq 0$,

$$(g(a) - w) \cdot N - \frac{r}{2}\kappa|\beta|^2 + \sum_y (d(y) \cdot N)\lambda(y|a) + \varepsilon' \geq 0, \tag{6}$$

and the IC constraints relaxed by ε' hold for any a' (such that $a'_j = a_j$ and $a'_i \in A_i$),

$$\begin{aligned} &g_i(a) - g_i(a') + \beta(\mu(a) - \mu(a'))T_i \\ &\quad + \sum_{y \in Y} d_i(y)(\lambda(y|a) - \lambda(y|a')) + \varepsilon' \geq 0, \end{aligned}$$

where N and T are the normal and tangential vectors to the curve at w.

In the definition, note that the term $r\kappa|\beta|^2/2$ in (6) takes into account value destruction induced by the tangential transfers of continuation values along the boundary of a curve of positive curvature κ. An analogous term arises due to Ito's lemma in continuous-time models.[21] Note that ε' relaxes both the IC constraints and the promise-keeping constraint.

To complete our argument, we need to present a continuity argument to show that a family of curves that satisfies property (iii′) must also satisfy property (iii) for all sufficiently small Δ. The bound on curvature in property (iii″) then ensures that the continuity argument works uniformly for all points on all curves.

The argument is by contradiction. Suppose that for arbitrarily small Δ, there exists a point w on one of the curves $C(s)$ that is generated by the convex hull of the curve using discrete-time instruments $\{a, \omega(x, (j_y))\}$. We would like to then show that w is ε'-generated by the curve using continuous-time instruments $\{a, \beta, d(y)\}$, with β and d defined by

$$\begin{aligned} \beta &= \frac{e^{-r\Delta}}{1 - e^{-r\Delta}} \int (\omega(x, 0) \cdot T) x f_a(x)\, dx \quad \text{and} \\ d(y) &= \frac{\Delta e^{-r\Delta}}{1 - e^{-r\Delta}} E_x[\omega(x, y)|a], \end{aligned}$$

[20]*Curvature* is defined as the rate, at which the tangential angle changes with arc length.

[21]See Sannikov (2007).

where 0 denotes the event in which no Poisson jump arrives, y is the event in which exactly one signal of type y arrives, and $f_a(x)$ is the density of the Brownian signal under action profile a.

Lemma B1 in Appendix B implies that

$$\begin{aligned}
&(g_i(a) - g_i(a')) + \frac{e^{-r\Delta}}{1 - e^{-r\Delta}}(E[\omega_i|a] - E[\omega_i|a'])\\
&\quad = (g_i(a) - g_i(a'))\\
&\qquad + \frac{e^{-r\Delta}}{1 - e^{-r\Delta}} T_i \int (\omega(x, 0) \cdot T) x (\mu(a) - \mu(a')) f_a(x)\, dx\\
&\qquad + \frac{e^{-r\Delta}}{1 - e^{-r\Delta}} \sum_y \Delta(\lambda(y|a) - \lambda(y|a')) E_x[\omega_i(x, y)|a] + O(\Delta^{0.4999})\\
&\quad = g_i(a) - g_i(a') + \beta(\mu(a) - \mu(a')) T_i\\
&\qquad + \sum_y d_i(y)(\lambda(y|a) - \lambda(y|a')) + O(\Delta^{0.4999})
\end{aligned}$$

and Lemma B2 implies that

$$(g(a) - w) \cdot N - \frac{r}{2}\kappa|\beta|^2 + \sum_y (d(y) \cdot N)\lambda(y|a) + O(\Delta) \geq 0 \tag{7}$$

with the terms $O(\Delta^\alpha)$ (with $\alpha > 0$) bounded in absolute value by $K\Delta^\alpha$ for some constant K, uniformly for all sufficiently small Δ, $s \in [\varepsilon, s^*]$, and $w \in C(s)$.

Thus, if there are arbitrarily small Δ for which the set $V(\Delta, r)$ sticks outside $C(\varepsilon)$, it follows that for all $\varepsilon' > 0$, there is a point v on one of the curves $C(s)$ that can be ε'-generated using continuous-time instruments by $C(s)$. This leads to a contradiction.[22]

5.1. *Converse of the Theorem*

Payoffs outside M cannot be achieved for small Δ. We now present a partial converse of this statement: for small Δ and r, one can attain any payoff inside the set M_-, defined below. Appendix O-D in the Supplemental Material shows that generically $M_- = M$ (and also provides a nongeneric example in which these two sets are different).[23]

[22]Assuming that a Nash equilibrium in pure strategies exists, we guaranteed that M is nonempty. However, for the case when M is empty, we could construct a family of curves $C(s)$, starting from an appropriately chosen *point* $C(\varepsilon) \in V$, to show that $V(\Delta, r)$ is empty as well for small Δ.

[23]By generically we mean here that for any *game structure* (the set of actions of each player, the set of possible Poisson jumps, and the number of dimensions of the Brownian signal), the

DEFINITION 3: Define the set $M(\varepsilon)$ analogously to M, but replacing the (IC) constraints in the program (2) by the tighter constraints

$$(\text{IC}\varepsilon) \qquad g_i(a) - g_i(a') + \beta(\mu(a) - \mu(a'))T_i + \sum_{y \in Y} d_i(y)(\lambda(y|a) - \lambda(y|a')) \geq \varepsilon,$$

that is, requiring the incentive constraints to be slack by ε. Let $M_- = \lim_{\varepsilon \downarrow 0} M(\varepsilon)$, that is, the limit of $M(\varepsilon)$ as we take ε to zero from below (tightening the ICε constraints less and less).

Clearly, $M(0) = M$ and $M(\varepsilon)$ is decreasing in the set-inclusion sense (for sufficiently large ε, $M(\varepsilon)$ is empty and for very negative ε, $M(\varepsilon)$ equals V).

We can now state formally our version of the converse of our main theorem:

THEOREM 2: *For any smooth convex set W in the interior of M_-, there exist r^* and Δ^* such that for all $r \leq r^*$ and $\Delta \leq \Delta^*$, any payoff profile in W is attainable in a SEp.*

The proof of Theorem 2 in Appendix O-C builds upon the methods of FLM. However, in contrast to FLM, we establish the result not only for any $r \leq r^*$, but also uniformly for all $\Delta \leq \Delta^*$. Moreover, we show that the set W can be generated while respecting the informational restrictions (a)–(d) from Section 3 more and more strictly as $\Delta \to 0$. We sketch the proof below.

In Proposition O-C1 in the Supplemental Material, we show that for any v on the boundary of W there exists a neighborhood of v of radius δ_v, a discount rate r_v, and period length Δ_v such that any extreme point of W in this neighborhood is generated by W for all discount rates and period lengths not exceeding r_v and Δ_v. These open neighborhoods form a cover of the boundary of W. Since the boundary is compact, any cover has a finite subcover, which implies that W is self-generating for sufficiently small r and Δ.

In the proof of Proposition O-C1, we reverse the steps from the proof of Theorem 1. We start with the continuous-time instruments (found in the construction of M) to build discrete-time instruments that generate the desired payoffs. In the process, to satisfy feasibility constraints, we might tighten the IC constraints by a term that converges to zero as $\Delta \to 0$, which explains why we need to use the set M_- instead of M. Since the discrete-time instruments are based on continuous-time ones, they satisfy informational restrictions (a)–(d) from Section 3 more and more precisely as $\Delta \to 0$.

statement is true everywhere except for a set of game parameters of measure 0. For a given game structure, there are finitely many game parameters that specify for each action profile payoff to each player, the mean of the Brownian signal and the intensity of each possible Poisson jump.

6. IMPATIENT PLAYERS, $r > 0$

We now illustrate that our results regarding the effective uses of information in games with frequent actions are relevant for any $r > 0$, not just in the limit as $r \to 0$. We show that if we restrict continuation payoffs to (a) provide incentives with Brownian signals only tangentially, (b) use Brownian and Poisson signals separately, and (c) not condition on multiple arrivals of Poisson signals, then the set of attainable payoffs cannot collapse beyond the set of ε-strict equilibria for all sufficiently small Δ. We comment on restriction (d), linearity, at the end of this section.

Let us define ε-strict equilibria via an operator $B(W, \varepsilon)$, which gives the set of value pairs generated by a convex closed set W, with the incentive constraints tightened by ε.

DEFINITION 4: A value pair $v \in \mathbb{R}^2$ is ε-generated by W if

$$v = (1 - e^{-r\Delta})g(a) + e^{-r\Delta}E[w|a],$$

where the action profile, a, and a function from signals to continuation values, w, satisfy the following conditions:

(i) *Feasibility*, that is, w takes values in W.

(ii) *IC constraints*, that is, $(g_i(a) - g_i(a')) + \frac{e^{-r\Delta}}{1-e^{-r\Delta}}(E[w_i|a] - E[w_i|a']) \geq \varepsilon$ for all deviations a' of players $i = 1, 2$.

Let $B(W, \varepsilon)$ be the convex hull of values $v \in \mathbb{R}^2$ that are ε-generated by W.

For $\varepsilon = 0$, $B(W, \varepsilon)$ is the standard set operator from APS90, whose largest fixed point is the set of all SEp payoffs $V(\Delta, r)$. Denote the largest fixed point of the operator $B(W, \varepsilon)$ by $V(\Delta, r, \varepsilon)$. By an argument analogous to Theorem 4 from APS90, $V(\Delta, r, \varepsilon)$ is compact. The set of *ε-strict* equilibrium payoffs $V(\Delta, r, \varepsilon)$ is weakly decreasing in ε (in the set inclusion sense). As a function of ε, $V(\Delta, r, \varepsilon)$ may have only countably many discontinuities, and we conjecture that for generic parameters of the stage game and generic r, it is continuous in ε at $\varepsilon = 0$.

The next definition formalizes our restrictions on the use of information.

DEFINITION 5: A maximal value pair $v \in \mathbb{R}^2$ restricted-generated by W *in the direction* N solves

$$\max_{a, w(x,y)} v \cdot N$$

subject to four constraints:

(i) *Promise keeping*, $v = (1 - e^{-r\Delta})g(a) + e^{-r\Delta}E[w|a]$.

(ii) *Feasibility*, that is, w takes values in W.

(iii) *Continuation values* satisfy informational restrictions (b) and (c),

$$w(x, y) = v + 1_0 \omega(x, 0) + \sum_{y \in Y} 1_y d(y).$$

(iv) *IC constraints* that respect restriction (a),

$$\begin{aligned} & g_i(a) - g_i(a') \\ & \quad + \frac{e^{-r\Delta}}{1 - e^{-r\Delta}} \biggl(E\biggl[T_i(\omega(x, 0) \cdot T) + \sum_{y \in Y} 1_y d_i(y) \Big| a \biggr] \\ & \quad - E\biggl[T_i(\omega(x, 0) \cdot T) + \sum_{y \in Y} 1_y d_i(y) \Big| a' \biggr] \biggr) \\ & \quad + \frac{e^{-r\Delta}}{1 - e^{-r\Delta}} \bigl(E\bigl[N_i(\omega(x, 0) \cdot N) | a \bigr] - E\bigl[N_i(\omega(x, 0) \cdot N) | a' \bigr] \bigr)_- \\ & \geq 0, \end{aligned}$$

where $(z)_-$ equals 0 if z is positive and z otherwise.[24]

Let $B_R(W)$ be the convex hull of values $v \in \mathbb{R}^2$; such that v is the maximal pair *restricted-generated* by W in some direction N.

Definition 5 imposes restrictions (a)–(c) on the use of information to provide incentives. We can show that the restrictions have a small effect on the provision of incentives and, if $V(\Delta, r, \varepsilon)$ is continuous at $\varepsilon = 0$, on the set of attainable payoffs:

THEOREM 3: *For every $\varepsilon > 0$ there exists $\Delta^* > 0$ such that for all $\Delta \leq \Delta^*$, $V(\Delta, r, \varepsilon) \subseteq B_R(V(\Delta, r, \varepsilon))$.*

In words, SEp with the provision of incentives restricted as in Definition 5 can attain a set of payoffs at least as large as the set of ε-strict SEp payoffs without any restrictions. To see that this follows from the theorem, note that standard APS90 arguments imply that if a set is restricted-self-generated (i.e., if $W \subseteq B_R(W)$), then all points in this set can be supported by SEp in which incentives are provided in the restricted ways. Therefore, $V(\Delta, r, \varepsilon) \subseteq B_R(V(\Delta, r, \varepsilon))$ implies that the set of SEp with restricted incentive provision is at least as large as $V(\Delta, r, \varepsilon)$.

PROOF OF THEOREM 3: Consider $(a, w = v + \omega)$ that maximize $v \cdot N$ subject to constraints (i) and (ii) from Definition 4, and let us, first, show that

[24]The notation $(\cdot)_-$ is used in these IC constraints to ignore the positive impact of the normal component of $\omega(x, 0)$ on these constraints.

the maximal value *restricted-generated* by $V(\Delta, r, \varepsilon)$ in the direction N improves upon v. Let $d(y) = E_x[\omega_i(x, y)|a]$ and consider the pair $(a, w_R) = (a, v + 1_0\omega(x, 0) + \sum_{y\in Y} 1_y d(y))$. Then, by an argument analogous to that in Lemma B1,

$$\begin{aligned}
&E[\omega_i|a] - E[\omega_i|a'] \\
&\quad = E\left[T_i(\omega(x,0)\cdot T) + \sum_{y\in Y} 1_y d_i(y)\big|a\right] \\
&\qquad - E\left[T_i(\omega(x,0)\cdot T) + \sum_{y\in Y} 1_y d_i(y)\big|a'\right] + O(\Delta^{1.4999}).
\end{aligned}$$

Thus (a, w_R) satisfies the IC constraint (iv) in Definition 5 when Δ is sufficiently small. Also, $v + d(y) \in V(\Delta, r, \varepsilon)$, so (a, w_R) satisfies the feasibility constraint (ii). Finally, relative to (a, w), (a, w_R) improves upon the objective function in Definition 5 by not destroying value following multiple arrivals of Poisson signals.

Now we are ready to argue that $B_R(V(\Delta, r, \varepsilon))$ contains $V(\Delta, r, \varepsilon)$. If not, then there is $v' \in V(\Delta, r, \varepsilon)$ that is not in $B_R(V(\Delta, r, \varepsilon))$. Since $B_R(V(\Delta, r, \varepsilon))$ is convex and closed, the separating hyperplane theorem implies that there is a hyperplane $\{w : (w - v')\cdot N = 0\}$ such that $(w - v')\cdot N < 0$ for all $w \in B_R(V(\Delta, r, \varepsilon))$. Then the maximal point v of $V(\Delta, r, \varepsilon)$ in the direction N cannot be improved upon by any point in $B_R(V(\Delta, r, \varepsilon))$, since $(v - v')\cdot N \geq 0$, a contradiction. *Q.E.D.*

REMARK: While Definition 5 does not incorporate the restriction on the linear use of Brownian signals, we conjecture that this restriction would not harm the set of attainable payoffs by a significant amount when Δ is small. To see the intuition, imagine a set W that ε-generates itself (that is, $W \subseteq B(W, \varepsilon)$) and that is smooth at all points on the boundary generated with the use of Brownian information (i.e., points where the Brownian information has an impact on the IC constraints).[25] Consider one such point v with a normal vector N. What is the optimal way to use Brownian information to maximize a vector of payoffs in the direction N? If $\kappa > 0$ is the curvature near v, and with the expectation that transfers $\omega(x, 0)$ are small (on the order of $\sqrt{\Delta}$ to provide incentives on the order of Δ), we are trying to minimize value destroyed

$$\int \frac{\kappa}{2}(\omega(x,0)\cdot T)^2 f_a(x)\,dx \tag{8}$$

[25]It is natural to allow W to have kinks at a Nash equilibrium payoff points, for example, as in the continuous-time games of Sannikov (2007).

subject to providing a given level of incentives $D(a')\Delta$ against each deviation a', that is,

$$\int (\omega(x,0)\cdot T)T_i(f_a(x)-f_{a'}(x))\,dx \geq D(a')\Delta.$$

Letting $\rho(a')$ be the Lagrange multiplier on the incentive constraint with respect to deviation a', the first-order condition is

$$\kappa\omega(x,0)\cdot T+\sum_{a'}\rho(a')T_i\left(1-\frac{f_{a'}(x)}{f_a(x)}\right)=0.$$

Thus, for approximation (8), optimal transfers are linear in the likelihood ratio. Moreover, likelihood ratios themselves are approximately linear in x in the range where x falls with probability close to 1, as one can see from the Taylor expansion

$$\begin{aligned}\frac{f_{a'}(x)}{f_a(x)} &= \exp\left(\left(x-\Delta\frac{\mu(a')+\mu(a)}{2}\right)(\mu(a')-\mu(a))\right)\\ &= 1+\left(x-\Delta\frac{\mu(a')+\mu(a)}{2}\right)(\mu(a')-\mu(a))\\ &\quad+\frac{1}{2}\left(x-\Delta\frac{\mu(a')+\mu(a)}{2}\right)^2(\mu(a')-\mu(a))^2+\cdots.\end{aligned}$$

The main difficulty in transforming this intuition into a general result for impatient players is the construction of such a set W that is close to $V(\Delta,r,\varepsilon)$. One idea is to construct W using a continuous-time game, since equilibrium payoff sets in continuous time tend to have smoothness properties whenever Brownian information is used to provide incentives (see Sannikov (2007)), and to show that W approximates $V(\Delta,r,\varepsilon)$ for small Δ. Since such an argument falls beyond the scope of the paper, we leave the linearity conjecture to future research.

7. CONCLUDING REMARKS

Identifiability

FLM have provided identifiability conditions on the noise structure of the stage game that are sufficient for establishing a folk theorem. In our setup we can provide analogous sufficient conditions on the Lévy process for the set M to coincide with the set V^*.

We say that an action profile a is enforceable on the hyperplane N if $D(a,N)=0$. Following the logic from FLM, we have $M=V^*$ if the following two conditions hold:

(i) All action profiles are enforceable on all regular hyperplanes, with $N_1, N_2 \neq 0$.

(ii) All action profiles with a best-response property for player i, including the profile that maximizes player i's payoff and the profile that min-maxes player i, are enforceable on the coordinate hyperplane with $T_i = 0$.

Let us derive sufficient conditions on signal structure for (i) and (ii) to hold. Denote by $G_i(a) \in \mathbb{R}^{|A_i|-1}$ the *gain* vector with entries $g_i(a_i', a_j) - g_i(a)$ for all deviations $a_i' \neq a_i$ of player i. Denote by $\Pi_i(a)$ the $(|A_i|-1) \times (k+|Y|)$ matrix with rows

$$(\mu(a_i', a_j) - \mu(a), \lambda(y|a_i', a_j) - \lambda(y|a), y \in Y)$$

for all deviations $a_i' \neq a_i$ of player i. Then action pair a is enforceable on the hyperplane parallel to T if and only if there are vectors $\beta \in \mathbb{R}^k$ and $d_T \in \mathbb{R}^{|Y|}$ that, together with $d_N = 0$, satisfy the conditions (IC) from (2), that is,

$$\begin{bmatrix} G_1(a) \\ G_2(a) \end{bmatrix} \leq \begin{bmatrix} T_1\Pi_1(a) \\ T_2\Pi_2(a) \end{bmatrix} \begin{pmatrix} \beta \\ d_T \end{pmatrix}. \tag{9}$$

PROPOSITION 1: *The following two conditions are sufficient for* (i) *and* (ii) *to hold*, *and thus for M to coincide with V^**:

Pairwise Identifiability: *The row-spaces of the matrices $\Pi_1(a)$ and $\Pi_2(a)$ intersect only at the origin*.

Individual Full Rank: *There is no linear dependence among the rows of the matrix $\Pi_i(a)$*.

PROOF: For the regular hyperplanes, (9) can always be solved for β and d_T with equality if there is no linear dependence among the rows of the matrix

$$\begin{bmatrix} T_1\Pi_1(a) \\ T_2\Pi_2(a) \end{bmatrix},$$

which is equivalent to having pairwise identifiability and individual full rank. Thus, (i) holds under these conditions.

Moreover, (ii) also holds. Indeed, consider a profile a with a best-response property for player i. Then individual full rank implies that equation

$$G_j(a) = T_j\Pi_j(a) \begin{pmatrix} \beta \\ d_T \end{pmatrix}$$

has a solution, and the best-response property of player i implies that

$$G_i(a) \leq 0 = T_i\Pi_i(a) \begin{pmatrix} \beta \\ d_T \end{pmatrix}.$$

This completes the proof. *Q.E.D.*

More Than Two Players

While a formal analysis of games with $n > 2$ players would be more complicated, let us argue informally that our methods and results extend to that setting as well. First, to define the set M, for a direction N in the n-dimensional payoff space, let

$$D(N) = \max_{a,B,d(y)} \left(g(a) + \sum_{y \in Y} d(y)\lambda(y|a) \right) \cdot N$$

$$\text{s.t.} \quad d(y) \cdot N \le 0, \quad B \cdot N = 0 \quad \text{and}$$

$$g_i(a) - g_i(a') + e_i B(\mu(a) - \mu(a')) + \sum_{y \in Y} d_i(y)(\lambda(y|a) - \lambda(y|a')) \ge 0$$

for all deviations $a' = (a_i', a_{-i})$ of each player $i = 1, \dots, n$, where $d(y) \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times k}$, and e_i represents the ith coordinate vector (with ith coordinate 1 and the rest 0). The maximal half-space in the direction N and the set M are defined in the same way as before:

$$H(N) = \{v \cdot N \le D(N)\} \quad \text{and} \quad M = \bigcap_N H(N).$$

With this generalized definition of M, our main theorem (see Section 5) holds for $n > 2$ players. To prove this result formally, we could construct a family of continuously expanding convex *surfaces* $C(s)$, $s \in [\varepsilon, s^*]$, around M, rather than curves. Surface $C(\varepsilon)$ is constructed by starting from an approximation of M as an intersection of finitely many half-spaces $H(N)$ (see Lemma O-A in the Supplemental Material), and by drawing a sphere of sufficiently large radius near each half-space (see Lemma A1 in Appendix A). Analogously to our proof for two players, this family of surfaces can be used to show that points outside M cannot be attained in equilibrium for sufficiently small Δ given any discount rate r.

Do Small Modeling Differences Matter?

Given our result about the dichotomy between continuous and discontinuous information, one may wonder what happens when Poisson signals arrive more frequently, and when they carry little information individually and, in the right limit, information approaches Brownian motion. Is there a discontinuity between Poisson and Brownian information in this limit? Does the choice of how to model information that arrives frequently but in small pieces lead to large differences in results?

Our intuition is that typically it does not: that Poisson jumps that arrive frequently and individually carry little information have similar properties to Brownian information when incentives are concerned, even if Δ is so small that multiple jumps per period are extremely rare. Suppose that $|\lambda(y|a) - \lambda(y|a')|$ is significantly smaller than $\lambda(y|a)$ (so that the right panel in Figure 1 would look similar to the left panel, which is the measure of how informative are individual arrivals). Then first, burning value upon an arrival of signal y destroys a lot of value but contributes very little to incentives. As a result, analogously to the case of continuous information, burning value conditional on this information is an ineffective way of providing incentives. Second, regarding linearity, to provide some very informal intuition, assume that the set of equilibrium payoffs has a well defined and continuous curvature. Suppose an arrival of jump y optimally triggers a tangential transition of continuation values from v to $v+\omega$. Because signal y carries little information, the transition ω should be small to keep down the cost of the tangential transfers. When v and $v+\omega$ are close and the curvature of the set of equilibrium payoffs is continuous, the optimization problems at these two points should be similar. If a second jump y arrives a few periods later, it should be followed by a similar transition (approximately to $v + 2\omega$). Thus, the cumulative transition becomes approximately linear in the number of Poisson arrivals.

Therefore, even if all events in the world are discrete, and even if Δ is so small that only one event can possibly occur per period, our intuition is that typically using Brownian motion to model signals that individually contain very little information may not have any major impact on results.

Final Remark

Repeated games are a useful abstraction—a system of simplifying assumptions that allows us to gain intuition about more complicated dynamic systems. One of these assumptions is the idea of a period—a friction that does not have a real counterpart in many applications and that one can question whether the simplifying assumptions of repeated games are adequate to study dynamic interactions.[26] In this paper we attempted to uncover fundamental principles of how incentives can be provided in repeated interactions that are robust to the assumption of fixed periods by allowing the players to act frequently. As in many other areas of economic theory (for example, bargaining, asset pricing), looking at the outcomes of the games as frictions disappear (i.e., as $\Delta \to 0$) has proven fruitful in developing new results.

[26]Disturbing examples appear in Abreu, Milgrom, and Pearce (1991) and Sannikov and Skrzypacz (2007), who showed that the scope of cooperation can change drastically when players are allowed to move frequently. Also, see Fudenberg and Olszewski (2009) for interesting new results about games, in which different players observe signals at different random time points.

The intuitions of our paper can be applied to any area concerned with a dynamic incentive provision. In particular, we envision applications for accounting (in the area of information release and incentives), finance (in the area of dynamic contracts), and industrial organization (in the study of dynamic collusion). Although we work with a repeated game model, the results can be directly applied to analysis of optimal self-enforcing contracts. Furthermore, the contributions can be translated to more complicated nonstationary environments with public state variables.

APPENDIX A: CONSTRUCTION OF THE FAMILY OF CURVES $C(s)$ CONTAINING M

In this appendix, for a given M and $\varepsilon > 0$, we construct a family of strictly convex curves $C(s)$, $s \in [\varepsilon, \bar{s}]$ around M, which expand continuously as s increases from ε to $\bar{s}$. Each curve is a union of a finite number of arcs that satisfies the following properties:

(i) M is contained in the inside of $C(\varepsilon)$, and the distance between any point on the curve $C(\varepsilon)$ and M is at most ε.

(ii) The set V is contained inside the curve $C(\bar{s})$.

(iii′) There exist $\varepsilon' > 0$ such that for any $s \in [\varepsilon, \bar{s}]$ and any point v on $C(s)$ with a normal N, v is not ε'-generated using continuous-time instruments by the corresponding arc.

Because the number of arcs is finite, we have another property:

(iii″) There exists $\kappa > 0$ such that the curvature of any curve $C(s)$ at any point is at least κ.

Recall that by definition of ε'-generation (Definition 2), property (iii′) means that there are no instruments $\{a, \beta, d\}$ that satisfy the conditions

$$(g(a) - v) \cdot N - \frac{r}{2}\kappa(v)|\beta|^2 + \sum_y (d(y) \cdot N)\lambda(y|a) + \varepsilon' \geq 0,$$

$$|d(y)| \leq \bar{V}, \quad d(y) \cdot N \leq 0$$

and

$$g_i(a) - g_i(a') + \beta(\mu(a) - \mu(a'))T_i + \sum_{y \in Y} d_i(y)(\lambda(y|a) - \lambda(y|a')) + \varepsilon' \geq 0.$$

For short, when condition (iii′) holds for v on an arc, then we say that this point is *unattainable*.

We start with a polygonal approximation of the set M as an intersection supporting half-spaces

$$M \subseteq \bigcap_{k=1}^{K} H(\tilde{N}_k)$$

with the property that the distance from any point of the polygon to M is at most $\varepsilon/2$. Such an approximation exists by Lemma O-A in the Supplemental Material.

The following lemma allows us to draw a circular arc outside each supporting half-space $H(\tilde{N}_k)$, such that for some $\varepsilon' = \varepsilon'_k > 0$, every point on the intersection of the circular arc with the set V is unattainable.

LEMMA A1: *Consider a supporting half-space $H(N)$ of the set M at point $v \in M$. Consider circles of various radii that pass through point $w = v + N\varepsilon/2$ tangentially to T, as illustrated in Figure 7. Then there is a sufficiently large radius ρ (thus, a sufficiently small curvature $\kappa = 1/\rho$) and a sufficiently small value of $\varepsilon' > 0$, such that not a single point of the arc of the circle with radius ρ inside the set V^* can be ε'-generated by the arc.*

PROOF: Take a decreasing sequence of positive numbers $\varepsilon_n \to 0$. If the lemma is false, then we can choose a sequence of radii $\rho_n \to \infty$ for which the arc R_n has an ε_n-generated point w_n by this arc (using instruments $\{a_n, \beta_n, d_n(y)\}$). Without loss of generality, we can assume that a stays fixed along the sequence (because we can always choose an appropriate subsequence).

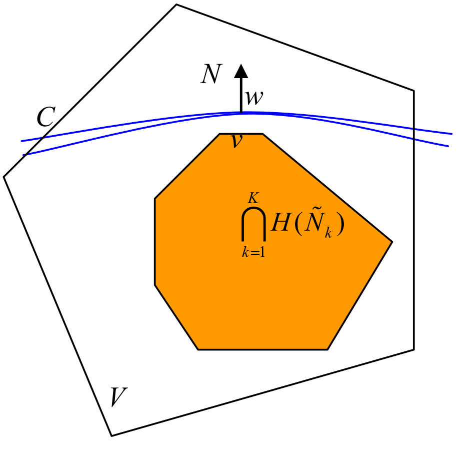

FIGURE 7.—Circular arcs of various radii near a supporting half-space $H(N)$.

Let us show that for the sequence $\{\beta_n, d_n(y)\}$, the *linear* inequalities

$$(g(a) - w) \cdot N + \sum_y (d_n(y) \cdot N)\lambda(y|a) \geq 0, \quad d_n(y) \cdot N \leq 0,$$

and

$$g_i(a) - g_i(a') + \beta_n(\mu(a) - \mu(a'))T_i + \sum_{y \in Y} d_{n,i}(y)(\lambda(y|a) - \lambda(y|a')) \geq 0$$

are satisfied arbitrarily closely as $n \to 0$. Then Lemma A2 below implies that there exists a pair (d, β) for which these inequalities hold exactly, which contradicts $H(N)$ being a supporting hyperplane of M (since $(g(a) - w) \cdot N < (g(a) - v) \cdot N$).

Note that for sufficiently large ρ_n, we have

$$|N - N_n| \leq 2\kappa_n \bar{V},$$

where N_n is the normal vector to point w_n of arc R_n. Also, since

$$(g(a) - w_n) \cdot N_n - \frac{r}{2}\kappa_n|\beta_n|^2 + \sum_y (d_n(y) \cdot N_n)\lambda(y|a) + \varepsilon_n' \geq 0 \quad \text{and}$$

$$d_n(y) \cdot N_n \leq 0,$$

it follows that

$$\frac{r}{2}\kappa_n|\beta_n|^2 \leq (g(a) - w_n) \cdot N_n + \varepsilon_n' \leq 2\bar{V}.$$

Therefore,

$$\begin{aligned}
&g_i(a) - g_i(a') + \beta_n(\mu(a) - \mu(a'))T_i + \sum_{y \in Y} d_{n,i}(y)(\lambda(y|a) - \lambda(y|a')) \\
&\quad \geq g_i(a) - g_i(a') + \beta_n(\mu(a) - \mu(a'))T_{n,i} \\
&\qquad + \sum_{y \in Y} d_{n,i}(y)(\lambda(y|a) - \lambda(y|a')) - |\beta_n||\mu(a) - \mu(a')||T - T_n| \\
&\quad \geq -\varepsilon_n' - \sqrt{\frac{4\bar{V}}{r\kappa_n}}|\mu(a) - \mu(a')|2\bar{V}\kappa_n \\
&\quad \to 0, \\
&d_n(y) \cdot N \leq d_n(y) \cdot N_n + |d_n(y)||N - N_n| \leq 2\bar{V}^2\kappa_n \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\underbrace{(g(a)-w)\cdot N}_{\geq (g(a)-w_n)\cdot N} + \sum_y (d_n(y)\cdot N)\lambda(y|a) \\
&\quad \geq (g(a)-w_n)\cdot N_n + \sum_y (d_n(y)\cdot N_n)\lambda(y|a) \\
&\qquad - \bar{V}|N-N_n| - \sum_y |d_n(y)||N-N_n|\lambda(y|a) \\
&\quad \geq -\varepsilon_n' - 2\bar{V}^2\kappa_n - 2\bar{V}^2\kappa_n \sum_y \lambda(y|a) \\
&\quad \to 0. \qquad\qquad \textit{Q.E.D.}
\end{aligned}$$

LEMMA A2: *Let Q be an $m \times n$ matrix and let $q \in \mathbb{R}^m$. Suppose that for all $\varepsilon \in \mathbb{R}^m$ such that $\varepsilon > 0$ there exists an $x_\varepsilon \in \mathbb{R}^n$ such that $Qx_\varepsilon \geq q - \varepsilon$. Then there is an $x^* \in \mathbb{R}^n$ such that $Qx^* \geq q$.*

PROOF[27]: The proof relies on Farkas' lemma: *There exists x such that $Qx \geq b$ if and only if for all $y \geq 0$ such that $y^T Q = 0$, we have $y^T b \leq 0$.*

Since for all $\varepsilon \in \mathbb{R}^m$ such that $\varepsilon > 0$ there exists an $x_\varepsilon \in \mathbb{R}^n$ such that $Qx_\varepsilon \geq q - \varepsilon$, Farkas' lemma implies that for all $y \geq 0$ such that $y^T Q = 0$, $y^T(q-\varepsilon) \leq 0$ for all $\varepsilon \geq 0$. Taking ε to 0, we find that for all $y \geq 0$ such that $y^T Q = 0$, $y^T q \leq 0$. Therefore, by Farkas' lemma again there exists $x^* \in \mathbb{R}^n$ such that $Qx^* \geq q$. *Q.E.D.*

Using Lemma A1, we construct circular arcs outside every face of the polygonal approximation such that every point of the intersection of any arc with the set V cannot be ε'-generated for $\varepsilon' = \min_k \varepsilon_k'$.

Denote by $C(\varepsilon)$ the union of these arcs. Note that the distance between $C(\varepsilon)$ and M is bounded from above by ε since the distance between the polygon $\bigcap_{k=1}^K H(\tilde{N}_k)$ and the set M is at most $\varepsilon/2$, and each arc is constructed through the point $w_k = v_k + \tilde{N}_k \varepsilon/2$ that is at distance $\varepsilon/2$ away from the polygon.

To construct the family of curves $C(s)$, $s \in [\varepsilon, \bar{s}]$ we translate the arcs out continuously until they bound the set V, as shown in Figure 8. Note that if we translate an arc out, every point of the arc remains unattainable (because translation does not change the curvature of the arc).

[27]We thank an anonymous referee for suggesting this beautiful short proof.

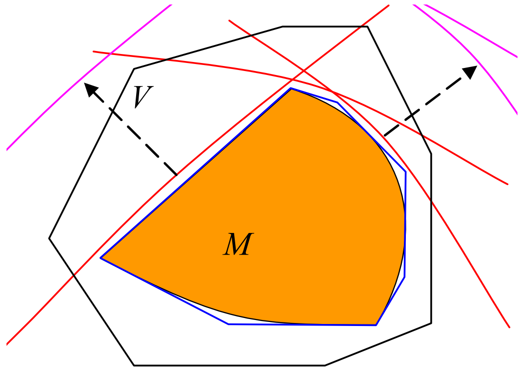

FIGURE 8.—Construction of circular arcs around supporting half-spaces of M.

APPENDIX B: LEMMAS

LEMMA B1: *If*

$$\left|\omega(x,(j_y))\right| \le \bar V, \quad |E[\omega|a]| \le \frac{1-e^{-r\Delta}}{e^{-r\Delta}}\bar V = O(\Delta), \quad \textit{and} \tag{10}$$

$$\omega \cdot N \le -\kappa|\omega \cdot T|^2/2$$

for some $\kappa > 0$, *then*

$$\begin{aligned} E[\omega_i|a] - E[\omega_i|a'] &= T_i \int (\omega(x,0)\cdot T)x \cdot (\mu(a)-\mu(a'))f_a(x)\,dx \\ &\quad + \Delta \sum_y (\lambda(y|a)-\lambda(y|a'))E_x[\omega_i(x,y)|a] + O(\Delta^{1.49999}), \end{aligned} \tag{11}$$

with the term $O(\Delta^{1.49999})$ *bounded in absolute value by* $K\Delta^{1.49999}$ *for some constant* K, *uniformly for all sufficiently small* Δ, a, *and* ω *that satisfy the above bounds.*[28]

PROOF: First,

$$\begin{aligned} &E[\omega|a] - E[\omega|a'] \\ &\quad = \int \omega(x,0)(f_a(x) - f_{a'}(x))\,dx \end{aligned}$$

[28]Expression (11) is the expected change in player i continuation payoffs when he considers a deviation to a_i'. Note that $\omega_i(x,(j_y)) = T_i(\omega(x,(j_y))\cdot T) + N_i(\omega(x,(j_y))\cdot N)$, where $(\omega(x,(j_y))\cdot T)$ is the tangential component of ω and $(\omega(x,(j_y))\cdot N)$ is the normal component.

$$
\begin{aligned}
&+\sum_y \big(\Pr[y|a]E_x[\omega(x,y)-\omega(x,0)|a] \\
&-\Pr[y|a']E_x[\omega(x,y)-\omega(x,0)|a']\big) \\
&+\underbrace{E\big[(\omega(x,(j_y))-\omega(x,0))1_{\sum j_y>1}|a\big]-E\big[(\omega(x,(j_y))-\omega(x,0))1_{\sum j_y>1}|a'\big]}_{O(\Delta^2)},
\end{aligned}
$$

because the probability of multiple jumps arriving is $O(\Delta^2)$.

Second, we have

$$
\begin{aligned}
&\int(\omega(x,0)\cdot T)(f_a(x)-f_{a'}(x))\,dx \\
&\quad=\int(\omega(x,0)\cdot T) \\
&\qquad\times\left(1-\exp\left(\left(\frac{\mu(a)+\mu(a')}{2}\Delta-x\right)(\mu(a)-\mu(a'))\right)\right)f_a(x)\,dx \\
&\quad=\int(\omega(x,0)\cdot T)x(\mu(a)-\mu(a'))f_a(x)\,dx \\
&\qquad+\int(\omega(x,0)\cdot T)\bigg(1-x(\mu(a)-\mu(a')) \\
&\qquad-\exp\left(\left(\frac{\mu(a)+\mu(a')}{2}\Delta-x\right)(\mu(a)-\mu(a'))\right)\bigg)f_a(x)\,dx.
\end{aligned}
$$

Now, using the inequality ($A^2+B^2\geq 2AB$) we bound the second term by

$$
\begin{aligned}
&2\bigg|\int(\omega(x,0)\cdot T)\bigg(1-x(\mu(a)-\mu(a')) \\
&\qquad-\exp\left(\left(\frac{\mu(a)+\mu(a')}{2}\Delta-x\right)(\mu(a)-\mu(a'))\right)\bigg)f_a(x)\,dx\bigg| \\
&\quad\leq\Delta^{1/2}\int(\omega(x,0)\cdot T)^2 f_a(x)\,dx \\
&\qquad+\Delta^{-1/2}\int\underbrace{\left(\exp\left(\left(\frac{\mu(a)+\mu(a')}{2}\Delta-x\right)(\mu(a)-\mu(a'))\right)-1+x(\mu(a)-\mu(a'))\right)^2}_{(\frac{\mu(a)+\mu(a')}{2}\Delta(\mu(a)-\mu(a'))+\frac{1}{2}(\frac{\mu(a)+\mu(a')}{2}\Delta-x)^2(\mu(a)-\mu(a'))^2+\cdots)^2} \\
&\qquad\times f_a(x)\,dx \\
&\quad=O(\Delta^{3/2}),
\end{aligned}
$$

where the last equality follows since

$$
\int(\omega(x,0)\cdot T)^2 f_a(x)\,dx\leq-\frac{1}{\kappa}\int(\omega(x,0)\cdot N)f_a(x)\,dx=O(\Delta) \tag{12}
$$

and the Taylor expansion of the second integrand

$$\left(\frac{\mu(a)+\mu(a')}{2}\Delta(\mu(a)-\mu(a')) + \frac{1}{2}\left(\frac{\mu(a)+\mu(a')}{2}\Delta - x\right)^2(\mu(a)-\mu(a'))^2 + \cdots\right)^2$$

delivers terms of orders Δ^2, Δx^2, $x^4, \ldots$, whose expectation under the density f_a of $N(\Delta\mu(a), \Delta I)$ is at least Δ^2.

This step shows that the nonlinear elements of $(\omega(x,0)\cdot T)$ are not important for provision of incentives. The intuition lies behind the condition $\kappa|\omega\cdot T|^2/2 \le |\omega\cdot N|$, which holds due to the curvature of $C(s)$. Because

$$|E[\omega\cdot N|a]| \le \frac{1-e^{-r\Delta}}{e^{-r\Delta}}\bar{V} = O(\Delta),$$

$\omega\cdot T$ is limited to be small. But then only the linear term in the Taylor expansion of $f_{a'}(x)/f_a(x)$ contributes significantly to incentives as Δ gets small.

Third, generalizing an argument from Sannikov and Skrzypacz (2007), Appendix O-B in the Supplemental Material shows that

$$\int(\omega(x,0)\cdot N)(f_a(x)-f_{a'}(x))\,dx \le O(\Delta^{1.49999})$$

whenever

$$\omega(x,0)\cdot N \in [-\bar{V}, 0] \quad \text{and} \quad |E[\omega\cdot N|a]| \le O(\Delta).$$

The meaning is that incentive provision by triggering the destruction of value with Brownian signals is inefficient. The destruction of value of order Δ creates incentives weaker than the order of $\Delta^{1.49999}$ (the result follows from properties of the Normal distribution).

Fourth, we can decompose

$$\begin{aligned}(13)\qquad & \Pr[y|a]E_x[\omega(x,y)-\omega(x,0)|a] - \Pr[y|a']E_x[\omega(x,y)-\omega(x,0)|a'] \\ &= \underbrace{(\Pr[y|a]-\Pr[y|a'])E_x[\omega(x,y)|a]}_{A} \\ &\quad - \underbrace{(\Pr[y|a]-\Pr[y|a'])E_x[\omega(x,0)|a]}_{B} \\ &\quad + \underbrace{\Pr[y|a']\int(\omega(x,y)-\omega(x,0))(f_a(x)-f_{a'}(x))\,dx}_{C}.\end{aligned}$$

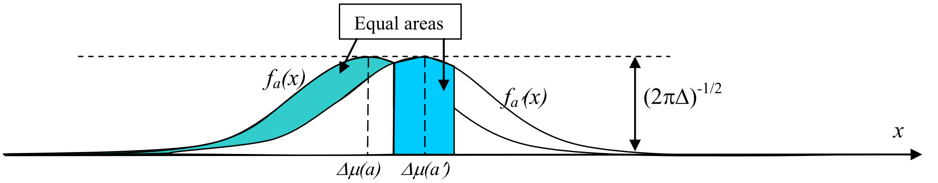

FIGURE 9.—Bound on the area between two density functions.[29]

We now bound the three terms on the right-hand side of (13).

Since $|\omega(x,y)-\omega(x,0)|\le\bar{V}$ and

(14) $$\int|f_a(x)-f_{a'}(x)|\,dx<2\Delta|\mu(a)-\mu(a')|\frac{1}{\sqrt{2\pi\Delta}}$$

(see Figure 9), it follows that

$$C=\Pr[y|a']\int(\omega(x,y)-\omega(x,0))(f_a(x)-f_{a'}(x))\,dx=O(\Delta^{3/2}).$$

For term B, $|E[\omega|a]|\le\frac{1-e^{-r\Delta}}{e^{-r\Delta}}\bar{V}$ implies that

$$\underbrace{E[\omega|a]}_{O(\Delta)}=\Pr[0|a]E_x[\omega(x,0)|a]+\underbrace{\sum_{j_y\neq 0}\Pr[(j_y)|a]E_x\big[\omega(x,(j_y))|a\big]}_{O(\Delta)}$$

$$\Rightarrow\quad E_x[\omega(x,0)|a]=O(\Delta).^{30}$$

So $B=O(\Delta^2)$. Finally, term A in (13) is

$$\Delta\sum_y(\lambda(y|a)-\lambda(y|a'))E[\omega(x,y)|a]+O(\Delta^2).$$

It follows that the entire expression in (13) is

$$\Delta\sum_y(\lambda(y|a)-\lambda(y|a'))E[\omega(x,y)|a]+O(\Delta^{3/2}).$$

Adding the four steps establishes the claim. Note that the term $O(\Delta^{1.49999})$ in (11) is bounded in absolute value by $K\Delta^{1.49999}$ for some K that depends only on $\bar{V}$ and κ. *Q.E.D.*

[29]For a multidimensional x, inequality (14) can be justified by integrating along the plane orthogonal to the line between $\mu(a)$ and $\mu(a')$ first, and then integrating along the line that connects $\mu(a)$ and $\mu(a')$ (which gives the one-dimensional integral illustrated in Figure 9).

[30]$\Pr[0]$ is the probability of no jump arriving; $\Pr[(j_y)]$ is the probability of (j_y) jumps arriving.

We now establish Lemma B2 to verify that the inequality (7) holds.

LEMMA B2: *Under conditions* (10), *we have*

$$E[\omega\cdot N|a]\le -P[0|a]\int\frac{\kappa}{2}(\omega(x,0)\cdot T)^2 f_a(x)\,dx$$
$$+\sum_y P[y|a]E_x[\omega(x,y)|a]\cdot N$$
$$\le\frac{1-e^{-r\Delta}}{e^{-r\Delta}}\left(-\frac{r\kappa}{2}|\beta|^2+\sum_y(d(y)\cdot N)\lambda(a)+O(\Delta)\right).$$

PROOF: The first inequality follows from the conditions $\omega(x,0)\cdot N\le -\kappa(\omega(x,0)\cdot T)^2/2$ and $\omega(x,(j_y))\cdot N\le 0$ whenever (j_y) involves more than one jump.[31] The second inequality follows if we show that

$$|\beta|^2\le\frac{e^{-r\Delta}P[0|a]}{r(1-e^{-r\Delta})}\int(\omega(x,0)\cdot T)^2 f_a(x)\,dx+O(\Delta)$$

and

$$d(y)\lambda(a)=\frac{e^{-r\Delta}P[y|a]}{1-e^{-r\Delta}}E[\omega(x,y)|a]+O(\Delta).$$

By Cauchy–Schwarz inequality,

$$\left(\int x\omega_T(x,0)f_a(x)\,dx\right)^2$$
$$\le\underbrace{\left(\int\omega_T(x,0)^2 f_a(x)\,dx\right)}_{O(\Delta)\text{ by (12)}}\underbrace{\left(\int x^2 f_a(x)\,dx\right)}_{E[x^2]=\mu(a)^2\Delta^2+\Delta}.$$

Therefore,

$$\beta^2\le\frac{e^{-2r\Delta}}{(1-e^{-r\Delta})^2}\left(\Delta\int\omega_T(x,0)^2 f_a(x)\,dx+O(\Delta^3)\right)$$
$$=\frac{e^{-r\Delta}P[0|a]}{r(1-e^{-r\Delta})}\int(\omega(x,0)\cdot T)^2 f_a(x)\,dx+O(\Delta),$$

[31]The curve $C(s)$ with the point w on the boundary is contained in the parabolic region $\{v:(v-w)\cdot N\le\kappa((v-w)\cdot T)^2\}$, since every point of $C(s)$ has curvature greater than κ, by condition (iii). Therefore, $\omega(x,0)\cdot N\le-\kappa(\omega(x,0)\cdot T)^2$.

where we used $P[0|a] = 1 - O(\Delta), e^{-r\Delta}\Delta/(1 - e^{-r\Delta}) = 1/r + O(\Delta)$, and $\int(\omega(x,0)\cdot T)^2 f_a(x)\,dx = O(\Delta)$.

Also,

$$d(y)\lambda(y|a) = \frac{\Delta e^{-r\Delta}\lambda(y|a)}{1-e^{-r\Delta}}E[\omega(x,y)|a]$$
$$= \frac{e^{-r\Delta}P[y|a]}{1-e^{-r\Delta}}E[\omega(x,y)|a] + O(\Delta)$$

since $P[y|a] = \Delta\lambda(y|a) + O(\Delta^2)$. *Q.E.D.*

Lemma B2 in combination with

$$\frac{1-e^{-r\Delta}}{e^{-r\Delta}}(v - g(a))\cdot N = E[\omega\cdot N|a]$$

implies (7).

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