

Monetary Exchange with Multilateral Matching*

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Abstract

This paper analyzes monetary exchange in a search model allowing for multilateral matches to be formed, according to a standard urn-ball process. We consider three physical environments: indivisible goods and money, divisible goods and indivisible money, and divisible goods and money. We compare the results with Kiyotaki and Wright (1993), Trejos and Wright (1995), and Lagos and Wright (2005) respectively. We find that the multilateral matching setting generates very simple and intuitive equilibrium allocations that are similar to those in the other papers, but which have important differences. In particular, surplus maximization can be achieved in this setting, in equilibrium, with a positive money supply. Moreover, with flexible prices and directed search, the first best allocation can be attained through price posting or through auctions with lotteries, but not through auctions without lotteries. Finally, analysis of the case of divisible goods and money can be performed without the assumption of large families (as in Shi (1997)) or the day and night structure of Lagos and Wright (2005).

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1 Introduction

Search theoretic models of monetary exchange have traditionally carried the assumptions that matches occur randomly and bilaterally, and that prices are determined by bilateral bargaining. These assumptions have proven to be useful in the development of models with indivisible money and goods (Kiyotaki and Wright (1993)), indivisible money and divisible goods (Shi (1995), Trejos and Wright (1995)), and models with divisible money and divisible goods (Shi (1997), Molico (2004), Lagos and Wright (2005)). However, to a certain extent, they run counter to the spirit of clean economic theorizing that motivates the research program (Wallace (2001)). The assumption of random matching removes the search process itself from the realm of choice theory by making it purely a technological phenomenon that agents face. Similarly, the assumption of bilateral bargaining imposes a particular price determination mechanism that has no justification from the theory of mechanism design (McAfee (1993)). Moreover, this particular choice of mechanism impels modelers to restrict attention to bilateral matches – something that is difficult to justify if one side of the market randomizes over another, except in the limit when the length of time period goes to zero.

In this paper we relax these assumptions, in a very standard monetary exchange environment based on the models presented in the above papers, to see how robust the central results of these models are, and to explore the usefulness of a theory based on multilateral matching and directed search. We use a standard urn-ball matching process generated by buyers randomizing over sellers, and justified as the directed search mixed strategy equilibrium recently utilized in the labor market literature. (See, for example, Montgomery (1991), Julien, Kennes, and King (2000), and Burdett, Shi, and Wright (2001).)¹ Three different physical environments are considered, following the historical development of the literature: two-sided indivisibility, indivisible money and divisible goods, and two-sided divisibility. We also consider two different price determination mechanisms: *ex ante* price

¹Rocheteau and Wright (2005) also analyse monetary models with directed search. However, directed search in their paper takes a different form, based on the competitive search framework developed by Moen (1997). In that setting the aggregate market is divided into submarkets, each of which is frictional, but where movement across submarkets is directed by market-makers who perform a role similar to Walrasian auctioneers.

posting and *ex post* bidding (auctions). In all cases, we characterize the steady state monetary equilibria, examine their efficiency properties, and explore the influence of the quantity of fiat money supplied. Overall, a key difference between these models and those with purely random matching is that the ratio of buyers to sellers (that is, market tightness) now influences the matching rate. The introduction of this new channel, through which monetary policy can affect the economy, significantly alters the positive and normative affects of the policy.

With indivisible money and goods, as in Kiyotaki and Wright (1993), we identify restrictions on the two key parameters (the discount factor and the supply of money) that are necessary and sufficient for the existence of monetary equilibria. When examining the issue of the optimal supply of money we find that, in this environment, the expected aggregate surplus of the economy is not strictly declining in the number of units of fiat money, as in Kiyotaki and Wright (1993). Rather, because the arrival rate for matches is no longer parametric (but now a function of the ratio of buyers and sellers), a positive optimal supply of money exists which maximizes expected surplus.

With indivisible money but divisible goods we consider directed search equilibria under different pricing mechanisms: *ex ante* price posting (as in Burdett, Shi, and Wright (2001) and *ex post* auctions (as in Julien, Kennes, and King (2000)). In both settings, the supply of money now affects the expected surplus through two distinct margins: *extensive* and *intensive*. The extensive margin is the same as in the model with indivisible goods – through its influence on market tightness, the money supply affects the number of matches. The intensive margin is the same as in Trejos and Wright (1995): the equilibrium price affects the marginal costs and benefits associated with production. With price posting, we find that there exists a quantity of money that maximizes expected surplus. Interestingly, by way of contrast, with *ex post* bidding, surplus maximization cannot be attained unless lotteries are used in the presence of bilateral matches. This somewhat surprising result comes about because of the price dispersion inherent in the auction mechanism: this implies that, with concave utility and convex costs, the intensive margin condition can not be satisfied.

With divisible money and goods, we restrict attention to the case of *ex ante* price posting and find that we can characterize a steady state distribution of money holdings with relative ease. This distribution has a simple

two-point support that resembles the equilibrium in the model with indivisible money. This shares some of the features of the distribution found by Molico (2004) in a model based on Trejos and Wright (1995) with divisible money, but is *much* simpler – and in some respects is similar to the one found in Lagos and Wright (2005) in quite a different setting.² Monetary policy in this environment has both the intensive and extensive margins mentioned above, but now has an extra dimension to consider: how changes in the money supply are distributed across agents. We find, for example, that increments to the money supply can be neutral if and only if they are distributed only to existing money holders.

The remainder of the paper is organized as follows. Section 2 introduces and analyses the model with two-sided indivisibility. Section 3 considers the model with divisible goods and indivisible money. Section 4 studies the case of two-sided divisibility. Section 5 presents a conclusion and suggestions for further research, and an appendix contains proofs of some of the propositions and an extra proposition not contained in the body of the paper.

2 Indivisible Goods and Money

2.1 The Model

There is a $[0,1]$ continuum of infinitely lived agents. Time is discrete and the agents have a rate of time preference $r > 0$ or, equivalently, a discount factor, $\beta = 1/(1+r)$. There is also a continuum of indivisible services (or non-storable goods, where these goods are produced only after an agreement to exchange). Agent i has the ability to produce one unit of one good. The unit production cost for any agent is $c \geq 0$.

Tastes are heterogenous and modelled as follows. Given any two agents i and j , we write iWj to mean ‘ i wants to consume the good that j produces’- that is i derives utility $u > c$ from consuming what j produces if iWj and he derives utility 0 from consuming what j produces otherwise. For any two agents selected at random, we assume $prob(iWi) = 0$ and $prob(jWi) = x$. To keep things simple, we also assume away the double coincidence of wants, (and, thereby, the possibility of direct barter) so $prob(jWi | iWj) = 0$.

²In particular, here, we do not use the day-night market structure employed in Lagos and Wright (2005).

Further, to focus on the impact of introducing multilateral matching, we assume away the absence of any single coincidence of wants; thus, we set $x = 1$.

In addition to the consumable goods, there is an object called *fiat money*. This money consists of a fixed quantity of $M \in [0, 1]$ indivisible units of a storable object. Initially one (normalized) unit of money is randomly allocated to M agents. We assume, in this section, that agents holding money cannot produce.³ Thus no one can ever acquire more than one unit of money, and hence all agents hold either 0 or 1 units of money.

We assume that all goods producers operate as *sellers* and that all money holders operate as *buyers*. Thus, in equilibria with monetary exchange, the number of buyers is given by M and the number of sellers is given by $(1 - M)$. Let ϕ denote the ratio of buyers to sellers (market tightness), thus: $\phi = M/(1 - M)$. In each period, sellers appear identical and buyers randomly choose which seller to visit - generating a simple urn-ball matching process. For any two agents that meet, let π_0 denote the probability that the seller will accept money in exchange for goods. Similarly, let π_1 denote the probability that a money holder will accept the good in return for the unit of money. Thus, monetary exchange occurs if and only if $\pi \equiv \pi_0\pi_1 > 0$. In this case, the probability of exchange for any given seller choosing π_0 is given by

$$\xi(\pi) = \pi(1 - e^{-\phi}) = \pi(1 - e^{-\frac{M}{1-M}}) \quad (1)$$

where $1 - e^{-\phi}$ is the probability a seller gets at least one buyer. Similarly, the probability of exchange for a money holder (buyer) is given by:

$$\psi(\pi) = \pi \left(\frac{1 - e^{-\phi}}{\phi} \right) = \pi \frac{1 - M}{M} (1 - e^{-\frac{M}{1-M}}) \quad (2)$$

where the probability $(1 - e^{-\phi})/\phi$ incorporates the fact that the good is rationed randomly to only one buyer in the event of multiple buyers.

Equations (1) and (2) illustrate an important difference between this framework and that of Kiyotaki and Wright (1993). In their model, agents encounter each other with the parametric Poisson arrival rate α ⁴ so that, for a seller, the probability of encountering a buyer is αM . Similarly, for a buyer, the probability of encountering a seller is $\alpha(1 - M)$. Thus, the associated

³This assumption is relaxed in Section 4, below.

⁴Using the notation from Rupert et al. (2000).

probabilities of monetary exchange, corresponding to the expressions in (1) and (2) respectively, are $\pi\alpha M$ and $\pi\alpha(1 - M)$. Trejos and Wright (1995), for example, normalize by effectively setting $\alpha = 1$. Here, this would not be an innocuous normalization, since the arrival rate is not parametric, but a function of $\phi = M/(1 - M)$. That is, in this framework, market tightness affects arrival rates and the equilibrium number of matches.

Let V_0 and V_1 be the value functions of agents with 0 and 1 units of money, respectively - i.e. sellers and buyers. Since we only consider stationary, symmetric equilibria, V_j does not depend on time or on the agent's name. The payoffs when trading occurs are $u + \beta V_0$ if a money holder, and $-c + \beta V_1$ if a producer. The value function of a producer is given by

$$V_0(\pi) = \max_{\pi_0} \{ \xi(\Pi)[\beta V_1(\pi') - c] + (1 - \xi(\Pi))\beta V_0(\pi') \} \quad (3)$$

where Π represents the strategies of all buyers and other sellers, which is taken as given. Similarly, π' indicates the equilibrium strategies next period. With a slight abuse of notation, the value function of a money holder is

$$V_1(\pi) = \max_{\pi_1} \{ \psi(\Pi)(\beta V_0(\pi') + u) + (1 - \psi(\Pi))(\beta V_1(\pi')) \} \quad (4)$$

We are now in a position to define the relevant equilibrium.

2.2 The Equilibrium

In this paper we restrict attention to steady state symmetric equilibria.

Definition 1 *A steady state symmetric monetary equilibrium is a pair $(V_0(\pi), V_1(\pi))$ satisfying the following conditions:*

- (i) *Equations (3) and (4) where $\pi = \pi'$;*
- (ii) *The monetary exchange constraints*

$$-c + \beta V_1(\pi') \geq \beta V_0(\pi') \quad (5)$$

$$u + \beta V_0(\pi') \geq \beta V_1(\pi') \quad (6)$$

Hereafter, we will refer to steady state symmetric equilibria simply as "equilibria". In these equilibria, $\pi = \Pi'$, and the value functions are:

$$V_1(\pi) = \frac{\psi(\pi)(u(1 - \beta(1 - \xi(\pi))) - \beta\xi(\pi)c)}{(1 - \beta)(1 - \beta(1 - \psi(\pi) - \xi(\pi)))} \quad (7)$$

and

$$V_0(\pi) = \frac{\xi(\pi)(\beta\psi(\pi)u - c(1 - \beta(1 - \psi(\pi))))}{(1 - \beta)(1 - \beta(1 - \psi(\pi) - \xi(\pi)))} \quad (8)$$

The difference between V_1 and V_0 is given by

$$\Delta = V_1(\pi) - V_0(\pi) = \frac{u\psi(\pi) + c\xi(\pi)}{1 - \beta(1 - \psi(\pi) - \xi(\pi))}. \quad (9)$$

Define the net gains from trading goods for money and money for goods as, respectively:

$$\begin{aligned} \Delta_0 &= \beta(V_1 - V_0) - c \\ \Delta_1 &= u - \beta(V_1 - V_0) \end{aligned} \quad (10)$$

Substituting in the steady state values for V_0 and V_1 , or for Δ , yields

$$\begin{aligned} \Delta_0 &\geq 0 \iff V_0 \geq 0 \\ \Delta_1 &\geq 0 \iff V_1 \geq 0. \end{aligned} \quad (11)$$

The equilibrium conditions are

$$\pi_j \begin{cases} = 1 \\ \in (0, 1) \\ = 0 \end{cases} \text{ if } \Delta_j \begin{cases} > 0 \\ = 0 \\ < 0. \end{cases} \quad (12)$$

Notice that, from (7), $V_1 > 0$ for all parameter values. From (11), this implies that $\Delta_1 > 0$, and from (12) this implies that $\pi_1 = 1$. Therefore, to determine where monetary equilibria exist, we need only focus on π_0 .

Proposition 1 *Let $\varphi(\phi, \beta) \equiv \frac{\beta\psi(\phi)}{1 - \beta(1 - \psi(\phi))}$ and $\psi(\phi) = (\frac{1 - e^{-\phi}}{\phi})$, then*

- a. $\pi_0 = 0$ exists as a non-monetary equilibrium everywhere in the parameter space.
- b. $\pi_0 = 1$ exists as a monetary equilibrium when $c \leq \varphi(\phi, \beta)u$.
- c. $\pi_0 = \hat{\pi} = \frac{(1 - \beta)c}{\beta\psi(\phi)(u - c)} \in (0, 1)$ exists as a monetary equilibrium when $c \leq \varphi(\phi, \beta)u$.

d. No monetary equilibria exist when $c > \varphi(\phi, \beta)u$.

Proof. Notice that Δ_0 has the same sign as $\pi_0 - \hat{\pi}$ where $\hat{\pi}$ is defined by

$$\frac{u}{c} = \frac{(1 - \beta)}{\beta\psi(\hat{\pi}, \phi)} + 1$$

or

$$\hat{\pi} = \frac{(1 - \beta)c}{\beta\psi(\phi)(u - c)}.$$

Then $\pi_0 = 0$ is always an equilibrium since $\hat{\pi} > 0$. Which implies $\pi_0 - \hat{\pi} < 0$ and hence, $\Delta_0 < 0$. The equilibrium condition is satisfied and there is an equilibrium where no one accepts fiat money.

Next, notice that $\hat{\pi} < 1$ if

$$\frac{(1 - \beta)c}{\beta\psi(\phi)(u - c)} < 1 \quad \text{or} \quad c < \frac{\beta\psi(\phi)u}{1 - (\beta - \psi(\phi))}.$$

This means that $\pi_0 = 1 > \hat{\pi}$ is an equilibrium. There exists a monetary equilibrium if the cost of producing a unit is small enough relative to other parameters. (This is standard in search models of money, since agents must incur a cost to obtain money, that cost must not be too high.) Since in any well defined game there is always an odd number of equilibria, there exists a mixed strategy equilibrium where $\pi_0 = \hat{\pi}$. If other agents accept money with probability $\hat{\pi}$, then one agent is indifferent between accepting or rejecting money and randomizing is an equilibrium. ■

In simple fixed-price search models of fiat money, such as this, it is standard to find parameter regions for which there is no monetary equilibrium. Consider the equilibrium with $\pi_0 = 1$. The associated trading constraint $\Delta_0 \geq 0$ is

$$\varphi(\phi, \beta) \equiv \frac{\beta\psi(\phi)}{(1 - \beta(1 - \psi(\phi)))} \geq \frac{c}{u}.$$

The LHS is strictly decreasing in ϕ , and strictly increasing in β , which leads to the following corollary.

Corollary 1 *For any given u and c , and assume $\pi = 1$, the unique equilibrium is non monetary when*

- a. for all β and for all $\phi > \bar{\phi}$ where $\bar{\phi} < \infty$ ($\bar{M} < 1$) is given by $\frac{u}{c} = \frac{(1 - \beta)}{\beta\psi(\bar{\phi})} + 1$;

b. for all ϕ and for all $\beta < \underline{\beta}$ where $\underline{\beta}$ is given by $\frac{u}{c} = \frac{(1-\beta)}{\underline{\beta}\psi(\phi)} + 1$.

Intuitively, too much money chasing too few goods and impatience eliminate the monetary equilibrium.

2.3 Monetary Policy

Consider a social planner who can choose M to maximize surplus, but must respect all other constraints imposed by the decentralized economy. This is essentially a static problem because the quantity of buyers (holders of money) is independent of any past decisions. Suppose that a monetary equilibrium exists whenever feasible. In this case, the objective of a central authority (social planner) who controls M is to maximize surplus:

$$\max_M Y = X(M)(u - c) \quad (13)$$

such that

$$\frac{\beta\psi(\phi)}{1 - \beta(1 - \psi(\phi))} \geq \frac{c}{u} \quad (14)$$

where the number of transactions each period is given by

$$X(M) = (1 - M) \left(1 - e^{-(M/(1-M))}\right) \quad (15)$$

and $(u - c)$ is a constant. Maximizing surplus is equivalent to maximizing the number of matches (that is, $X(M)$ in (15)), which is achieved if the quantity of money satisfies: $M^*/(1 - M^*) = 1.146$, or $M^* = 0.533$. In principle, the money acceptance constraint may affect the decision on how much money to allocate to this economy. In particular, from the Corollary above, there exists an $\bar{M} < 1$ beyond which there is only a non-monetary equilibrium. The question then becomes whether or not $M^* < \bar{M}$. From the Corollary, there is also a $\underline{\beta}$, below which there is only a non-monetary equilibrium. To assess the significance of these constraints on the determination of optimal stock of money, Figure 1 below illustrates both the number of matches and the LHS of (14) for different values of M and β : In this picture, the thick black line at the bottom is the number of matches as a function of M , which peaks when $M^* = 0.533$. The middle (grey) line is the value of the LHS of (14) if β is small ($\beta = .9$), corresponding to very lengthy (annual)

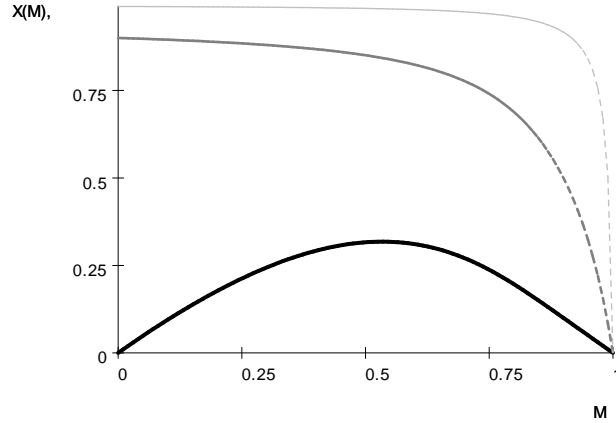


Figure 1: Matching function and $\varphi(\phi, \beta)$.

transactions. The top (light grey) line is the value of the LHS of (14) if β is moderately large ($\beta = .99$), corresponding to (roughly) weekly transactions. Clearly, at the value of $M^* = 0.533$, the LHS of (14) is above 0.75 for $\beta = .9$ and very close to 1.0 for $\beta = .99$. Thus, the cost/utility ratio c/u would have to be very high in order for the constraint to be binding. Therefore, in this model of monetary exchange, with indivisible goods and money, under reasonable parameter values, the monetary acceptance constraint does not influence the optimal choice of monetary policy, and we can reasonably presume that optimal monetary balances are those that maximize the number of transactions.

2.3.1 A Comparison with Kiyotaki and Wright (1993)

In Kiyotaki and Wright (1993), agents face a parametric Poisson arrival rate α so, in a monetary equilibrium, the number of transactions is given by

$$\tilde{X}(M) = \alpha(1 - M)$$

and the surplus is therefore

$$Y = \alpha(1 - M)(u - c)$$

which is strictly decreasing in M . Kiyotaki and Wright therefore focus attention, instead, on maximizing a social welfare function

$$W = MV_1 + (1 - M)V_0$$

which has a non-zero optimal M .

3 Divisible Goods and Indivisible Money

We now relax the assumption of indivisible goods, along the lines of Shi (1995) and Trejos and Wright (1995), and consider two alternative pricing mechanisms. Under the first pricing method, producers announce the terms of trade prior to matching. In this setting, prices are determined *ex ante* through price posting. Under the second pricing method, the terms of trade are determined by local market conditions, which are revealed only after matching. In this setting, prices are determined *ex post* through a bidding game (or auction). As before, we assume that money holders are always buyers and thus there is never a chance of direct trade.

Goods are now perfectly divisible. Let $u(q)$ be the utility of consuming q units of one's consumption good and $c(q)$ the disutility of producing q units of one's production good. Following Trejos and Wright (1995), we assume $u(0) = c(0) = 0$, $u'(0) > c'(0) = 0$, $u'(q) > 0$, $c'(q) > 0$, $u''(q) \leq 0$, and $c''(q) \geq 0$, for $q > 0$, with at least one of the weak inequalities strict. For future reference, we define q^* by $u'(q^*) = c'(q^*)$. Also, there is a $\hat{q} > 0$ such that $u(\hat{q}) = c(\hat{q})$. It is important to understand, though, what is meant by quantity in this setting. In Trejos and Wright (1995), q is interpreted as either the quality of a service or the quantity of a perishable good. In this model, the service interpretation is valid, but the goods interpretation requires another assumption: due to the possibility of multilateral matching, we need to restrict sellers to serve only one buyer at a time.

3.1 Ex ante pricing (price posting)

In this subsection, we consider a model that is similar to the (non-monetary) price posting model presented in Burdett, Shi and Wright (2001). The main difference is that, in that model, sellers have one indivisible unit of a good to sell, which can be priced according to a perfectly divisible numeraire. Here, instead, sellers have a perfectly divisible good that they can produce and sell, in exchange for a unit of the indivisible fiat money. Each seller announces, in advance of matching, the quantity level, q , at which they will

produce, implying a nominal price $p = 1/q$. In both cases, sellers compete in prices; however, the indivisibility is now on the other side of the market, and sellers are attempting to acquire fiat money. In effect then, here, sellers bid for money.⁵

The equilibrium quantity (price) announcement is solved by a simple game for which we find a Nash equilibrium. We start by assuming that all sellers but one announce a quantity level q , but one seller considers announcing a level $\tilde{q} \neq q$. Buyers observe the announcements and select a trading partner from among the deviating and non-deviating sellers using a mixed strategy. The Nash equilibrium is a value q from which no unilateral deviation is profitable and buyers select among the sellers using the same mixed strategy.

To be precise, we provide the following definition.

Definition 2 *An equilibrium quantity announcement is a q_e satisfying*

$$q_e = \arg \max_{\tilde{q}} \tilde{V}_0(\tilde{q}, q)$$

where \tilde{q} is a deviation for a seller, taking as given all other sellers' strategies q , and \tilde{V}_0 is the value associated with the deviation. In any deviation, a seller correctly anticipates that buyers' mixed strategies solve

$$\tilde{V}_1(\tilde{q}, q) = V_1(\tilde{q}, q).$$

where \tilde{V}_1 is the value for a buyer from selecting a deviating seller.

Therefore, in equilibrium, $q_e = \tilde{q} = q$.

3.1.1 The non-deviant sellers

The present values to non-deviant sellers, and to buyers selecting non-deviant sellers are:

$$V_0(\tilde{q}, q) = \xi (\beta V_1(q'_e)) - c(q) + (1 - \xi)\beta V_0(q'_e) \quad (16)$$

⁵It is worth noting that this pricing environment, in common with Shi (1995) and Trejos and Wright (1995), requires a lot of commitment, and the following results lean heavily on this assumption. Alternatively, one could consider a setting in which sellers could physically produce and announce truthfully their production ex ante in order to attract buyers. The difference in that setting is that the production cost is sunk in the sense of being unavoidable in the case of no trade. It turns out that there is no monetary equilibrium under this assumption. The explanation is that the hold up problem is so severe that sellers will not produce at all or the equilibrium quantity is zero. We conjecture that this is also the case in Shi (1995) and Trejos and Wright (1995).

and

$$V_1(\tilde{q}, q) = \psi [u(q) + \beta V_0(q_e')] + (1 - \psi)\beta V_1(q_e') \quad (17)$$

respectively, where q_e' is the equilibrium quantity in the next period.

The probabilities in the above value functions are affected by buyers' mixed strategies, captured by the buyer/seller ratio ϕ , which is a function of \tilde{q} and q . To save on notation we write $\phi \equiv \phi(\tilde{q}, q)$. The probabilities of trade for sellers and buyers, respectively, are given by:

$$\xi = 1 - e^{-\phi} \quad (18)$$

and

$$\psi = \frac{1 - e^{-\phi}}{\phi}. \quad (19)$$

Here, also, to keep notation to a minimum, we restrict attention to monetary equilibria where $\pi = 1$. The expressions above can characterize both a non-monetary and a monetary equilibrium. In particular, if $q = 0$, then buyers receive nothing in exchange for money and if $q > 0$, sellers are trading goods for money.

3.1.2 The deviant seller

The value function for a buyer selecting the deviant seller is

$$\tilde{V}_1(\tilde{q}, q) = \tilde{\psi} [u(\tilde{q}) + \beta V_0(q_e')] + (1 - \tilde{\psi})\beta V_1(q_e') \quad (20)$$

where $\tilde{\psi} = \frac{1 - e^{-\tilde{\phi}}}{\tilde{\phi}}$ is the buyer's probability of trade and $\tilde{\phi} = \phi(\tilde{q}, q)$ is the buyer/seller ratio summarizing the buyer's mixed strategies over the deviating seller and non-deviating sellers. Buyers mixed strategies are chosen such that

$$\tilde{V}_1(\tilde{q}, q) = V_1(\tilde{q}, q) \quad (21)$$

where

$$V_1(\tilde{q}, q) = \psi [u(\tilde{q}) + \beta V_0(q_e')] + (1 - \psi)\beta V_1(q_e'). \quad (22)$$

The deviant seller's value function is

$$\tilde{V}_0(\tilde{q}, q) = \tilde{\xi} (\beta V_1(q_e') - c(\tilde{q})) + (1 - \tilde{\xi})\beta V_0(q')$$

where $\tilde{\xi} = 1 - e^{-\tilde{\phi}}$ is the probability of trade.

The best deviation solves:

$$\begin{aligned} \max_{\tilde{q} \in \mathbb{R}_+} \tilde{V}_0(\tilde{q}, q) \\ \text{s.t.} \end{aligned} \quad (23)$$

$$\begin{aligned} (i) \quad \tilde{V}_1(\tilde{q}, q) &= V_1(\tilde{q}, q) \\ (ii) \quad -c(\tilde{q}) + \beta V_1(q'_e) &\geq \beta V_0(q'_e) \\ (iii) \quad u(\tilde{q}) + \beta V_0(q'_e) &\geq \beta V_1(q'_e) \end{aligned}$$

Constraint (i) represents the fact that, in a deviation, a seller anticipate buyers' behavior correctly in their mixed strategies. Using (20) and (21), this can be re-written as:

$$(i)' \quad \tilde{\psi} = \frac{[u(q) - \beta \Delta(q'_e)]}{[u(\tilde{q}) - \beta \Delta(q'_e)]} \psi \quad (24)$$

where $\Delta(q'_e) = V_1(q'_e) - V_0(q'_e)$.

Constraints (ii) and (iii) are the exchange constraints. These two constraints simplify to:

$$(ii)' \quad \tilde{q}_{\max} \leq c^{-1}(\beta \Delta(q'_e)) \quad (25)$$

$$(iii)' \quad \tilde{q}_{\min} \geq u^{-1}(\beta \Delta(q'_e)) \quad (26)$$

The deviant seller's problem then is reduced to:

$$\begin{aligned} \max_{\tilde{q} \in [\tilde{q}_{\min}, \tilde{q}_{\max}]} \tilde{V}_0(\tilde{q}, q) \\ \text{s.t.} \quad \tilde{\psi} = \frac{[u(q) - \beta \Delta(q'_e)]}{[u(\tilde{q}) - \beta \Delta(q'_e)]} \psi \end{aligned} \quad (27)$$

Notice that the constraint summarizes buyers' reaction functions with respect to the deviating and non-deviating sellers' choices, and hence, is always binding.

In a symmetric equilibrium $\tilde{q} = q = q_e$, $\tilde{\xi} = \xi$, $\tilde{\psi} = \psi$, and $\tilde{\phi} = \phi$.⁶ In steady state $q_e = q'_e$. To simplify notation, for the remainder of this section, we write the equilibrium q_e as q only. (The subscript was introduced only to differentiate equilibrium from quantities chosen by non-deviant sellers.)

⁶The symmetric equilibrium is well-known to be unique in this type of environment. See, for example, Burdett, Shi, and Wright (2001).

Proposition 2 *In the symmetric steady-state equilibrium, the quantity announcement by sellers is characterized by:*⁷

$$\frac{c'(q)}{u'(q)} = \frac{(\xi - 1)}{(1 - \psi - \xi)} \frac{[\beta\Delta(q) - c(q)]}{[u(q) - \beta\Delta(q)]}. \quad (28)$$

Proof. See Appendix. ■

The steady state symmetric equilibrium quantity implies $\tilde{V}_0(\tilde{q}, q) = V_0(\tilde{q}, q) = V_0(q)$ and $\tilde{V}_1(\tilde{q}, q) = V_1(\tilde{q}, q) = V_1(q)$. Using this in (16) and (17), the steady state values become

$$V_0(q) = \frac{\xi (\beta\psi u(q) - (1 - \beta(1 - \psi)) c(q))}{(1 - \beta)(1 - \beta(1 - \psi - \xi))} \quad (29)$$

and

$$V_1(q) = \frac{\psi (\beta\xi u(q) - (1 - \beta(1 - \xi)) c(q))}{(1 - \beta)(1 - \beta(1 - \psi - \xi))}. \quad (30)$$

It follows immediately that, as long as $q > 0$,

$$\Delta(q) = V_1(q) - V_0(q) = \frac{\psi u(q) + \xi c(q)}{(1 - \beta(1 - \psi - \xi))} > 0 \quad (31)$$

for all parameter values of β and ϕ . The link between the trading constraints (or net gains from trade) and the values at the steady state are established as

$$\begin{aligned} \Delta_0(q) &= \beta(\Delta(q)) - c(q) \geq 0 \iff V_0(q) \geq 0 \\ \Delta_1(q) &= u(q) - \beta(\Delta(q)) \geq 0 \iff V_1(q) \geq 0. \end{aligned} \quad (32)$$

As in the previous section, but now with production, $V_1(q) \geq 0$ for all parameter values. The buyers always want to participate in this market, a consequence of the fiat money assumption. Therefore we need only focus on the sellers' value function to determine the condition for existence of a steady state monetary equilibrium (i.e. when $q > 0$).

Definition 3 *A symmetric steady state equilibrium is a triple (q, V_0, V_1) satisfying the following conditions:*

⁷Astute readers may notice that equation (28) shares a similarity with the equilibrium condition in standard search models of money using pairwise matching and bargaining (e.g., Trejos and Wright (1995)).

- (i) $\tilde{q} = q$ is the symmetric Nash equilibrium which solves the seller's problem (27) of the ex ante game characterized by equation (28) taking $V_0(q)$ and $V_1(q)$ as given;
- (ii) the value functions V_0 and V_1 satisfy (29) and (30), taking q as given;
- (iii) the sellers exchange constraint evaluated at q is satisfied:

$$\Delta_0(q) \geq 0 \tag{33}$$

From (25) and (32) any steady state equilibrium value of $q \in [0, q_{\max}]$ is both a necessary and sufficient condition for $V_0(q) \geq 0$, for given β, ϕ .

3.1.3 Equilibrium

The existence and equilibrium properties of the model are now stated and derived. As in the previous section, the steady state equilibrium is referred to simply as an equilibrium.

Proposition 3 *For any $\beta \in (0, 1)$ and $M \in (0, 1)$, in the steady state, there exists a non-monetary equilibrium and a unique monetary equilibrium with $q \in (0, \hat{q})$.*⁸

Proof. First, the non-monetary equilibrium $q = V_0 = V_1 = 0$ is easily established since it satisfies all the conditions from the definition of an equilibrium. To show existence of the monetary equilibrium, write the following transformation function representing the first derivative of the seller's objective function evaluated at the steady state symmetric equilibrium q :

$$T(q) = g(\phi)[\beta\Delta(q) - c(q)]u'(q) - [u(q) - \beta\Delta(q)]c'(q)$$

where $\Delta(q)$ is the steady state value from equation (31) and $g(\phi) = \left(\frac{1-\xi}{\psi+\xi-1}\right)$. It implies that $T(q) = 0$ represents the first-order condition and, in the steady state, the equation (28). To show existence, observe that $T(0) = 0$, and $T(q)$, is continuous since $u(\bullet)$ and $c(\bullet)$ are continuous, and recall that

⁸The assumption that $c'(0) = 0$ (which is standard in these models) is important here. If, instead, $c'(0) > 0$ then monetary equilibria exist only in a restricted region of the parameter space – as in the previous section. We consider this case in Proposition 10, given in the appendix.

only $q \in [0, q_{\max}]$, are equilibrium steady state candidates. Now given the assumption that $c'(0) = 0$,

$$T'(0) = g(\phi)[\beta\Delta'(0)]u'(0) > 0$$

for all $\beta \in (0, 1)$ and $\phi < \infty$. Next, when $q = q_{\max}$ then $V_0(q_{\max}) = 0 \iff \beta\Delta(q_{\max}) - c(q_{\max}) = 0$, and since $V_1(q_{\max}) > 0$, we find

$$T(q_{\max}) = -[u(q_{\max}) - \beta\Delta]c'(q_{\max}) < 0,$$

which shows that the equilibrium is such that $q < q_{\max} < \hat{q}$. By continuity of $T(q)$ there exist a $q \in (0, q_{\max})$ such that $T(q) = 0$.

Uniqueness is shown by setting $T(q) = 0$ and obtaining the first-order condition

$$\frac{c'(q)}{u'(q)} = g(\phi) \frac{[\beta\Delta(q) - c(q)]}{[u(q) - \beta\Delta(q)]}. \quad (34)$$

The LHS is weakly increasing (non-decreasing) in q and the RHS is strictly decreasing in q for any ϕ and β . It is easily shown using L'Hopital's rule that evaluated at $q = 0$, the $LHS|_{q=0} < RHS|_{q=0}$ is equivalent to $T'(0) > 0$. Therefore, there exists a unique value of q for which the first-order condition is satisfied, and hence, a unique monetary equilibrium. ■

An example of this equilibrium with $u(q) = q$, $c(q) = q^2$ and $\phi = 1$ with $\beta = .99$ is illustrated in Figure 2, where the utility function and the cost function form the lens (equal at both $q = 0$ and $q = 1$), V_0 and V_1 are the strictly concave functions, with V_1 slightly higher than V_0 , and the function $T(q)$ is the dotted line. The equilibrium occurs where this dotted line intersects the horizontal axis from above.

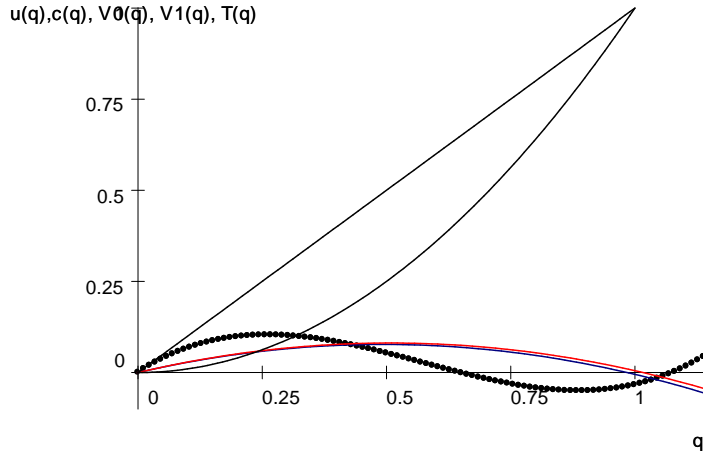


Figure 2: The Ex Ante Pricing Equilibrium

Using the Implicit Function Theorem with (34) we write the equilibrium as $q(\phi, \beta)$ and derive some properties of the monetary equilibrium.

Lemma 1 *For any given $\beta \in (0, 1)$, the equilibrium $q \in (0, q_{\max})$ is decreasing in ϕ (hence M) for all $\phi \in (0, \infty)$, with $\lim_{\phi \rightarrow 0} q_{\max} = \hat{q}$ and $q(0) = 0$. For any given ϕ , the equilibrium q is strictly increasing in β .*

Proof. (See Appendix) ■

The above results suggest that there exist parameter values for which the equilibrium can be efficient at the intensive margin. The following proposition establishes that.

Proposition 4 *For any given β , there exists a ϕ^* for which the monetary equilibrium is efficient: $q(\phi^*, \beta) = q^*$ where $u'(q^*) = c'(q^*)$. Otherwise the monetary equilibrium is inefficient as follows: if $\phi \gtrless \phi^*$ then $q \lesseqgtr q^*$.*

Proof. First we must show that $q^* < q_{\max}$. We know that

$$V_0(q_{\max}) = 0 \Leftrightarrow D_0 = 0$$

where $D_0 = [\beta\psi u(q) - (1 - \beta(1 - \psi))c(q)]$. At q^* , the first-order condition is $1 = g(\phi)\frac{D_0}{D_1}$ or $D_0 = D_1/g(\phi) > 0$. By concavity of V_0 , it follows that $V_0(q^*) > 0$, and hence, $q^* < q_{\max}$. The remainder of the proof follows from Lemma 1. ■

3.2 Ex post pricing (auctions)

In the previous subsection the terms of trade were determined ex ante: prior to the matching allocation. As noted, this pricing mechanism assumes commitment on the part of the sellers. Although one could find ways to rationalize this commitment, it is interesting to investigate alternatives where the terms of trade are determined ex post. Since we allow for the possibility of multilateral matches to form, the natural ex post mechanism to consider is auctions. In this environment, and throughout this section of the paper,

it is common knowledge that each buyer has only one unit of fiat money. Bidding, here, therefore takes the following structure. Because the margin of adjustment is on the sellers' side (the quantity choice of the divisible good) a bid from a buyer is proposition of a quantity required in exchange for his unit of fiat money. It follows that, within a multilateral match, where one seller is matched with several buyers, competition between the buyers reduces the minimal quantity of output acceptable for the unit of money. (In effect, this is a procurement auction, where the unit of money corresponds to the fixed budget of the buyer, and the quantity level is analogous to the cost of input.) In all other respects, the model remains the same.

The matching process determines the number, n , of buyers matched with a particular seller. As is usual with auction mechanisms, the outcome depends on the number of buyers bidding. Let q be the vector of possible outcomes from the auction in any period. Due to the common knowledge assumption, the bidding outcome is reduced to two possible cases: when $n = 1$ (a pairwise match) and when $n > 1$ (a multilateral match). Let the outcome of the auction be q_p when $n = 1$ and q_m when $n > 1$. The value functions for sellers and buyers, respectively are:

$$V_0(q) = \xi_p[-c(q_p) + \beta V_1(q')] + \xi_m[-c(q_m) + \beta V_1(q')] + (1 - \xi_m - \xi_p)\beta V_0(q') \quad (35)$$

where, for a seller, $\xi_p = \phi e^{-\phi}$ is the probability of a pairwise match, $\xi_m = 1 - \phi e^{-\phi} - e^{-\phi}$ is the probability of a multilateral match, $1 - \xi_m - \xi_p$ is the probability of no match, and $q = (q_p, q_m)$ is the vector of possible outcomes. Similarly, the value function for a buyer is given by:

$$V_1(q) = \psi_p[u(q_p) + \beta V_0(q')] + \psi_m[u(q_m) + \beta V_0(q')] + (1 - \psi_m - \psi_p)\beta V_1(q') \quad (36)$$

where, for a buyer, $\psi_p = e^{-\phi}$ is the probability of a pairwise match for a buyer, $\psi_m = \frac{1 - e^{-\phi}}{\phi}$ is the joint probability of a multilateral match and winning the bidding game, and $1 - \psi_m - \psi_p$ is the probability of not winning the bidding game.

We are now ready to consider the determination of quantities in equilibrium. From the structure of the auction, it should be clear that the quantity

resulting from a pairwise match will be higher than from a multilateral match (which we demonstrate in Lemma 2 below). In any pairwise matching event, the buyer is bidding alone and thus proposes a (high) quantity level solving

$$\bar{q} = \arg \max_q \{u(q) + \beta V_0(q')\} \quad (37)$$

s.t.

$$-c(q) + \beta V_1(q') \geq \beta V_0(q') \quad (38)$$

where the constraint is required for the seller to participate in trade.⁹ This has the obvious solution:

$$\bar{q} = c^{-1}(\beta \Delta') \quad (39)$$

where $\Delta' = V_1(q') - V_0(q')$ as before. Thus, $q_p = \bar{q}$.

In a multilateral matching event, the seller has multiple buyers and thus the buyers propose a quantity level that cannot be beat by other buyers. This is a simple Bertrand pricing game - the outcome of a simple auction. Therefore, the equilibrium bid is one where each buyer is indifferent between trading and not trading (i.e., remaining a buyer in the next period). Since all bids are the same in this scenario in equilibrium, whether or not a particular buyer wins the auction, he receives the same expected payoff. The equilibrium quantity can therefore be found from the following problem:

$$\underline{q} = \arg \max_q \{-c(q) + \beta V_1(q')\} \quad (40)$$

s.t.

$$u(q) + \beta V_0(q') \geq \beta V_1(q') \quad (41)$$

This has the solution:

$$\underline{q} = u^{-1}(\beta \Delta'). \quad (42)$$

⁹In a pairwise match such as this, one could think of alternative ways of determining the price, (in particular: bargaining). Here, to be consistent with auctions, in the event of pairwise matches, the buyer has the power to push the seller to her outside option with the minimum acceptable quantity demanded or required. Interestingly, Halko, Kultti, and Niinimäki (2004) show that this mechanism is evolutionary stable, while the hybrid mechanism with bargaining is not. In section 3.2.1, below, we also consider the introduction of lotteries with pairwise matches.

Thus, $q_m = \underline{q}$. The following lemma verifies the existence of price dispersion in any candidate equilibrium and, as argued above, prices with multilateral matching are higher than with pairwise matching.¹⁰

Lemma 2 *For all $\Delta' > 0$, $(\underline{q}, \bar{q}) \in (0, \hat{q})^2$ and $\underline{q} < \bar{q}$, where \hat{q} is defined by $u(\hat{q}) = c(\hat{q})$. Otherwise, $\Delta' = 0$ if and only if $\underline{q} = \bar{q} = 0$ and $\Delta' = \hat{\Delta} > 0$, if and only if $\underline{q} = \bar{q} = \hat{q}$.*

Proof. In appendix. ■

In the steady state, $q = q'$, and using (39) and (42) in (35) and (36), we find the steady state values:

$$V_0(q) = \frac{\xi_m (\beta \psi_p u(\bar{q}) - (1 - \beta (1 - \psi_p)) c(\underline{q}))}{(1 - \beta) (1 - \beta (1 - \psi_p - \xi_m))} \quad (43)$$

and

$$V_1(q) = \frac{\psi_p ((1 - \beta (1 - \xi_m)) u(\bar{q}) - \beta \xi_m c(\underline{q}))}{(1 - \beta) (1 - \beta (1 - \psi_p - \xi_m))}. \quad (44)$$

It follows immediately that as long as $\underline{q} > 0$,

$$\Delta(q) = V_1(q) - V_0(q) = \frac{u(\bar{q})\psi_p + c(\underline{q})\xi_m}{1 - \beta(1 - \psi_p - \xi_m)} > 0 \quad (45)$$

and, using Lemma 2, it follows that $V_1(q) > 0$ for all parameter values. Once again, this implies that, for existence of a steady state equilibrium we need only establish conditions under which $V_0(q) \geq 0$.

Definition 4 *A steady state equilibrium is a tuple $(\underline{q}, \bar{q}, V_0, V_1)$ satisfying the following conditions:*

1. \underline{q} and \bar{q} satisfy (39) and (42) respectively, taking V_0 and V_1 as given;
2. V_0 and V_1 satisfy (43) and (44).

The existence and uniqueness is stated in the following proposition.

¹⁰Notice that price dispersion occurs in this auction setting where the buyer has the bargaining power in pairwise matches and the seller has the power with multilateral matches. Hence, here, the bargaining power within a match is determined by the matching process.

Proposition 5 For any $\beta \in (0, 1)$ and $M \in (0, 1)$, in the steady state, there exists a non-monetary equilibrium and a unique monetary equilibrium with $0 < \underline{q} < \bar{q} < \hat{q}$.¹¹

Proof. First note that \underline{q} and \bar{q} are maximized values such that respectively a seller and a buyer is indifferent between trading or not. Therefore, we do not have to worry about participation constraints. For any $\beta \in (0, 1)$ and $M \in (0, 1)$ the existence of a non-monetary equilibrium is immediate since $\underline{q} = \bar{q} = V_0 = V_1 = 0$ satisfies all the equilibrium conditions. To show the existence of a monetary equilibrium with $0 < \underline{q} < \bar{q}$, insert the steady state value functions into (39) and (42) to get

$$c(\bar{q}) = \beta \frac{\psi_p u(\bar{q}) + \xi_m c(\underline{q})}{1 - \beta(1 - \psi_b - \xi_m)} \quad (46)$$

and

$$u(\underline{q}) = c(\bar{q}). \quad (47)$$

Therefore, quantities are linked by a strictly convex function $\underline{q} = u^{-1}(c(\bar{q}))$. Using this into (46) gives

$$c(\bar{q}) = \beta \frac{\psi_p u(\bar{q}) + \xi_m c(u^{-1}(c(\bar{q})))}{1 - \beta(1 - \psi_p - \xi_m)} \equiv H(\bar{q}).$$

Define $T(\bar{q}) = H(\bar{q}) - c(\bar{q})$. Clearly $T(\bar{q})$ is continuous, $T(0) = 0$, and $T'(0) = \beta \frac{\psi_p u'(0)}{1 - \beta(1 - \psi_p - \xi_m)} > 0$ for all parameter values satisfying the assumptions of the proposition. Observe that $T(\hat{q}) = -u(\hat{q}) \left(\frac{(1-\beta)}{1 - \beta(1 - \psi_p - \xi_m)} \right) < 0$ meaning that $\bar{q} = \hat{q}$ cannot be an equilibrium. By continuity and the Weierstrass Intermediate Value Theorem, there exist a $\bar{q} > 0$, such that $T(\bar{q}) = 0$ and $\underline{q} = u^{-1}(c(\bar{q})) > 0$. This implies $\Delta > 0$ and from Lemma 2 we have $\underline{q} < \bar{q}$. Therefore, $0 < \underline{q} < \bar{q} < \hat{q}$.

To show uniqueness, observe that $H(\bar{q})$ is strictly increasing and a linear combination of a concave and a convex function. This implies that there is a unique inflexion point below which $H(\bar{q})$ is concave and above which $H(\bar{q})$ is convex. Therefore, the function $T(\bar{q})$ has also a unique concave and a unique convex portion. Since $T(0) = 0$, $T'(0) > 0$ and $T(\hat{q}) < 0$, it implies

¹¹As in the ex ante pricing model, changing the assumptions to allow for $c'(0) > 0$ restricts the parameter space for which a monetary equilibrium exist. The proof is similar to the one provided for the ex ante case given in Proposition 10 in the appendix.

that there must be a unique value $\bar{q} \in (0, \hat{q})$ such that $T(\bar{q}) = 0$. Since $u(\underline{q}) = c(\bar{q})$ then $\underline{q} > 0$ is also unique. ■

Figure 3 provides a numerical example with $u(\bar{q}) = \bar{q}$ and $c(\bar{q}) = \bar{q}^2$ with different values of ϕ . The horizontal axis shows values of \bar{q} . As in Figure 2, the lens is formed by the utility and cost functions. The $T(\bar{q})$ function is represented by the dotted line, and the equilibrium value of \bar{q} occurs where this function cuts the horizontal axis from above.

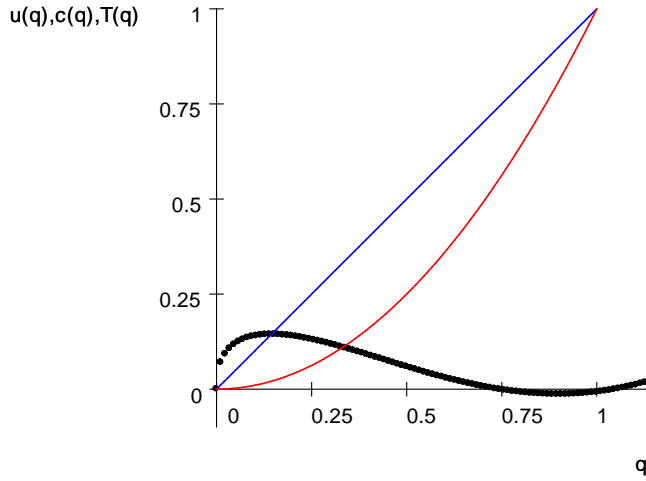


Figure 3: The Ex Post Pricing Equilibrium:

Using the Implicit Function Theorem, we now derive key properties of the monetary equilibrium.

Lemma 3 *For any given $\beta \in (0, 1)$, the equilibrium values $(\underline{q}, \bar{q}) \in (0, \hat{q})^2$ are decreasing in ϕ (hence M) for all $0 < \phi < \infty$, with $\lim_{\phi \rightarrow 0} \underline{q} = \lim_{\phi \rightarrow 0} \bar{q} = \hat{q}$ and $\underline{q}(0) = \bar{q}(0) = 0$. For any given ϕ , the equilibrium values (\underline{q}, \bar{q}) are strictly increasing in β .*

Proof. (See Appendix) ■

Proposition 6 *The steady state equilibrium of the ex post pricing game is inefficient.*

Proof. From the assumptions on $u(q)$ and $c(q)$, we know that q^* is unique. From Lemma 2, we know that, in any monetary equilibrium $\underline{q} < \bar{q}$. It follows immediately that we cannot have $\underline{q} = \bar{q} = q^*$. ■

It is worth noting that this proposition does not rule out the existence of some efficient matches (for example, if either $\underline{q} = q^*$ or $\bar{q} = q^*$) however, in equilibrium, we can never have all matches efficient. The inefficiency may entail either underproduction in all matches (with high ϕ or low β), overproduction in all matches (with low ϕ or high β), or underproduction in some and overproduction in others. Interestingly, even if the *average* production level is equal to the efficient level (q^*), the price dispersion inherent in auctions guarantees that the equilibrium will always be inefficient

It is also worth noting that this inefficiency is not due to the usual holdup problem common in search models with pairwise matching and bargaining. In particular, the equilibrium in this model can be rationalized as a directed search equilibrium. In models of this type, allowing sellers to announce reserve prices ex ante, and buyers directing their search using mixed strategies, the equilibrium reserve price equals the sellers' outside option and buyers perfectly randomize as they do here (see Julien, Kennes, and King (2000)).

3.2.1 Using a lottery in pairwise matching events

In light of the inefficiency result of the above ex post pricing mechanism, we now consider the following alternative. Here, in multilateral matching events, the mechanism is precisely the same as above; however, in pairwise matching events, we allow the buyer to propose a lottery. In particular, in pairwise matches, the buyer proposes a contract asking the seller to produce a quantity \bar{q} and a lottery in which the seller receives the buyer's money with probability $\tau \in [0, 1]$.¹² In this case, the buyer solves

$$\begin{aligned} \max_{\bar{q}, \tau} & \{u(\bar{q}) + \tau\beta V_0 + (1 - \tau)\beta V_1\} \\ \text{s.t.} & \\ & -c(\bar{q}) + \tau\beta V_1 + (1 - \tau)\beta V_0 \geq \beta V_0 \\ & \tau \leq 1. \end{aligned}$$

¹²See Berensten, Molico and Wright (2002) for the implications of using lottery on money in a model with pairwise matching and bargaining.

The introduction of this mechanism produces the following results.

Proposition 7 *For $\beta < 1$, there exists a $\bar{\phi}$ such that:*

- (i) *for all $\phi > \bar{\phi}$, there exists a non-monetary and a unique monetary equilibrium with $\tau = 1$. The monetary equilibrium has the characteristic $0 < \underline{q} < \bar{q} < \hat{q}$.*
- (ii) *for all $\underline{\phi} \leq \phi \leq \bar{\phi}$, there exists a non-monetary equilibrium and two monetary equilibria with $\tau \in (0, 1)$. One monetary equilibrium has relatively high τ and $\underline{q}^0 < \bar{q} = q^*$. The other equilibrium has relatively low τ and $\underline{q}^1 > \bar{q} = q^*$.*
- (iii) *for all $\phi < \underline{\phi}$, there exists a non-monetary equilibrium and a unique monetary equilibrium where $0 = \underline{q} < \bar{q} = q^*$.*

Proof. See appendix. ■

The first part of the proposition says that if money supply is high enough, offering a lottery is not an equilibrium choice. The second part shows that for a middle range of the money supply, there are multiple equilibria, and the quantity exchanged in pairwise matches is efficient. The last part demonstrates that for low enough money supply, exchanges occur only under pairwise matches with a lottery on money. Moreover, these exchanges are efficient.

This mechanism shares some features with the ex ante pricing implications as exemplified in the following corollary.

Corollary 2 *As $\beta \rightarrow 1$, there exist a ϕ^* such that the unique monetary equilibrium converges to $\underline{q} = q^*$.*

This corollary follows again from the lemmas in the proof of Proposition 7 in the appendix where one can observe that as $\beta \rightarrow 1$, then $F'(\underline{q}) \rightarrow 0$. This result is similar to the ex ante price posting outcome as stated in Proposition 4. Namely, an efficient monetary equilibrium is possible with no price dispersion.

3.3 Optimal monetary policy

With divisible and endogenous goods production, any discussion of optimal policy must consider not only the number of matches ($X(M) = (1 - M) (1 - e^{-(M/(1-M))})$ as in Section 2) but also the surplus from each match, as measured by $u(q(M)) - c(q(M))$. Within any period, expected surplus is the product of the two.

We consider two different cases. In the first, the policymaker can control both M and q directly and independently. In the second, the policymaker can control only M , and must accept the equilibrium response of $q(M)$.

3.3.1 Choosing M and q Independently

The policymaker's problem is:

$$\max_{\{M,q\}} Y = (1 - M) \left(1 - e^{-(M/(1-M))}\right) [u(q) - c(q)].$$

The solution to this problem is given by:

$$M^* = 0.533$$

$$u'(q^*) = c'(q^*)$$

These two equations characterize optimality along the extensive margin and the intensive margin respectively. Clearly, optimality along the extensive margin is precisely the same as in Section 2 – here, the policymaker seeks to maximize the number of matches, as before. Similarly, optimality along the intensive margin is precisely the efficiency condition considered above.

3.3.2 Choosing M when $q = q(M)$

The policymaker's problem is:

$$\max_{\{M\}} Y = (1 - M) \left(1 - e^{-(M/(1-M))}\right) [u(q(M)) - c(q(M))]$$

Assuming concavity of the objective function, the first-order condition is:

$$X'(M) (u(q(M)) - c(q(M))) + X(M) (u'(q(M)) - c'(q(M))) q'(M) = 0$$

where the first term is the impact on a change in the stock of money on the extensive margin and the second the impact on the intensive margin. Optimality along the extensive margin is given $X'(M) = 0$ and the efficient intensive margin is given by $u'(q(M)) = c'(q(M))$. In general, there is no reason to presume that optimality along these two margins will coincide.

The function $q(M)$, of course, depends on the specific pricing mechanism used. With *ex ante* price posting, this is determined by (34). In principle, then, with price posting, surplus maximization is possible through judicious monetary policy. With *ex post* auctions, as pointed out above, since optimality along the intensive margin is possible only in the presence of lotteries, surplus maximization is possible only in that case.¹³

4 Divisible Goods and Money

We now relax the assumption that money is indivisible. We assume, instead, that any agent can hold any amount $m \in \mathbb{R}_+$. We restrict attention to the *ex ante* price posting mechanism, since this is simple to work with and (as shown above) admits equilibria that maximize expected surplus. In all other respects, the model and general notation remain the same as in Section 3.1. In Section 3.1, as in Trejos and Wright (1995), agents are forced to alternate between acting as buyers and sellers after each trade because of the indivisibility of money. Once a buyer has made a trade, by disposing of his unit of money, he has no choice but to become a seller next period. With divisible money, it is not clear that a buyer who traded would act a seller next period. For example, if the price paid is only half his money holdings, he could, in principle, afford to act as a buyer for two consecutive periods.

¹³As mentioned above, Trejos and Wright (1995) do not consider the problem of surplus maximization in their model. Rather, they consider the problem of welfare maximization with the equilibrium values of buyers and sellers in the social welfare function, weighted by the proportions of each, with parametric matching rates unaffected by the choice of M .

Moreover, an agent may choose to accumulate money holdings by acting as a seller over consecutive periods. In this section, we allow agents to choose whether to act as sellers ($i = 0$) or buyers ($i = 1$) at the beginning of each period. We show that, in effect, the Trejos and Wright (1995) setting with alternating identities is a symmetric stationary Markov strategy equilibrium with a stationary degenerate distribution of money holdings.

We assume that there is a set of sellers $S \subseteq [0, 1]$ with mass μ_0 and a set of buyers $B \subseteq [0, 1]$ with mass μ_1 , such that $\mu_0 + \mu_1 = 1$. We consider the distribution $(m_i, \mu_i), i = 0, 1$, where m_i is the money holding of agent of type i , and μ_i is the fraction (measure) of agent i holding money m_i , to be a state variable. To keep the analysis simple, we consider the following initial distribution of money holdings:

$$(m_i, \mu_i) = \begin{cases} (m_1, \mu_1) = (m, \mu) \\ (m_0, \mu_0) = (0, 1 - \mu) \end{cases}$$

That is, a fraction μ of agents holds the same number of units of money $m > 0$, and the remaining fraction $1 - \mu$ holds zero units. (In other words, the initial distribution of money holdings is degenerate.) We will show that this initial distribution can be a steady state Markov equilibrium distribution. To proceed, we need to keep the notation general to account for the possibility for agents of type i to change their money holding.

In a way analogous to that used in Section 3.1, each seller announces, in advance of matching, a quantity level $q(m_0, m_1)$ and a required money transfer $d(m_0, m_1)$, which potentially can depend on money holdings by sellers and buyers. As a consequence, the implied price is given by $p = d/q$. We will refer to the pair $\gamma(m_0, m_1) \equiv [q(m_0, m_1), d(m_0, m_1)]$ as a contract, but to simplify notation we write $\gamma \equiv [q, d]$.

The equilibrium γ announcement is solved by a simple game for which we find a Nash equilibrium. We start by assuming that all sellers but one announce a contract γ , and one seller considers announcing a level $\tilde{\gamma} \neq \gamma$. Agents who decide to act as buyers observe the announcements and select a trading partner from among the deviating and non-deviating sellers using a mixed strategy. The added difficulty here is the fact that a part of the deviating contract, namely the money transfer \tilde{d} , affects the value in subsequent periods. We therefore focus on Markov strategies: in deviating, a seller only cares about the impact of his deviation on the value next period. The Markov equilibrium is a value γ from which no unilateral deviation is

profitable and buyers select among the sellers using the same mixed strategy.

The value functions for sellers and buyers are now written as $V_0(m_0; \gamma)$ and $V_1(m_1; \gamma)$, where m_i is the state variable. We simplify the notation by omitting the dependence of the values on distribution of money holdings which is also a state variable.

To be precise, we provide the following definition.

Definition 5 *A Markov equilibrium contract announcement is a γ_e satisfying*

$$\gamma_e = \arg \max_{\tilde{\gamma}} \tilde{V}_0(m_0; \tilde{\gamma}, \gamma)$$

where $\tilde{\gamma}$ is a deviation for a seller, taking as given all other sellers' strategies γ , and the agent's choice of type next period as given. \tilde{V}_0 is the value associated with the deviation. In any deviation, a seller correctly anticipates that buyers' mixed strategies solve

$$\tilde{V}_1(m_1; \tilde{\gamma}, \gamma) = V_1(m_1; \tilde{\gamma}, \gamma).$$

where \tilde{V}_1 is the value for a buyer from selecting a deviating seller.

Therefore, in equilibrium, $\gamma_e = \tilde{\gamma} = \gamma$.

4.1 The non-deviant sellers

The present values for non-deviant sellers, and to buyers selecting non-deviant sellers are, respectively:

$$V_0(m_0; \tilde{\gamma}, \gamma) = \xi [-c(q) + \beta V(m_0 + d; \gamma'_e)] + (1 - \xi)\beta V_0(m_0; \gamma'_e) \quad (48)$$

and

$$V_1(m_1; \tilde{\gamma}, \gamma) = \psi [u(q) + \beta V(m_1 - d; \gamma'_e)] + (1 - \psi)\beta V_1(m_1; \gamma'_e) \quad (49)$$

where γ'_e is the equilibrium contract in the next period, and $V(\bullet; \gamma'_e)$ is the value for an agent next period (which will depend on the choice of being a buyer or a seller).¹⁴

The probabilities in the above value functions are the same as in Section 3.1. However, here, the buyers to sellers ratio is $\phi = \frac{\mu}{1-\mu}$. Also, again, we

¹⁴Embedded in equations (48) and (49) is the assumption that agents do not change their type until they perform transactions, as in Trejos and Wright (1995).

restrict attention to monetary equilibria where $\pi = 1$. The expressions above can characterize both non-monetary and monetary equilibria. In particular, if $q = 0$, then buyers receive nothing in exchange for money and if $q > 0$, sellers are trading goods for money.

4.2 The deviant seller

The value function for a buyer selecting the deviant seller is

$$\tilde{V}_1(m_1; \tilde{\gamma}, \gamma) = \tilde{\psi} \left[u(\tilde{q}) + \beta V(m_1 - \tilde{d}; \gamma'_e) \right] + (1 - \tilde{\psi}) \beta V_1(m_1; \gamma'_e) \quad (50)$$

where the probabilities are the same as in Section 3.1. Buyers mixed strategies are chosen such that

$$\tilde{V}_1(m_1; \tilde{\gamma}, \gamma) = V_1(m_1; \tilde{\gamma}, \gamma) \quad (51)$$

where

$$V_1(m_1; \tilde{\gamma}, \gamma) = \psi \left[u(q) + \beta V(m_1 - d; \gamma'_e) \right] + (1 - \psi) \beta V_1(m_1; \gamma'_e). \quad (52)$$

The deviant seller's value function is

$$\tilde{V}_0(m_0; \tilde{\gamma}, \gamma) = \tilde{\xi} \left(\beta V(m_0 + \tilde{d}; \gamma'_e) - c(\tilde{q}) \right) + (1 - \tilde{\xi}) \beta V_0(m_0; \gamma'_e)$$

where once more, the probabilities are the same as in Section 3.1. Since agents are allowed to choose whether to act as a buyer or a seller each period, the value is defined as

$$V(m'_i; \gamma'_e) = \max_{\sigma_h \in [0,1]} \{ \sigma_h V_0(m'_i; \gamma'_e) + (1 - \sigma_h) V_1(m'_i; \gamma'_e) \} \quad (53)$$

where σ_h is the agent's Markov strategy as a function of the history of play last period summarized by $h \in \{0, 1\}$, meaning that the agent was a seller, $h = 0$, or a buyer, $h = 1$, and m'_i is the money holding next period which depends on whether the agent was a seller or a buyer.

The best deviation solves:

$$\begin{aligned} & \max_{\tilde{\gamma} \in \mathbb{R}_+} \tilde{V}_0(m_0; \tilde{\gamma}, \gamma) \\ & \text{s.t.} \end{aligned} \quad (54)$$

$$\begin{aligned}
(i) \quad & \tilde{V}_1(m_1; \tilde{\gamma}, \gamma) = V_1(m_1; \tilde{\gamma}, \gamma) \\
(ii) \quad & -c(\tilde{q}) + \beta V(m_0 + \tilde{d}; \gamma'_e) \geq \beta V_0(m_0; \gamma'_e) \\
(iii) \quad & u(\tilde{q}) + \beta V(m_1 - \tilde{d}; \gamma'_e) \geq \beta V_1(m_1; \gamma'_e) \\
(iv) \quad & \tilde{d} \leq m_1
\end{aligned}$$

Constraint (i) represents the fact that, in a deviation, buyers anticipate sellers' behavior correctly in their mixed strategies. Using (50) and (51), this can be re-written as:

$$(i)' \quad \tilde{\psi} = \frac{[u(q) - \beta \Delta(m_1, m_1 - d; \gamma'_e)]}{[u(\tilde{q}) - \beta \Delta(m_1, m_1 - \tilde{d}; \gamma'_e)]} \psi \quad (55)$$

The trading constraints (ii) and (iii) simplify to:

$$\begin{aligned}
(ii)' \quad \tilde{q}_{\max} & \leq c^{-1} \left(\beta \Delta \left(m_0, m_0 + \tilde{d}; \gamma'_e \right) \right) \\
(iii)' \quad \tilde{q}_{\min} & \geq u^{-1} \left(\beta \Delta \left(m_1, m_1 - \tilde{d}; \gamma'_e \right) \right)
\end{aligned} \quad (56)$$

The deviant seller's problem then is reduced to:

$$\begin{aligned}
& \max_{\{\tilde{q} \in [\tilde{q}_{\min}, \tilde{q}_{\max}], \tilde{d}\}} \tilde{V}_0(m_0; \tilde{\gamma}, \gamma) \\
& \text{s.t.}
\end{aligned} \quad (57)$$

$$(i)'' \quad \tilde{\psi} = \frac{[u(q) - \beta \Delta(m_1, m_1 - d; \gamma'_e)]}{[u(\tilde{q}) - \beta \Delta(m_1, m_1 - \tilde{d}; \gamma'_e)]} \psi \quad (58)$$

$$(ii)'' \quad \tilde{d} \leq m_1 \quad (59)$$

In a symmetric Markov equilibrium $\tilde{\gamma} = \gamma = \gamma_e$, $\tilde{\xi} = \xi$, $\tilde{\psi} = \psi$, and $\tilde{\phi} = \phi$. To simplify notation, for the remainder of this section, we write the equilibrium γ_e as γ only.

Proposition 8 *The symmetric Markov equilibrium contract announcement $\gamma = (q, d)$ by sellers is characterized by:*

1.

$$\frac{c'(q)}{u'(q)} = \frac{(\xi - 1)}{(1 - \psi - \xi)} \frac{[\beta(V(m_0 + d; \gamma') - V_0(m_0; \gamma')) - c(q)]}{[u(q) - \beta(V_1(m_1; \gamma') - V(m_1 - d; \gamma'))]}. \quad (60)$$

2. $d(m) = m_1 = m$

Proof. See Appendix. ■

In a symmetric equilibrium we have $\tilde{V}_0(m_0; \tilde{\gamma}, \gamma) = V_0(m_0; \tilde{\gamma}, \gamma) = V_0(m_0; \gamma)$ and $\tilde{V}_1(m_1; \tilde{\gamma}, \gamma) = V_1(m_1; \tilde{\gamma}, \gamma) = V_1(m_1; \gamma)$. Furthermore, from Proposition 8 and the initial distribution assumption, we have, in equilibrium:

$$\begin{aligned} m_0 &= 0 \\ m_1 &= m \end{aligned}$$

We are now ready to analyze the decision to act as sellers of buyers at the beginning of a period given the symmetric equilibrium contract announcements. Rewrite the value function for an agent who decided to act as a seller using (48) and (49), and (53) at the symmetric equilibrium:

$$\begin{aligned} V_0(0; \gamma) &= \xi \left[-c(q) + \beta \left(\max_{\sigma_0 \in [0,1]} \{ \sigma_0 V_0(m; \gamma') + (1 - \sigma_0) V_1(m; \gamma') \} \right) \right] \\ &\quad + (1 - \xi) \beta V_0(0; \gamma') \end{aligned} \quad (61)$$

and for an agent who decided to act as a buyer:

$$\begin{aligned} V_1(m; \gamma) &= \psi \left[u(q) + \beta \left(\max_{\sigma_1 \in [0,1]} \{ \sigma_1 V_0(0; \gamma') + (1 - \sigma_1) V_1(0; \gamma') \} \right) \right] \\ &\quad + (1 - \psi) \beta V_1(m; \gamma') \end{aligned} \quad (62)$$

It is quite clear that deciding to be a buyer next period without money is a dominated strategy, that is $V_1(0; \gamma) = 0$, and $\sigma_1 = 1$. We are left with

$$V_1(m; \gamma) = \psi [u(q) + \beta V_0(0; \gamma')] + (1 - \psi) \beta V_1(m; \gamma'). \quad (63)$$

Therefore, any agent who was initially a seller will become a buyer upon a successful trade. The decision to be a buyer or a seller next period, given that an agent is a seller this period is not as straightforward. If a seller chooses to be a buyer next period, $\sigma_0 = 0$, he receives

$$V_0(0; \gamma) = \xi [-c(q) + \beta V_1(m; \gamma')] + (1 - \xi) \beta V_0(0; \gamma'). \quad (64)$$

However, if he chooses to be a seller next period, $\sigma_0 = 1$, he receives

$$V_0(0; \gamma) = \xi [-c(q) + \beta V_0(m; \gamma')] + (1 - \xi)\beta V_0(0; \gamma'). \quad (65)$$

We focus on steady state Markov strategies, hence, $\sigma_0 = \sigma'_0$ between any two periods. Suppose every agent chooses $\sigma_0 = 0$, but one seller considers a deviation $\tilde{\sigma}_0 \in (0, 1]$. Consider first $\tilde{\sigma}_0 = 1$. This implies the agent is a seller forever, bears the cost of production in any successful trade, and never gets to consume. Clearly, this is a dominated strategy. Since we only have one pure strategy equilibrium, $\sigma_0 = 0$, and any well defined games such as this one have an odd number of equilibria, there cannot be a mixed strategy equilibrium $\sigma_0 \in (0, 1)$.

In the steady state $\gamma = \gamma'$, and the value functions are similar to the ones derived in Section 3.1, but where buyers hold m units of money. That is,

$$V_0(0; \gamma) = \frac{\xi (\beta \psi u(q) - (1 - \beta(1 - \psi)) c(q))}{(1 - \beta)(1 - \beta(1 - \psi - \xi))} \quad (66)$$

and

$$V_1(m; \gamma) = \frac{\psi (\beta \xi u(q) - (1 - \beta(1 - \xi)) c(q))}{(1 - \beta)(1 - \beta(1 - \psi - \xi))}. \quad (67)$$

Definition 6 *A symmetric steady state Markov equilibrium is a tuple $(q, d, V_0, V_1, \sigma_h)$ satisfying the following conditions:*

- (i) $\tilde{q} = q$ and $\tilde{d} = d = m$ is the symmetric Markov equilibrium which solves the seller's problem (57) of the ex ante game characterized by equation (60) taking $V_0(0; \gamma)$ and $V_1(m; \gamma)$, and σ_h as given;
- (ii) the value functions V_0 and V_1 satisfy (66) and (67), taking γ as given;
- (iii) the sellers exchange constraint evaluated at q and d is satisfied;
- (iv) the strategies σ_h solves (53), taking q, d, V_0 and V_1 as given.

4.3 Equilibrium

The existence and equilibrium properties are now stated and derived. As in previous sections, the steady state equilibrium is referred to simply as an equilibrium.

Proposition 9 For any $\beta \in (0, 1)$ and $\mu \in (0, 1)$, in the steady state, there exists a non-monetary Markov equilibrium and a unique monetary Markov equilibrium with $q \in (0, \hat{q})$ and $d = m$.

The proof of this proposition is essentially the same of Proposition 3 and is omitted. One interesting aspect of the monetary equilibrium is that, given the number of money-holders, output is independent of the amount of money they hold.

Corollary 3 In the monetary equilibrium, given a fixed proportion of money holders holding equal amounts, output is independent of the quantity of money.

Proof. This can be observed by substituting the steady state values into (60) giving the first-order condition for q that is independent of m . ■

Therefore, when considering steady state equilibria, changing the amount of money holding proportionately to τm , for any $\tau \in \mathbb{R}_+$ would have no real effect, only a nominal one via an increase in equilibrium d , and hence, p . Thus, in this model, money is *neutral* in this sense. Molico (2004) finds similar results when analyzing proportional transfers in the Trejos and Wright (1995) environment using numerical computation and a non-degenerate distribution of money, where all agents were allocate initially a positive amount.

It is easy to see that results similar to Lemma 1 and Proposition 4 also hold. Here, output is decreasing in μ , the fraction of agents holding m units of money as in Section 3.1 and increasing on β . Moreover, as in Proposition 4, there exists a ratio ϕ^* such that $q(\phi^*) = q^*$, giving efficiency on the intensive margin.

However, there is no reason why this ratio should coincide with $\mu^* = .533$, the fraction of agents holding money which maximizes the amount of trade, the extensive margin, that we found in previous sections. Therefore, there is still potentially a trade off between the intensive and the extensive margin. For instance, one can compare the implications of different initial allocations of money. Let $(\mu^l, m^h)'$ and $(\mu^h, m^l)''$ be two possible initial allocations where $\mu^l < \mu^h$ and $m^l < m^h$. In equilibrium we obtain $q^l > q^h$, and hence, $p' = \frac{m^h}{q^l}$ and $p'' = \frac{m^l}{q^h}$. It might then be possible to choose the allocations such that $p' = p''$, but with $q^l > q^h$ with the first allocation.

Therefore, if the goal is to achieve efficiency at the intensive margin by increasing output, it could be done without or minimal nominal effects by giving more money to less agents initially. However, this could potentially mean moving away from the efficient extensive margin. The existence of this kind of trade off can be shown to hinge on the shape of the utility and cost function, i.e. tastes and technology, using different functional forms. For example, $u(q) = q$, and $c(q) = q^2$, with $\mu = .533$, yields $\phi = 1.14$ and $q(\phi) = q^*$, where efficiency is achieved on all margins. This is illustrated in Figure 4 where, once again, the function $T(q)$ is given by the dotted line, and the equilibrium occurs where this cuts the horizontal axis from above.

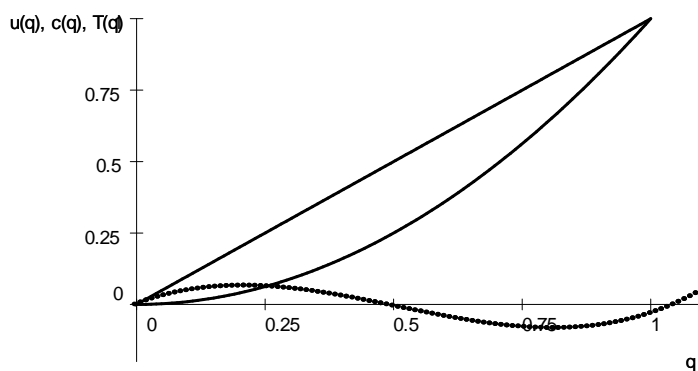


Figure 4: The Ex Ante Pricing Equilibrium with Divisible Money

Finally, we can discuss the potential implications of using a lump sum transfer η to all agents. In this case the implications would depend on the initial fraction of agents who did not hold money prior to the transfer. If the fraction is small enough, the fraction of agents choosing to act as seller would remain the same, with sellers offering contracts in equilibrium with $d = m + \eta$. The lump sum transfer would not affect the extensive margin, and only have a nominal effect since output is independent of m . On the other hand, if the fraction of agents not holding money prior to the transfer is large initially, the transfer may affect the decision to act as buyers, affecting the monetary equilibrium.

Overall, then, the effects of monetary policy depend crucially on how any increments (or reductions) in the money supply are distributed. If the proportion of money holders is kept constant, then this policy will be

neutral. However, in general, monetary policy that changes the number of money holders (and, hence, buyers) can have real effects.

5 Conclusions

The monetary exchange framework developed by Kiyotaki and Wright (1993), Trejos and Wright (1995) and others is flexible enough to be able to encompass multilateral matching and directed search, but still maintain its central message: the existence of money as a medium of exchange depends on key parameters such as peoples' discount factors and the quantity of money itself. We have found, though, that by doing this we have changed some of the key policy implications of these models. In particular, this framework points to the influence of money on the buyer-seller ratio and, thereby, the matching rate. This introduces an extra channel for monetary policy that allows for non-zero money balances to maximize expected surplus. With divisible money, yet another consideration appears: the precise effects of increasing the money supply depend on who the money is distributed to. This contrasts with results in other models with divisible money (Shi (1997) and Lagos and Wright (2005)) but is similar to the results found in Molico (2004). Interestingly, we also find that the indivisible money assumption in Trejos and Wright's (1995) model may not be as important as it first appeared – since an equivalent distribution exists as a steady state equilibrium with divisible money.

We also found the unexpected conclusion that auctions are incompatible with expected surplus maximization, unless they allow for lotteries in the event of bilateral matches. The equivalence between auctions and posted prices, with large markets, that appears with fixed production (for example, in the labor market literature cited above) does not hold in this monetary framework. This is due to the fact that quantity adjusts with price in these monetary models, optimal quantities (along the intensive margin) are unique, and auctions induce price dispersion in equilibrium. In most directed search labor market models, the quantity is fixed, so the intensive margin is not a concern.

Modelling divisible money with multilateral matching and directed search leads to a very tractable stationary distribution of money holdings – allowing for simple analytical solutions, without introducing the extra modelling

structures used by Shi (1997) or Lagos and Wright (2005). In this paper, we restricted attention to steady state equilibria, and found that increases in the money supply would be neutral only under proportional transfers to all agents. Thus, arguably, money is neutral in the long run if new money holdings are distributed in this way. It remains to be seen, though, what occurs in this model if new money holdings are, for example, randomly distributed across the entire population. We conjecture that this would lead to real effects along the transition path but only nominal effects in the long run.

6 Appendix A

Proof of Proposition 2. The first-order condition of the deviant seller without considering the constraint from equation (27) is

$$\tilde{\xi}_q[\beta\Delta(q'_e) - c(\tilde{q})] = \tilde{\xi}c'(\tilde{q}) \quad (68)$$

where $\tilde{\xi}_q$ is the derivative with respect to \tilde{q} and represents the impact of a deviation on the probability of trade. This also embeds the buyers reaction through their mixed strategies as summarized by $\tilde{\phi}$. In order to obtain an expression for this derivative, differentiate the probabilities of trade for a seller and a buyer:

$$\tilde{\xi}_q = \tilde{\xi}_\phi \tilde{\phi}_q$$

and

$$\tilde{\psi}_q = \tilde{\psi}_\phi \tilde{\phi}_q.$$

where $\tilde{\xi}_\phi$ and $\tilde{\psi}_\phi$ refer to the derivative with respect to $\tilde{\phi}$, and $\tilde{\phi}_q$ reflects the impact \tilde{q} on $\tilde{\phi}$. These two derivatives allows a link between the impact of a deviation on the buyers' probability of trade with the one for the deviant seller as follows:

$$\tilde{\xi}_q = \frac{\tilde{\xi}_\phi}{\tilde{\psi}_\phi} \tilde{\psi}_q. \quad (69)$$

Now to obtain a meaningful expression for $\tilde{\psi}_q$, differentiate the constraint in (27) to get

$$\tilde{\psi}_q = -\frac{[u(q) - \beta\Delta(q'_e)]}{[u(\tilde{q}) - \beta\Delta(q'_e)]^2} \psi u'(\tilde{q})$$

and using the constraint itself it simplifies to

$$\tilde{\psi}_q = -\frac{\tilde{\psi}}{[u(\tilde{q}) - \beta\Delta(q'_e)]}u'(\tilde{q}).$$

Notice that as one would expect $\tilde{\psi}_q < 0$. Using this into (69) and into the first-order condition (68) yields

$$-\frac{\tilde{\xi}_\phi \tilde{\psi}}{\tilde{\psi}_\phi} \frac{[\beta\Delta(q'_e) - c(\tilde{q})]u'(\tilde{q})}{[u(\tilde{q}) - \beta\Delta(q'_e)]} = \tilde{\xi}c'(\tilde{q}).$$

In a symmetric equilibrium, $\tilde{q} = q = q_e$, which implies $\tilde{\phi} = \phi$, $\tilde{\xi} = \xi$, and $\tilde{\psi} = \psi$. Furthermore, in steady state $q_e = q'_e$ and referring to q_e as simply q we find

$$-\frac{\xi_\phi \psi}{\psi_\phi \xi} \frac{[\beta\Delta(q) - c(q)]}{[u(q) - \beta\Delta(q)]} = \frac{c'(q)}{u'(q)}.$$

It is easy to verify that $-\frac{\xi_\phi \psi}{\psi_\phi \xi} = \frac{(\xi-1)}{(1-\psi-\xi)} \geq 0$. ■

Proof of Lemma 1. Using the Implicit Function Theorem write $q(\phi, \beta) = q$ as the equilibrium. Let $D_0 = [\beta\psi u(q) - (1 - \beta(1 - \psi))c(q)] > 0$ and $D_1 = [(1 - \beta(1 - \xi))u(q) - \beta\xi c(q)] > 0$, which respectively holds if $V_0 > 0$ and $V_1 > 0$. Using the value of Δ from (31) we can rewrite (34) as

$$\frac{c'(q)}{u'(q)} = g(\phi) \frac{[D_0]}{[D_1]}.$$

Take β as given and totally differentiate this equation to get

$$\begin{aligned} \left[\frac{c''u' - c'u''}{u'^2} \right] dq &= g_\phi \frac{[D_0]}{[D_1]} d\phi + g(\phi) \left[\frac{D_{0\phi}D_1 - D_0D_{1\phi}}{(D_1)^2} \right] d\phi \\ &\quad + g(\phi) \left[\frac{D'_0D_1 - D_0D'_1}{(D_1)^2} \right] dq. \end{aligned}$$

Collecting terms

$$dq = \frac{\left(g_\phi \frac{[D_0]}{[D_1]} + g(\phi) \left[\frac{D_{0\phi}D_1 - D_0D_{1\phi}}{(D_1)^2} \right] \right)}{\left(\left[\frac{c''u' - c'u''}{u'^2} \right] - g(\phi) \left[\frac{D'_0D_1 - D_0D'_1}{(D_1)^2} \right] \right)} d\phi.$$

From the definitions of we find $D_{0\phi} = \beta\psi_\phi(u(q) - c(q)) < 0$ and $D_{1\phi} = \beta\xi_\phi(u(q) - c(q)) > 0$, and it is easy to show that $g_\phi < 0$. Therefore, the numerator of the above equation is always negative. Next, the first term in the denominator is always positive since $u'' \leq 0$. It remains to show that the term $\left[\frac{D'_0D_1 - D_0D'_1}{(D_1)^2} \right] < 0$ for all $q \in (0, q_{\max})$. One can verify that this is

the case for all β and $\phi \in (0, \bar{\phi})$ if and only if $\frac{u'}{c'} < \frac{u}{c}$ which holds given the properties of the functions u and c . Finally, notice that there is discontinuity $\phi = 0$. Given the above demonstration, $\lim_{\phi \rightarrow 0} q(\phi, \beta) = q_{\max}$, but at $\phi = 0$, no monetary equilibrium can exist and $q(0) = 0$.

To show the impact of β on q consider the following derivatives $D_{0\beta} = \psi(u - c) > 0$ and $D_{1\beta} = \xi(u - c) > 0$. Totally differentiate the first-order condition to get

$$\begin{aligned} \left[\frac{c'u' - c'u''}{u'^2} \right] dq &= g(\phi) \left[\frac{D_{0\beta}D_1 - D_0D_{1\beta}}{(D_1)^2} \right] d\beta \\ &= g(\phi)(u - c) \frac{(\psi D_1 - \xi D_0)}{(D_1)^2} d\beta \end{aligned}$$

where the sign of $(\psi D_1 - \xi D_0)$ as the same sign as $\Delta = V_1 - V_0 > 0$. ■

Proposition 10 *Assume $c'(0) > 0$,*

- (i) *For any ϕ (hence M), there exists a $\underline{\beta} > 0$, such that for all $\beta < \underline{\beta}$ the unique equilibrium is a non-monetary equilibrium;*
- (ii) *For any $\beta < 1$, there exists a $\bar{\phi}$ such that for all $\phi > \bar{\phi}$, the unique equilibrium is a non-monetary equilibrium;*
- (iii) *For all other parameter values, there exists a non-monetary equilibrium and a unique monetary equilibrium.*

Proof. First the non-monetary equilibrium $q = V_0 = V_1 = 0$ is easily established since it satisfies all the conditions from the definition of an equilibrium. Write the following transformation function representing the first derivative of the seller's objective function evaluated at the steady state symmetric equilibrium q :

$$T(q) = g(\phi)[\beta\Delta(q) - c(q)]u'(q) - [u(q) - \beta\Delta(q)]c'(q)$$

where $\Delta(q)$ is the steady state value from equation (31) and $g(\phi) = \left(\frac{1-\xi}{\psi+\xi-1} \right)$. It implies that $T(q) = 0$ represent the first-order condition, and in steady state the equation (28). To show existence, observe that $T(0) = 0$, and $T(q)$, is continuous since $u(\bullet)$ and $c(\bullet)$ are continuous, and recall that only $q \in [0, q_{\max}]$, are equilibrium steady state candidates. Now consider

$$T'(0) = g(\phi)[\beta\Delta'(0) - c'(0)]u'(0) - [u'(0) - \beta\Delta'(0)]c'(0).$$

and it can be shown that $[\beta\Delta'(0) - c'(0)] > 0$ if and only if $V_0'(0) > 0$ and $[u'(0) - \beta\Delta'(0)] > 0$ if and only if $V_1'(0) > 0$, which they are. It follows that $T'(0) = 0$ if and only if

$$\underline{\beta} = \frac{(g(\phi) + 1)c'(0)u'(0)}{\Delta'(0)|_{\underline{\beta}}[g(\phi)u'(0) + c'(0)]} > 0 \text{ for all } \phi$$

or

$$g(\bar{\phi})[\beta\Delta'(0)|_{\bar{\phi}} - c'(0)]u'(0) = [u'(0) - \beta\Delta'(0)|_{\bar{\phi}}]c'(0) \text{ for all } \beta < 1.$$

Noting that $g'(\phi) < 0$, with $\frac{\partial T'(0)}{\partial \beta} > 0$ and $\frac{\partial T'(0)}{\partial \phi} < 0$, for all $\beta < \underline{\beta}$ or $\phi > \bar{\phi}$, $T'(0) < 0$ and there cannot be a value of $q > 0$ satisfying the first-order condition. Only non-monetary equilibrium exist. Otherwise, we have $T'(0) > 0$ and the rest of the proof proceed as in Proposition 5. ■

Proof of Lemma 2. Assume $\Delta' > 0$, then possible outcomes within a match implies $c(\bar{q}) = u(\underline{q}) > 0$, and hence, $\underline{q} > 0$. Under the assumptions about $u(\bullet)$ and $c(\bullet)$, $u^{-1}(c(\bar{q})) = \underline{q}$ is a strictly convex (one to one) equilibrium relationship, and note that $u(\bar{q}) > c(\bar{q}) = u(\underline{q})$ for $\bar{q} \in (0, \hat{q})$. If $\Delta' = 0$, clearly $\underline{q} = \bar{q} = 0$. Since $c(\bar{q}) = \beta\Delta'$ implies \bar{q} is increasing in Δ' , there exist a $\hat{\Delta}'$ such that $\bar{q} = \hat{q}$. ■

Proof of Lemma 3. First we must show that \bar{q} is decreasing in ϕ . Using the definition of $T(\bar{q})$ from the proof of Proposition 5

$$T(\bar{q}) = \beta \frac{\psi_p u(\bar{q}) + \xi_m c(u^{-1}(c(\bar{q})))}{1 - \beta(1 - \psi_p - \xi_m)} - c(\bar{q})$$

and the fact that we have uniqueness, it must be that $T'(\bar{q}) < 0$ at the equilibrium $\bar{q} > 0$. Totally differentiating $T(\bar{q})$ with respect to \bar{q} and ϕ , we need to show that

$$\frac{d\bar{q}}{d\phi} = -\frac{T_\phi}{T'(\bar{q})} < 0.$$

Therefore, we are left to demonstrate that $T_\phi < 0$. Taking the derivative and using $\underline{q} = u^{-1}(c(\bar{q}))$ into the equilibrium values (43) and (44) we find

$$T_\phi = \frac{\beta}{1 - \beta(1 - \psi_p - \xi_m)} \psi_{m\phi} \frac{(1 - \beta)V_1(\bar{q})}{\psi_m} - \xi_{p\phi} \frac{(1 - \beta)V_0(\bar{q})}{\xi_p} < 0$$

since $\psi_{m\phi} < 0$ and in equilibrium it must be that $V_1(\bar{q}) > 0$ and $V_0(\bar{q}) > 0$.

Next we must show \bar{q} is increasing in β . Totally differentiating $T(\bar{q})$ again with respect to \bar{q} and β , we need to show that

$$\frac{d\bar{q}}{d\beta} = -\frac{T_\beta}{T'(\bar{q})} > 0,$$

which holds since $T_\beta = (1 - \beta) \frac{\psi_p u(\bar{q}) + \xi_m c(u^{-1}(c(\bar{q})))}{(1 - \beta(1 - \psi_p - \xi_m))^2} > 0$. ■

Proof of Proposition 7.

Set up the Lagrangian

$$\mathcal{L} = \max_{\bar{q}, \tau} \{u(\bar{q}) + \beta V_1 - \tau \beta \Delta + \lambda_0(\tau \beta \Delta - c(\bar{q})) + \lambda_1(1 - \tau)\}$$

with first-order conditions

$$\begin{aligned} u'(\bar{q}) &= \lambda_0 c'(\bar{q}) \\ (\lambda_0 - 1)\beta \Delta &= \lambda_1 \end{aligned}$$

and complementary slackness conditions

$$\begin{aligned} \lambda_0(\tau \beta \Delta - c(\bar{q})) &= 0 \\ \lambda_1(\tau - 1) &= 0 \end{aligned}$$

Possibilities:

(1) $\lambda_0 > 0, \lambda_1 > 0$

$$\begin{aligned} \beta \Delta &= c(\bar{q}) \\ \tau &= 1 \end{aligned}$$

This is the case of the standard auction in the paper. Note that $u'(\bar{q}) = \lambda_0 c'(\bar{q})$, but $\lambda_0 > 0$ can be bigger or smaller than 1, so consistent with our findings.

(2) $\lambda_0 = 0, \lambda_1 > 0$ or $\lambda_0 = \lambda_1 = 0$

$$u'(\bar{q}) = 0 \text{ and } \beta \Delta = 0$$

cannot be an equilibrium.

(3) $\lambda_0 > 0, \lambda_1 = 0$. From $(\lambda_0 - 1)\beta \Delta = \lambda_1 = 0$ or $\lambda_0 = 1$.

$$\begin{aligned} u'(\bar{q}) &= c'(\bar{q}) \Rightarrow \bar{q} = q^* \\ \tau &= \frac{c(q^*)}{\beta \Delta} \end{aligned}$$

This joined with the $n > 1$ case, we have $u(\underline{q}) = \beta\Delta$ and equilibrium $\tau = \frac{c(q^*)}{u(\underline{q})}$ once we find equilibrium \underline{q} . Using the value for Δ we have

$$u(\underline{q}) = \beta \frac{\psi u(q^*) + \xi c(\underline{q})}{1 - \beta(1 - \frac{c(q^*)}{u(\underline{q})}\psi - \xi)}$$

Formulate

$$F(\underline{q}) = \beta\psi(u(q^*) - c(q^*)) - \beta\xi(u(\underline{q}) - c(\underline{q})) - (1 - \beta)u(\underline{q})$$

in equilibrium $F(\underline{q}) = 0$. First note that $F(0) = \beta\psi(u(q^*) - c(q^*)) > 0$ and $F'(\underline{q}) = \beta\xi c'(\underline{q}) - (1 - \beta(1 - \xi))u'(\underline{q}) \geq 0$. Since $\beta\xi < (1 - \beta(1 - \xi))$ for all $\beta < 1$, when $\underline{q} < q^*$, $u'(\underline{q}) > c'(\underline{q})$ and $F'(\underline{q}) < 0$. However, for $\underline{q} > q^*$, $u'(\underline{q}) > c'(\underline{q})$, then $F'(\underline{q}) > 0$. In fact $F(\underline{q})$ reaches a minimum and $F(\hat{q}) > 0$. Therefore, there exist two values of \underline{q} such that $F(\underline{q}) = 0$. Hence, multiple equilibria.

Lemma 4 *The equilibrium $\underline{q}^0 < q^*$ when $\tau \in (0, 1)$ is strictly decreasing in ϕ and increasing in β .*

Proof of the Lemma. Setting $F(\underline{q}^0) = 0$ and totally differentiating with respect to \underline{q}^0 and ϕ yields

$$\frac{d\underline{q}^0}{d\phi} = \frac{(\beta\psi_\phi(u(q^*) - c(q^*)) - \beta\xi_\phi(u(\underline{q}^0) - c(\underline{q}^0)))}{-F'(\underline{q}^0)} < 0$$

since $\psi_\phi < 0$, $\xi_\phi > 0$ and $F'(\underline{q}^0) < 0$ for $\underline{q} < q^*$. Similarly, but with respect to \underline{q}^0 and β yields

$$\frac{d\underline{q}^0}{d\beta} = \frac{\psi(u(q^*) - c(q^*)) + \xi c(\underline{q}) + (1 - \xi)u(\underline{q})}{-F'(\underline{q})} > 0.$$

■

Lemma 5 *The equilibrium $\underline{q}^1 < q^*$ when $\tau \in (0, 1)$ is strictly increasing in ϕ and in β .*

Proof of the Lemma. Similar to the proof of the previous lemma but noticing that $F'(\underline{q}^1) > 0$ for $\underline{q}^1 > q^*$. ■

The rest of the proof follows from these Lemmas and Lemma 1 in the paper. ■

Proof of Proposition 8.

Consider the Lagrangian for the deviant seller's problem in (57)

$$\mathcal{L} = \max_{\{\tilde{q} \in [\tilde{q}_{\min}, \tilde{q}_{\max}], \tilde{d}\}} \tilde{V}_0(m_0; \tilde{\gamma}, \gamma) + \lambda_0 \left(\tilde{\psi} - \frac{[u(q) - \beta\Delta(m_1, m_1 - d; \gamma'_e)]}{[u(\tilde{q}) - \beta\Delta(m_1, m_1 - \tilde{d}; \gamma'_e)]} \psi \right) + \lambda_1(m_1 - \tilde{d}) \quad (70)$$

with the associated first-order conditions with respect to:

$$\begin{aligned} (\tilde{q}) : \quad & \frac{\partial \tilde{V}_0(m_0; \tilde{\gamma}, \gamma)}{\partial \tilde{q}} + \lambda_0 \left(\tilde{\psi}_{\tilde{q}} + \frac{[u(q) - \beta\Delta(m_1, m_1 - d; \gamma'_e)]}{[u(\tilde{q}) - \beta\Delta(m_1, m_1 - \tilde{d}; \gamma'_e)]^2} \psi u'(\tilde{q}) \right) = 0 \\ (\tilde{d}) : \quad & \frac{\partial \tilde{V}_0(m_0; \tilde{\gamma}, \gamma)}{\partial \tilde{d}} + \lambda_0 \left(\tilde{\psi}_{\tilde{d}} - \frac{[u(q) - \beta\Delta(m_1, m_1 - d; \gamma'_e)]}{[u(\tilde{q}) - \beta\Delta(m_1, m_1 - \tilde{d}; \gamma'_e)]^2} \psi \frac{\partial V(m_1 - d; \gamma'_e)}{\partial \tilde{d}} \right) = \lambda_1 \\ (\lambda_0) : \quad & \tilde{\psi} - \frac{[u(q) - \beta\Delta(m_1, m_1 - d; \gamma'_e)]}{[u(\tilde{q}) - \beta\Delta(m_1, m_1 - \tilde{d}; \gamma'_e)]} \psi = 0 \\ (\lambda_1) : \quad & m_1 - \tilde{d} = 0. \end{aligned}$$

It should be clear that $\lambda_0 > 0$ since the associated constraint is always binding for any deviations. Otherwise, buyers would not be using mixed strategies. From Kuhn-Tucker conditions, the first two conditions become:

$$\begin{aligned} \frac{\partial \tilde{V}_0(m_0; \tilde{\gamma}, \gamma)}{\partial \tilde{q}} &= \tilde{\xi}_q [\beta\Delta(m_0, m_0 + \tilde{d}; \gamma'_e) - c(\tilde{q})] - \tilde{\xi} c'(\tilde{q}) = 0 \quad (71) \\ \frac{\partial \tilde{V}_0(m_0; \tilde{\gamma}, \gamma)}{\partial \tilde{d}} &= \lambda_1 \end{aligned}$$

where it is easily shown that $\frac{\partial \tilde{V}_0(m_0; \tilde{\gamma}, \gamma)}{\partial \tilde{d}} > 0$, hence, $\lambda_1 > 0$, and we have $\tilde{d} = m_1$.

The remainder of the proof follows the one for Proposition 2 where we exploit link between the probabilities of trade given by (69) but now

$$\tilde{\psi}_{\tilde{q}} = - \frac{[u(q) - \beta\Delta(m_1, m_1 - d; \gamma'_e)]}{[u(\tilde{q}) - \beta\Delta(m_1, m_1 - \tilde{d}; \gamma'_e)]^2} \psi u'(\tilde{q})$$

which simplifies to

$$\tilde{\psi}_{\tilde{q}} = - \frac{\tilde{\psi}}{[u(\tilde{q}) - \beta\Delta(m_1, m_1 - \tilde{d}; \gamma'_e)]} u'(\tilde{q}).$$

Using this into (69) and into the first-order condition (71) yields

$$-\frac{\tilde{\xi}_\phi \tilde{\psi}}{\tilde{\psi}_\phi} \frac{[\beta \Delta(m_0, m_0 + \tilde{d}; \gamma'_e) - c(\tilde{q})] u'(\tilde{q})}{[u(\tilde{q}) - \beta \Delta(m_1, m_1 - \tilde{d}; \gamma'_e)]} = \tilde{\xi} c'(\tilde{q}).$$

In a symmetric equilibrium, $\tilde{q} = q = q_e$, which implies $\tilde{\phi} = \phi$, $\tilde{\xi} = \xi$, and $\tilde{\psi} = \psi$. It is easy to verify that $-\frac{\xi_\phi \psi}{\psi_\phi \xi} = \frac{(\xi-1)}{(1-\psi-\xi)} \geq 0$. ■

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