On Flexibility and Productivity

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Abstract

We introduce a joint model of labor market search and firm size dynamics to explain the differential in labor market and productivity outcomes between the U.S. and the European Union. At the core, our model is a hybrid of the labor market search model by Mortensen and Pissarides (1994) and the model of the size distribution firms by Lucas (1978). Around this core, however, we add several layers that we use to add rigidities that affect the 'flexibility' with which resources are allocated in our model economy. The first layer that we add is creative destruction. That is, we relate the need for job reallocations to the growth rate of the economy. In each period better firms enter while inferior firms exit, in the spirit of Jovanovic (1982). Hence, contrary to Mortensen and Pissarides (1994), exit of firms, and the destruction of the jobs that they offer, is thus endogenous in our model. The second layer that we add is the occupational choice of workers that are without a job. That is, in equilibrium workers endogenously decide whether to look for a job or to become an entrepreneur based on the quality of a business idea that they have. The third layer is the dynamic hiring and firing decisions of firms. Similar to Hopenhayn and Rogerson (1993), the firm dynamics in our model economy are in large part driven by the dynamic hiring and firing decisions made by the existing firms. We use this model to identify which types of rigidities have the biggest distortionary effect on the allocation of resources both in terms of labor as well as in terms of productivity.

1 Introduction

We introduce a joint model of labor market search and firm size dynamics with many sources of rigidities. This model jointly explains the differential in labor market and productivity outcomes between the U.S. and the European Union.

Differences in labor market outcomes between the E.U. and the U.S. account for the bulk of per capita income differences between the U.S. and Europe. E.U. GDP per capita in 2002 was 73% of that in the U.S.[1]. Of this 27% difference, 19% is due to different labor market outcomes. Lower labor force participation in Europe reduces E.U. per capita GDP by 6% relative to the U.S., while higher unemployment reduces it by another 2%. The remainder of the 19% is due to European workers working fewer hours than their American counterparts.

On the productivity side, Europe's relative productivity level, in terms of output per hour worked, increased from 77% of that of the U.S. in 1979 to 94% in 1994. However, it has since declined to 85% (Gordon and Dew-Becker, 2005). Recent evidence on differences in firm size dynamics between Europe and the U.S. gives a more detailed picture of the source of these productivity disparities.

No matter whether the discussion is about labor markets or about productivity differentials between Europe and the U.S., a lack of flexibility in European labor and product markets is often presented as the culprit for Europe's dismal economic performance in the last decade. The problem with this explanation is that the word 'flexibility' is used in many different contexts. As a consequence, 'flexibility' has become somewhat of an empty placeholder for everything that is wrong with European economies.

In order to understand how labor and product market regulations affect the flexibility with which resources are allocated and reallocated, we introduce a model of labor market search and firm size dynamics in which we allow for a broad set of rigidities. We use this model to identify which types of rigidities have the biggest distortionary effect on the allocation of resources both in terms of labor as well as in terms of productivity

At the core, our model is a hybrid of the labor market search model by Mortensen and Pissarides (1994) and the model of the size distribution firms by Lucas (1978). Around this core, however, we add several layers that we use to add rigidities that affect the 'flexibility' with which resources are allocated in our model economy.

The first layer that we add is creative destruction. That is, we relate the need for job reallocations to the growth rate of the economy. In each period better firms enter while inferior firms exit, in the spirit of Jovanovic (1982). Hence, contrary to Mortensen and Pissarides (1994), exit of firms, and the destruction of the jobs that they offer, is thus endogenous in our model.

[1]OECD National Accounts and Labor Force Statistics.

The second layer that we add is the occupational choice of workers that are without a job. That is, in equilibrium workers endogenously decide whether to look for a job or to become an entrepreneur based on the quality of a business idea that they have.

The third layer is the dynamic hiring and firing decisions of firms. Similar to Hopenhayn and Rogerson (1993), the firm dynamics in our model economy are in large part driven by the dynamic hiring and firing decisions made by the existing firms. These hiring and firing decisions are dynamic because fixed costs for hiring and firing imply that firms do not equate the marginal product of labor to the real wage in each period, but instead expand and decrease their size only when the marginal product of labor is substantially different from the wage paid. In this sense, our model is very similar to Bentolila and Bertola's (1993) model of Eurosclerosis.

2 Model

2.1 Couch potato: Business plan aha-erlebniss

Let V_t^{CP} be the value of being a couch potato at the beginning of period t.

$$V_t^{CP} = \int \max\left\{V_t^E\left(z_t, 0\right) - \eta_t, V_t^U\right\} dF_t^B\left(z\right) \tag{1}$$

Here $V_t^E\left(z, n\right)$ is the value at time t of being an entrepeneur that operates a firm at productivity level z with n employees. The parameter η_t reflects the entrepeneur's entry cost. V_t^U is the value of being unemployed and looking for a job in period t. The distribution $F_t^B\left(z\right)$ reflects the state of technology in period t. It is the distribution from which business plans in period t are drawn.

2.2 Entrepeneur

The value at the beginning of time t of operating a business at productivity level z_t and that is of size n_{t-1} is determined by the optimal hiring decision. That is, the entrepreneur first decides on how many workers to hire or fire and then produces output. Output per worker is assumed to equal z_t. The entrepreneur faces decreasing returns to scale through a per worker overhead operating cost $c_t\left(z_t, n_t\right)$. The cost of changing its employment level from n_{t-1} to n_t is given by the hiring cost function, $h_t\left(n_t, n_{t-1}, z_t\right)$. Productivity is subject to shocks that are reflect by the conditional distribution $Q_t^z\left(z_{t+1} \left| z_t\right.\right)$. After the entrepreneur observes the productivity level for the next period, he/she can decide to either stay in the market or exit and become a couch potato in the next period.

$$V_t^E(z_t, n_{t-1}) = \max_{n_t}\{n_t[z_t - w_t(z_t,n_t) - \tau_t - c_t(z_t,n_t)] - \tau_t - h_t(n_t, n_{t-1}, z_t) \tag{2}$$

$$+\frac{1}{1+r}\int \max\left\{V_{t+1}^E(z_{t+1},n_t), V_{t+1}^{CP}\right\} dQ_t^z(z_{t+1}|z_t)\Big\} \tag{3}$$

Here $w_t(z_t, n_t)$ is the wage which is determined in a wage bargaining process explained in much more detail below. τ_t denotes the taxes levied to pay for the unemployment benefit. We include these taxes to take into account the tax pressures on firms and workers in countries with more extensive welfare programs. In the current version, we model these taxes as lump-sum on each job.

2.3 Unemployed individual

The way we determine the value of being unemployed is very similar to all the search models that I do not know references to. The main difference is that, if the unemployed individual does not get an offer or not accept an offer, at the beginning of the next period it will experience another business plan aha-erlebniss. That is, its reservation level is determined by that value of being a couch potato rather than of being unemployed in the next period. Let θ_t represent the ratio of vacancies to unemployed individuals. Just like in Mortensen and Pissarides, we will assume that the probability of getting a job offer is $\theta_t q(\theta_t)$. Furthermore, unemployed individuals are paid a per period benefit equal to b_t.

The value of being unemployed can be written as

$$\begin{aligned} V_t^U &= \theta_t q(\theta_t)\int \max\left\{V_t^J(z_t,n_t), b_t - \tau_t + \frac{1}{1+r}V_{t+1}^{CP}\right\} dF_t^O(z_t,n_t) \\ &\quad + (1-\theta_t q(\theta_t))\left(b_t - \tau_t + \frac{1}{1+r}V_{t+1}^{CP}\right) \end{aligned} \tag{4}$$

Here $F_t^O(z_t, n_t)$ is the endogenously determined distribution of job offers.

2.4 Worker

What remains in terms of value functions is the value of being employed in a firm that operates at productivity level z_t and of size n_t. The flow pay offs for such a worker are the wage payments in the current period, $w_t(z_t, n_t)$. What complicates this value is that there are two sides to this match. A match can severed by both the firm as well as the worker. If either of the sides decides to call it quits the match ends. Lay offs occur randomly among workers in a firm. That is, if a firm decides to lay workers off, i.e. $n_t < n_{t-1}$. Then the probability of a worker being let go in period t is $(n_{t-1} - n_t)/\, n_{t-1}$. Let $\nu_t(z_t, n_{t-1})$ be the optimal firm size decision that follows from (2). We will first calculate the probability of being laid off at the beginning of next period, in a firm with productivity level z_{t+1} and at a firm of size (before the hiring firing decision) n_t. This probability is

$$P^F_{t+1}(z_{t+1}, n_t) = I\left[\nu_{t+1}(z_{t+1}, n_t) < n_t\right] \left(\frac{n_t - \nu_{t+1}(z_{t+1}, n_t)}{n_t} \right) \tag{5}$$

which allows us to write the probability of being employed at productivity level z_t and at a firm of size n_t as

$$\begin{aligned} V^J_t(z_t, n_t) &= w_t(z_t, n_t) \\ &+ \frac{1}{1+r} \int \left(1 - P^F_{t+1}(z_{t+1}, n_t)\right) V^J_{t+1}(z_{t+1}, \nu_{t+1}(z_{t+1}, n_t))\, dQ^z_t(z_{t+1}\,|z_t) \\ &+ \frac{1}{1+r} \int P^F_{t+1}(z_{t+1}, n_t)\, V^{CP}_{t+1} dQ^z_t(z_{t+1}\,|z_t) \end{aligned} \tag{6}$$

Note that this assumes that there is no on the job search. That is, to change jobs a worker first has to become a couch potato.

2.5 Wage bargaining

Wage bargaining is assumed to satisfy standard Nash bargaining principles. This requires us to first define the threat levels of the workers and the entrepreneur. Because, all workers are ex ante the same we will assume that they will equally share their bargained part of the surplus. Both the workers' and the entrepreneur's outside values are becoming a couch potato. Hence, the surplus for the entrepreneur that manages a firm of productivity level z_t and after the new size n_t has been realized equals

$$\begin{aligned} S^E_t(z_t, n_t) &= n_t\left[z_t - w_t(z_t, n_t) - \tau_t - c_t(z_t, n_t)\right] - \tau_t \\ &+ \frac{1}{1+r} \int \max\left\{V^E_{t+1}(z_{t+1}, n_t), V^{CP}_{t+1}\right\} dQ^z_t(z_{t+1}\,|z_t) - V^{CP}_t \end{aligned} \tag{7}$$

The surplus of the worker is

$$S^J_t = n_t\left[V^J_t(z_t, n_t) - V^{CP}_t\right] \tag{8}$$

So the Nash-bargaining equilibrium wage is that $w_t(z_t, n_t)$ which maximizes

$$\left(S^E_t\right)^{1-\beta} \left(S^J_t\right)^{\beta} \text{ where } 0 < \beta < 1 \tag{9}$$

Here β reflects the bargaining power of workers. This implies that the equilibrium wage satisfies

$$-\left(1-\beta\right)\frac{\frac{\partial S_t^E}{\partial w_t(z_t,n_t)}}{S_t^E}=\beta\frac{\frac{\partial S_t^J}{\partial w_t(z_t,n_t)}}{S_t^J} \tag{10}$$

which simplifies to

$$(1-\beta)S_t^J=\beta S_t^E \tag{11}$$

The Nash-bargaining equilibrium wage $w_t\left(z_t,n_t\right)$ can be solved as

$$\begin{aligned} w_t\left(z_t,n_t\right) &= \beta\left[z_t-\left(1+\frac{1}{n_t}\right)\tau_t-c_t(z_t,n_t)\right]+\frac{\beta}{n_t(1+r)}\int\max\left\{V_{t+1}^E\left(z_{t+1},n_t\right),V_{t+1}^{CP}\right\}dQ_t^z\left(z_{t+1}\left|z_t\right.\right) \\ &-\frac{(1-\beta)}{n_t(1+r)}\int\left(1-P_{t+1}^F\left(z_{t+1},n_t\right)\right)\max\left\{V_{t+1}^J\left(z_{t+1},\nu_{t+1}\left(z_{t+1},n_t\right)\right),V_{t+1}^{CP}\right\}dQ_t^z\left(z_{t+1}\left|z_t\right.\right) \\ &+\frac{(1-\beta)}{n_t(1+r)}\int P_{t+1}^F\left(z_{t+1},n_t\right)V_{t+1}^{CP}dQ_t^z\left(z_{t+1}\left|z_t\right.\right)+\frac{(1-2\beta)}{n_t}V_t^{CP} \end{aligned} \tag{12}$$

2.6 Matching technology

The matching technology that we assume is the same as in Mortensen and Pissarides (199?) and is

$$q\left(\theta\right)=\theta^{1-\alpha} \tag{13}$$

2.7 Equilibrium dynamics

In every period we have to determine the size distribution of firms. How many workers get fired, leave their jobs. How many couch potatoes there are. How many unemployed individuals there are. How many jobs are created. The job offer distribution.

Let $\Phi_t\left(z_t,n_t\right)$ be the number of firms operating at productivity level z_t and of size n_t (after hiring and firing decision) in period t and let Φ_t be total number of firms at time t. Then

$$\Phi_t=\int\int\Phi_t\left(z_t,n_t\right)dz_tdn_t \tag{14}$$

and let $\phi_t\left(z_t,n_t\right)=\Phi_t\left(z_t,n_t\right)/\Phi_t$ be the fraction of firms of productivity level z_t and size n_t. Then the total number of workers is

$$N_t=\int\int\Phi_t\left(z_t,n_t\right)n_tdz_tdn_t \tag{15}$$

Average labor productivity at time t is equal to

$$ALP_t=\frac{1}{N_t+\Phi_t}\int\int\phi_t\left(z_t,n_t\right)\left(n_tz_t-c_t\left(z_t,n_t\right)\right)dz_tdn_t \tag{16}$$

which is total value added produced divided by the number of workers and entrepreneurs. The size distribution of firms is given by

$$\phi_t^n(n_t) = \int_{z_t} \phi_t(z_t, n_t)\, dz_t \tag{17}$$

The number of workers that get laid off equals

$$N_t^F = \int\int \Phi_{t-1}(z_{t-1}, n_{t-1})\, I[\nu_t(z_t, n_{t-1}) < n_{t-1}]\, (n_{t-1} - \nu_t(z_t, n_{t-1}))\, dn_t dQ_{t-1}^z(z_t | z_{t-1}) \tag{18}$$

The number of jobs created in firms of type (z_t, n_t) is determined by first identifying which types of firms will end up hiring n_t workers. Define the set

$$N(z_t, n_t) = \{n_{t-1} | \nu_t(z_t, n_{t-1}) = n_t\} \tag{19}$$

$$N_t^C(z_t, n_t) = \int\int_{N(z_t, n_t)} \Phi_{t-1}(z_{t-1}, n_{t-1})\, I[\nu_t(z_t, n_{t-1}) > n_{t-1}]\, (\nu_t(z_t, n_{t-1}) - n_{t-1})\, dn_{t-1} dQ_{t-1}^z(z_t | z_{t-1}) \tag{20}$$

$$N_t^C = \int\int N_t^C(z_t, n_t)\, dz_t dn_t \tag{21}$$

Hence

$$F_t^O(z_t, n_t) = N_t^C(z_t, n_t) / N_t^C \tag{22}$$

In equilibrium there are no firms for which workers decide to leave and are then replaced by new hires. This is because the new hires face the same marginal trade off as the workers that leave and will thus make the same decision. This is a consequence of the no-on-the-job-search-decision assumption.

We will assume that the government uses its current tax revenue generated through the τ_t's to finance next period's unemployment benefits. Everyone, including people who receive the unemployment benefits has to pay taxes. In that case, this condition is

$$\tau_t = (1 + r)\, b_{t+1}(1 - N_{t+1} - \Phi_{t+1}) \tag{23}$$

and can be plugged into the contraction mapping.

2.8 Hiring and firing costs

2.9 Technological progress

We will include exogenous technological progress in this model by assuming some exogenously moving technology frontier

$$\overline{z}_t = \overline{z}_0 (1+g)^t \text{ where } 0 < g < r \tag{24}$$

where g is the exogenous growth rate of the technological frontier. In every period, $\overline{z}_t$ reflects the maximum productivity level of a firm at time t. That is, the support of the distributions of productivity levels of business plans, i.e. $F_t^B(z)$, as well as of the transitional distribution for existing firms, i.e. $Q_t^z(z_{t+1} | z_t)$ for all z_t, at time t is $[0, \overline{z}_t]$.

In the rest of this paper, we express our whole model in a state variable for a firm that is its productivity level relative to the frontier. Hence, for each firm, we write

$$z_t = \xi_t \overline{z}_t \tag{25}$$

and consider the evolution of ξ_t rather than z_t.

The reason for this transformation is that this allows us to transform the value function equations into a system that is stationary for which we can define a proper steady state, i.e. a balanced growth path in this case to be precise. To this stationary system, we can then apply our standard solution methods. We detrend all value functions in the sense that

$$\widetilde{V}_t = \frac{V_t}{\overline{z}_t} \tag{26}$$

For the equation of the value function of the couch potato, this transformation yields

$$\widetilde{V}_t^{CP} = \int \max\left\{\widetilde{V}_t^E(\xi_t, 0) - \widetilde{\eta}_t, \widetilde{V}_t^U\right\} d\widetilde{F}_t^B(\xi) \tag{27}$$

where

$$\widetilde{\eta}_t = \frac{\eta_t}{\overline{z}_t}$$

The distribution of business plan productivity levels, i.e. $\widetilde{F}_t^B(\xi)$, is now expressed in terms of their productivity relative to the frontier.

The value of being an entrepreneur can now be expressed as

$$\begin{aligned} \widetilde{V}_t^E(\xi_t, n_{t-1}) \quad = \quad & \max_{n_t}\Big\{n_t\left[\xi_t - \widetilde{w}_t(\xi_t, n_t) - \widetilde{\tau}_t - \widetilde{c}_t(\xi_t, n_t)\right] - \widetilde{\tau}_t - \widetilde{h}_t(n_t, n_{t-1}, \xi_t) \\ & + \frac{1+g}{1+r}\int \max\left\{\widetilde{V}_{t+1}^E(\xi_{t+1}, n_t), \widetilde{V}_{t+1}^{CP}\right\} d\widetilde{Q}_t^z(\xi_{t+1} | \xi_t)\Big\} \end{aligned} \tag{28}$$

while the value of being unemployed reads

$$\widetilde{V}_t^U = \theta_t q(\theta_t) \int \max\left\{\widetilde{V}_t^J(z_t, n_t), \widetilde{b}_t - \widetilde{\tau}_t + \frac{1+g}{1+r}\widetilde{V}_{t+1}^{CP}\right\} d\widetilde{F}_t^O(z_t, n_t) + (1-\theta_t q(\theta_t))\left(\widetilde{b}_t - \widetilde{\tau}_t + \frac{1+g}{1+r}\widetilde{V}_{t+1}^{CP}\right) \tag{29}$$

Here

$$\widetilde{b}_t = \frac{b_t}{\overline{z}_t} \tag{30}$$

and $\widetilde{F}_t^O(\xi_t, n_t)$ is the endogenously determined distribution of job offers. The value of being in a job reads

$$\begin{aligned}\widetilde{V}_t^J(\xi_t, n_t) &= \xi_t(z_t, n_t) \\ &+ \frac{1+g}{1+r}\int\left(1 - \widetilde{P}_{t+1}^F(\xi_{t+1}, n_t)\right)\max\left\{\widetilde{V}_{t+1}^J(\xi_{t+1}, \nu_{t+1}(\xi_{t+1}, n_t)), \widetilde{V}_{t+1}^{CP}\right\} dQ_t^z(\xi_{t+1}|\xi_t) \\ &+ \frac{1+g}{1+r}\int \widetilde{P}_{t+1}^F(\xi_{t+1}, n_t)\widetilde{V}_{t+1}^{CP} d\widetilde{Q}_t^z(\xi_{t+1}|\xi_t)\end{aligned}$$

The wage bargaining process results in a wage that equals

$$\begin{aligned}\widetilde{w}_t(\xi_t, n_t) &= \beta\left[\xi_t - \left(1 + \frac{1}{n_t}\right)\widetilde{\tau}_t - \widetilde{c}_t(\xi_t, n_t)\right] \\ &+ \frac{\beta(1+g)}{n_t(1+r)}\int \max\left\{\widetilde{V}_{t+1}^E(\xi_{t+1}, n_t), \widetilde{V}_{t+1}^{CP}\right\} d\widetilde{Q}_t^z(\xi_{t+1}|\xi_t) \\ &- \frac{(1-\beta)(1+g)}{n_t(1+r)}\int\left(1 - \widetilde{P}_{t+1}^F(\xi_{t+1}, n_t)\right)\max\left\{\widetilde{V}_{t+1}^J(\xi_{t+1}, \nu_{t+1}(\xi_{t+1}, n_t)), \widetilde{V}_{t+1}^{CP}\right\} d\widetilde{Q}_t^z(\xi_{t+1}|\xi_t) \\ &+ \frac{(1-\beta)(1+g)}{n_t(1+r)}\int \widetilde{P}_{t+1}^F(\xi_{t+1}, n_t)\widetilde{V}_{t+1}^{CP} d\widetilde{Q}_t^z(\xi_{t+1}|\xi_t) + \frac{(1-2\beta)}{n_t}\widetilde{V}_t^{CP}\end{aligned}$$

The detrended version of the balanced budget constraint is

$$\widetilde{\tau}_t = (1+g)(1+r)\widetilde{b}_{t+1}(1 - N_{t+1} - \Phi_{t+1}) \tag{31}$$

So, the question is how we implement the solution method of this model. Our guess is to use a contraction mapping approach. For this, we discretize the state space, such that

$$\xi_t \in \{0, x_1, \ldots, x_r, 1\} = \Xi \tag{32}$$

and

$$n_t \in \{0, l_1, \ldots, l_q\} = \Lambda \tag{33}$$

This means that the state space consists of the cross-product of these two spaces, i.e. (ξ_t, n_t) consists of $(r+2) \times (q+1)$ points. All the decisions made in this economy depend on the following unknown value functions

$$\widetilde{V}_t^{CP} \text{ and } \widetilde{V}_t^{U} \tag{34}$$

$$\widetilde{V}_t^{E} \; : \; \Xi \times \Lambda \to \mathbb{R} \tag{35}$$

$$\widetilde{V}_t^{J} \; : \; \Xi \times \Lambda \to \mathbb{R} \tag{36}$$

2.10 Calibration

The matching function and the bargaining power β are calibrated following Shimer (2005). Both α and β are set to 0.72. (These numbers lie toward the upper end of the estimates.) g is set to 2.5%.

- $\frac{N_t}{\Phi_t + N_t}$ should be between 0.85 and 0.95. (Chang (2000) and Guner, Ventura, and Yi (2005))
- Job destruction rate ($\frac{N_t^F}{N_t}$) is 8.4% (annual).
- Job creation rate ($\frac{N_t^C}{N_t}$) is 8.3% (annual).
- Average firm size ($\int n_t \phi_t^n(n_t) dn_t$) is around 15. (Guner, Ventura, and Yi (2005))
- 90% of total firms has less than 20 employees. (Nickell)
- Firm turnover rate (entry+exit) is around 20%.]

3 Lucas Model (1978)

As a baseline for the analysis of optimal firm size with rigidities, let's put the Lucas (1978) model into a dynamic setting and describe the steady state distribution of firm size.

As a condition for full employment in equilibrium, Lucas assumes that there are perfect labor markets. Therefore, there are no forms of unemployment insurance here, and the value of being unemployed in such an economy is zero, $V^U = 0$. Accordingly, for a given level of productivity per worker, the firm chooses the number of workers that maximizes its flow of profits at each period of time.

We specify the cost function as $c(n) = a \cdot n^2$, and the profit function as $\pi(\xi) = n(\xi - w) - \xi \cdot c(n)$, where the firm faces decreasing returns to scale through $c(n)$, or the firm's per worker overhead operating cost. Then the price taking, profit maximizing firm will hire labor until the marginal product of labor is equal to the wage. In order to obtain the optimal number of workers for a firm operating at productivity level ξ and wage w, solve $\frac{\partial}{\partial n}\pi = 0$:

$$\begin{aligned} \frac{\partial}{\partial n}[(\xi - w)n - \xi \cdot c(n)] &= 0 \\ (\xi - w) - \xi \cdot c'(n) &= 0 \\ (\xi - w) - \xi \cdot (2an) &= 0 \end{aligned}$$

Then optimal firm size is:

$$n^*(\xi) = \left\{ \begin{array}{cc} \frac{1}{2a}\left(\frac{\xi - w}{\xi}\right) & \text{if } \xi > w \\ 0 & \text{if } \xi < w. \end{array} \right\} = \max\left[0,\ \frac{1}{2a}\left(\frac{\xi - w}{\xi}\right)\right]$$

Next, Lucas assumes that entrepreneurs are selected from a probability distribution $F^B(\xi) : R^+ \to [0, 1]$, where $\underline{\xi}$ is the minimum managerial talent required in order to become an entrepreneur. Thus, all agents with a draw less than $\underline{\xi}$ will be workers. In order to capture the dynamics of entrepreneurial selection, we will make the further assumption that entrepreneurs face a conditional probability distribution $Q^z(\xi'|\xi)$ that determines whether they will remain entrepreneurs in the next period. For continuing firms (i.e. those with productivity draws $\xi' > \underline{\xi}$), $Q^z(\xi'|\xi)$ governs whether the firm will face increased or decreased productivity. If the current entrepreneur observes some productivity level $\xi' < \underline{\xi}$ given their current productivity ξ, then they will become a worker and their firm will exit the market. This suggests that $\underline{\xi}$ acts as a mechanism that regulates the entry condition for firms. In equilibrium with continuous values of productivity growth and labor growth, then, an agent will be indifferent between being a worker and being an entrepreneur that operates with productivity $\underline{\xi}$. To proceed, consider the continuous case, then consider the discrete case.

3.1 The Continuous Case

In order to formalize the entry condition, consider the value functions of the worker, V^J, and the entrepreneur, V^E. In the current period, the worker obtains a flow payoff of the wage. After observing his draw of managerial talent in the next period, he can decide whether to become an entrepreneur with productivity ξ_{t+1} or remain a worker. According to the entry condition, then, we would find that for productivity levels less than $\underline{\xi}$ the agent will remain a worker.

$$\begin{aligned} V_t^J &= w_t + \frac{1}{1+r}\int_0^1 \max\{V_{t+1}^E(\xi_{t+1}), V_{t+1}^J\}\ dF^B(\xi) \\ &= w_t + \frac{1}{1+r}F^B(\underline{\xi})V_{t+1}^J + \frac{1}{1+r}\int_{\underline{\xi}}^1 V_{t+1}^E(\xi_{t+1})\ dF^B(\xi) \end{aligned}$$

In the current period, the entrepreneur obtains his flow of profits. In the next period he observes his productivity draw and decides whether he will become a worker or an entrepreneur with firm productivity ξ_{t+1}. Also note the implicit assumption that entrepreneurs of existing firms must have a greater value of continuing than becoming workers for at least one value of ξ, or else there are no firms in equilibrium.

$$\begin{aligned} V_t^E(\xi_t) &= \pi_t(\xi_t) + \frac{1}{1+r}\int_0^1 \max\{V_{t+1}^E(\xi_{t+1}), V_t^J\}\ dQ^z(\xi_{t+1}|\xi_t) \\ &= \pi_t(\xi_t) + \frac{1}{1+r}\left[\int_0^{\underline{\xi}} dQ^z(\xi_{t+1}|\xi_t)\right]V_{t+1}^J + \frac{1}{1+r}\int_{\underline{\xi}}^1 V_{t+1}^E(\xi_{t+1})\ dQ^z(\xi_{t+1}|\xi_t) \end{aligned}$$

With the specification of these value functions, we are in a position to posit the entry condition as: $V^E(\underline{\xi}) = V^J$. Technically, in continuous space the probability that a random variable takes on a particular value is zero, $P(X = a) = \int_a^a f(x)dx = 0$. In order to solve this model we will look at a discretization of the reals and specify a range of values that includes $\underline{\xi}$. This is explored in the next section.

Now we consider the steady state distribution of firms over productivity levels. Define Φ as the number of firms (entrepreneurs) in the economy, and $\Phi(\xi)$ as the distribution of firms over productivity levels. In equilibrium, the number of firms at a given level of productivity will be composed of continuing firms according to the conditional distribution $Q^z(\xi_{t+1}|\xi_t)$ and entrants according to the distribution $F^B(\xi_{t+1})$.

$$\Phi(\xi_{t+1}) = \int_{\underline{\xi}}^{1} \Phi(\xi_t)\ dQ^z(\xi_{t+1}|\xi_t) + NF^B(\xi_{t+1}) \tag{37}$$

Labor supply will be given by the fraction of agents that are not entrepreneurs:

$$N = 1 - \int_{\underline{\xi}}^{1} \Phi(\xi_t)\ d\xi \tag{38}$$

Thus, in equilibrium labor demand equals labor supply:

$$\int_{\underline{\xi}}^{1} \Phi(\xi_t)n(\xi_t)\ d\xi \ = \ N \tag{39}$$

3.2 The Discrete Case

In the discrete case, we take ξ and segment it into a grid. The equilibrium condition requires that labor supply and demand are equal, as in (39). In order to check for this, given $\underline{\xi}$ and the implied wage, we must solve for the firm productivity distribution, $\Phi(\xi)$. Notice that in discrete space, (37) and (38) constitute a linear system, which we can solve as:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & -Q^z(\underline{\xi}|\xi_1) & \dots & -Q^z(\overline{\xi}|\xi_1) & -F^B(\xi_1) \\ 0 & 1 & \dots & 0 & -Q^z(\underline{\xi}|\xi_2) & \dots & -Q^z(\overline{\xi}|\xi_2) & -F^B(\xi_2) \\ 0 & 0 & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \vdots & & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1-Q^z(\underline{\xi}|\underline{\xi}) & \dots & -Q^z(\overline{\xi}|\underline{\xi}) & -F^B(\underline{\xi}) \\ \vdots & & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & -Q^z(\underline{\xi}|\overline{\xi}) & \dots & 1-Q^z(\overline{\xi}|\overline{\xi}) & -F^B(\overline{\xi}) \\ 0 & \dots & \dots & 0 & 1 & \dots & 1 & 1 \end{pmatrix} \begin{pmatrix} \Phi(\xi_1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \Phi(\overline{\xi}) \\ N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

To solve for the wage, take some $\underline{\xi}$ from the ξ-grid and set up the system of linear equations, $A(\underline{\xi})\ V \ = \ B(w)$, which is produced below:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & -\frac{1}{1+r}Q^z(\underline{\xi}|\xi_1) & \dots & -\frac{1}{1+r}Q^z(\bar{\xi}|\xi_1) & -\frac{1}{1+r}\left[\int_0^{\underline{\xi}} Q^z(\xi'|\xi_1)\right] \\ 0 & 1 & \dots & 0 & -\frac{1}{1+r}Q^z(\underline{\xi}|\xi_2) & \dots & -\frac{1}{1+r}Q^z(\bar{\xi}|\xi_2) & -\frac{1}{1+r}\left[\int_0^{\underline{\xi}} Q^z(\xi'|\xi_2)\right] \\ 0 & 0 & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \vdots & & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1-\frac{1}{1+r}Q^z(\underline{\xi}|\underline{\xi}) & \dots & -\frac{1}{1+r}Q^z(\bar{\xi}|\underline{\xi}) & -\frac{1}{1+r}\left[\int_0^{\underline{\xi}} Q^z(\xi'|\underline{\xi})\right] \\ \vdots & & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & -\frac{1}{1+r}Q^z(\underline{\xi}|\bar{\xi}) & \dots & 1-\frac{1}{1+r}Q^z(\bar{\xi}|\bar{\xi}) & -\frac{1}{1+r}\left[\int_0^{\underline{\xi}} Q^z(\xi'|\bar{\xi})\right] \\ 0 & \dots & \dots & 0 & -\frac{1}{1+r}F^B(\underline{\xi}) & \dots & -\frac{1}{1+r}F^B(\bar{\xi}) & 1-\frac{1}{1+r}F^B(\underline{\xi}) \end{pmatrix} \begin{pmatrix} V^E(\xi_1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ V^E(\bar{\xi}) \\ V^J \end{pmatrix} = \begin{pmatrix} \pi(\xi_1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \pi(\bar{\xi}) \\ w \end{pmatrix}$$

Given the $A(\underline{\xi})$ and $B(w)$ matrices in the system, we can compute $V = A^{-1}(\underline{\xi})\ B(w)$ and then exploit the free entry condition, $V^E(\underline{\xi}) = V^J$, to back out the wage that brings the system into equilibrium. To do this, we solve for the wage that satisfies:

$$[0 \ \dots \ 0 \ \ \underset{\underset{\underline{\xi}^{th}}{\uparrow}}{1} \ \ 0 \ \dots \ 0 \ -1] \quad A^{-1}(\underline{\xi}) \quad B(w) \quad = \quad 0 \tag{40}$$

By substituting the optimal firm size decision $n^*(\xi)$ into the entry condition, we see that for any $\underline{\xi}$ on the grid, (40) will be decreasing in the wage,

$$V^E(\underline{\xi}) - V^J = 0$$

$$\pi(\underline{\xi}, w) - w + \frac{1}{1+r}\left[\left(\int_0^{\underline{\xi}} Q^z(\xi'|\underline{\xi}) - F^B(\underline{\xi})\right)V^J + \left(\int_{\underline{\xi}}^1 V^E(\xi')\ dQ^z(\xi'|\underline{\xi}) - \int_{\underline{\xi}}^1 V^E(\xi')\ dF^B(\xi)\right)\right] = 0$$

$$\left[\max\left(0, \frac{1}{2a}\frac{\underline{\xi}-w}{\underline{\xi}}\right)\right](\underline{\xi}-w) - \underline{\xi}a\left[\max\left(0, \frac{1}{2a}\frac{\underline{\xi}-w}{\underline{\xi}}\right)\right]^2 - w + \psi = 0$$

Thus,

$$V^E(\underline{\xi}) - V^J = 0 \implies \begin{cases} \left(\underline{\xi}-w\right)^2 + 4a\underline{\xi}(\psi - w) = 0 & \text{if } \underline{\xi} > w; \\ \psi - w = 0 & \text{if } \underline{\xi} < w. \end{cases}$$

where ψ is a constant corresponding to the third term in the second equation immediately above. Since this is a non-linear system that is decreasing in the independent variable, we will use Newton-Raphson approximation to obtain the equilibrium value of the wage for each threshold level, $\underline{\xi}$.

Because $\underline{\xi}$ is a portion of a discrete grid, satisfaction of the free entry condition implies a sum of indifferent agents that is greater than or equal to one. In the continuous case, there is only one such "marginal manager." Therefore we can consider two segmentations that divide agents into workers and entrepreneurs:

(A) If $\xi_0 \leq \underline{\xi}$, then the agent is a worker.
If $\xi_0 > \underline{\xi}$, then the agent is an entrepreneur.

(B) If $\xi_0 < \underline{\xi}$, then the agent is a worker.
If $\xi_0 \geq \underline{\xi}$, then the agent is an entrepreneur.

where ξ_0 is a given agent's productivity draw in the grid. As stated, the idea is to assign all the indifferent agents to be either workers or entrepreneurs, respectively. Under each assignment we can find the solutions to the model. Since these assignments are extreme segmentations, we expect that the "true" solutions are found within the interval established by these two limiting cases. For large enough grids, we can find adequately small solution intervals.

So, the steady state level of threshold productivity, $\underline{\xi}$ is found at the intersection of labor supply and demand. But since labor supply increases when indifferent agents become workers and decreases when indifferent agents become entrepreneurs, and since labor demand decreases when indifferent agents become workers and increases when indifferent agents become entrepreneurs, the segmentation of indifferent agents along the lines of (A) and (B) imply different levels of threshold productivity. Additionally, if indifferent agents become entrepreneurs, the distribution of firm productivity is spread out over more existing firms than when indifferent agents become workers.

3.3 Introducing Imperfect Labor Markets with Rigidities

Lucas (1978) has modelled an economy with perfect labor markets. In order to get at the question of how differences in "flexibility" affect firm size across productivity levels, we wish to make several augmentations to the Lucas framework. First, and most obviously, we wish to implant rigidities into this economy in the form of taxation, hiring and firing costs, unemployment benefits, and entry costs. We see immediately that (a) the value of employment is no longer zero, (b) with frictions in the labor market there exists a job offer distribution governing the transition from unemployment to employment and vice-versa, and (c) firms will respond to productivity shocks by hiring and firing employees. Second, under the condition that the value of employment is non-zero, an agent no longer decides between two options: working or managing. In this case, the inclusion of an entry cost requires an agent to obtain a business idea that he values more than staying unemployed. The agent may wish take advantage of unemployment until he draws a management position from the distribution of business ideas, $F^B(\xi)$. For this we will introduce another agent type that formalizes this position.

3.4 Some Preliminary Results of the Dynamic Lucas Model

Parameters

Cost Parameter	a	0.50
Population Mass		1.00
Steady State Labor Supply	L^S	0.69
Steady State Num. Firms		0.31
Steady State Wage	w^*	0.46
Steady State ξ Threshold	$\underline{\xi}^*$	0.62

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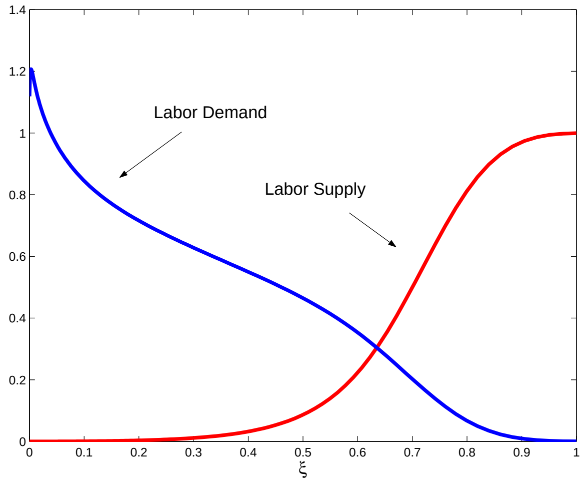
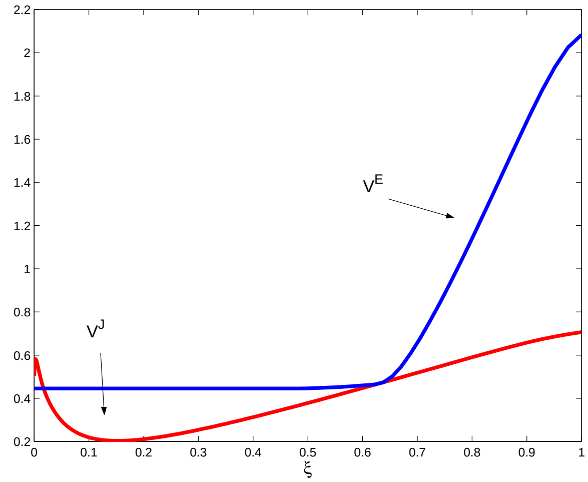
(a) Labor Supply and Demand



ξ- correspondence

(aa)

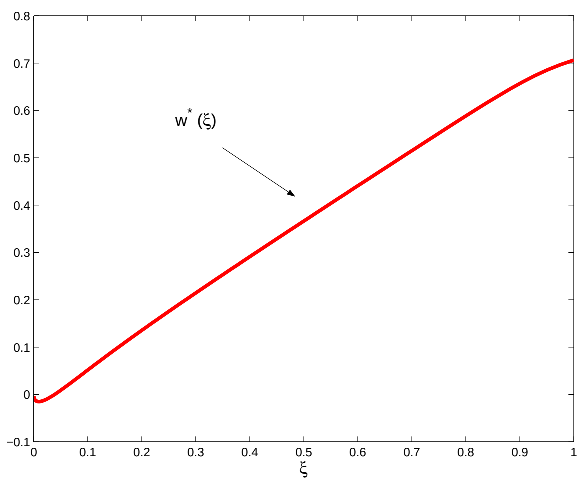
(b) Value Functions, evaluated at threshold



ξ- correspondence

(bb)

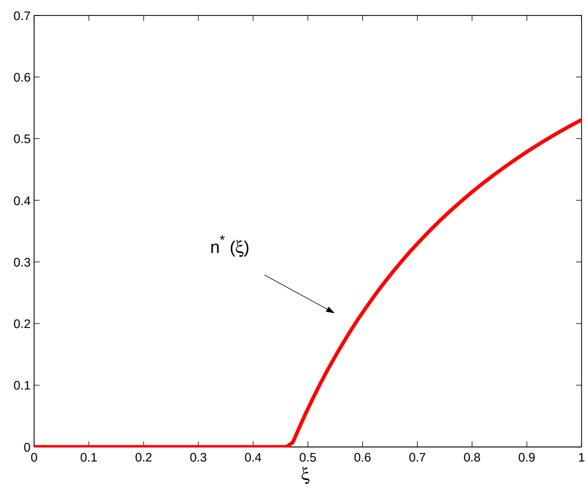
(c) Wage



ξ- correspondence

(cc)

(d) Optimal Firm Size



ξ- correspondence

(dd)