A Finite-Life Private-Information Theory of Unsecured Debt [1]

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Abstract

We present a theory of unsecured consumer debt that recognizes a debtor's legal right to default. Our theory does not rely on psychological costs of default (or stigma) nor does it rely on enforcement mechanisms that arise in repeated-game settings. Our theory is based on private information about a person's *type* and on a person's incentive to *signal* his type by avoiding default in the credit market.

People in our model differ with respect to patience and in the likelihood of an insurable (i.e., observable) loss. More patient types are assumed to be less likely to suffer the loss and these (correlated) differences in patience and probability of loss are private information. Crucially, in our model the opportunity to repay debt occurs at an earlier date than the opportunity to purchase insurance. This means that debtors who are also good insurance risks have an opportunity to signal their type by repaying debt. The signal works because the benefit of a lower cost of insurance comes in the future and the type who is a good insurance risk is also the type who values the future more.

Our theory is consistent with the well-known fact that the cost of some kinds of insurance is positively related to the presence of derogatory information in a person's *credit* history. Our theory also provides an example of the more general point that people avoid opportunistic behavior in one social context because doing so has positive spillover effects in other social contexts.

1 Motivation

In this paper we construct a theory of unsecured debt that recognizes a debtor's legal right to default - i.e. it recognizes the presence of a right to bankruptcy. The question we address is the following: how can *unsecured* consumer debt coexist with the unilateral right of a debtor to invoke bankruptcy?

We propose a theory of unsecured debt that is based on the existence of private information about a person's *type* and on the fact that some debtors have the incentive to forego bankruptcy in order to *signal* their type. The theory formalizes the intuitive notion that the type of a person is likely to be relevant to trading partners in many exchange situations and by resisting opportunistic behavior in one exchange context, a person may signal valuable information about his type to trading partners in other exchange contexts.

Specifically, people in our model differ with respect to patience and in the likelihood of an insurable loss. More patient types are also less likely to suffer the loss and these (correlated) differences in patience and probability of loss are private information. To formalize the information spillovers between markets, we assume the opportunity to repay debt occurs at an earlier date in our model than the opportunity to purchase insurance. This means that debtors who are also good insurance risks have the opportunity and incentive to signal their type by repaying debt. The signal works because the benefit of a lower cost of insurance comes in the future and the type who is a good insurance risk is also the type who values the future more. In other words, patience makes for better insurance risk but it also makes a person value the future more (relative to the present) and, therefore, lowers the opportunity cost of giving out signals of being a better insurance risk – a signal that takes the form of debt repayment. In our model, no credit is extended to a person if type is public information or if all types are equally risky with respect to the insurable loss. In either case, there is no reward to curtailing opportunistic behavior in the credit market.

It is worth noting that our theory of unsecured debt is distinct from some other approaches to explaining unsecured debt when enforcement is imperfect. Our theory does not rely on enforcement mechanisms that depend on exclusion from the asset (loan or deposit) markets following default. Indeed, our model has finite-lived households to whom the opportunity to borrow (and repay) is presented only *once*. Also, our theory does not depend on any psychological cost of failing to honor debt contracts - there is no *stigma* attached to bankruptcy.

Our paper is closely related to three earlier works. The first is Diamond's (1989) well-known paper on acquisition of reputation in private debt markets. Diamond considers a situation where there are two types of indivisible investment projects: one is safe and the other is risky (including a state of the world where the project fails yielding zero output). Some infinitely-lived risk-neutral entrepreneurs own the first type of project, others the second, and a third group choose which project to undertake. To start the project requires some input which the entrepreneurs borrow in a competitive loan market populated by one-period lived lenders. The key friction is that project type is unobservable to the lender. Since an entrepreneur's loss is bounded below by defaulting in the case of project failure, entrepreneurs in the third group have an incentive to choose the risky project. In this environment, an entrepreneur's payment history reveals information about his type and hence the terms of credit offered to an entrepreneur will depend on that history. Since default only happens when the project fails, the only choice for an entrepreneur from the third group is which type of project to undertake.

The second two papers study sovereign debt. Cole, Dow and English (1995) focus on the fact that because sovereign repayments cannot be enforced, governments might signal their willingness to repay future loans by settling old debts. Governments in their model can be one of two types, one being more myopic than the other. The government's type follows a Markov process and type is unobservable. Governments borrow in order to finance a stochastic project yielding nonstorable output. As in the Diamond paper, lenders offer one period contracts which are contingent on a lender's belief as to the type of borrower he faces, which is updated according to observed behavior in the asset market. The authors construct equilibria where the less myopic government pays off debt, the more myopic one defaults on its debt, and when a myopic government changes state it makes a payment to signal the change. There are similarities between this paper and Chatterjee, et. al. (2005) except for the differences associated with the legal system. For instance, a bankruptcy results in a *discharge* of existing debt and individuals do not have the option of making a payment on discharged debt in the future.[1]

Cole and Kehoe (1998) is the most closely related paper. In particular, they study a model where a sovereign's actions in the debt market affect its reputation in a different market (e.g. the labor market). There are two different types of risk neutral governments, where one receives a large disutility (i.e. stigma) when it reneges on previous contracts. The country's nonstorable goods production technology has deterministically fluctuating productivity and the government must borrow to finance the project. One period risk neutral lenders offer contracts that depend on the payment history of borrowers to maximize expected profits and beliefs about type are updated according to Bayes' rule wherever possible. The authors show that if there are no reputational spillovers, there can be no loan market activity but once default in the credit market tarnishes the government's reputation in another market, it is actually possible to support large amounts of debt when borrowers are sufficiently patient. One key difference between our framework and theirs is that there is no stigma associated with default. Hence we needn't revert to complicated limiting arguments (taking the prior over stigma sensitive governments to zero) to study existence of equilibria.

Our theory is motivated by some features of the U.S. economy. First, bankruptcy is both legal and pervasive. Second, credit histories are used both by the credit industry and by the insurance industry (auto and home insurance) as indicators of good performance. People with superior credit histories are offered credit and insurance at a cheaper price.[2] In turn,

[1]Given the choice between Chapter 7 and 13, individuals would choose to file Chapter 13 only if they wished to keep assets they would lose under a Chapter 7 filing. Since there is only one asset in our model, borrowers have negative net worth and Chapter 7 is always the preferred means to file for bankruptcy.

[2]According analysts (see http://www.bankrate.com/brm/news/insurance/credit-scores1.asp), consumers

they default less frequently and file substantially fewer insurance claims.

2 Environment

There is a single good. There are 3 periods, denoted $t = 1, 2$, and 3 and a unit measure of people. We will describe people who live in this economy, the legal environment they face, the market arrangement and the timing of events.

2.1 People

There are two types of people, denoted $i = g, b$. The measure of type g is $0 < \gamma < 1$. The preferences of each type is given by

$$E\,\theta_1 u(c_1) + \beta^i\,\theta_2 u(c_2) + \beta^{i^2} u(c_3) \tag{1}$$

where c_t is consumption in period t of the single good and $\theta_t \in \Theta \subset R$ is a preference shock drawn independently for each person from the probability space $\{\Theta, \mathcal{B}(\Theta), F_t\}$. The draws are also assumed to be i.i.d. over time. The period utility function $u(c)$ is taken to be strictly concave and twice continuously differentiable. Observe that a person's type is assumed to affect the person's discount factor. We assume that $1 > \beta^g > \beta^b > 0$.

The endowment of all people is assumed to be a constant over time and given by $e > 0$.

In period 3, a person of type i faces a probability $\pi^i > 0$ of experiencing a loss in wealth $L > 0$. Thus, the probability of loss is taken to be exogenous but type-dependent. Specifically, $0 < \pi^g < \pi^b < 1$. We will denote the loss incurred by a person in period 3 by $z \in \{0, L\}$. We we will denote the (discrete) probability space for z for type i by $\{\{0, L\}, \mathcal{P}(\{0, L\}), Z^i\}$.

with bad credit scores pay 20 to 50 percent more in auto insurance premiums than consumers with high credit scores. And, two-thirds of policy holders have lower premiums because of their good credit records.

Finally, unless revealed by a person's actions, a person's type and the realizations of his preference shocks are assumed to be private information.

2.2 Legal Environment

A key feature of the environment is the existence of a bankruptcy law. This law gives people the right to disavow their financial obligations. As is generally true for actual bankruptcy law, this "right to bankruptcy" is assumed to be inalienable – meaning that a debtor cannot waive his or her right to bankruptcy at the time of taking out a loan. For simplicity, we assume that invoking the right to bankruptcy does not cost the debtor any fees or expenses.

2.3 Market Arrangements

There are two sets of markets. In one market people borrow from or lend to banks and in the other market people purchase insurance against the loss L. Since household type is private information, and type will matter for the propensity of a person to declare bankruptcy or suffer a loss, banks and insurance companies must make an assessment of a person's type when selling a loan or an insurance policy.[3] We will study a market structure that permits the terms of financial contracts to depend on such an assessment. In what follows, we will use σ as the generic symbol to denote the probability that a person is type g. Then σ is the assessment of a person's type or, more succinctly, a person's *type score*. A person's type score will evolve over time because a person's actions in the asset and insurance markets can (and will) reveal information about a person's true type. We imagine that there is an information processing agency (resembling real-world credit bureaus) that keeps track of a person's actions in the asset and insurance markets – i.e. keeps track of a person's financial

[3] If the preference shocks are correlated over time then banks may also have an incentive to form an assessment of the preference shock hitting a person because this information will be valuable in predicting future default. Since we have assumed that shocks are *i.i.d*, this assessment is not necessary.

history.

The asset market operates in periods 1 and 2 - there is obviously no role for an asset market in period 3 since no one lives beyond period 3. In period 1, the asset market offers one-period bond contracts $y \in A$, where A is a compact subset of R. These bond contracts are offered at prices $q_1(y, \sigma)$. A person with assessment σ who purchases the contract y pays $q_1(y, \sigma) \cdot y$ in period 1 and receives y in period 2. A positive y signifies a deposit and a negative y signifies a loan. If $y < 0$, the person promises to repay y in period 2 conditional on not declaring bankruptcy. In period 2 we assume that people are permitted to borrow but not save. The restriction on period 2 saving is made for tractability and is discussed later in the paper.[4] Thus the period 2 asset market offers one-period bond contracts $y \in A \cap R_-$ at prices $q_2(y, \sigma)$. Again, default on period 2 loans is possible. Finally, we assume that banks have access to a world credit market in which they borrow or save at the interest rate r.[5]

The insurance market in period 3 operates as follows. Insurers offer contracts x in I, where I is a compact subset of R. A person with assessment σ who purchases the contract x pays $p(x, \sigma) \cdot x$ as premium and, in the event of loss, collects the indemnity x. This notation emphasizes the symmetry between the loan and insurance markets. Just as the probability of default on a loan will depend on the size of the loan, so too can the probability of loss on an insurance contract depend on the amount of insurance purchased – for the usual moral hazard reasons. However, in this paper we abstract from issues of moral hazard issues – for each type of person, the probability of loss cannot be affected by any action that the person can take. Therefore, in equilibrium, the price of insurance will depend only σ and not x.

[4] In an actual Chapter 7 bankruptcy proceeding, a person is required to relinquish available assets - with some exceptions - to creditors when obtaining discharge of debt. If the exemptions are generous, a person can discharge debt *and* accumulate substantial assets in the period of bankruptcy. By assuming that no saving is permitted in period 2 we avoid having to take a stand on this point.

[5] Alternatively, we could imagine that banks have access to a storage technology that allows them to transform 1 unit of output in period 1 into $(1+r)$ units of output in period 2. This would require incorporating the restriction that aggregate consumer assets cannot be negative.

A financial firm takes the set of contracts and prices $\{q_1(y,\sigma), y \in A\}$, $\{q_2(y,\sigma), y \in A \cap R_-\}$ and $\{p(x,\sigma), x \in I\}$ as given. Any given contract is viewed as a distinct financial product. There is free-entry in the provision of each of these financial products. In equilibrium each of these financial products will fetch zero profits in expectation.

An important feature of the environment is the possibility that a person's actions in the asset and insurance markets may reveal information about a person's type. The possibility of information transmission is captured by the following three *belief-updating* functions.

- The function $s = \Psi_1(\ell)$ gives the person's type score at the end of period 1 if the person chooses asset level ℓ.
- The function $s' = \Psi_2(d, \ell', \ell, s)$ gives a person's type score at the end of period 2 if he starts period 2 with asset ℓ and type-score s and chooses a bankruptcy decision d ($d = 1$ means file for bankruptcy and $d = 0$ means no filing) and an asset level ℓ'.
- The function $s'' = \Psi_3(x, \ell', s')$ gives a person's type score if he starts period 3 with debt ℓ' and type-score s' and chooses an insurance level x.

Market participants take these functions as given just as they take the pricing functions $q_t(y,\sigma)$ and $p(x,\sigma)$ and the sets A and I as given. In this sense, the functions Ψ_t are also part of the market arrangement.[6]

2.4 The Timing of Events

The timing of events in each period is as follows.

At the start of period 1, people learn their type i and the realization of the preference shock θ_1. Then they choose how much to borrow or save. Then they consume and period 1 ends.

[6] Just as we do not model the Walrasian auctioneer, we do not model the "credit scoring agency" explicitly. But of course, the existence of auctions and credit scoring companies motivate our market arrangement.

At the start of period 2 people learn the realization of their preference shock θ_2. If a person borrowed in period 1, the person then chooses whether to default or not. After the default decision is made, people choose whether to borrow or not. Then they consume and period 2 comes to an end.

At the start of period 3 people purchase insurance. Next, the shock z is realized and the person receives the insurance payment if $z = L$. Then, if a person borrowed in period 2 the person chooses whether to default or not. Finally, people consume and die.[7]

3 Decision Problems

In this section we describe the decision problem of people, insurers and banks.

3.1 People

It's convenient to start with the final period and work backwards.

3.1.1 Period 3

After insurers have paid off, a person's decision problem is to choose whether to pay back any loans. Therefore, the post-insurance decision problem person, regardless of type, is

$$\max_{d \in \{0,1\}} u(e + \ell'(1-d) - z - p(x, s'') \cdot x + x \cdot 1_{\{z=L\}})$$

where $\ell' \leq 0$ is the person's debt position at the start of period 3 (recall that no saving is permitted in period 2) and s'' is the person's type-score following his purchase of insurance. We will denote the decision rule for this problem by $d^i(x, z, \ell', s'')$. This decision rule of course

[7] An alternative model where periods 2 and 3 are lumped together and the timing is a default decision followed by an insurance choice cannot support the type of equilibrium we describe later in the paper. Therefore, a three period model seems the most parsimonious environment for the purposes of this paper.

takes a very simple form: if $\ell' = 0$ then $d^i(x, z, \ell', s'') = 0$ and if $\ell' < 0$ then $d^i(x, z, \ell', s'') = 1$. Because the default decision is taken *after* the insurance purchase has been made there is no cost to defaulting on a loan.

At the start of period 3, a type i person's insurance decision problem is as follows.

$$V_3^i(\ell', s') = \max_{x \in I(s')} \pi^i u \left[e + \ell'(1 - d^i(x, z, \ell', s'')) - p(x, s'') \cdot x - L + x\right]$$
$$+ (1 - \pi^i) u \left[e + \ell'(1 - d^i(x, 0, \ell', s'')) - p(x, s'') \cdot x\right]]$$

s.t.

$s'' = \Psi_3(x, \ell', s')$ and$I(s') \subseteq I$

Observe that the set of insurance choices available to a person is allowed to depend on a person's beginning of period 3 risk-assessment s'. Thus, for some s' only a strict subset of insurance contracts may be available. We will denote a type i person's decision rule regarding insurance purchase as $x^i(\ell', s')$.

3.1.2 Period 2

At the start of period 2, people learn their preference shock θ_2. If a person of type i is a debtor and chooses to default, the person's utility is given by

$$V_2^{i\,1}(\theta_2, \ell, s) = \max_{\ell' \in A \cap R_-} \theta_2 u(e - q_2(\ell', s') \cdot \ell') + \beta^i V_3^i(\ell', s')$$

s.t.

$e - q_2(\ell', s') \cdot \ell' \geq 0$

$s' = \Psi_2(1, \ell', \ell, s)$

where ℓ is the person's beginning-of-period 2 asset position and s is the person's beginning-period-of-period 2 risk assessment. If the person of type i chooses not to default, then the person's utility is given by

$$V_2^{i\,0}(\theta_2, \ell, s) = \max_{\ell' \in A \cap R_-} \theta_2 u(e + \ell - q_2(\ell', s') \cdot \ell') + \beta^i V_3^i(\ell', s')$$

s.t.

$$e + \ell - q_2(\ell', s') \cdot \ell' \geq 0$$

$$s' = \Psi_2(0, \ell', \ell, s)$$

Therefore,

$$V_2^i(\theta_2, \ell, s) = \max\left\{V_2^{i\,1}(\theta_2, \ell, s),\ V_2^{i\,0}(\theta_2, \ell, s)\right\}$$

We will denote a type i's period 2 decision rules as $d^i(\theta_2, \ell, s)$ and $\ell'^i(\theta_2, \ell, s)$.

3.1.3 Period 1

At the start of period 1 people learn their type i and their preference shock θ_1. The decision problem of a person of type i is then

$$V_1^i(\theta_1) = \max_{\ell \in A} \theta_1 u(e - q_1(\ell, s) \cdot \ell) + \beta^i \int_\Theta V_2^i(\theta_2, \ell, s) dF_2(\theta_2)$$

s.t.

$$e - q_1(\ell, s) \cdot \ell \geq 0$$

$$s = \Psi_1(\ell)$$

We will denote a type i's period 1 decision rule as $\ell^i(\theta_1)$.

3.2 Insurers

Insurers face a set of insurance contracts I and prices $\{p(x, \sigma), x \in I\}$. The decision problem of insurers is to choose how many of these different types of contracts to sell. Clearly insurers will participate in selling any contract $x \in I$ that makes non-negative profits in expectation.

That is if

$$p(x,\sigma) \cdot x \geq \sigma \cdot \pi^g \cdot x + (1-\sigma) \cdot \pi^b \cdot x.$$

Eliminating x, this condition reduces to

$$p(x,\sigma) \geq \pi^b - \sigma[\pi^b - \pi^g].$$

3.3 Banks

In period $t = 1, 2$, banks face a set of loan contracts A and prices $\{q_1(y,\sigma), y \in A\}$ and $\{q_2(y,\sigma), y \in A \cap R_-\}$, respectively. As in the case of insurers, the decision problem of banks is to choose how many of these different types of contracts to sell. And, as in the case of insurers, banks will participate in selling only those contracts that make non-negative profits in expectation.

For $y < 0$, non-negative profits requires

$$q_t(y,\sigma) \cdot y \geq \sigma \left[1 - \mu_t^g(y,\sigma)\right] \frac{y}{(1+r)} + (1-\sigma) \left[1 - \mu_t^b(y,\sigma))\right] \frac{y}{(1+r)},$$

where r is the risk-free rate available to banks and $\mu_t^i(y,\sigma)$ is the period-t probability that a person of type i whose type-score is σ will default on a loan of size y. Eliminating y, yields the condition

$$q_t(y,\sigma) \geq \left(\sigma \left[1 - \mu_t^g(y,\sigma)\right] + (1-\sigma) \left[1 - \mu_t^b(y,\sigma)\right]\right) (1+r)^{-1}.$$

For $y > 0$, non-negative profits require that

$$q_1(y,\sigma) \geq (1+r)^{-1}.$$

4 Equilibrium

We can now give the definition of a competitive equilibrium.

Definition A competitive equilibrium is (i) a set of loan prices $q_t^*(y,\sigma)$, (ii) a set of insurance prices $p^*(x,\sigma)$, (iii) a set of default probabilities $\mu_t^{i*}(y,\sigma)$, (iv) a set of decision rules $\ell^{i*}(\theta_1)$, $\ell'^{i*}(\theta_2,\ell,s)$, $d^{i*}(\theta_2,\ell,s)$, $x^{i*}(\ell',s')$ and $d^{i*}(x,z,\ell',s')$ and (v) a set of belief-updating functions $\Psi_1^*(\ell)$, $\Psi_2^*(d,\ell',\ell,s)$, and $\Psi_3^*(x,\ell',s')$ such that

1. For any given σ, each loan $y \in A$ earns zero profits.

   (a) For $y \geq 0$ this requires

   $$q_1^*(y,\sigma) = (1+r)^{-1}$$

   (b) For $y < 0$, this requires

   $$q_t^*(y,\sigma) = \left(\sigma\left[1-\mu_t^{g*}(y,\sigma)\right] + (1-\sigma)\left[1-\mu_t^{b*}(y,\sigma))\right]\right)(1+r)^{-1}$$

2. For any given σ, each insurance contract $x \in I$ earns zero profits. This requires

   $$p^*(x,\sigma) = \pi^b - \sigma[\pi^b - \pi^g].$$

3. Default probabilities are consistent with decision rules.

   (a) For $y < 0$ and $t = 1$ this requires

   $$\mu_1^{i*}(y,\sigma) = \int_\Theta d^{i*}(\theta_2,y,\sigma)dF_2(\theta_2)$$

   (b) For $y < 0$ and $t = 2$ this requires

   $$\mu_2^{i*}(y,\sigma) = \int d^{i*}(x^{i*}(y,\sigma),z,y,\Psi_3^*(x^{i*}(y,\sigma),y,\sigma)dZ^i(z)$$

4. The decision rules solve each household type's optimization problem given the pricing functions $q_t^*(y,\sigma)$, $p^*(x,\sigma)$, and the belief-updating functions $\Psi_1^*(\ell)$, $\Psi_2^*(d,\ell',\ell,s)$, and $\Psi_3^*(x,\ell',s')$.

5. Updating functions are consistent with decision rules and satisfy Bayes' Rule whenever possible. To state these conditions, define $H_1^{i*}(\ell) = \{\theta_1 : \ell^{i*}(\theta_1) = \ell\}$ and $H_2^{i*}(d, \ell'; \ell, s) = \{\theta_2 : \ell'^{i*}(\theta_2, \ell, s) = \ell'$ and $d^{i*}(\theta_2, \ell, s) = d\}$. Then,

   (a) For $\Psi_1^*(\ell)$ this requires

   $$\Psi_1^*(\ell) = \frac{\gamma \int 1_{\{\theta_1 \in H_1^{g*}(\ell)\}} dF_1(\theta_1)}{\gamma \int 1_{\{\theta_1 \in H_1^{g*}(\ell)\}} dF_1(\theta_1) + (1-\gamma) \int 1_{\{\theta_1 \in H_1^{b*}(\ell)\}} dF_1(\theta_1)}$$

   provided the denominator is positive – that is, provided a positive measure of people choose ℓ in period 1.

   (b) For $\Psi_2^*(d, \ell', \ell, s)$ this requires

   $$\Psi_2^*(d, \ell', \ell, s) = \frac{s \int 1_{\{\theta_2 \in H_2^{g*}(d,\ell';\ell,s)\}} dF_2(\theta_2)}{s \int 1_{\{\theta_2 \in H_2^{g*}(d,\ell';\ell,s)\}} dF_2(\theta_2) + (1-s) \int 1_{\{\theta_2 \in H_2^{b*}(d,\ell';\ell,s)\}} dF_2(\theta_2)}$$

   provided, again, the denominator is positive.

   (c) For $\Psi_3^*(x, \ell', s')$ this requires

   $$\Psi_3^*(x, \ell', s') = \frac{s' \cdot 1_{\{x^{g*}(\ell',s')=x\}}}{s' \cdot 1_{\{x^{g*}(\ell',s')=x\}} + (1-s') \cdot 1_{\{x^{b*}(\ell',s')=x\}}}$$

   provided, again, the denominator is positive.

We can use this definition to establish a simple property of every competitive equilibrium, namely, that there cannot be any lending in period 2.

Proposition 1. For all σ, $q_2^*(y, \sigma) = 0$ for any $y < 0$ and $\ell'^{i*}(\theta_2, \ell, s) = 0$.

Proof. Observe that, as noted earlier, $d^{i*}(x, z, y, s'') = 1$ whenever $y < 0$. Hence, it follows from part 3(b) that $\mu_2^{i*}(y, \sigma) = 1$ for $i = g, b$ and from part 1(b) that $q_2^*(y, \sigma) = 0$. Since loan prices are zero, $\ell'^{i*}(\theta_2, \ell, s) = 0$ is consistent with household optimization.

$\square$

This result comes from the assumption that the default decision is taken *after* the insurance purchase is made. Since there are no other market transactions that people engage in after the insurance market closes, there is no reason whatsoever to repay a loan. If the insurance purchase followed the default decision, the debtor would have to take account of any adverse change in the market's assessment of his or her type resulting from opportunistic behavior. Indeed, this is the channel through which it may be possible to constrain opportunistic behavior with regard to loans taken out in period 1 because for these loans the insurance purchase will follow the default decision. It is to this possibility that we now turn.

5 An Environment with Binary Choices

In this section, we consider a simple but illuminating special case of the general environment described in the previous section. First, Θ is restricted to be $\{1, \overline{\theta}\}$, where $\overline{\theta} > 1$. We assume that $F_1(1) = 1 - \phi$ and $F_2(1) = 1$, – that is, the period 2 distribution of θ is degenerate at $\theta = 1$ but there is probability $\phi > 0$ that $\theta > 1$ in period 1. Second, asset choice is restricted to a loan and deposit of a single size a – that is $A = \{-a, 0, a\}$. Third, insurance choice is restricted to full insurance only – that is $I = \{0, L\}$.

We consider an equilibrium with the following properties. All type b households take out a loan regardless of their preference shock. Among type g households only those with the high preference shock take out a loan and others save. All type g households who borrow pay back their loans in period 2 and purchase insurance at the cheapest price in period 3. All type b borrowers default in period 2 and purchase insurance at the highest price in period 3. The equilibrium clarifies the precise conditions under which "good behavior" in the credit market can be supported by superior treatment of "well-behaved" borrowers in the insurance market.

To establish an equilibrium with these features, the following five assumptions on primitives are sufficient.

Assumption 1 Sufficiently High Risk Aversion:

$$\pi^g u(e-L)+(1-\pi^g)u(e)<u(e-\pi^b L).$$

The inequality asserts that type g households prefer to purchase insurance even if the insurance is offered at a price that is appropriate for the type b households. Clearly this requirement is a restriction on curvature of the u function – the function must be sufficiently concave.

Assumption 2 Type g are Sufficiently Patient and Type b are Sufficiently Impatient:

$$\beta^g > \frac{u(e)-u(e-a)}{u(e-\pi^g\cdot L)-u(e-\pi^b\cdot L)}$$
$$\beta^b < \min\left\{\beta^g, 1, \frac{u\left(e+\frac{\phi\gamma a}{(\phi\gamma+1-\gamma)(1+r)}\right)-u(e)}{u(e-\pi^g\cdot L)-u(e-\pi^b\cdot L)}\right\}.$$

Note that the terms on the r.h.s. of these inequalities are positive so it is always possible to choose strictly positive β^i to satisfy them. It is possible that β^g would need to be larger than 1 but in the finite horizon context this is not an issue. Note however that we require β^b to be strictly less than 1.

Assumption 3 Sufficiently Many People with Low Urgency to Consume:

$$\left\{u\left(e+\frac{\phi\gamma a}{(\phi\gamma+1-\gamma)(1+r)}\right)-u(e)\right\}\leq \beta^g\left[u(e)-u(e-a)\right].$$

Observe that for any given u function and any admissible values of other parameters, the inequality is satisfied strictly for $\phi=0$. Therefore, by continuity, the inequality can always be satisfied for a positive ϕ sufficiently small.

Assumption 4 Sufficiently High Urgency to Consume:

$$\overline{\theta}\left\{u\left(e+\frac{\phi\gamma a}{(\phi\gamma+1-\gamma)(1+r)}\right)-u(e)\right\}\geq \beta^g\left[u(e)-u(e-a)\right].$$

Since $0 < \gamma < 1$, the term $(\phi\gamma a)/(\phi\gamma + 1 - \gamma)(1 + r))$ is positive. Therefore for any given u function and any (admissible) values of the other parameters, a $\overline{\theta} > 1$ can be chosen sufficiently high to satisfy the inequality.

Then we have the following.

Proposition 2 Given Assumptions 1-4 the following functions constitute an competitive equilibrium

- Pricing Functions and Default Probabilities

  1. For all σ, $q_1^*(a, \sigma) = 1/(1 + r)$, $q_1^*(-a, \sigma) = \sigma/(1 + r)$, and $q_2^*(-a, \sigma) = 0$.
  2. For all σ, $p^*(x, \sigma) = \pi^b - \sigma[\pi^b - \pi^g]$ for $x \in \{0, L\}$
  3. For all σ, $\mu_1^{g*}(-a, \sigma) = 0$, $\mu_1^{b*}(-a, \sigma) = 1$, and $\mu_2^{i*}(-a, \sigma) = 1$.

- Decision Rules

  1. $\ell^{b*}(\theta_1) = -a$ for all $\theta_1 \in \Theta$, $\ell^{g*}(\theta_1) = -a$ if $\theta_1 = \overline{\theta}$ and a otherwise, and $\ell'^{i*}(\theta_2, \ell, s) = 0$ for all (ℓ, s) and i.
  2. $d^{b*}(\theta_2, \ell, s) = 1$ if $\ell = -a$ and 0 otherwise, $d^{g*}(\theta_2, \ell, s) = 0$, and $d^{i*}(x, z, \ell', s'') = 1$.
  3. $x^{i*}(\ell', s') = L$

- Belief-Updating Functions

  1. $\Psi_1^*(-a) = (\phi\gamma)/(\phi\gamma + 1 - \gamma)$, $\Psi_1^*(0) = \gamma$, and $\Psi_1^*(a) = 1$
  2. For $\ell' \in \{0, -a\}$, $\Psi_2^*(0, \ell', a, s) = s$, $\Psi_2^*(0, \ell', 0, s) = s$, $\Psi_2^*(0, \ell', -a, s) = 1$ and $\Psi_2^*(1, \ell', -a, s) = 0$,
  3. For $\ell' \in \{0, -a\}$, $\Psi_3^*(0, \ell', s') = s'$ and $\Psi_3^*(L, \ell', s') = s'$.

Proof The proof involves checking that these functions satisfy each of the conditions in parts 1 through 6 of the definition of a competitive equilibrium.

It is easy to verify that given the default probabilities, the asset-market zero profit conditions in part 1 are satisfied. Also, the insurance-market zero profit condition in part 2 is (trivially) satisfied. It is also easy to verify that default probabilities are consistent with default decision rules, as required by the conditions in part 3.

Next we will verify that given the decision rules, the belief-updating functions satisfy Bayes' Rule whenever applicable. Recall that the fraction of type g in the population is $\gamma \in (0,1)$ and any person has a probability $\phi > 0$ of drawing $\theta_1 = \overline{\theta}$. Consider first the function $\Psi_1^*(\ell)$. The decision rules $\ell^{i*}(\theta_1)$ imply that

$$\begin{aligned}
&\gamma \int_\Theta 1_{\{\theta_1 \in H_1^{g*}(-a)\}} dF_1(\theta_1) = \phi\gamma \\
&\gamma \int_\Theta 1_{\{\theta_1 \in H_1^{g*}(-a)\}} dF_1(\theta_1) + (1-\gamma) \int_\Theta 1_{\{\theta_1 \in H_1^{b*}(-a)\}} dF_1(\theta_1) = \phi\gamma + (1-\gamma),
\end{aligned}$$

and

$$\begin{aligned}
&\gamma \int_\Theta 1_{\{\theta_1 \in H_1^{g*}(a)\}} dF_1(\theta_1) = (1-\phi)\gamma \\
&\gamma \int_\Theta 1_{\{\theta_1 \in H_1^{g*}(0)\}} dF_1(\theta_1) + (1-\gamma) \int_\Theta 1_{\{\theta_1 \in H_1^{b*}(0))\}} dF_1(\theta_1) = (1-\phi)\gamma
\end{aligned}$$

Therefore, for $\ell \in \{-a, a\}$, $\Psi_1^*(\ell)$ satisfies the conditions in part 5(a). But for $\ell = 0$, the conditions in 5(a) lead the indeterminacy (0/0) because $\int_\Theta 1_{\{\theta_1 \in H_1^{i*}(0)\}} dF_1(\theta_1) = 0$ for all $i = \{g, b\}$. In this case, Bayes Rule is not applicable and *any* assignment of beliefs is legitimate. We assume that a choice of $\ell = 0$ is uninformative about a person's true type.

Now consider the function $\Psi_2^*(d, \ell', \ell, s)$. For $(\ell, s) \in \{0, a\} \times [0, 1]$, the decision rules imply that everyone chooses $(d, \ell') = (0, 0)$ regardless of their θ_2 (actually there is no uncertainty with regard to θ_2 which always takes the value 1).

Therefore, for $(\ell, s) \in \{0, a\} \times [0,1]$

$$s \int 1_{\{\theta_2 \in H_2^{g*}(0,0;\ell,s)\}} dF_2(\theta_2) = s$$
$$s \int 1_{\{\theta_2 \in H_2^{g*}(0,0;\ell,s)\}} dF_2(\theta_2) + (1-s) \int 1_{\{\theta_2 \in H_2^{b*}(0,0;\ell,s)\}} dF_2(\theta_2) = 1$$

For $\ell = -a$, the decision rules imply that type g choose $(d, \ell') = (0,0)$ regardless of s and θ_2 and type b choose $(d, \ell') = (1,0)$ regardless of s and θ_2. Therefore

$$s \int 1_{\{\theta_2 \in H_2^{g*}(0,0;-a,s)\}} dF_2(\theta_2) = s$$
$$s \int 1_{\{\theta_2 \in H_2^{g*}(0,0;-a,s)\}} dF_2(\theta_2) + (1-s) \int 1_{\{\theta_2 \in H_2^{b*}(0,0;-a,s)\}} dF_2(\theta_2) = s$$

and

$$s \int 1_{\{\theta_2 \in H_2^{g*}(1,0;-a,s)\}} dF_2(\theta_2) = 0$$
$$s \int 1_{\{\theta_2 \in H_2^{g*}(1,0;-a,s)\}} dF_2(\theta_2) + (1-s) \int 1_{\{\theta_2 \in H_2^{b*}(1,0;-a,s)\}} dF_2(\theta_2) = (1-s)$$

Finally, for all (ℓ, s) and d and i, it is the case that $\int_\Theta 1_{\{\theta_2 \in H_2^{i*}(d,-a;\ell,s)\}} dF_2(\theta_2) = 0$, since the decision rules imply that no one chooses $(d, -a)$.

These conditions taken together imply that for $s \in (0,1)$ and $\ell' = 0$, $\Psi_2^*(d, \ell', \ell, s)$ satisfies all the conditions in part 5(b). For $s = 1$ and $(d, \ell') = (1,0)$ the requirement in 5(b) leads to the indeterminacy (0/0). The value assigned is reasonable in the sense that $\Psi_2^*(1,0,-a,1) = \lim_{s\to 1} \Psi_2^*(1,0,-a,s)$. Similarly, for $s = 0$ and $(d, \ell') = (0,0)$ the requirement in 5(b) leads to an indeterminacy. In this case, $\Psi_2^*(0,0,-a,0) = \lim_{s\to 0} \Psi_2^*(0,0,-a,s)$. Finally, for $\ell' = -a$, the requirement also leads to an indeterminacy – in this case we assume that this action is uninformative about a person's type and set $\Psi_2^*(d, -a, \ell, s) = s$ for all (ℓ, s) and d.

Finally, consider the function $\Psi_3^*(x, \ell', s')$. For any $\ell' \in A \cap R_-$ and any s', the decision rules imply that everyone chooses $x = L$. Therefore, for $(\ell', s') \in$

$\{A \cup R_{-}\} \times [0,1]$,

$$s' \cdot 1_{\{x^{g*}(\ell',s')=L\}} = s'$$
$$s' \cdot 1_{\{x^{g*}(\ell',s')=L\}} + (1-s') \cdot 1_{\{x^{b*}\ell',s')=L\}} = 1$$

and

$$s' \cdot 1_{\{x^{g*}(\ell',s')=0\}} = 0$$
$$s' \cdot 1_{\{x^{g*}(\ell',s')=0\}} + (1-s') \cdot 1_{\{x^{b*}(\ell',s'0=)\}} = 0$$

These expressions imply that for all $(\ell', s') \in \{A \cap R_{-}\} \times (0,1)$, $\Psi_3^*(L, \ell', s')$ satisfies the requirements in part 5(c). For s' equal to 0 or 1, the requirements lead to (0/0). In these cases, we assume that $\lim_{s' \to 0 \text{ or } 1} = \Psi_3^*(L, \ell', s') = s'$. For $x = 0$, (and all (ℓ', s')) the requirements also lead to the indeterminacy (0/0). In this case we assume that the action is uninformative about a person's true type, i.e., $\Psi_3^*(0, \ell', s') = s'$.

Next, we need to verify that the decision rules are the *optimal* decision rules given the pricing functions q_t^* and p^* and the belief-updating functions Ψ_t^*. Consider first the optimal choice of insurance in period 3. For a household of type i in state (ℓ', s'), the expected utility from choosing $x = 0$ is $\pi^i u(e-L) + (1-\pi^i)u(e)$ and expected utility from choosing $x = L$ is $u(e - [\pi^b - s'(\pi^b - \pi^g)]L)$. But by Assumption 1,

$$u(e - [\pi^b - s'(\pi^b - \pi^g)]L) > \pi^i u(e-L) + (1-\pi^i)u(e) \text{ for all } s'.$$

Hence $x(\ell', s') = L$ is the optimal decision rule. Consequently,

$$V_3^i(\ell', s') = u(e - [\pi^b - s'(\pi^b - \pi^g)]L)$$

The optimal value in the final period is independent of ℓ' and increasing in s'. Observe that independence from ℓ' follows from the fact that people cannot save and that if anyone arrived in the final period with debt then it is always optimal to default on that debt.

Next consider the decision rules in period 2. First, consider people (of any type) with $(\ell, s) \in \{0, a\} \times [0, 1]$. Observe that for such people $(0, 0)$ weakly dominates $(0, -a)$. This follows because (i) choosing $-a$ does not affect period-2 budget constraint since loan prices are zero, (ii) does not lead to an s' that is different from the s' that results from choosing $(d, 0)$ and (iii) $V_3^i(\ell', s')$ is independent of ℓ'. Therefore, it is (weakly) optimal for such people to choose $(d, \ell') = (0, 0)$. Consider now a person of type i with $\ell = -a$ and assessment s. If this person chooses $(d, \ell') = (1, 0)$ then by the belief-updating function Ψ_2^* he will start period 3 with $s' = 0$ and if he chooses $(d, \ell) = (0, 0)$ then by the belief-updating function he will start period 3 with $s' = 1$. Recognizing that θ_2 is always 1 (i.e there are no preference shocks in period 2), the utilities from default and no-default are given by

$$V_2^{i\,1}(1, -a, s) = u(e) + \beta^i\, u(e - \pi^b L)$$
$$V_2^{i\,0}(1, -a, s) = u(e - a) + \beta^i\, u(e - \pi^g L).$$

By Assumption 2, it follows that the optimal choice for a type-g household with debt is $(d, \ell') = (0, 0)$ regardless of s. Assumption 2 also implies that the optimal choice for a type-b household with debt is $(d, \ell') = (1, 0)$ regardless of s. To see this note that if $y < e$ and $\zeta < 1$ then for any strictly concave function u, $u(y + a) - u(y) > u(e + \zeta a) - u(e)$. The result follows by taking $y = e - a$ and $\zeta = (\phi\gamma)/(\phi\gamma + (1 - \gamma))$.

Finally, consider the decision rules in period 1. Households of either type must decide whether to borrow or not. Each household faces the equilibrium price schedule given by

$$q_1^*(-a, \sigma) = \sigma/(1 + r), \quad q_1^*(0, \sigma) = 1/(1 + r)$$

and the belief-updating function given by

$$\Psi_1^*(\ell) = \begin{cases} (\phi\gamma)/(\phi\gamma + 1 - \gamma) & \text{if} \quad \ell = -a \\ 1 & \text{if} \quad \ell = 0 \end{cases}$$

From these equations it follows that anyone who does not borrow in period 1 will start period 2 with $s' = 1$ and anyone who borrows in period 1 will do so at the price $(\phi\gamma)/[(1+r)\cdot(\phi\gamma+1-\gamma)]$ and will start period 2 with $s' = (\phi\gamma)/(\phi\gamma+1-\gamma)$.

Consider first a household of type g with $\theta_1 = 1$. From our earlier demonstration that indebted type-g households always choose to pay back in period 2 and (consequently) purchase insurance at the price π^g in period 3, it follows that the life-time utility of such a household from choosing to borrow is

$$V_1^g(1)_{\ell=-a} = u\left(e + a\frac{\phi\gamma}{(1+r)(\phi\gamma+1-\gamma)}\right) + \beta^g\, u(e-a) + (\beta^g)^2\, u(e - \pi^g L).$$

On the other hand, if such a household chooses not to borrow then he will start period 2 with $s' = 1$ and consequently will, again, purchase insurance in the period 3 at the price π^g. Therefore, the lifetime utility of such a household from choosing not to borrow is

$$V_1^g(1)_{\ell=0} = u(e) + \beta^g\, u(e) + (\beta^g)^2\, u(e - \pi^g L)$$

It follows from Assumption 3 that a type-g household with $\theta_1 = 1$ will choose not to borrow. Now consider a type-g household with $\theta_1 = \overline{\theta}$. For such a household the lifetime utility from choosing to borrow is

$$V_1^g(\overline{\theta})_{\ell=-a} = \overline{\theta} u\left(e + a\frac{\phi\gamma}{(1+r)(\phi\gamma+1-\gamma)}\right) + \beta^g\, u(e-a) + (\beta^g)^2\, u(e - \pi^g L)$$

and the lifetime utility of such a household from choosing not to borrow is

$$V_1^g(\overline{\theta})_{\ell=0} = \overline{\theta} u(e) + \beta^g\, u(e) + (\beta^g)^2\, u(e - \pi^g L)$$

It follows from Assumption 4 that such a household will choose to borrow. Hence $\ell^g(\theta_1) = -a$ for $\theta_1 = \overline{\theta}$ and $\ell^g(\theta_1) = 0$ for $\theta = 1$ is the optimal decision rule.

Consider next a household of type b with $\theta_1 = 1$. From our earlier demonstration that an indebted household of type b defaults regardless of his s and (consequently) purchases insurance in period 3 at the price π^b, it follows that the lifetime utility of such a household from choosing to borrow is

$$V_1^b(1)_{\ell=-a} = u\left(e + a\frac{\phi\gamma}{(1+r)(\phi\gamma+1-\gamma)}\right) + \beta^b\, u(e) + (\beta^b)^2\, u(e - \pi^b L)$$

On the other hand, if such a household chooses not to borrow then he will start period 2 with $s' = 1$ and will consequently purchase insurance in the period 3 at the price π^g. Therefore, the lifetime utility of such a household from choosing not to borrow is

$$V_1^b(1)_{\ell=0} = u(e) + \beta^b\, u(e) + (\beta^b)^2\, u(e - \pi^g L)$$

It follows from Assumption 2, in particular the fact that $\beta^b < 1$, that the optimal choice for this household is to borrow. Furthermore, it should be clear that if a type-b household with $\theta_1 = 1$ finds it optimal to borrow then a type-b household with $\theta_1 = \overline{\theta} > 1$ would also find it optimal to borrow. Hence the optimal decision rule for a type-b household is $\ell^b(\theta_1) = -a$ for all $\theta_1 \in \Theta$.

□

Corollary. If type is observable, or if $\pi^g = \pi^b$, then $q_1^*(\ell, \sigma) = 0$.

Proof. If type is observable then in period 3, full insurance is available to type g and b at prices π^g and π^b respectively. In period 2, it is a strictly optimal for both types to default since defaulting yields utility $u(e) + \beta^i u(e - \pi^i L)$ and not defaulting yields utility $u(e-a) + \beta^i u(e - \pi^i L)$ where $a > 0$. Since both types default with certainty, the price of loans is zero. If type is not observable but $\pi^g = \pi^b = \pi$, then again full insurance is available to both types at the price π and once again it is strictly optimal to for both types to default in period 2.

□

It is worth noting that the equilibrium has properties that match features of the data. If we interpret σ as a person's credit score (i.e., a credit score is simply an assessment that a person is of type-g) then the equilibrium implies that (i) when people default their credit scores decline and (ii) people with a low credit scores get worse insurance rates and are, on average, more likely to file a claim.

6 An Environment with Signalling in the Insurance Market

When people are only offered a choice between full insurance and no insurance, a standard means of signalling one's type is unavailable. Namely, the good insurance risks do not have the option of signalling their low-risk status by accepting limited insurance and thereby receiving insurance at a cheaper rate. Of course, such mechanisms are ubiquitous in insurance markets. It is important, therefore, to investigate if the inability of people to signal their type in the insurance market is critical to the results derived in Proposition 2. The upshot of this section is that the possibility of signalling one's type by taking limited insurance makes it harder, but not impossible, for the period 1 loan market to function.

The microeconomic literature on the provision of insurance indicates that for a population with two hidden types, competition among insurers will result in one of two kinds of equilibrium - pooling or separating. In a pooling equilibrium insurers offer one full-insurance contract at a price that reflects the composition of low- and high-risk types in the population. In a separating equilibrium insurers offer two contracts, one with limited insurance at a low price designed to attract only the low-risk types and another with full insurance at a higher price for the remaining high-risk types.

An important insight of the microeconomic insurance literature is that competitive insurers have an incentive to break away from pooling contracts. In a pooling contract on which insurers make zero profits, the low-risk types must subsidize the high-risk types since both types pay the same price but the high-risk types have a higher probability of loss. Because of this subsidy from low-risk to high-risk types, insurers have an incentive to entice away the low-risk types by offering them less-than-full-insurance at a price that is below the pooling contract price but above the price that would be actuarially fair for low-risk types. Since low-risk types receive insurance at an actuarially *unfair* price in a pooling contract, a profitable limited-insurance contract generally exists. However, in a significant early contribution to this literature, Wilson (1977) noted that this logic overlooks an important point, namely, that once the low-risk types are enticed away by an attractively priced limited-insurance contract, the *pooling* contract becomes unavailable to the high-risk types. Consequently, for the enticement strategy to succeed the high-risk types must not find it in their interest to pool with the low-risk types (and accept limited insurance) when the alternative is purchasing full insurance at a *non-subsidized* price. This incentive of high-risk types to "re-pool" with the low-risk types is most intense when the pool contains relatively *few* high-risk types. In this case the price of insurance in the limited-insurance contract cannot be too much below the price of insurance in the original pooling contract and therefore the insurance offered at the lower price cannot be too much below full insurance. Such a contract will appear quite attractive to a high-risk type whose alternative is to purchase somewhat more insurance at a potentially *much* higher price.

In the context of this paper, Wilson's insight translates into the following observation. The kinds of insurance opportunities a person will face will depend on the person's type-score (or risk assessment) s'. A type-g person with a low s' is in a pool that is mostly composed of type-b people. Since the type-g's are the low-risk types, the standard separating contract argument suggests that the type-g people in this pool can be profitably offered the option of cheap but limited insurance. In contrast, a type-g person with a high s' is in a pool composed of mostly the low-risk types and Wilson's insight suggests that their insurance choices will be limited only to the appropriate pooling contract.

We now develop this observation in formal detail. The focus is to determine the *best* separating insurance contract, if any, that can be offered to attract the low-risk types in a given pool of people with type-score σ. In what follows we will find it convenient to denote an *insurance contract* as a pair (X, m), where X is the indemnity, m is the price per unit of insurance (so that the premium on the contract is $m \cdot X$). We will denote the utility of a type i from purchasing a contract (X, m) by $W^i(X, m) = \pi^i u(e - mX - L + X) + (1 - \pi^i)u(y - mX)$.[8]

Lemma 1 There exists a unique $0 < X < L$ such that $W^b(X, \pi^g) = W^b(L, \pi^b)$.

Proof Since $\pi^g < \pi^b$ it's clear that $W^b(X = L, \pi^g) > W^b(L, \pi^b)$. And, by virtue of the strict concavity of u, no-insurance is worse than full-insurance at an actuarially fair price so $W^b(X = 0, \pi^g) < W^b(L, \pi^b)$. Clearly $W^b(X, \pi^g)$ is a continuous function of $X \in [0, L]$. Therefore the existence of $X \in (0, L)$ follows from the Intermediate Value Theorem. Uniqueness of X follows from the fact that $W^b(X, \pi^g)$ is strictly increasing in X since insurance is being offered at a price that is lower than the probability of loss.

$\square$

Since $W^b(X, \pi^g)$ is strictly increasing in X, X has the interpretation of being the *most generous* actuarially fair insurance that can be offered to type-g people who are in a pool of people with type-score $\sigma \in [0, 1]$ without necessarily attracting the type-b people in the pool.

Lemma 2 Let $m(\sigma) = \pi^b - \sigma[\pi^b - \pi^g]$. Then, for any $\sigma \in [0, 1]$ there exists a unique $x(\sigma) \in [0, L]$ such that $[W^g(x(\sigma), \pi^g) - W^g(L, m(\sigma))] \cdot x(\sigma) = 0$. Furthermore $x(\sigma)$ is a continuous and weakly increasing in σ.

Proof First, observe that $W^g(X, \pi^g)$ is clearly continuous in X and, because the price of the insurance is actuarially fair, it is strictly increasing in X. Consider first the case where $\sigma = 1$. Clearly a unique $x(1)$ exists and is equal to

[8] Since no savings is permitted in period 2, the beginning of period resources of every consumer is e.

L. Next consider any $\sigma \in [0, 1)$. Then $W^g(X = L, \pi^g) > W^g(L, m(\sigma))$ because $\pi^g < m(\sigma)$. Now two cases can arise: (i) $W^g(X = 0, \pi^g) < W^g(L, m(\sigma))$, in which case a unique $x(\sigma) \in (0, L)$ exists by the continuity and monotonicity of $W^g(X, \pi^g)$; or (ii) $W^g(X = 0, \pi^g) \geq W^g(L, m(\sigma))$ in which case a unique $x(\sigma)$ exists and is equal to 0.

From the continuity of $W^g(X, \pi^g)$ with respect to X, the continuity of $m(\sigma)$ with respect to σ, and the continuity of $W^g(L, m)$ with respect to m, it follows that $x(\sigma)$ is a continuous function of $\sigma \in [0, 1]$. Furthermore, $W^g(L, m(\sigma))$ is strictly increasing in σ because $m(\sigma)$ is strictly decreasing in σ. Since $W^g(X, \pi^g)$ is also strictly increasing in X, one may easily verify that the $x(\sigma)$ has the following form: either $x(\sigma)$ is strictly increasing over the entire range $\sigma \in [0, 1]$, or, $x(\sigma) = 0$ for all $\sigma \leq \sigma^0$ and strictly increasing for all $\sigma > \sigma^0$ where σ^0 is some value in $[0, 1]$. Hence $x(\sigma)$ is weakly increasing in σ.

$\square$

Since $W^g(X, \pi^g)$ is strictly increasing in X, $x(\sigma)$ has the interpretation of being the *least generous* actuarially fair insurance that can be offered to type-g people who are in a pool of people with wealth y and type-score σ without giving the type-g in the pool a strict incentive to choose the full-insurance contract offered at the price $m(\sigma)$.

Given the interpretations of X and $x(\sigma)$, it follows that if $X < x(\sigma)$, the low-risk (type-g) people who are in a pool of people with assessment σ *cannot* be offered a separating contract they would actually want to take. The reason is because the least generous separating contract that type-g would weakly prefer over the full-insurance pooling contract requires a *greater* level of insurance than is consistent with keeping the type-b people from also accepting the same contract. In contrast, if $x(\sigma) = X$ then a separating contract that type-g would (weakly!) prefer over the pooling contract exists and is given by (X, π^g). Similarly, if $x(\sigma) < X$ then infinitely many separating contracts exist, including the contract (X, π^g).

We can now state the following important result.

Proposition 3 There exists a unique $\sigma^* \in (0,1)$ such that type-g people who are in a pool composed of people with risk assessment $\sigma \geq \sigma^*$ cannot be offered a separating contract but type-g people who are in a pool composed of people with risk assessment $\sigma < \sigma^*$ can be offered a separating contract and the best separating contract that can be offered to them is (X, π^g).

Proof We will prove the Proposition by showing that there is a unique $\sigma^* \in (0,1)$ that satisfies the equation $x(\sigma^*) = X$. From Lemmas 1 and 2 we know $x(\sigma = 1) = L > X$. Now consider $x(0)$ which solves

$$\pi^g u(e - \pi^g x(0) - L + x(0)) + (1 - \pi^g) u(e - \pi^g x(0)) = u(e - \pi^b L). \tag{2}$$

We know that X solves

$$\pi^b u(e - \pi^g X - L + X) + (1 - \pi^b) u(e - \pi^g X) = u(e - \pi^b L). \tag{3}$$

Since $X < L$ (by Lemma 1), we know that $u(e - \pi^b L) < u(e - \pi^g X))$. Therefore, $u(e - \pi^b L)$ being the average of the two terms in (3), it follows that

$$u(e - \pi^g X(y) - L + X) < u(e - \pi^b L) < u(e - \pi^g X).$$

Therefore, since $(1 - \pi^g) > (1 - \pi^b)$,

$$\pi^g u(e - \pi^g X(y) - L + X) + (1 - \pi^g) u(e - \pi^g X) > u(e - \pi^b L). \tag{4}$$

Hence (2) and (4) imply $x(0) < X$. By Lemma 2, $x(\sigma)$ is continuous and monotone in σ. Therefore there must exist a unique $\sigma^* \in (0,1)$ such that $x(\sigma^*) = X$. Since $X > 0$, it follows from Lemma 2 that $x(\sigma) - X$ is strictly increasing in σ at σ^*.

When $x(\sigma) < X$ then any contract $(\widetilde{X}, \pi^g)$, $\widetilde{X} \in [x(\sigma), X]$ is a separating contract. However among the set of separating contracts (X, π^g) gives the highest utility to type-g people. This follows because u is strictly concave and the insurance is offered at a price that is actuarially fair for type-g people.

$\square$

To summarize, there is a $\sigma^* \in (0,1)$ such that people with $s' \geq \sigma^*$ can only be offered the full-insurance pooling contract at a price that depends on s'. People with assessment $s' < \sigma^*$ can be offered the opportunity to buy limited insurance at a cheaper price and the most attractive limited-insurance contract that can be offered low-risk type-g agents is $X < L$ at a price π^g.

We now proceed to incorporate this logic of pooling and separation into the framework of the paper. We do this by *expanding* the set I to include a choice of limited insurance as well. Specifically, $I = \{0, X, L\}$. That is, we now allow a limited insurance option $0 < X < L$, where X is the most generous limited insurance contract that can be offered to people (when such a contract can be offered at all). The set A remains $\{-a, 0\}$. In what follows, Assumptions 1, 3 and 4 will be maintained but Assumption 2 is modified as follows.

Assumption 2' Type g are Sufficiently Patience and Type b are Sufficiently Impatient

$$\beta^g > \frac{u(e) - u(e-a)}{u(e - \pi^g L) - [\pi^g u(y - \pi^g X(e) - L + X) + (1 - \pi^g) u(y - \pi^b X]}$$

$$\beta^b < \min \left\{ 1, \beta^g, \frac{u\left(e + \frac{\phi\gamma a}{(\phi\gamma + 1 - \gamma)(1+r)}\right) - u(e)}{u(e - \pi^g \cdot L) - u(e - \pi^b \cdot L)} \right\}$$

Proposition 4 Under the Assumptions 1, 2', 3 and 4, the following functions constitute a competitive equilibrium

- Pricing Functions and Default Probabilities

  1. For all σ, $q_1^*(-a, \sigma) = \sigma/(1+r)$, $q_2^*(-a, \sigma) = 0$, and $q_t^*(0, \sigma) = 1/(1+r)$
  2. For all σ, $p^*(x, \sigma) = \pi^b - \sigma[\pi^b - \pi^g]$ for $x \in I$
  3. For all σ, $\mu_1^{g*}(-a, \sigma) = 0$, $\mu_1^{b*}(-a, \sigma) = 1$, and $\mu_2^{i*}(-a, \sigma) = 1$.

- Decision Rules

  1. $\ell^{b*}(\theta_1) = -a$ for all $\theta \in \Theta$, $\ell^{g*}(\theta_1) = -a$ if $\theta_1 = \overline{\theta}$ and 0 otherwise, and for all s, $\ell'^{i*}(\theta_2, \ell, s) = 0$.
  2. $d^{b*}(\theta_2, \ell, s) = 1$ if $\ell = -a$ and 0 otherwise, $d^{g*}(\theta_2, \ell, s) = 0$, and $d^{i*}(x, z, \ell', s') = 1$.
  3. $x^{i*}(\ell', s') = L$ for $s' \geq \sigma^*$, $x^{g*}(\ell', s') = X$ for $s' < \sigma^*$, and $x^{b*}(\ell', s') = L$ for $s' < \sigma^*$

- Belief-Updating Functions

  1. $\Psi_1^*(-a) = (\phi\gamma)/(\phi\gamma + 1 - \gamma)$, $\Psi_1^*(0) = 1$
  2. For $\ell' \in \{0, -a\}$, $\Psi_2^*(0, \ell', 0, s) = s$, $\Psi_2^*(0, \ell', -a, s) = 1$, $\Psi_2^*(1, \ell', -a, s) = 0$,
  3. For $\ell' \in \{0, -a\}$

$$\Psi_3^*(x, \ell', s') = \begin{cases} s' & \text{if } s' \geq \sigma^* \text{ and } x \in \{0, L\} \\ 1 & \text{if } s' < \sigma^* \text{ and } x = X \\ s' & \text{if } s' < \sigma^* \text{ and } x = 0 \\ 0 & \text{if } s' < \sigma^* \text{ and } x = L \end{cases}$$

Proof Observe that the only differences between the statements of Proposition 2 and 3 is in period 3, namely the decision rule $x^{i*}(\ell', s')$ and the specification of the $\Psi_3^*(x, \ell', s')$. Therefore, to establish this proposition we only need to establish two facts. First, the updating function Ψ_3^* meets the Bayes' Rule requirements and, second, that *all* the decision rules are optimal given the pricing functions and belief-updating functions.

Consider first (ℓ', s') for which $s' > \sigma^*$. The decision rules imply that both types choose $x = L$. Therefore,

$$s' \cdot 1_{\{x^{g*}(\ell', s') = L\}} + (1 - s') \cdot 1_{\{x^{b*}\ell', s') = L\}} = 1$$

$$s' \cdot 1_{\{x^{g*}(\ell', s') = L\}} = s'$$

and

$$s' \cdot 1_{\{x^{g*}(\ell',s')=0\}} + (1-s') \cdot 1_{\{x^{b*}(\ell',s')=0)\}} = 0$$
$$s' \cdot 1_{\{x^{g*}(\ell',s')=0\}} = 0$$

These expressions imply that for (ℓ', s') such that $s' \geq \sigma^*$, $\Psi_3^*(x = L, \ell', s')$ satisfies the requirements in part 5(c). For $x = 0$, however, the requirements lead to an indeterminacy (0/0). We assume that off-equilibrium choice is uninformative about a person's type. That is $\Psi_3^*(x = 0, \ell', s') = s'$.

Next, consider the case (ℓ', s) for which $s' < \sigma^*$. The decision rules imply that type-g choose $x = X$ and type-b choose $x = L$. Therefore,

$$s' \cdot 1_{\{x^{g*}(\ell',s')=L\}} + (1-s') \cdot 1_{\{x^{b*}\ell',s')=L\}} = 1 - s'$$
$$s' \cdot 1_{\{x^{g*}(\ell',s')=L\}} = 0,$$

and

$$s' \cdot 1_{\{x^{g*}(\ell',s')=X\}} + (1-s') \cdot 1_{\{x^{b*}(\ell',s'=X)\}} = s'$$
$$s' \cdot 1_{\{x^{g*}(\ell',s')=X(e)\}} = s',$$

and for x equal to 0,

$$s' \cdot 1_{\{x^{g*}(\ell',s')=0\}} + (1-s') \cdot 1_{\{x^{b*}(\ell',s')=0)\}} = 0$$
$$s' \cdot 1_{\{x^{g*}(\ell',s')=0\}} = 0.$$

These expression imply that the requirements in part 5(c) are met for $x = L$ and $x = X$ but lead to the indeterminacy (0/0) for $x = 0$. In the proposed equilibrium it assumed that $\Psi_3^*(x = 0, \ell', s') = s'$.

Now we establish that the decision rules are indeed optimal. We begin with the decision rules in period 3. Note that Proposition 1 (which is true for any Ψ_3^* function) still applies. Hence for $\ell' < 0$, $d^i(x, z, \ell', s'') = 1$ is still the optimal decision. Therefore, using $A = \{-a, 0\}$ and the insurance pricing $p^*(x, \sigma)$, a type-i household's insurance choice problem reduces to:

$V_3^i(\ell', s') =$

$\max_x \pi^i u(e - [\pi^b - s''(\pi^b - \pi^g)]x - L + x) + (1 - \pi^i)u(e - [\pi^b - s''(\pi^b - \pi^g)]x)$

s.t

$x \in \{0, L\}$ if $s' > \sigma^*$ and $x \in I$ otherwise

$s'' = \Psi_3^*(x, \ell', s')$.

Consider first households with $s' \geq \sigma^*$. For these households $\Psi_3^*(x, \ell', s') = s'$ for $x \in \{0, L\}$. By Assumption 1 it follows that the optimal insurance choice for these households, regardless of type, is $x = L$. Consider next a household with $s' < \sigma^*$. If this household chooses $x \leq X$ then Ψ_3^* implies that the household's $s'' = 1$. By the pricing function p^*, the household will face the price π^g. Now suppose the household is of type-g. Since the price π^g is actuarially fair for him, *conditional* on choosing $x \leq X$, it is optimal for him to choose $x = X$. If the household is of type-b, then the price π^g is *better than* actuarially fair. So, conditional on choosing $x \leq X$, it also optimal for a type-b household to choose $x = X$. On the other hand, for any household that chooses $x = L$, Ψ_3^* implies that the household's $s'' = 0$. By the pricing function p^*, it follows that the household will face the price π^b. Consequently, the choice of insurance reduces to a choice between the contract (X, π^g) and the contract (L, π^b) regardless of type. Now we know from Lemma 1 that a type-b person is indifferent between these two contracts. And, by Lemma 2 we know that a type-g household with $s' < \sigma^*$ strictly prefers the contract (X, π^g) to (L, π^b). Therefore $x^*(\ell', s')$ is the optimal decision rule.

Next consider the decision rules in period 2. As before, it is weakly optimal for a household with $\ell = 0$ to choose $(d, \ell') = (0, 0)$. Consider then a household of type i with $\ell = -a$ and assessment s. If this household chooses $(d', \ell) = (1, 0)$

then by the belief-updating function Ψ_2^* he would start period 3 with $s' = 0$ and if he chooses $(d', \ell) = (0, 0)$ then by the belief-updating function he would start period 3 with $s' = 1$. Recognizing that θ_2 is always 1 (no preference shocks in period 2), the utilities for a type-g household from default and no-default are given by

$$V_2^{g\,1}(1, -a, s) = u(e) + \beta^i\, \pi^g u(e - \pi^g X(e) + X - L) + (1 - \pi^g) u(e - \pi^g X)$$
$$V_2^{g\,0}(1, -a, s) = u(e - a) + \beta^i\, u(e - \pi^g L).$$

By Assumption 2', it follows that the optimal choice for a type-g household with debt is $(d, \ell') = (0, 0)$ regardless of s. The utilities for type-b household from default and no-default are given by

$$V_2^{b\,1}(1, -a, s) = u(e) + \beta^i\, u(e - \pi^b L)$$
$$V_2^{b\,0}(1, -a, s) = u(e - a) + \beta^i\, u(e - \pi^g L).$$

As before, Assumption 2' continues to imply that the optimal choice for a type-b household with debt is $(d, \ell') = (1, 0)$ regardless of s. Finally, the proof establishing that $\ell^*(\theta_1)$ is the optimal decision rule is *identical* to the one given in Proposition 2 and is therefore not repeated here.

□

It is worth noting that even though we permitted the insurance industry to offer separating contracts when profitable, separating contracts are not used in equilibrium. It is still the case that households choose to reveal their type in the credit market. A fraction $(1 - \phi)$ of type-g households reveal their type in period 1 by choosing not to borrow and the remaining fraction $\phi > 0$ of type-g households borrow but reveal their type in period 2 by choosing not to default. All type-b households borrow but reveal their type in period 2 by choosing to default. Thus by the time people arrive in the insurance market, insurers are fully informed about each person's type and competition ensures that each person is offered *full* insurance at the appropriate actuarially fair price.

Nevertheless the fact that a separating contract could be chosen by a type-g in situations where such a contract is desirable does have implications for the operation of the credit market. In particular, when a type-g person contemplates default he is aware that even though he will arrive in the insurance market with $s' = 0$, he does not have to purchase full insurance at the price π^b. At that stage he will have the option of purchasing limited insurance $x = X$ at the price π^g – an option that is strictly better than purchasing $x = L$ at the price π^b. Consequently, the pay-off from default for a type-g person is *higher* as a result of the possibility of separating insurance contracts. This, in turn, means that it becomes *harder* to sustain good behavior in the credit market. This fact is evident in the new Assumption 2' – the term on the r.h.s in the inequality for β^g is now *larger* relative to the corresponding term in Assumption 2. Given a value of β^g, there are values for a – the loan size– for which Assumption 2 is satisfied but *not* Assumption 2'. Therefore, the possibility of separating contracts in the insurance market *reduces* the set of loan sizes for which it is possible to sustain debt.

7 Conclusion

As the argument in Propositions 2 and 4 make clear, the logic of debt repayment in this model relies on two things – the good types (type g) have a lower probability of loss and therefore have an incentive to separate themselves from the type b in the insurance market (this is why "looking good" is valuable to the good types) and the bad types (type b) do not have an incentive to mimic the good types because the rewards to "looking good" come in the future and the bad types do not care sufficiently about the future.

In closing we comment on a wider motivation for considering problems of the sort analyzed in this paper. The fundamental aspect of our environment is that people have private information about some personal characteristics that are relevant to their trading partners. We know from previous work that in such environments with *adverse selection*, competitive equilibria need not exist and even if one exists it need not be Pareto Optimal. Yet, there is little doubt that both lenders and insurers view adverse selection as one of their most challenging business problem. More broadly, the issue of adverse selection arises in many exchange contexts. We view this paper as taking a modest step in the direction of formulating adverse selection problems in the language of recursive competitive equilibrium. In this regard, we believe that separating the "learning problem" which is characteristic of these environments from the "equilibrium pricing problem" is conceptually useful. The examples worked out in this paper illustrate how we can use the recursive belief-updating functions to accomplish this separation. In our companion work (Chatterjee *et. al.*(2005b)) we are using this approach to analyze unsecured borrowing and lending with adverse selection in a more standard infinite-horizon macro model. We have found that under certain conditions competitive equilibria exist and can be computed. Furthermore the equilibrium has properties that match the data – for instance, interest rates depend negatively on credit scores and people who default see their credit scores decline.

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