Stochastic impulse control with discounted and ergodic optimization criteria: A comparative study for the control of risky holdings

Yiannis Kamarianakis[a,b*], Anastasios Xepapadeas[a]

[a]*Department of Economics, University of Crete, Rethymnon, Crete, Greece*
[b]*Regional Analysis Division, Institute of Applied and Computational Mathematics, Foundation for Research and Technology, Heraklion, Crete, Greece*

Abstract

We consider a single-asset investment fund that in the absence of transactions costs would hold a constant amount of wealth in the risky asset. In the presence of market frictions wealth is allowed to fluctuate within a control band: Its upper (lower) boundary is chosen so that gains (losses) from adjustments to the target minus (plus) fixed plus proportional transaction costs maximize (minimize) a power utility function. We compare stochastic impulse control policies derived via ergodic and discounted optimization criteria. For the solution of the ergodic problem we use basic tools from the theory of diffusions whereas the discounted problem is solved after being characterized as a system of quasi-variational inequalities. For both versions of the problem, derivation of the control bands pertains to the numerical solution of a system of nonlinear equations. We solve numerous such systems and present an extensive comparative sensitivity analysis with respect to the parameters that characterize investor's preferences and market behavior.

JEL classification: C61; G11

Keywords: Transaction costs; stochastic impulse control; ergodic criteria

* Corresponding author. Present address: Regional Analysis Division, Institute of Applied and Computational Mathematics, Foundation for Research and Technology, Vasilika Vouton, GR 71110, Heraklion, Crete, Greece.
Tel: +30 2810 391771, Fax: +30 2810 391761, E-mail: kamarian@iacm.forth.gr

1. Introduction

This article compares control bands derived via stochastic impulse control with ergodic criteria that maximize expected profits (or minimize expected costs) per unit time and discounted criteria that maximize expected discounted profits (respectively minimize expected discounted costs) over lifetime. To our knowledge this is the first research effort that attempts such a comparison; the vast majority of the literature related to stochastic impulse control problems examines discounted criteria. However, in some problems[1] the choice of a discounting rate does not have a clear economic interpretation. Moreover, comparing with previous comparative statics analyses (e.g. Cadenillas and Zapatero, 1999, Suzuki and Pliska, 2004, Cadenillas *et al.* 2006) this is the first time that sensitivity of the control bands with respect to the discounting rate is examined.

As a means for the attempted comparison, we use the following problem. An investor, in the absence of market frictions would aim to keep a constant amount (not proportion!), say χ, of her wealth in a risky investment[2]. In the presence of transactions costs, to avoid continuous rebalancing (and thus ruin) she seeks for an optimal control band (L, U) with $L<\chi<U$ for her risky holdings. Wealth is allowed to fluctuate freely within the band and as soon as it reaches the boundaries it is adjusted to the target level. The upper (lower) boundary is chosen so that wealth obtained (lost) from adjustments to the target minus (plus) transaction costs maximizes (minimizes) a power utility function.

---

[1] Like for instance when controlling an FX rate, see Jack and Zervos (2006), Melas and Zervos (2006).

[2] Merton (1990, chapter 4) showed that such policies are optimal for investors that maximize exponential utility of lifetime consumption. In Browne (1995) it is shown that such investment strategies minimize the probability of ruin for investors that face an uncontrollable stochastic cash flow. Browne (1998) displayed that constant amount of wealth in the risky asset is also optimal for investors that aim to maximize the mean rate of return on risky investment (defined as the mean excess return from investment above the risk free rate).

Similarly to the vast majority of related literature, we assume that instantaneous asset returns follow a diffusion process with constant mean and volatility. Moreover, we do not consider any finite fuel constraints for financing adjustments to the target level from the lower boundary. Since our analysis takes into account fixed plus proportional transaction costs per intervention, derivation of optimal policies requires stochastic impulse control methods. A related problem derives control bands for some target asset proportions; see Suzuki and Pliska (2004), Tamoura (2006), Kamarianakis and Xepapadeas (2006).

During the past decade, numerous research efforts developed the stochastic impulse control theory and applied it to problems emerging in economics and finance. Important theoretical contributions include Harrison *et al.* (1983), Bensoussan and Lions (1984), Dixit (1991), Dumas (1991) and Korn (1997, 1998). Buckley and Korn (1999) and Baccarin (2002) applied the theory to the cash management problem, Plehn-Dujowich (2005) derived control bands for optimal price adjustment in the presence of menu costs, Jeanblanc-Pique (1993), Mundaca and Oksendal (1998) and Cadenillas and Zapatero (1999, 2000) controlled an exchange rate, Suzuki and Pliska (2004), Tamoura (2006) and Kamarianakis and Xepapadeas (2006) applied the theory to control the risky fraction process of a portfolio, Jeanblanc-Pique and Shiryaev (1995) and Cadenillas *et al.* (2005, 2006) derived optimal dividend policies, Zakamouline (2006) performed utility-based European option pricing and hedging and the list of applications is far from being complete. The aforementioned research efforts have focused on the stochastic impulse control problem with discounted optimization criteria; recently, Jack and Zervos (2006) considered the stochastic impulse control problem with ergodic optimization criteria. The objective there was to minimize a long-term expected criterion as well as a long term pathwise criterion that penalize both deviations of the state process from a given nominal point and the use of impulsive control effort *per unit time*.

The article is organized as follows. In the second section we display the model and present the alternative discounted/ergodic stochastic impulse control objectives. In the third section we examine how optimal rebalancing strategies can be computed via standard diffusion theory for investors that maximize long run benefit minus cost *per unit time*. The method presented in Jack and Zervos (2006) cannot be applied here since the assumptions they state do not hold for our problem; instead we adopt a more computationally intensive approach that follows the methodology presented in Karlin and Taylor (1981, section 15.4) for a simple example[3]. The fourth section illustrates the solution of the discounted problem, which is characterized as a system of quasi-variational inequalities. In the fifth section we perform a sensitivity analysis and show the dependence of optimal policy to the type of optimization objective. Our results indicate linear dependence of the lower band of the considered (discounted) problem to the discount rate. Moreover we discover significant differences between policies derived with ergodic/discounted criteria; the magnitude of these differences varies with parameters that characterize market behavior and investor's preferences. For the range of discount rates considered in our application the investor behaves in a less risk averse manner when adopting the ergodic criterion. We conclude with some final remarks and directions for further research in section six.

[3] Karlin and Taylor examined a simple cash management problem with cash dynamics following a Brownian motion with no drift. The optimization objective minimized tracking error plus (fixed) transaction costs per unit time. The setting of their problem allowed them to obtain analytical expressions for the boundaries of the control band and the rebalancing point. For more complex dynamics (e.g. geometric Brownian motion, mean reverting processes) this approach is computationally very intensive and analytical expressions for the control boundaries are impossible to obtain. To our knowledge this is the first time this method is used to solve a stochastic impulse control problem with an ergodic optimization criterion after Karlin and Taylor's seminal contribution.

2. Problem formulation

Consider a complete probability space (*Ω, F, P*) endowed with a filtration (F_t), which is the *P*-augmentation of the filtration generated by a one-dimensional Brownian motion *W*. Our state variable is the risky wealth *X* of an investor or a firm, that in the absence of interventions follows a geometric Brownian motion process. In the presence of interventions the state dynamics are described by the following generalized Ito equation

$$X_t = \chi + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dW_s - \sum_{n=1}^{\infty} I_{\{\tau_n < t\}} \xi_n \ , \tag{2.1}$$

where μ is a non-negative constant representing the expected rate of return, σ^2 is the variance of the process with σ assumed positive, ξ_n is the magnitude of the *n*th intervention and χ represents the initial holdings which are strictly positive and for simplicity are assumed to coincide with the target holdings. It is worth noting that the expected value of $X(t)$ is

$$EX(t) = \chi \exp(\mu t) \, , \tag{2.2}$$

its variance is given by

$$VarX(t) = \chi^2 \exp(2\mu t)\left(\exp(2\sigma^2 t) - 1\right) \tag{2.3}$$

and we observe that while the expected value of $X(t)$ in (2.2) increases at a rate $\exp(\mu t)$ the standard deviation (square root of variance in equation 2.3) increases even faster.

Turning to the specification of the objective function and transaction costs we define the functions

$$g_1(\xi) := -K_1 + \frac{1}{\gamma_1}(k_1\xi)^{\gamma_1} I_{\{\xi>0\}}, \qquad g_2(\xi) := -K_2 - \frac{1}{\gamma_2}(k_2\xi)^{\gamma_2} I_{\{\xi<0\}} \tag{2.4}$$

where ξ represents the magnitude of intervention, K_1 and K_2 are positive constants that represent fixed costs per intervention (independent of the size of transaction), $k_1=1-\kappa_1$ with κ_1 representing proportional costs for rebalancing from $\chi+\xi>\chi$ to χ, $k_2=1+\kappa_2$ with κ_2 representing proportional costs for rebalancing from $\chi+\xi<\chi$ to χ and $\gamma_1, \gamma_2 \in (0,1]$. Specification (2.4) allows for different utilities for positive/negative interventions and different fixed and proportional costs according to the type of intervention. A similar choice regarding the form of the objective function has been adopted in Cadenillas et al. (2006) for a dividend allocation problem. Because of the fixed cost components, it suffices to consider trading strategies of the form $\{(\tau_n, \xi_n)\}$, where τ_n is the time of the nth transaction and ξ_n is the magnitude of the nth intervention. $\{(\tau_n, \xi_n)\}$ must satisfy some standard technical requirements: τ_n is a stopping time, $\tau_n < \tau_{n+1}$, $\tau_n \to \infty$ as $n \to \infty$, and ξ_n is F_{τ_n}-measurable.

To proceed with formulating the ergodic optimization problem assume U and L be fixed subject to $-\infty < L < U < \infty$, and define $T(s) = T_s$ be the hitting time of s for the X process. Throughout the paper we let

$$T^* = T_{U,L} = \min\{T(U), T(L)\} = T(U) \wedge T(L) \tag{2.5}$$

be the first time the process reaches *U* or *L* and define the following quantities for the risky wealth process *X:*

$$v_1(x) = \Pr\{T(U) < T(L) | X(0) = x\} \qquad L < x < U \ , \tag{2.6}$$

the probability the process reaches *U* before *L* starting from *x,* and

$$v_2(x) = E\left[T^* | X(0) = x\right], \qquad L < x < U \ , \tag{2.7}$$

the mean time to reach *U* or *L* starting from *x*. Now the ergodic problem is formulated as follows.

Problem 2.1 *The investor wants to maximize long-run profits per unit time. In particular, the investor wants to select the pair (T, ξ) that maximizes the expression*

$$J_1(\chi, T, \xi) = z(\chi, L, U) = \frac{f(\chi, L, U)}{v_2(\chi)} \tag{2.8}$$

with the numerator in (2.8) *defined as follows:*

$$f(\chi, L, U) = v_1(\chi) g_1(U - \chi) + (1 - v_1(\chi)) g_2(L - \chi). \tag{2.9}$$

Consider a cycle to be from one intervention returning the level of the risky holdings back to χ, to the next such intervention; the numerator weights expected gains and losses as expressed by the corresponding utility functions in (2.4) by the probability of reaching the upper or lower boundary during a transaction cycle and the denominator is the expected duration of a transaction cycle.

Next, we formulate the discounted optimization problem.

Problem 2.2 *The investor wants to maximize profits over lifetime. In particular, the investor wants to select the pair (T, ξ) that maximizes the functional J_2 defined by*

$$J_2(x,T,\xi) := E_x\left[\sum_{n=1}^{\infty} e^{-\lambda\tau_n}\left(g_1(\xi_n)I_{\{\xi_n>0\}} + g_2(-\xi_n)I_{\{\xi_n<0\}}\right)I_{\{\tau_n<\infty\}}\right] \tag{2.10}$$

with λ representing the discount rate and g_1, g_2 *as defined in* (*2.4*).

3. Maximization of long run profits per unit time

To solve problem (2.1) we use basic tools from the theory of diffusions (see Karlin and Taylor, 1981 and Borodin and Salminen, 2002). Similar to a large number of stochastic impulse control problems formulated as quasi-variational inequalities (QVI), derivation of the control bands pertains to the solution of a system of nonlinear equations. Unfortunately, these nonlinear equations turn out to be significantly more complex compared to the ones derived from the QVI approach. Nevertheless, one may relatively easily derive them using any software that performs symbolic calculations and solve the resultant system using standard routines that perform algorithms like Newton-Raphson or one of its descendants.

To calculate the numerator and denominator in (2.8) we note that v_1 and v_2 in (2.6) and (2.7) need to satisfy the following differential equations

$$\mu x \frac{dv_1}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2 v_1}{dx^2} = 0 \text{ for } L < x < U, \; v_1(L) = 0, \quad v_1(U) = 1; \tag{3.1}$$

$$\mu x \frac{dv_2}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2 v_2}{dx^2} = -1 \text{ for } L < x < U, \; v_2(L) = v_2(U) = 0. \tag{3.2}$$

To solve these problems, let the scale function of the X process be denoted as

$$S(x) = \int^x s(t)dt, \tag{3.3}$$

where

$$s(x) = \exp\left\{ -\int^x \left[\frac{2\mu}{\sigma^2 t} \right] dt \right\}. \tag{3.4}$$

Let also

$$m(x) = \frac{1}{\sigma^2 x^2 s(x)} \tag{3.5}$$

denote the speed density of the process. The solution to (3.1) is

$$v_1(x) = \frac{S(x) - S(L)}{S(U) - S(L)} \quad \text{for } L \le x \le U, \tag{3.6}$$

and the solution to (3.2) is formulated as follows

$$v_2(x) = 2\left\{v_1(x)\int_x^U \left[S(U) - S(t)\right]m(t)dt + \left[1 - v_1(x)\right]\int_L^x \left[S(t) - S(L)\right]m(t)dt\right\}. \qquad (3.7)$$

Now the scale function for the geometric Brownian motion (2.1) and the corresponding speed measure are expressed by

$$S(x) = \frac{x^{1-a}}{1-a} \qquad (3.8)$$

and

$$m(x) = \frac{x^{a-2}}{\sigma^2} \qquad (3.9)$$

with

$$a = \frac{2\mu}{\sigma^2}\,. \qquad (3.10)$$

A transaction cycle for the X process starts at the exogenously defined target level χ and finishes the first time the process reaches L or U. One may easily calculate expected profit minus cost during a cycle by substituting (3.6) and (3.8) in (2.9). In words, the expected profit minus cost per transaction cycle is comprised by two components: the profit of reaching first U and depositing U-χ (expressed by utility g_1), weighted by the probability that X reaches U first, starting from χ and the cost of reaching first L and injecting χ-L to reach χ (expressed by the utility g_2) weighted by the probability that X reaches L first. For the expected cycle time, using (3.7)-(3.10), we obtain

$$v_2(x) = 2\frac{x\log(x)\left(U^a L - UL^a\right) + U^a \log(U)\left(L^a x^{-a-2} - Lx\right) + L^a \log(L)\left(Ux - U^a x^{-a-2}\right)}{x\left(UL^a - U^a L\right)(a-1)\sigma^2}$$

(3.11)

To find the (L,U) pair that maximizes (2.8), one has to find all the local maxima of z. Thus, one should take the corresponding derivatives, find the (L,U) pairs that equate them to zero and select among them the ones for which the Hessian is negative definite. The expressions for the second derivatives are lengthy, thus we chose to present here only the first derivatives; all calculations were performed via MATLAB's Symbolic Math toolbox. The derivatives of z in (2.8) with respect to U and L are formulated as follows:

$$\frac{dz}{dU} = \frac{A_1 - A_2 A_3 A_4}{A_5} \tag{3.12}$$

and

$$\frac{dz}{dL} = \frac{A_1' - A_2 A_3' A_4'}{A_5} \tag{3.13}$$

with

$$A_1 = v_1(x)\left(\frac{k_1^{\gamma_1}(U-x)^{\gamma_1}}{(U-x)} - \frac{U^{-a}\left(g_1(U-x) - g_2(x-L)\right)}{S(U)-S(L)}\right) \tag{3.14}$$

$$A_1{}' = \left(1 - v_1(x)\right)\left(\frac{k_2^{\gamma_2}(x-L)^{\gamma_2}}{(x-L)} - \frac{L^{-a}\left(g_1(U-x) - g_2(x-L)\right)}{S(U)-S(L)}\right) \tag{3.15}$$

$$A_2 = \left(v_1(x)g_1(U-x) + (1-v_1(x))g_2(x-L)\right) \tag{3.16}$$

$$A_3 = \frac{-A_6}{(S(U)-S(L))x} + \frac{2\mu(\log(U)-\log(x))(a-1)-\sigma^2}{U} + U^{-2a}x^{-2a+1}\sigma - \frac{aA_6}{U^{-2a+1}x} - \frac{A_7 U^{-2a}}{(S(U)-S(L))L^{-2a}x} \tag{3.17}$$

$$A_3' = \frac{-A_7}{(S(U)-S(L))x} + \frac{2\mu(\log(x)-\log(L))(a-1)-\sigma^2}{L} + L^{-2a}x^{-2a+1}\sigma - \frac{aA_7}{L^{-2a+1}x} - \frac{A_6 L^{-2a}}{(S(U)-S(L))U^{-2a}x} \tag{3.18}$$

$$A_4 = \frac{2v_1(x)}{\left(2\mu-\sigma^2\right)^2} \tag{3.19}$$

$$A_4{}' = \frac{2(1-v_1(x))}{\left(2\mu-\sigma^2\right)^2} \tag{3.20}$$

$$A_5 = 2\left(\frac{v_1(x)A_6}{\left(2\mu-\sigma^2\right)^2 U^{-2a}x} - \frac{(1-v_1(x))A_7}{\left(2\mu-\sigma^2\right)^2 L^{-2a}x}\right) \tag{3.21}$$

$$A_6 = xU^{-2a}\left(2\mu(\log(U)-\log(x)) - \sigma^2(\log(U)-\log(x)) - \sigma^2\right) + Ux^{-2a}\sigma^2 \tag{3.22}$$

$$A_7 = xL^{-2a}\left(2\mu(\log(x)-\log(L)) - \sigma^2(\log(x)-\log(L)) - \sigma^2\right) - Lx^{-2a}\sigma^2. \tag{3.23}$$

The derivatives form a system of two nonlinear equations, which can be solved computationally using the Newton-Raphson algorithm or one of its successors; numerical results are presented at the fifth section. The method of this section may seem quite tedious in terms of computations but in contrast to the QVI approach it can easily accommodate constraints (e.g. via Lagrange multipliers). Hence in contrast to the QVI approach one may easily search for (L, U) that (for instance) maximize (2.8) and at the same time the probability of reaching U first starting from χ is at least p.

4. Maximization of discounted profits minus costs over lifetime

This section characterizes the stochastic impulse control problem as a system of quasivariational inequalities following the arguments demonstrated in Cadenillas *et al*. (2006).

Admissible strategies

Since we want to maximize the functional J_2 in problem 2.2 we should consider only those strategies for which J_2 is well defined and finite. In order that

$$E_x\left[\sum_{n=1}^{\infty} e^{-\lambda\tau_n}\left(g_1(\xi_n)I_{\{\xi_n>0\}} + g_2(-\xi_n)I_{\{\xi_n<0\}}\right)I_{\{\tau_n<\infty\}}\right] \tag{4.1}$$

be well defined and finite, we need that

$$E\left[\sum_{n=1}^{\infty} e^{-\lambda\tau_n} I_{\{\tau_n<\infty\}}\right] < \infty \text{ and } E\left[\sum_{n=1}^{\infty} e^{-\lambda\tau_n}\left|\xi_n\right| I_{\{\tau_n<\infty\}}\right] < \infty . \tag{4.2}$$

To obtain the inequality on the left-hand-side, we need that

$$\forall T \in [0,\infty): \qquad P\left\{\lim_{n\to\infty} \tau_n \le T\right\} = 0 . \tag{4.3}$$

To obtain the inequality on the right-hand-side, we need that

$$\lim_{T\to\infty} E\left[e^{-\lambda T} X(T+)\right] = 0 \tag{4.4}$$

and

$$E\left[\int_0^\infty e^{-\lambda t} X(t)dt\right] < \infty . \tag{4.5}$$

The last two conditions are implied from the formula of integration by parts (see, for instance, section VI.38 of Rogers and Williams (1987)) which postulates that for every $0 < s \le t < \infty$,

$$E\left[e^{-\lambda t} X(t+)\right] - E\left[e^{-\lambda s} X(s+)\right] = (\mu - \lambda) E\left[\int_s^t e^{-\lambda u} X(u)du\right] + E\left[\sum_{n=1}^{\infty} e^{-\lambda \tau_n} \xi_n I_{\{s<\tau_n \le t\}}\right]. \tag{4.6}$$

DEFINITION 4.1 [Admissible controls]. We shall say that an impulse control is admissible if the conditions (4.3)-(4.5) are satisfied. We shall denote by A(x) the class of admissible impulse controls.

The Value Function

Let us denote by V the value function. That is, for every $x \in (0,\infty)$,

$$V(x) := \sup\left\{J_2(x;T,\xi); (T,\xi) \in \mathrm{A}(x)\right\}. \tag{4.7}$$

For a function $\phi : [0,\infty) \to \Re$ we define the *maximum utility operator M* by

$$M\phi(x) := \sup\{\phi(x-\xi) + g_1(\xi)I_{\{\xi>0\}} + g_2(-\xi)I_{\{\xi<0\}} : \xi \in \Re, x-\xi \in (0,\infty)\}. \qquad (4.8)$$

$MV(x)$ represents the value of the strategy that consists in choosing the best immediate intervention and then selecting optimally the times and the amounts of the future control actions. Let us consider the differential operator $\Im$ defined by

$$\Im\psi(x) := \frac{1}{2}\sigma^2 x^2 \frac{d^2\psi(x)}{dx^2} + \mu x \frac{d\psi(x)}{dx} - \lambda\psi(x). \qquad (4.9)$$

Now we intend to find the value function and an associated optimal strategy.

Suppose there exists an optimal strategy for each initial point. Then, if the process starts at x and follows the optimal strategy, the expected utility associated with this optimal strategy is $V(x)$. On the other hand, if the process starts at x, makes immediately the best immediate intervention, and then follows an optimal strategy, then the expected utility associated with this strategy is $MV(x)$. Since the first strategy is optimal, its associated expected utility is greater or equal than the expected utility associated with the second strategy. Furthermore, when these two expected utilities are equal, it is optimal to intervene. Hence, $V(x) \geq MV(x)$, with equality when it is optimal to intervene. In the continuation region, that is, when there are not interventions, we must have $\Im V(x) = 0$ (this is an heuristic application of the dynamic programming principle to the problem we are considering). These intuitive observations can be applied to give a characterization of the value function. We formalize this intuition in the next two definitions and theorem.

DEFINITION 4.2 (QVI) *We say that a function* $v:(0,\infty)\to\Re$ *satisfies the quasi-variational inequalities for Problem 2.2 if for every* $x\in(0,\infty)$:

$$\Im v(x)\le 0\,, \tag{4.10}$$

$$v(x)\ge Mv(x)\,, \tag{4.11}$$

$$\big(v(x)-Mv(x)\big)\big(\Im v(x)\big)=0 \tag{4.12}$$

Quasi-variational inequalities have been studied, for instance, in Bensoussan and Lions (1984), Perthame (1984a, 1984b) and Baccarin (2004) but the theory developed in those references cannot be applied directly to the above QVI.

A solution v of the QVI separates the interval $(0,\infty)$ into two disjoint regions: a continuation region

$$C:=\{x\in(0,\infty):v(x)>Mv(x)\quad and\quad \Im v(x)=0\}$$

and an intervention region

$$\Sigma:=\{x\in(0,\infty):v(x)=Mv(x)\quad and\quad \Im v(x)<0\}.$$

From a solution to the QVI it is possible to construct the following stochastic impulse control.

DEFINITION 4.3 *Let v be a solution of the QVI. The following stochastic impulse control*

$$\left(T^{v},\xi^{v}\right)=\left(\tau_{1}^{v},\tau_{2}^{v},\ldots,\tau_{n}^{v},\ldots;\xi_{1}^{v},\xi_{2}^{v}\ldots,\xi_{n}^{v},\ldots\right)$$

is called the QVI-control associated with v (if it exists):

$$\tau_{1}^{v}:=\inf\left\{t\geq 0: v\left(X^{v}(t)\right)=Mv\left(X^{v}(t)\right)\right\}$$
$$\xi_{1}^{v}:=\arg\sup\left\{v\left(X^{v}(\tau_{1}^{v})-\xi\right)+g_{1}(\xi)I_{\{\xi>0\}}+g_{2}(\xi)I_{\{\xi<0\}}:\xi\in\Re, X^{v}(\tau_{1}^{v})-\xi\in\left(0,\infty\right)\right\}$$

and, for every $n\geq 2$:

$$\tau_{n}^{v}:=\inf\left\{t\geq \tau_{n-1}^{v}: v\left(X^{v}(t)\right)=Mv\left(X^{v}(t)\right)\right\}$$
$$\xi_{n}^{v}:=\arg\sup\left\{v\left(X^{v}(\tau_{n}^{v})-\xi\right)+g_{1}(\xi)I_{\{\xi>0\}}+g_{2}(-\xi)I_{\{\xi<0\}}:\xi\in\Re, X^{v}(\tau_{n}^{v})-\xi\in\left(0,\infty\right)\right\}$$

where $\tau_{0}^{v}:=0$ *and* $\xi_{0}^{v}:=0$.

This means that the investor intervenes whenever *v* and *Mv* coincide and the size of her control actions solve the optimization problem corresponding to *Mv*(*x*).

Korn (1997, Theorem 3.2) has developed a general sufficient condition of optimality for stochastic impulse control problems, and applied it to some examples. In each example, he shows that an admissible control satisfies that sufficient condition, and is therefore optimal. We have developed the following version of Theorem 3.2 of Korn (1997). This version is suitable for the application that we consider in this paper.

THEOREM 4.1 *Let* $v \in C^1([0,\infty);\Re)$ *be a solution of the QVI and let* $a,b \in (0,\infty)$ *be such that* $v \in C^2([0,\infty)-\{a,b\};\Re)$. *Suppose that for every* $y \geq a$ *:*

$$v(x) = \mu_1 + \lambda_1 (x-\chi)^{\gamma_1}, \tag{4.13}$$

and for every $0 < x \leq b < a$

$$v(x) = \mu_2 - \lambda_2 (\chi - x)^{\gamma_2}, \tag{4.14}$$

where $\mu_1, \mu_2 \in \Re$, *and* $\lambda_1, \lambda_2 \in (0,\infty)$. *Then, for every* $x \in (0,\infty)$*:*

$$V(x) \leq v(x). \tag{4.15}$$

Furthermore, if the QVI-control (T^v, ξ^v) *corresponding to v is admissible, then it is an optimal stochastic impulse control and for every* $x \in (0,\infty)$*:*

$$V(x) = v(x) = J_2(x; T^v, \xi^v). \tag{4.16}$$

Proof. See Appendix.

The solution of the QVI

We conjecture that there exists an optimal solution (T,ξ) characterized by two parameters *L*, *U* with $0 < L < \chi < U < \infty$ such that the optimal strategy is to stay in the band $[L,U]$ and jump to χ when reaching the boundaries. That is we conjecture that for every $i \in \mathrm{N}$:

$$\tau_i = \inf\{t > \tau_{i-1} : X_t \notin (L,U)\} \tag{4.17}$$

and

$$X_{\tau_{i+}} = X_{\tau_i} + \xi_i = \chi\left(I_{X_{\tau_i}=U} + I_{X_{\tau_i}=L}\right). \tag{4.18}$$

Thus, the value function would satisfy

$$\forall x \in [U,\infty): \quad V(x) = v(\chi) - K_1 + \frac{1}{\gamma_1}\left[k_1(x-\chi)\right]^{\gamma_1} \tag{4.19}$$

and

$$\forall x \in (0,L]: \quad V(x) = v(\chi) - K_2 - \frac{1}{\gamma_2}\left[k_2(\chi - x)\right]^{\gamma_2} . \tag{4.20}$$

If V were differentiable in $\{L,U\}$, then from (4.19), (4.20) we would get

$$V'(U) = k_1^{\gamma_1}(U-\chi)^{\gamma_1-1} \tag{4.21}$$

and

$$V'(L) = k_2^{\gamma_2}(\chi - L)^{\gamma_2-1} . \tag{4.22}$$

We also conjecture that the continuation region is the interval (L,U), so

$$\forall x \in [L,U]: \Im v(x) = \frac{1}{2}\sigma^2 x^2 \frac{d^2 v(x)}{dx^2} + \mu x \frac{dv(x)}{dx} - \lambda v(x) = 0 . \tag{4.23}$$

Applying standard methods of ordinary equations, we see that the general solution to (4.23) is given by

$$v(x) = C_1 x^{a_1} + C_2 x^{a_2} \tag{4.24}$$

where C_1,C_2 are unknown constants and

$$a_{1,2} = -\frac{-2\mu + \sigma^2 \pm \left(4\mu^2 - 4\mu\sigma^2 + \sigma^4 + 8\lambda\sigma^2\right)^{1/2}}{2\sigma^2}\,. \tag{4.25}$$

In summary, we conjecture that the solution is described by (4.17)-(4.18) and the four unknowns L, U, C_1, C_2 are a solution to a system of four nonlinear equations

$$h(U) = h(\chi) - K_1 + \frac{1}{\gamma_1} k_1^{\gamma_1} (U - \chi)^{\gamma_1} \tag{4.26}$$

$$h(L) = h(\chi) - K_2 - \frac{1}{\gamma_2} k_2^{\gamma_2} (\chi - L)^{\gamma_2} \tag{4.27}$$

$$h'(U) = k_1^{\gamma_1} (U - \chi)^{\gamma_1 - 1} \tag{4.28}$$

$$h'(L) = k_2^{\gamma_2} (\chi - L)^{\gamma_2 - 1}\ . \tag{4.29}$$

where

$$h(x) = C_1 x^{a_1} + C_2 x^{a_2}\,. \tag{4.30}$$

The above are proved rigorously in the following theorem.

THEOREM 4.2. *Let L, U with* $L < x < U < \infty$ *be a solution of the system of equations (4.26)-(4.29). Let us define the function* $V : (0, \infty) \rightarrow \Re$ *by*

$$V(x) := \begin{cases} h(\chi) - K_1 + \dfrac{1}{\gamma_1} k_1^{\gamma_1} (x - \chi)^{\gamma_1} & \text{if} \quad x > U \\ h(x) & \text{if} \quad L \le x \le U \\ h(\chi) - K_2 - \dfrac{1}{\gamma_2} k_2^{\gamma_2} (\chi - x)^{\gamma_2} & \text{if} \quad x < L \end{cases} \tag{4.31}$$

If for every x>U

$$\frac{1}{2}\sigma^2 x^2 k_1^{\gamma_1} (\gamma_1 - 1)(x - \chi)^{\gamma_1 - 2} + \mu x k_1^{\gamma_1} (x - \chi)^{\gamma_1 - 1} - \lambda \left[h(\chi) - K_1 + \frac{k_1^{\gamma_1}}{\gamma_1} (x - \chi)^{\gamma_1} \right] < 0, \tag{4.32}$$

for every x<L

$$\frac{1}{2}\sigma^2 x^2 k_2^{\gamma_2} (\gamma_2 - 1)(\chi - x)^{\gamma_2 - 2} - \mu x k_2^{\gamma_2} (\chi - x)^{\gamma_2 - 1} - \lambda \left[h(\chi) - K_2 - \frac{k_2^{\gamma_2}}{\gamma_2} (\chi - x)^{\gamma_2} \right] < 0, \tag{4.33}$$

the function $\Phi_x : [\chi, U] \rightarrow \Re$ *defined by*

$$\Phi_\chi(x) := h(x) + K_1 - \frac{k_1^{\gamma_1}}{\gamma_1} (x - \chi)^{\gamma_1} \text{ is decreasing in } [\chi, U] \tag{4.34}$$

and the function $\Psi_\chi : [L, \chi] \rightarrow \Re$ *defined by*

$$\Psi_{\chi}(x) := h(x) + K_2 + \frac{k_2^{\gamma_2}}{\gamma_2}(\chi - x)^{\gamma_2} \text{ is increasing in } [L, \chi] \tag{4.35}$$

then v is the value function of problem 2.2. That is

$$v(x) = V(x) = \sup\{J_2(x; T, \xi); (T, \xi) \in \mathrm{A}(x)\}$$

and the optimal strategy is given by (4.17), (4.18).

Proof. See Appendix.

5. Numerical illustration

In this section, we provide numerical solutions for the ergodic and discounted stochastic impulse control problems considered at sections 3 and 4. For the ergodic problem one has to find the solutions to a system of two nonlinear equations that correspond to the derivatives of z in (2.8) with respect to L and U. This can be achieved via performing an algorithm like the Newton-Raphson or one of its descendants several times for different starting values. The (L, U) pairs for which the derivatives equal zero are local maxima if the Hessian of the system at these points is negative definite. The process of finding global maxima can be guided by a three-dimensional plot of z in (2.8) as depicted below. For the discounted problem of the fourth section we provide numerical solutions for the system of nonlinear equations (4.26)-(4.29) and derive the four unknowns: the two outer boundaries L and U and the two constants C_1 and C_2 in (4.23) that characterize the evolution of the value function within the control band. The reader should note that both nonlinear systems are complex and their convergence is sensitive to the initial values provided as starting points. For the sensitivity analysis conducted at the second part of this section, we first found appropriate initial values for a baseline experiment and then, for each perturbation of the parameters, we plugged as starting values the outcomes of the previous run. MATLAB codes are available upon request from the authors.

5.1 A specific example

We first consider the following data for market characteristics and investor's preferences:

$$\mu = 0.1 \quad \sigma = 0.2 \quad \lambda = 0.01 \quad K_1 = 0.05 \quad K_2 = 0.05 \quad k_1 = 0.95 \quad k_2 = 1.1 \quad \chi = 10 \quad \gamma_1 = 0.8 \quad \gamma_2 = 0.9 \,.$$

For the ergodic problem we find as possible solutions the pairs (9.0796, 10.1348) and (6.4081, 16.3629). Calculation of the Hessian at these points indicates that the first pair is the global maximum whereas the second is a saddle point; a three-dimensional plot of the z function in (2.8) complies with the previous arguments (figure 1).

For the discounted problem we find two possible optimal quadruplets:

U=10.1338 L= 9.0116 C_1=157.2394 C_2=534.1197

and

U=19.2442 L= 4.8551 C_1=52.4051 C_2=45.9706, with only the first one satisfying conditions (4.32)-(4.35). For both problems errors are of the order 10^{-8}. We observe that for the discounting rate of the benchmark example, the investor that adopts the discounted criterion intervenes earlier (later) than the one that adopts the ergodic criterion in the right (left) side of χ. The probability of reaching U first, starting from χ is 0.9089 for the control band of the discounted problem and 0.9004 for the ergodic problem respectively. Apparently, given the discounting rate of the benchmark example, the investor that adopts the ergodic criterion behaves in a less conservative manner.

<<Figure 1>>

5.2 Sensitivity Analysis

To conduct sensitivity analysis, we use as baseline values the ones used in the previous example and perturb each parameter separately to uncover how optimal strategies are affected. Table 1 presents control bands for the discounted problem for varying discount rate at two levels of volatility. In both cases the upper boundaries seem to be almost unaffected from the changes in the discount rate; on the contrary, lower boundaries depend linearly to the discount rate. Indeed, for $\sigma = 0.2$ we get perfect fit (regression's R^2 equals one) from the regression line $L = 9.014 - 0.234 \cdot \lambda$ whereas for $\sigma = 0.3$ the perfect fitting line is expressed as $L = 8.968 - 0.126 \cdot \lambda$. Thus, for increasing levels of volatility lower control boundaries become less sensitive to the discount rate. As the discount rate decreases, control bands of the discounted problem come closer to the ones of the ergodic problem.

<<Table 1>>

Sensitivity of the control bands with respect to the cost parameters is examined in tables 2 and 3. The intuition is clear: the investor rebalances more often with lower transaction costs. We also observe that the control boundaries from the ergodic and discounted optimization criteria differ more as transaction costs increase with lower boundaries differing more than the upper ones. Sensitivity analysis with respect to γ_1 and γ_2 is performed at tables 4 and 5. As γ_1 increases, wealth obtained from adjustments to the target level from the upper boundary becomes more important to the investor; the upper boundaries increase and the lower ones decrease faster for both ergodic/discounted problems (figure 2). The differences between ergodic/discounted control boundaries increase as γ_1 increases. As expected, the opposite findings are true for increasing levels of γ_2. Figure 2 displays the ergodic/discounted control boundaries for varying γ_1 and γ_2. Our findings suggest that the perfect fitting curves are third order polynomials. The corresponding curves for γ_1, γ_2 ranging between 0.8 and 0.9 are formulated as

$$\begin{aligned} U_{DISC} &= -119 + 473.4 \cdot \gamma_1 - 580.5 \cdot \gamma_1^2 + 258.2 \cdot \gamma_1^3 \\ U_{ERG} &= -181.1 + 697.9 \cdot \gamma_1 - 850.9 \cdot \gamma_1^2 + 346.8 \cdot \gamma_1^3 \\ L_{DISC} &= -1029 + 3707 \cdot \gamma_2 - 4379 \cdot \gamma_2^2 + 1708 \cdot \gamma_2^3 \\ L_{ERG} &= -648.9 + 2370 \cdot \gamma_2 - 2818 \cdot \gamma_2^2 + 1105 \cdot \gamma_2^3 \end{aligned} . \tag{5.1}$$

Tables 6-8 examine the sensitivity with respect to volatility, expected return and target level respectively. As volatility increases the left part of the control band becomes wider whereas the right part gets narrower. On the other hand both boundaries increase with increasing levels of μ and χ and the difference between discounted/ergodic control bands are larger as μ increases. The control boundaries can be expressed as (perfect fitting) regression lines for the range of μ values considered in our numerical experiment:

$$\begin{aligned} U_{DISC} &= 10.13 + 0.0108 \cdot \mu \\ U_{ERG} &= 10.13 + 0.02 \cdot \mu \\ L_{DISC} &= 8.931 + 0.806 \cdot \mu \\ L_{ERG} &= 8.954 + 1.249 \cdot \mu \end{aligned} \quad . \tag{5.2}$$

The slopes of the regression lines indicate a faster rate of change for the ergodic boundaries as μ increases.

<< Tables 2-8 >>
<<Figure 2>>

6. Concluding Remarks

Stochastic impulse control problems examined in the literature during the past decade have focused on discounted criteria over lifetime. However in some problems the choice of a discounting rate does not have a clear economic interpretation; in such cases a manager would prefer to adopt an ergodic criterion that optimizes an objective function per unit time. This work uses an investment problem as a means to compare control bands derived by ergodic/discounted criteria. Sensitivity analysis of the optimal policies indicates that the magnitude of the differences between ergodic/discounted control bands is a function of the coefficients that characterize investor’s preferences and market behavior.

A research question that emerges next is related to the constrained stochastic impulse control problem. For instance, in the investment problem considered in this paper, one may place finite fuel constraints on the financing of positive impulses from a bank account. Another application is related to the optimal dividend policy or harvesting policy where a manager may desire rebalancing points (levels of cash reservoir or population levels after dividend payout or harvesting) from which a lower benchmark can be reached with a certain probability (say less than 0.05) to avoid bankruptcy/species extinction.

Appendix

Proof of Theorem 4.1

The differentiability of v implies its continuity, and therefore its boundedness in the compact interval [b,a]. Furthermore, v' is bounded in $(0,\infty)$ because it is continuous in [b,a], for every $x \in [a,\infty): v'(x) \in \left[0, \lambda_1\gamma_1(U-\chi)^{\gamma_1 - 1}\right]$ and for every $x \in (0,b]: v'(x) \in \left[-\lambda_2\gamma_2(\chi - L)^{\gamma_2 - 1}, -\lambda_2\gamma_2(\chi)^{\gamma_2 - 1}\right]$. Let (T,ξ) be an admissible policy, and denote by $X = X^{(T,\xi)}$ the trajectory determined by (T,ξ). We observe that condition (4.4), the boundedness of v in the compact interval $[b,a]$ and its boundedness by a linear function in $(0,b) \cup (a,\infty)$ imply that

$$\lim_{T\to\infty} E\left[e^{-\lambda T} v(X(T+))\right] = 0 . \tag{A1}$$

Furthermore, condition (4.5) and the boundedness of v' imply that

$$E\left[\int_0^\infty \left\{e^{-\lambda t} X(t) v'(X(t))\right\}^2 dt\right] < \infty . \tag{A2}$$

We may write, for every t>0 and $n \in \mathrm{N}$,

$$\begin{aligned} & e^{-\lambda(t\wedge\tau_n)} v\left(X_{(t\wedge\tau_n)+}\right) - v(X_0) \\ & = \sum_{i=1}^{n} \left\{e^{-\lambda(t\wedge\tau_i)} v\left(X_{(t\wedge\tau_i)}\right) - e^{-\lambda(t\wedge\tau_{i-1})} v\left(X_{(t\wedge\tau_{i-1})+}\right)\right\} + \sum_{i=1}^{n} I_{\{\tau_i \le t\}} e^{-\lambda\tau_i} \left\{v\left(X_{\tau_i +}\right) - v\left(X_{\tau_i}\right)\right\} \end{aligned}$$

Since X is a continuous semimartingale in the stochastic interval $\left(\tau_{i-1}, \tau_i\right]$ and v is twice continuously differentiable in $\left(0,\infty\right)-\{a,b\}$, we may apply an appropriate version of Ito's formula (Rogers and Williams, 1987, section IV.45). Thus, for every $i \in \mathrm{N}$,

$$
\begin{aligned}
& e^{-\lambda(t\wedge\tau_i)} v\left(X_{t\wedge\tau_i}\right) - e^{-\lambda(t\wedge\tau_{i-1})} v\left(X_{(t\wedge\tau_{i-1})+}\right) \\
&= \int_{(t\wedge\tau_{i-1}, t\wedge\tau_i]} e^{-\lambda s}\left\{ v'(X_s)\mu X_s + \frac{1}{2}\sigma^2 v''(X_s) X_s^2 - \lambda v(X_s) \right\} + \int_{(t\wedge\tau_{i-1}, t\wedge\tau_i]} e^{-\lambda s} v'(X_s)\sigma X_s dW_s \\
&= \int_{(t\wedge\tau_{i-1}, t\wedge\tau_i]} e^{-\lambda s} \Im v(X_s) ds + \int_{(t\wedge\tau_{i-1}, t\wedge\tau_i]} e^{-\lambda s} v'(X_s)\sigma X_s dW_s
\end{aligned}
$$

Now according to inequality (4.10),

$$
e^{-\lambda(t\wedge\tau_i)} v\left(X_{t\wedge\tau_i}\right) - e^{-\lambda(t\wedge\tau_{i-1})} v\left(X_{(t\wedge\tau_{i-1})+}\right) \le \int_{(t\wedge\tau_{i-1}, t\wedge\tau_i]} e^{-\lambda s} v'(X_s)\sigma X_s dW_s ,
$$

and this inequality becomes an equality for the QVI-control associated with v. According to inequality (4.11), in the event $\{\tau_i \le t\}$ we have

$$
e^{-\lambda\tau_i}\left\{ v\left(X_{\tau_i +}\right) - v\left(X_{\tau_i}\right) \right\} \le -e^{-\lambda\tau_i}\left\{ g_1(\xi_i) I_{\{\xi_i > 0\}} + g_2(\xi_i) I_{\{\xi_i < 0\}} \right\}
$$

and this inequality becomes an equality for the QVI-control associated with v. Combining the above inequalities and taking expectations, we obtain

$$
\begin{aligned}
& v(x) - E_x\left[e^{-\lambda(t\wedge\tau_n)} v\left(X_{(t\wedge\tau_n)+}\right) \right] \\
& \ge E_x\left[\sum_{i=1}^{\infty} \left\{ \left\{ I_{\{\tau_i \le t\}} e^{-\lambda\tau_i} \left\{ g_1(\xi_i) I_{\{\xi_i > 0\}} + g_2(\xi_i) I_{\{\xi_i < 0\}} \right\} \right\} - \int_{(t\wedge\tau_{i-1}, t\wedge\tau_i]} e^{-\lambda s} v'(X_s)\sigma X_s dW_s \right\} \right]
\end{aligned}
$$

with equality for the QVI-control associated with *v*. From condition (4.3),

$$\lim_{n\to\infty}\left\{v(x)-E_x\left[e^{-\lambda(t\wedge\tau_n)}v\left(X_{(t\wedge\tau_n)+}\right)\right]\right\}=v(x)-E_x\left[e^{-\lambda t}v\left(X_{t+}\right)\right]$$

and according to (A2),

$$\lim_{n\to\infty}E_x\left[\int_0^{t\wedge\tau_n}e^{-\lambda s}v'\left(X_s\right)\sigma X_s dW_s\right]=0\ .$$

Thus,

$$v(x)-E_x\left[e^{-\lambda t}v\left(X_{t+}\right)\right]\geq E_x\left\{I_{\{\tau_i\leq t\}}e^{-\lambda\tau_i}\left\{g_1\left(\xi_i\right)I_{\{\xi_i>0\}}+g_2\left(\xi_i\right)I_{\{\xi_i<0\}}\right\}\right\}$$

with equality for the QVI-control associated with *v.* According to (A1)

$$\lim_{t\to\infty}\left\{v(x)-E_x\left[e^{-\lambda t}v\left(X_{t+}\right)\right]\right\}=v(x)$$

and according to the monotone convergence theorem

$$\begin{aligned}&\lim_{t\to\infty}E_x\left\{I_{\{\tau_i\leq t\}}e^{-\lambda\tau_i}\left\{g_1\left(\xi_i\right)I_{\{\xi_i>0\}}+g_2\left(\xi_i\right)I_{\{\xi_i<0\}}\right\}\right\}\\&=E_x\left\{I_{\{\tau_i<\infty\}}e^{-\lambda\tau_i}\left\{g_1\left(\xi_i\right)I_{\{\xi_i>0\}}+g_2\left(\xi_i\right)I_{\{\xi_i<0\}}\right\}\right\}\end{aligned}.$$

Hence,

$$v(x)\geq E_x\left\{I_{\{\tau_i<\infty\}}e^{-\lambda\tau_i}\left\{g_1\left(\xi_i\right)I_{\{\xi_i>0\}}+g_2\left(\xi_i\right)I_{\{\xi_i<0\}}\right\}\right\}$$

with equality for the QVI-control associated with *v*. Therefore for every $\left(T,\xi\right)\in \mathrm{A}(x)$:

$$v(x)\geq J\left(x;T,\xi\right),$$

with equality for the QVI-control associated with *v*.☐

Proof of Theorem 4.2

We observe that if *V* were a solution to the QVI then, according to theorem 4.1, *V* would be the value function and the optimal strategy would be given by (4.17)-(4.18). Indeed, *V* is twice continuously differentiable in $\left(0,L\right)\cup\left(L,U\right)\cup\left(U,\infty\right)$ and once continuously differentiable in {*L*, *U*}. Furthermore, *V* has the form (4.13) in $\left(U,\infty\right)$ and (4.14) in $\left(0,L\right)$. In addition, the QVI-control associated with *V* is admissible, because the trajectory *X* generated by the QVI-control associated with *V* behaves like a geometric Brownian motion in each random interval $\left(\tau_n,\tau_{n+1}\right)$ and satisfies $P\left\{\forall t\in\left(0,\infty\right): X(t)\in\left[L,U\right]\right\}=1$. Thus, the conditions (4.3)-(4.5) would be satisfied, and the QVI-control associated to *V* would be admissible. Hence it only remains to verify that *V* is a solution to the QVI.

By construction of *h*, we have for every $L\leq x\leq U$:

$$\Im V(x)=\Im h(x)=0.$$

According to condition (4.32), for every $x>U$:

$$\Im V(x)=\frac{1}{2}\sigma^2x^2k_1^{\gamma_1}(\gamma_1-1)(x-\chi)^{\gamma_1-2}+\mu xk_1^{\gamma_1}(x-\chi)^{\gamma_1-1}-\lambda\left[h(\chi)-K_1+\frac{k^{\gamma_1}}{\gamma_1}(x-\chi)^{\gamma_1}\right]<0$$

and according to condition (4.33), for every *x<L*

$$\Im V(x)=\frac{1}{2}\sigma^2x^2k_2^{\gamma_2}(\gamma_2-1)(\chi-x)^{\gamma_2-2}-\mu xk_2^{\gamma_2}(\chi-x)^{\gamma_2-1}-\lambda\left[h(\chi)-K_2-\frac{k_2^{\gamma_2}}{\gamma_2}(\chi-x)^{\gamma_2}\right]<0\,.$$

Thus $\Im V(x)$ is zero in $[L,U]$ and negative in $(0,L)\cup(U,\infty)$, so inequality (4.10) is satisfied.

We note that

$$MV(x)=\begin{cases} h(\chi)-K_1+\dfrac{1}{\gamma_1}k_1^{\gamma_1}(x-\chi)^{\gamma_1} & if \quad x\geq U \\ h(x)-\min(K_1,K_2) & if \quad L<x<U \\ h(\chi)-K_2-\dfrac{1}{\gamma_2}k_2^{\gamma_2}(\chi-x)^{\gamma_2} & if \quad x\leq L \end{cases}$$

and observe that

$$\forall\chi\in[L,U]:\qquad v(\chi)-Mv(\chi)=\min(K_1,K_2)>0\,.$$

Moreover,

$$\forall x\in(\chi,U]:\qquad v(x)-Mv(x)=h(x)-h(\chi)+K_1-\frac{k_1^{\gamma_1}}{\gamma_1}(x-\chi)^{\gamma_1}\,,$$

and according to (4.34) the function *v*-*Mv* is decreasing in (χ, *U*) with *v*(*U*)-*Mv*(*U*) =0, so *v*-*Mv* is positive in (χ,*U*]. Additionally,

$$\forall x \in [L, \chi): \qquad v(x) - Mv(x) = h(x) - h(\chi) + K_2 + \frac{k_2^{\gamma_2}}{\gamma_2}(\chi - x)^{\gamma_2}$$

and according to (4.35) the function *v*-*Mv* is increasing in (*L*,χ) with *v*(*L*)-*Mv*(*L*) =0, so *v*-*Mv* is positive in [*L*,χ). Thus *v*-*Mv* equals zero in the intervention region $(-\infty, L] \cup [U, \infty)$ and is positive in the continuation region (*L*,*U*), so inequalities (4.10)-(4.12) are satisfied. Hence *v* is a solution of the QVI and this proves the theorem.□

References

Baccarin, S., 2002. Optimal impulse control for cash management with quadratic holding-penalty costs. Decisions in Economics and Finance 25, 19-32.

Baccarin, S., 2004. On the existence of an optimal policy for a problem of impulse control, working paper, Universita di Torino.

Bensoussan, A., Lions, J., 1984. Impulse control and quasi-variational inequalities Paris: Gauthier Villars.

Borodin, A.N., Salminen, P., 2002. Handbook of Brownian motion-Facts and Formulae, Second edition, Birkhauser Verlag.

Browne, S., 1995. Optimal Investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. Mathematics of Operations Research 20, 4, 937-958.

Browne, S., 1998. The return from investment from proportional portfolio strategies. Advances in Applied Probability 30, 216-238.

Buckley, I., Korn, R., 1998. Optimal cash management and transaction costs. International journal of Theoretical and Applied Finance 1, 3, 315-330.

Cadenillas, A., Zapatero, F., 1999. Optimal central bank intervention in the foreign exchange market. Journal of Economic Theory 87, 218-242.

Cadenillas, A., Zapatero, F., 2000. Classical and impulse stochastic control of the exchange rate using interest rates and reserves. Mathematical Finance 10, 141-156.

Cadenillas, A., Sarkar, S., Zapatero, F., 2006. Optimal dividend policy with a mean reverting cash reservoir, to appear in Mathematical Finance.

Cadenillas, A., Choulli, T., Taksar, M., Zhang, L., 2006. Classical and impulse stochastic control for the optimization of the dividend and risk policies of an insurance firm. Mathematical Finance 16, 1, 181-202.

Dixit, A., 1991. A simplified treatment of the theory of optimal regulation of Brownian motion. Journal of Economic Dynamics and Control 15, 657-673.

Dumas, B., 1991. Super contact and related optimality conditions. Journal of Economic Dynamics and Control, 15, 675-685.

Harrison, J.M., Sellke, T.M. Taylor, A.J., 1983. Impulse control of Brownian motion. Mathematics of Operations Research 8, 454-466.

Jack, A., Zervos, M., 2006. Impulse control of one-dimensional ito diffusions with an expected and a pathwise ergodic criterion. Applied Mathematics and Optimization 54, 1, 71-93.

Jeanblanc-Pique, M., 1993. Impulse control method and exchange rate. Mathematical Finance 3, 161-177.

Jeanblanc-Pique, M., Shiryaev, A.N., 1995. Optimization of the flow of dividends. Russian Mathematical Surveys 50, 257-277.

Kamarianakis, Y, Xepapadeas, A., 2006. Control Bands for tracking constant portfolio allocations with fixed and proportional transaction costs. Preprint.

Karlin, S., Taylor, H.M., 1981. A second course in Stochastic Processes: Academic Press.

Korn, R., 1997 Optimal Impulse Control when the Control Consequences are random. Mathematics of Operations Research 22, 3, 639-667.

Korn, R., 1998. Portfolio optimization with strictly positive transaction costs and impulse control. Finance and Stochastics 2, 85-114.

Melas, D., Zervos, M., 2006. Optimal control of Ito diffusions with an ergodic criterion and external parameters. Preprint.

Merton, R.C., 1990. Continuous-Time Finance. Oxford, U.K.: Blackwell.

Mundaca, G., Oksendal, B., 1998. Optimal stochastic intervention control with application to the exchange rate. Journal of Mathematical Economics 29, 225-243.

Perthame, B., 1984a. Continuous and impulsive control of diffusion processes in $\Re^n$. Nonlinear Analysis, Theory, Methods & Applications 8, 1227-1239.

Perthame, B., 1984b. Quasi-variational inequalities and Hamilton-Jacobi-Bellman equations in a bounded region. Communications in Partial Differential Equations 9, 561-595.

Plehn-Dujowich, J.M., 2005. The optimality of a control band policy. Review of Economic Dynamics 8, 877-901.

Rogers, L.C.G., Williams, D., 1987. Diffusions, Markov Processes, and Martingales, Vol. 2. Wiley, New York.

Suzuki, K., Pliska, S.R., 2004. Optimal tracking for asset allocation with fixed and proportional transaction costs. Quantitative Finance 4, 233-243.

Tamoura, T., 2004. Maximizing the growth rate of a portfolio with fixed and proportional transaction costs. Applied Mathematics & Optimization 54, 95-116.

Zakamouline, V.I., 2006. European option pricing and hedging with both fixed and proportional transaction costs, Journal of Economic Dynamics and Control 30, 1, 1-25.

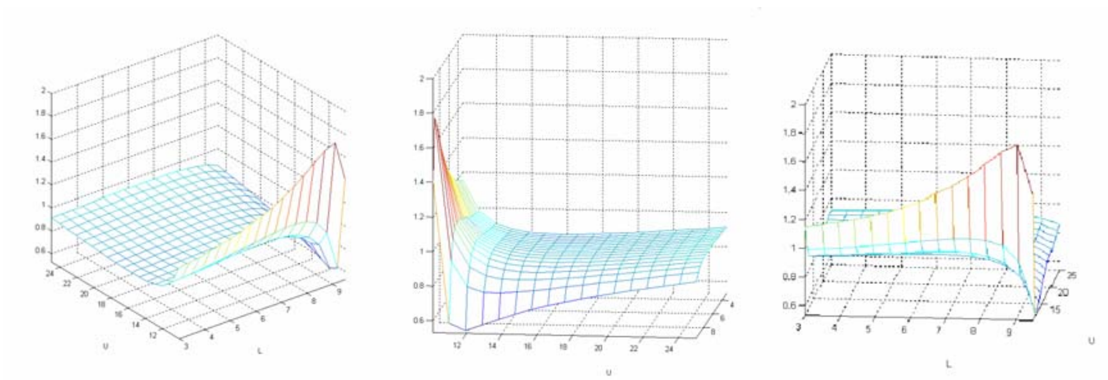

Figure 1. 3-dimensional views of the z function in (2.8) for the parameters of the benchmark example.

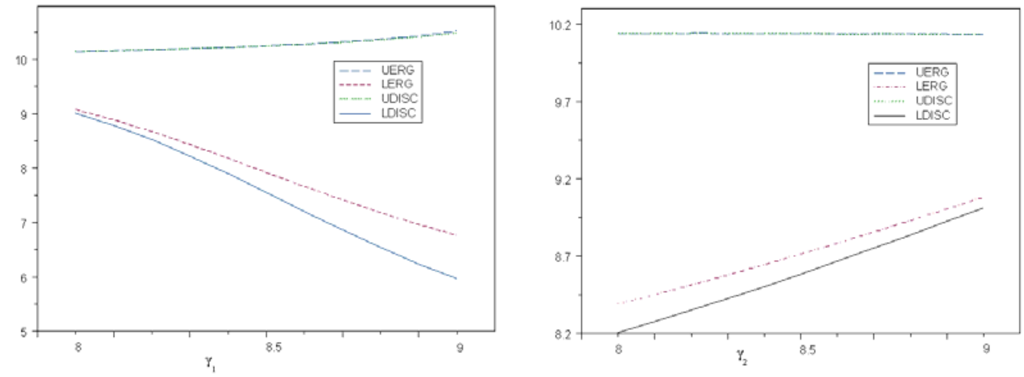

Figure 2. Control bands for the discounted and ergodic problems for varying levels of γ_1 and γ_2.

Table 1. Control bands for the discounted problem for varying discount rate for two levels of volatility.

λ	$L \mid \sigma = 0.2$	$U \mid \sigma = 0.2$	$L \mid \sigma = 0.3$	$U \mid \sigma = 0.3$
0.005	9.0127	10.1338	8.9671	10.1332
0.010	9.0116	10.1338	8.9664	10.1332
0.015	9.0104	10.1338	8.9658	10.1332
0.020	9.0093	10.1337	8.9652	10.1332
0.025	9.0081	10.1337	8.9646	10.1332
0.030	9.0069	10.1337	8.9639	10.1332
0.035	9.0057	10.1337	8.9633	10.1332
0.040	9.0046	10.1337	8.9627	10.1332
0.045	9.0034	10.1337	8.9620	10.1332
0.050	9.0022	10.1336	8.9614	10.1331

Table 2. Control bands for the discounted and ergodic problems for varying levels of fixed costs.

K_1	K_2	L_{ERG}	U_{ERG}	L_{DISC}	U_{DISC}
0.09	0.09	7.9845	10.2968	7.6466	10.2912
0.08	0.08	8.2411	10.2529	7.9816	10.2489
0.07	0.07	8.5131	10.2112	8.3280	10.2085
0.06	0.06	8.7956	10.1718	8.6759	10.1701
0.05	0.05	9.0796	10.1348	9.0116	10.1338

Table 3. Control bands for the discounted and ergodic problems for varying levels of proportional costs.

k_1	k_2	L_{ERG}	U_{ERG}	L_{DISC}	U_{DISC}
0.95	1.10	9.0796	10.1348	9.0116	10.1338
0.96	1.09	9.1524	10.1324	9.0956	10.1314
0.97	1.08	9.2180	10.1301	9.1702	10.1291
0.98	1.07	9.2768	10.1278	9.2365	10.1268
0.99	1.06	9.3297	10.1254	9.2955	10.1245
0.99	1.05	9.3507	10.1249	9.3188	10.1241
0.99	1.04	9.3708	10.1244	9.3708	10.1234
0.99	1.03	9.3902	10.1239	9.3623	10.1230
0.99	1.02	9.4087	10.1234	9.3826	10.1225

Table 4. Control bands for the discounted and ergodic problems for varying levels of γ_1.

γ_1	L_{ERG}	U_{ERG}	L_{DISC}	U_{DISC}
0.80	9.0796	10.1348	9.0116	10.1338
0.81	8.8899	10.1511	8.7875	10.1498
0.82	8.6739	10.1696	8.5233	10.1678
0.83	8.4361	10.1907	8.2226	10.1882
0.84	8.1837	10.2148	7.8937	10.2114
0.85	7.9248	10.2429	7.5485	10.2380
0.86	7.6674	10.2761	7.1999	10.2691
0.87	7.4182	10.3161	6.8597	10.3059
0.88	7.1826	10.3657	6.5373	10.3504
0.89	6.9651	10.4298	6.2392	10.4056
0.90	6.7693	10.5186	5.9700	10.4769

Table 5. Control bands for the discounted and ergodic problems for varying levels of γ_2.

γ_2	L_{ERG}	U_{ERG}	L_{DISC}	U_{DISC}
0.80	8.3896	10.1381	8.2058	10.1367
0.81	8.4502	10.1379	8.2770	10.1365
0.82	8.5129	10.1377	8.3506	10.1364
0.83	8.5775	10.1375	8.4265	10.1361
0.84	8.6442	10.1372	8.5047	10.1359
0.85	8.7129	10.1369	8.5853	10.1357
0.86	8.7835	10.1366	8.6679	10.1354
0.87	8.8558	10.1362	8.7524	10.1351
0.88	8.9296	10.1358	8.8383	10.1347
0.89	9.0044	10.1353	8.9250	10.1343
0.90	9.0796	10.1348	9.0116	10.1338

Table 6. Control bands for the discounted and ergodic problems for varying levels of σ.

σ	L_{ERG}	U_{ERG}	L_{DISC}	U_{DISC}
0.30	9.0053	10.1337	8.9664	10.1332
0.25	9.0342	10.1341	8.9833	10.1334
0.20	9.0796	10.1348	9.0116	10.1338

Table 7. Control bands for the discounted and ergodic problems for varying levels of μ.

μ	L_{ERG}	U_{ERG}	L_{DISC}	U_{DISC}
0.12	9.1014	10.1352	9.0264	10.1340
0.11	9.0907	10.1350	9.0191	10.1339
0.10	9.0796	10.1348	9.0116	10.1338
0.09	9.0679	10.1346	9.0038	10.1337
0.08	9.0555	10.1344	8.9958	10.1336
0.07	9.0424	10.1342	8.9875	10.1335
0.06	9.0286	10.1340	8.9789	10.1334
0.05	9.0139	10.1338	8.9700	10.1332

Table 8. Control bands for the discounted and ergodic problems for varying levels of χ.

χ	L_{ERG}	U_{ERG}	L_{DISC}	U_{DISC}
10.02	9.0992	10.1548	9.0313	10.1538
10.00	9.0796	10.1348	9.0116	10.1338
9.98	9.0599	10.1148	8.9919	10.1138
9.95	9.0305	10.0848	8.9623	10.0838