Controlling the risky fraction process with an ergodic criterion

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Abstract

This article examines a tracking problem, similar to the one presented in Pliska and Suzuki (Quantitative Finance, 2004): an investor would keep constant proportions of her wealth in different assets if markets were frictionless; however, in the presence of fixed and proportional transaction costs her implementation problem is to keep asset proportions close to the target levels whilst avoiding too much intervention costs. Instead of minimizing discounted tracking error plus transaction costs over an infinite horizon, the optimization objective here is minimization of long run tracking error plus intervention costs per unit time. This ergodic problem is treated via combining basic tools from diffusion theory and nonlinear optimization techniques. A comparative sensitivity analysis of the ergodic and discounted problems is undertaken.

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1.Introduction

This article examines the implementation problem of an investor who, in the absence of market frictions would adopt an investment strategy that places constant proportions of wealth in different assets. The investor may be passive, e.g. aiming to track an index (see Grinold and Kahn, 1995 chapter 5, and Focardi and Fabozzi, 2004 chapter 19) or active in the sense that constant proportions may be derived from a utility-maximizing or a probability-maximizing objective[1] under the frictionless market hypothesis. The problem is: what trading strategy should be implemented in the presence of fixed and proportional transaction costs? Our market setting consists of two assets, one riskless and one risky whose dynamics follow a geometric Brownian motion process[2], and the investor's objective is to *minimize long run tracking error plus transaction costs per unit time*.

This kind of tracking problem has been examined in Leland (2000) and Pliska and Suzuki (2004). Both articles essentially considered as state dynamics the risky fraction process (see section 2) and derived control bands that characterize transaction/no-transaction regions. The former used an approximation for the dynamics of the risky fraction process while assuming only proportional costs per adjustment whereas the latter analyzed the original risky fraction process and assumed both fixed and proportional costs per intervention. In both articles the investor's objective was to minimize discounted tracking error plus transaction costs over lifetime. However, for the considered problem a discounted optimization criterion does not have a clear economic interpretation e.g. it's not necessarily true that a given amount of tracking error is preferable to occur tomorrow than to occur today[3]. Thus it is more appropriate to adopt an ergodic optimization objective that minimizes long run tracking error plus transaction costs per unit time.

The plan of the paper is as follows. In section 2 we formulate our model and present a precise statement of the portfolio manager's optimization objective. In section 3 we solve the ergodic problem via combining basic tools from diffusion theory and nonlinear optimization techniques. Jack and Zervos (2006) and Melas and Zervos (2006) solved a similar problem via characterizing it as a system of quasi-variational inequalities (QVI) but their methods cannot be applied here since the assumptions they state do not hold for the state dynamics and the objective function of our problem. Instead we adopt a more computationally intensive approach that follows the methodology presented in Karlin and Taylor (1981, section 15.4) for a simple example[4]. To compare the optimal policies derived with the ergodic criterion

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[1] A number of optimization objectives result into constant proportion policies under the frictionless market hypothesis. Examples include HARA utility maximization, see Merton (1971), minimization of the time to reach a goal and maximization of the probability to reach a target as in Browne (1999, 2000).

[2] Our methods can be modified in a straightforward way to analyze a market with two risky assets, similar to Pliska and Suzuki (2004).

[3] For a similar argument concerning control of an exchange rate see Jack and Zervos (2006) and Melas and Zervos (2006).

[4] Karlin and Taylor examined a simple cash management problem with cash dynamics following a Brownian motion with no drift. The optimization objective minimized tracking

with the ones derived with a discounted optimization criterion we also solve the corresponding discounted problem at the fourth section. Our approach is similar with the one presented in Pliska and Suzuki (2004) with two differences. First, our optimization objective solely penalizes tracking error in contrast to the aforementioned article that includes a term related to excess return with respect to the benchmark strategy. Second, we adopt Nagai's (2005) transformation for the risky fraction process which considerably simplifies computations as the nonlinear system that needs to be solved does not contain any hypergeometric functions. Section 5 presents a comparative sensitivity analysis for the ergodic and discounted problems and the sixth section concludes the paper.

2. Problem Formulation

We consider the simple two-asset market model, in which the set of securities consists of one riskless asset, whose price $S^0(t)$ is described by the following ordinary differential equation:

$$dS^0(t) = rS^0(t)dt, \qquad S^0(0) = s^0, \tag{2.1}$$

and one risky asset with price $S^1(t)$ that is governed by the stochastic differential equation:

$$dS^1(t) = S^1(t)\left(\mu dt + \sigma dW_t\right), \qquad S^1(0) = s^1 \tag{2.2}$$

where W_t is a standard Wiener process defined on a filtered probability space (Ω, F, P, F_t). We assume that F_t satisfies the usual conditions, namely it is right continuous and F_0 includes all *P*-null sets in *F*, and that $\sigma^2 > 0$. Let $(p^0(t), p^1(t))$ be the shareholding process, to be chosen by the portfolio manager, each component of which represents the number of shares for the *i*-th asset at time *t*. It is required to be a piecewise constant, adapted process. Denote by $V(t) := \sum_{i=0}^{1} p^i(t)S^i(t)$ the wealth process or value process, which is strictly positive for all $t \geq 0$. Now we may define the risky fraction process $b^i(t)$ by setting

error plus (fixed) transaction costs per unit time. The setting of their problem allowed them to obtain analytical expressions for the boundaries of the control band and the rebalancing point. For more complex dynamics (e.g. geometric Brownian motion, mean reverting processes) this approach is computationally very intensive and analytical expressions for the control boundaries are impossible to obtain. To our knowledge this is the first time this method is used to solve a stochastic impulse control problem with an ergodic optimization criterion after Karlin and Taylor's seminal contribution. Alvarez (2004) and Alvarez and Virtanen (2004) developed similar ideas to derive optimal harvesting/dividend allocation policies with a discounted criterion.

$$b^i(t) = \frac{p^i(t)S^i(t)}{V(t)}, \qquad i = 0,1 \tag{2.3}$$

and for later use we set $b(t)=b^1(t)$. We prohibit short selling and borrowing so for each t we require $b(t) \geq 0$ and $b(t) \leq 1$. Under the condition of self-financing $V(t)$ satisfies

$$dV(t) = V(t)\left(\sum_{i=0}^{1} b^i(t)\frac{dS^i(t)}{S^i(t)}\right) = b^0(t)rdt + b^1(t)\left(\mu dt + \sigma dW_t\right), \ \ V(0) = v \tag{2.4}$$

which may be alternatively expressed as

$$\frac{dV(t)}{V(t)} = \left(r + b(t)(\mu - r)\right)dt + b(t)\sigma dW_t, \ \ V(0) = v \ . \tag{2.5}$$

The risky fraction process was first studied by Morton and Pliska (1995). Using Ito's formula, they showed that, for the two-asset case, it evolves according to the following stochastic differential equation

$$db_t = b_t(1 - b_t)\left(\mu - r - \sigma^2 b_t\right)dt + b_t\left(1 - b_t\right)\sigma dW_t \, . \tag{2.6}$$

To ease calculations in later sections, we adopt the 1-1 transformation proposed recently by Nagai (2005), defined by

$$y \ = \psi(b) := \log b - \log(1 - b) \tag{2.7}$$

with the corresponding inverse mapping *φ:*

$$\phi(y) := \frac{\exp y}{1 + \exp y}\, . \tag{2.8}$$

Using once again Ito's formula, the evolution of y is formulated as a Brownian motion with constant drift

$$dy_t \ = \kappa dt + \sigma dW_t \tag{2.9}$$

where

$$\kappa = \mu - r - \frac{\sigma^2}{2}\, .$$

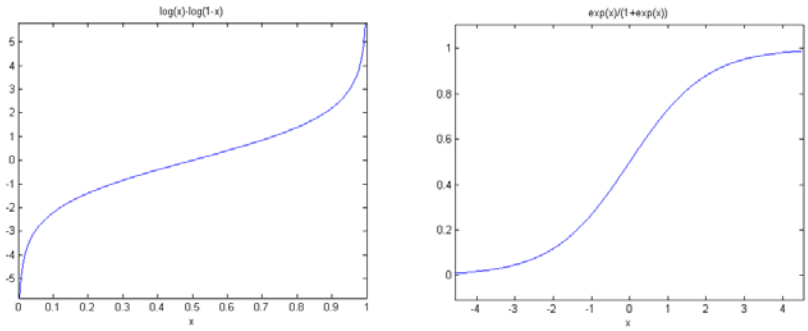

Figure 1. Nagai's transformation and its inverse for the two-dimensional case.

Let $\tilde{b}$ and $\tilde{y}$ denote the target proportion of wealth in the risky asset in the original[5] and the transformed scale respectively. If the transformed risky proportion is y and a transaction is made resulting the new risky proportion y^1, then the transaction cost incurred at that time is[6]

$$c(y, y^1) := K + k\left|y - y^1\right| \tag{2.10}$$

where K and k are two suitably chosen (so that the scale transformation is accounted for), strictly positive scalars. Thus, the linear component is proportional to the change in transformed proportions and not, as is common in much of the transaction cost literature, proportional to the dollar amount of the transaction. Because of the fixed cost component, it suffices to consider trading strategies of the form $\{(\tau_n, y_n)\}$, where τ_n is the time of the nth transaction and y_n the risky proportion that results from the nth transaction. $\{(\tau_n, y_n)\}$ must satisfy some standard technical requirements: τ_n is a stopping time, $\tau_n < \tau_{n+1}$, $\tau_n \to \infty$ as $n \to \infty$, and y_n is F_{τ_n}-measurable. The advantage for such a specification for transaction costs is that it facilitates computations; a disadvantage is that for practical applications a portfolio manager should recalibrate transaction cost parameters for investors of different wealth levels.

Deviations of the target proportion involve an opportunity cost since part of wealth in not optimally invested. We therefore suppose that holding stock at level y_t for the transformed risky fraction process incurs quadratic opportunity costs[7] (in the original scale) that are expressed in the transformed scale by

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[5] The target levels of the risky fraction process in the original scale may for instance be equal to $\frac{\mu - r}{\sigma^2}$, the risky asset proportion that maximizes log utility and the portfolio's exponential growth rate, $\frac{\mu - r}{(1-\gamma)\sigma^2}$, $\{\gamma \in \Re : \gamma < 1, \gamma \neq 0\}$, the risky asset proportion that maximizes HARA utility with risk aversion parameter γ, $f\frac{\mu - r}{\sigma^2}$ with $f \in (0,1]$, an efficient fractional Kelly strategy that maximizes capital growth and at the same time achieves a given probability of maintaining an accumulated risk free return (e.g., see Li, 1993), or a target proportion that follows an index as in Leland (2000) and Pliska and Suzuki (2004).

[6] Specification of the transaction cost is essentially the same as in Pliska and Suzuki (2004).

[7] Similar to Grinold and Kahn (1995), Leland (2000) and Suzuki and Pliska (2004). This choice is consistent with the findings of Cover (1991) and Rogers (2001) who observed that

$$g(y_t) = \lambda\left(e^{(y_t - \tilde{y})} - 1\right)^2 \qquad (2.11)$$

where λ is a constant chosen by the portfolio manager to reflect his/her loss preferences. To proceed with formulating the ergodic optimization problem assume U and L be fixed subject to $-\infty < L < U < \infty$, and define $T(s) = T_s$ be the hitting time of s for the y process. Throughout the paper we let

$$T^* = T_{U,L} = \min\{T(U), T(L)\} = T(U) \wedge T(L) \qquad (2.12)$$

be the first time the process reaches U or L and define the following quantities for y:

$$v_1(y) = \Pr\{T(U) < T(L) | y(0) = y\} \qquad L < y < U, \qquad (2.13)$$

the probability the process reaches U before L starting from y,

$$v_2(y) = E\left[T^* | y(0) = y\right], \qquad L < y < U, \qquad (2.14)$$

the mean time to reach U or L starting from y and

$$v_3(y) = E\left[\int_0^{T^*} g(Y(t)) | Y(0) = y\right], \qquad L < y < U. \qquad (2.15)$$

Now consider the following control band policy for the transformed risky fraction process: "If the transformed risky fraction process reaches level U above the target level $\tilde{y}$, reduce its level to u. This transaction incurs a cost of $K + k(U - u)$. If the transformed risky fraction process reaches level L below the target level $\tilde{y}$, increase its level to l. This transaction incurs a cost of $K + k(l - L)$." Define a cycle to be from one intervention returning the level to l or u from L or U, to the next such intervention; the long-run cost per unit time will be the expected cost per cycle divided by the expected cycle time. The expected cycle time is expressed as

$$A(L,l,u,U) = (v_1(u) + v_1(l))v_2(u) + (2 - v_1(u) - v_1(l))v_2(l) \qquad (2.16)$$

whereas the expected cost per transaction cycle is comprised by the sum of the expected transaction cost per cycle plus the expected opportunity cost/tracking error per cycle. The former is expressed as

$$B(L,l,u,U) = K + (v_1(u) + v_1(l))k(U - u) + (2 - v_1(u) - v_1(l))k(l - L) \qquad (2.17)$$

with $v_1(.)$ given by (2.13). In words, the expected transaction cost per cycle is comprised by five components: a fixed part, two parts proportional to the

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the payoff of a fixed proportion rule is quite insensitive to the chosen proportion in a neighborhood of the Merton proportion.

difference between the upper boundary and the upper rebalancing point weighted by the probabilities of reaching the upper boundary from the upper and lower rebalancing points and two parts proportional to the difference between the lower boundary and the lower rebalancing point weighted by the probabilities of reaching the lower boundary from the upper and lower rebalancing points. Similar to (2.16) expected opportunity cost/tracking error per cycle is expressed as

$$C(L,l,u,U) = (v_1(u) + v_1(l))v_3(u) + (2 - v_1(u) - v_1(l))v_3(l)\,. \tag{2.18}$$

Now the ergodic problem is formulated as follows

Problem 2.1 *The portfolio manager aims to find the inner and outer control band boundaries that minimize long run (opportunity plus transaction) cost per unit time. In particular, the investor aims to select the quadruplet (L, l, u, U) that minimizes the expression*

$$h(L,l,u,U) = \frac{B(L,l,u,U) + C(L,l,u,U)}{A(L,l,u,U)}\,. \tag{2.19}$$

In the discounted problem, the objective is to minimize the expected discounted squared tracking error plus transaction costs over an infinite planning horizon. Under an admissible trading strategy $\{(\tau_n, y_n)\}$ and given an initial proportion vector $b(0)=b_0$, the objective function in this case is given by

$$J(y_0,\{(y_n,b_n)\}) := \mathrm{E}_{y_0}^{\{(\tau_n,y_n)\}}\left[\lambda\int_0^\infty e^{-\beta t} g(y(t))dt + \sum_{n=1}^{\infty} e^{-\beta\tau_n} c(y(\tau_{n-}),y_n)I_{\{\tau_n<\infty\}}\right], \tag{2.20}$$

where $\beta > 0$ is the discount rate, λ is a constant chosen by the portfolio manager to reflect his/her loss preferences and $c(y(\tau_{n-}),y_n)$ as in (2.10). In (2.20) the first term measures discounted tracking error/opportunity costs over lifetime and the second discounted transaction costs. The discounted problem can now be formulated as follows:

Problem 2.2 *The portfolio manager seeks an admissible trading strategy that minimizes discounted tracking error plus transaction costs over lifetime. Hence, she would like to compute the value function*

$$J(y_0) := \inf_{\{(\tau_n,y_n)\}} J(y_0,\{(\tau_n,y_n)\}) \tag{2.21}$$

where the infimum is taken, over all admissible trading strategies, and find the trading strategy that attains this infimum.

In section 4 it will be shown that the optimal strategy for problem 2.2 pertains to the estimation of a control band with two outer (L, U) and two inner boundaries (l, u).

3. The ergodic problem

To solve problem 2.1, we use basic tools from the theory of diffusions (see Karlin and Taylor, 1981, or Borodin and Salminen, 2002) combined with nonlinear optimization techniques. To calculate the numerator and denominator in (2.19) we note that v_1, v_2, and v_3 in (2.13)-(2.15) need to satisfy the following differential equations

$$\kappa \frac{dv_1}{dy} + \frac{\sigma^2}{2}\frac{d^2 v_1}{dy^2} = 0 \quad \text{for} \quad L < y < U\,, \ v_1(L) = 0, \quad v_1(U) = 1\,; \tag{3.1}$$

$$\kappa \frac{dv_2}{dy} + \frac{\sigma^2}{2}\frac{d^2 v_2}{dy^2} = -1 \ \text{for} \quad L < y < U\,, \ v_2(L) = v_2(U) = 0\,; \tag{3.2}$$

$$\kappa \frac{dv_3}{dy} + \frac{\sigma^2}{2}\frac{d^2 v_3}{dy^2} = -g(y) \ \text{for} \ L < y < U\,, \ v_3(L) = v_3(U) = 0\,. \tag{3.3}$$

To solve these problems, let the scale function of the y process be denoted as

$$S(y) = \int^{y} s(\eta)d\eta\,, \tag{3.4}$$

where

$$s(y) = \exp\left\{-\int^{y}\left[2\kappa/\sigma^2\right]d\xi\right\}. \tag{3.5}$$

Let also

$$m(y) = 1/\left[\sigma^2 s(y)\right] \tag{3.6}$$

denote the speed density of the process. The solution to (3.1) is given by

$$v_1(y) = \frac{S(y) - S(L)}{S(U) - S(L)} \quad \text{for} \ L \le y \le U\,. \tag{3.7}$$

It is straightforward to observe that (3.3) is a special case of (3.2) with g equal to the indicator function. The solutions to (3.2), (3.3) are formulated as follows:

$$v_2(y) = 2\left\{v_1(y)\int_y^U \left[S(U) - S(\xi)\right]m(\xi)d\xi + \left[1 - v_1(y)\right]\int_L^y \left[S(\xi) - S(L)\right]m(\xi)d\xi\right\} \tag{3.8}$$

$$v_3(y) = 2\left\{v_1(y)\int_y^U \left[S(U) - S(\xi)\right]m(\xi)g(\xi)d\xi + \left[1 - v_1(y)\right]\int_L^y \left[S(\xi) - S(L)\right]m(\xi)g(\xi)d\xi\right\} \tag{3.9}$$

The scale function for the Brownian motion with drift (2.9) and the corresponding speed measure are expressed as

$$S(y) = \exp\left(-2\kappa y / \sigma^2\right) \tag{3.10}$$

and

$$m(y) = \exp\left(2\kappa y / \sigma^2\right) / \sigma^2 \tag{3.11}$$

respectively. The expected time to reach the outer boundaries (L, U) starting from y is

$$\begin{aligned} v_2(y) &= 2\frac{U(S(y)-S(L)) + y(S(L)-S(U)) + L(S(U)-S(y))}{(S(L)-S(U))\sigma^2} \\ &= \frac{2y}{\sigma^2} - \frac{2U}{\sigma^2}v_1(y) - \frac{2L}{\sigma^2}(1-v_1(y)) \end{aligned} \tag{3.12}$$

and the expected tracking error of the process (starting at y) till it reaches one of the outer boundaries is formulated as

$$v_3(y) = -v_1(y)f(U) - (1-v_1(y))f(L) \tag{3.13}$$

with

$$\begin{aligned} f(x) = {} & \delta\left(\kappa^3\left(4(x-\tilde{y}) + 2e^{2(x-\tilde{y})} - 8e^{x-\tilde{y}}\right) + \sigma^2\kappa^2\left(6(x-\tilde{y}) + e^{2(x-\tilde{y})} - 8e^{x-\tilde{y}} + 7\right)\right) \\ & + \delta\left(\sigma^4\kappa(2(x-\tilde{y})) + \sigma^6(1 - S(x-\tilde{y}))\right) \end{aligned} \tag{3.14}$$

and

$$\delta = \lambda\left(\sigma^2\kappa(\sigma^2+\kappa)(\sigma^2+2\kappa)\right)^{-1}. \tag{3.15}$$

Hence one may calculate the expression in (2.19) by substituting (3.7)-(3.14) to (2.16)-(2.18).

To find the quadruplet that minimizes (2.19) one may directly employ a nonlinear minimization algorithm like the quasi-Newton method (Fletcher, 1980) the simplex search method (Lagarias *et al.,* 1998) or a genetic algorithm (Dorsey and Mayer, 1995). Alternatively, one may take the corresponding derivatives, find the quadruplets that equate them to zero via using an algorithm for solving nonlinear equations like the Newton-Raphson and select among them the ones for which the Hessian is positive definite. We will present results based on the simplex search method at the fifth seaction.

Remark 3.1. The scale function and speed measure of the (untransformed) risky fraction process are formulated as

$$S(b) = b^{-2a}(1-b)^{-2(a+1)} \tag{3.16}$$

and

$$m(b) = \left(\sigma^2\left(2a(\log(b) - \log(b-1))\right) - \log(b-1)\right)^{-1} \tag{3.17}$$

with

$$a = \frac{\mu - r}{\sigma^2}.$$

Thus, computation of the integrals in (3.8), (3.9) is practically intractable for the problem in the original scale.

Remark 3.2 Of particular interest (because of its computational tractability) is the following simplified version of problem 2.1 that abolishes the inner control boundaries resulting in a policy that adjusts the transformed risky fraction process to the target as soon as it reaches the outer boundaries of the control band. A similar practice has been adopted in Korn (2004).

Problem 2.1' *The investor aims to find outer control band boundaries that minimize long run (opportunity plus transaction) cost per unit time. In particular, the investor aims to select the pair (L, U) that minimizes the expression*

$$\tilde{h}(L,U) = \frac{B(L,U) + C(L,U)}{A(L,U)}. \tag{3.18}$$

where

$$A(L,U) = v_2(\tilde{y}), \tag{3.19}$$

$$B(L,U) = K + v_1(\tilde{y})k(U - \tilde{y}) + (1 - v_1(\tilde{y}))k(\tilde{y} - L) \tag{3.20}$$

$$C(L,U) = v_3(\tilde{y}). \tag{3.21}$$

4. The Discounted Problem

In this section, we show how to solve the portfolio manager's discounted problem (2.2) via characterizing it as a QVI. The problem could have been approached as in Pliska and Suzuki (2004) with a change in the objective function. Here, for computational simplicity we work with the transformed risky fraction process. Impulse control problems similar to ours (deviating mainly in the objective function) have been applied in Cadenillas and Zapatero (1999) for optimal control of an exchange rate, Buckley and Korn (1998) and Baccarin (2002) for cash management and in Plehn-Dujowich (2005) for optimal price changes for a firm that faces menu costs.

4.1 Admissible Rebalancing Strategies

Since we want to minimize the functional J in (2.20), we should consider only those strategies for which J is well defined and finite. In order that

$$E\left[\int_0^\infty e^{-\beta t} g(y(t))\right] = E\left[\int_0^\infty e^{-\beta t} e^{2(y(t)-\tilde{y})} dt\right] - 2E\left[\int_0^\infty e^{-\beta t} e^{(y(t)-\tilde{y})} dt\right] + \frac{1}{\beta} \tag{4.1}$$

be well defined and finite, we need that the two expected values on the right-hand-side be finite. It is straightforward to see that the condition

$$E\left[\int_0^\infty e^{-\beta t} e^{2(y(t)-\tilde{y})} dt\right] < \infty \tag{4.2}$$

implies

$$E\left[\int_0^\infty e^{-\beta t} e^{(y(t)-\tilde{y})} dt\right] < \infty . \tag{4.3}$$

Now in order that

$$E\left[\sum_{n=1}^{\infty} e^{-\beta\tau_n} c\big(y(\tau_n -), y_n\big) I_{\{\tau_n < \infty\}}\right] < \infty \tag{4.4}$$

we need that

$$E\left[\sum_{n=1}^{\infty} e^{-\beta\tau_n} I_{\{\tau_n < \infty\}}\right] < \infty \text{ and } E\left[\sum_{n=1}^{\infty} e^{-\beta\tau_n} \left|y(\tau_n -) - y(\tau_n)\right| I_{\{\tau_n < \infty\}}\right] < \infty . \tag{4.5}$$

To obtain the inequality on the left-hand-side, we need that

$$\forall T \in [0,\infty): \qquad P\left\{\lim_{n\to\infty} \tau_n < T\right\} = 0. \tag{4.6}$$

To obtain the inequality on the right-hand-side, we need that

$$\lim_{T\to\infty} E\left[e^{-\beta T} y(T+)\right] = 0 \tag{4.7}$$

which holds true for the state dynamics of our problem.

DEFINITION 4.1 (Admissible controls): We shall say that an impulse control is admissible if the conditions (4.2), (4.6) are satisfied.

4.2 Solution via a Quasi-Variational Inequality

Let $J(.)$ denote the value function. That is for every $y_0 \in (-\infty,\infty)$,

$$J(y_0) := \inf_{\{(\tau_n, y_n)\} \in \mathrm{A}(y_0)} J\big(y_0, \{(\tau_n, y_n)\}\big) \tag{4.8}$$

and $\mathrm{A}(y_0)$ denotes the set of admissible strategies when the transformed risky fraction process starts from y_0. Define the minimum cost switching operator M, associated with any such function $J(.)$ and the transaction cost function $c(.,.)$ by taking

$$MJ(y) := \inf_z \{J(z) + c(y,z)\}. \tag{4.9}$$

$MJ(y)$ represents the value of the strategy that consists in choosing the best immediate intervention. Recall equation (2.9) satisfied by the transformed risky fraction process and define the second order partial differential operator L by taking

$$LJ(y) := \frac{1}{2}\sigma^2 J''(y) + \kappa J'(y) - \beta J(y). \tag{4.10}$$

Suppose there exists an optimal strategy for each initial point. Then, if the process starts at y_0 and follows the optimal strategy, the cost function associated with this optimal strategy is $J(y_0)$. On the other hand, if the process starts at y_0, selects the best immediate intervention, and then follows an optimal strategy, then the cost associated with this strategy is $MJ(y_0)$. Since the first strategy is optimal, its cost function is smaller than the cost function associated with the second strategy. Furthermore, these two costs are equal when it is optimal to jump. Hence, $J(y) \le MJ(y)$, with equality when it is optimal to intervene. In the continuation region, that is when the portfolio manager does not intervene, we must have $LJ(y) = -g(y)$.

By standard methods for impulse control problems (e.g. see Bensoussan (1982), Bensoussan and Lions (1984), Korn (1998, 1999)) we are led to the following quasi-variational inequality:

$$\min\{Lv(y) + g(y), Mv(y) - v(y)\} = 0. \tag{4.11}$$

Indeed, if v is a twice continuously differentiable function satisfying this QVI as well as the technical growth conditions depicted in the first part of this section, then

$$v(y) \le J\left(y, \left\{\left(\tau_n, y^n\right)\right\}\right) \tag{4.12}$$

for all $y \in \Re$ and all admissible strategies $\left\{\left(\tau_n, y^n\right)\right\}$. If, moreover, the strategy corresponding to v is admissible, then it is an optimal strategy and $v(.)$ is identical to the value function $J(.)$. The proof of this 'verification theorem' is lengthy, technical, and reasonably standard (e.g. see Korn (1998) or Bielecki and Pliska (2000)), so it will be omitted. The construction of the strategy corresponding to a solution v goes as follows. With $\tau_0 = 0$ and $Y(0-) = y_0$ one has

$$\tau_n := \inf\{t \ge \tau_{n-1} : v(y(t-)) = Mv(y(t-))\} \tag{4.13}$$

and

$$y^n = \arg\min_{z \in A}\{v(z) + c(y(\tau_n -), z)\}. \tag{4.14}$$

Note that v defines a continuation region

$$C := \{y \in \Re : Mv(y) > v(y)\}, \tag{4.15}$$

as no transactions occur as long as $y(t) \in C$. But if $y(t) \in \partial C$ (e.g., if $y(t)$ hits the boundary of C), then a transaction immediately occurs, shifting the risky fraction process according to (4.14).

The infimum operator M, for our problem is

$$Mv(y) = \inf_{z \in \Re} \{ v(z) + K + k|y - z| \} \tag{4.16}$$

thus qvi (4.11) becomes

$$0 = \min\left\{ \frac{\sigma^2}{2} v''(y) + \kappa v'(y) - \beta v(y) + \lambda(\exp(y - \tilde{y}) - 1)^2, \inf_{z \in A} \{ v(z) - v(y) + K + k|y - z| \} \right\} \tag{4.17}$$

We now explain how this qvi can be solved. The ordinary differential equation corresponding to (4.17) has a general solution of the form

$$v(y) = C_1 e^{-x_1 y} + C_2 e^{-x_2 y} + g(y) \tag{4.18}$$

where g is the particular solution of the differential equation given by

$$g(y) = \lambda \frac{A_1 + A_2 \exp(y - \tilde{y}) + A_3 \exp(2(y - \tilde{y}))}{A_4} \tag{4.19}$$

where

$$\begin{aligned} A_1 &= 6\kappa(\sigma^2 - \beta) - 5\sigma^2\beta + 2(\sigma^2 + \beta^2) + 4\kappa^2 \\ A_2 &= 4\beta(2\sigma^2 + 2\kappa - \beta) \\ A_3 &= -\beta(2\kappa + \sigma^2 - 2\beta) \\ A_4 &= (2\kappa + 2\sigma^2 - \beta)\beta(\sigma^2 + 2\kappa - 2\beta) \end{aligned} . \tag{4.20}$$

Here C_1 and C_2 are constants depending on boundary conditions and x_1, x_2 are formulated as follows

$$x_{1,2} = \frac{\kappa \pm \sqrt{\kappa^2 + 2\sigma^2\beta}}{\sigma^2} . \tag{4.21}$$

For most values of the data parameters, it can be shown that there exist four parameters satisfying $L<l<u<U$ such that the solution of the qvi (4.17) will be of the form

$$v(y) = \begin{cases} -ky + \{ v(y) + kl + K \} & y \in (-\infty, L] \\ v(y) & y \in (L, U) \\ ky + \{ v(y) - ku + K \} & y \in [U, \infty) \end{cases} \tag{4.22}$$

Here (L, U) is the continuation region. For $y \in (-\infty, L]$ one should immediately rebalance to $y=l$, and for $y \in [U, \infty)$ one should immediately rebalance to $y=u$. It remains to determine the values of the six parameters C_1,

C_2, *L, l, u* and *U*. On that purpose, one should solve a system of six nonlinear equations. To derive these equations we note that the function *v*(.) must be continuous at *y*=*l*, so

$$v(L) = -kL + v(l) + kl + K\ . \tag{4.23}$$

Similarly, we get a second equation for continuity at *y*=*u*,

$$v(U) = kU + v(u) - ku + K\ . \tag{4.24}$$

The derivatives at *y*=*L* and *U* must be continuous, so

$$v'(L) = -k \tag{4.25}$$

and

$$v'(U) = k\ . \tag{4.26}$$

Since $y = l$ minimizes $v(y) + K + k(y - L)$ the first order necessary condition gives

$$v'(l) = -k \tag{4.27}$$

and similarly the final equation is

$$v'(u) = k\ . \tag{4.28}$$

The system of six equations can readily be solved by MATLAB for the six parameters; a detailed numerical illustration presented at the sixth section.

5. Numerical Illustration

In this section, we provide numerical solutions for the control problems considered in the third and fourth parts of the article. The reader should note that the associated nonlinear systems are quite complex and thus sensitive to the initial values provided as starting points for their solutions. For the sensitivity analysis conducted at the second part of this section, we first found appropriate initial values for a baseline experiment and then, for each perturbation of the parameters, we plugged in as initial values the outcomes of the previous run. MATLAB codes are available upon request from the authors.

5.1 A Specific Example

We first consider the following data for market characteristics and investor's preferences

κ=0.1, σ=0.2, λ=1, β=0.05, $\tilde{y}$=0.5, k=0.05, K=0.005.

For problem 2.1, application of the Nelder-Mead simplex method gives

L_1=0.4519, l_1=0.4731, u_1=0.5018, U_1=0.5446.

For the simplified version of the ergodic problem, that is problem 2.1', the outer boundaries of the control band are

L_1'= 0.435, U_1'=0.5433,

after transforming back to the original scale[8]. The corresponding solutions for the discounted problem are

L_2=0.4338, l_2=0.4746, u_2=0.5023, U_2=0.5456, *C1*=-43.7633, *C2*= -0.0388

and errors are of the order 10^{-8}.

Figure 2. The value function for the minimization of discounted lifetime costs problem.

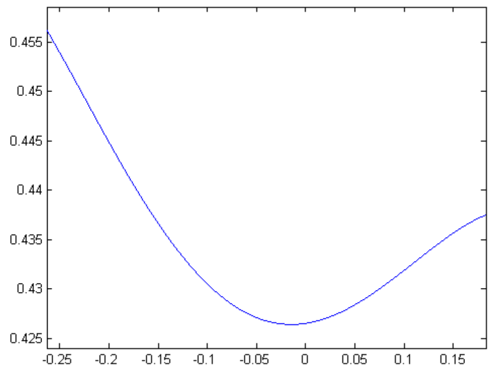

Figure 2 depicts the value function corresponding to this parameter selection for the discounted problem. The value function is depicted in the (transformed) continuation region (*TL*, *TU*). Outside this region, the value function is linear with slope $-k$ in the intervention region $(-\infty, TL]$ and a linear function with slope k in the intervention region $[TU, \infty)$. From (2.6) one observes that Merton's optimal proportion for the problem of maximizing the portfolio's exponential growth rate is an equilibrium point for the risky fraction process. In this example, Merton's proportion is much larger than the target proportion; thus, the risky fraction process is expected to force portfolio holdings to the right of the no-transaction region. For this reason the minimum point of the value function is located to the left of the target[9] proportion. When the investor intervenes on the weak side of the target (which corresponds to selling stock in this particular example) it is optimal to bring the portfolio

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[8] For problem 2.1' the derivatives of (3.18) w.r.t. L_1' and U_1' were calculated via MATLAB's symbolic MATH toolbox. The solutions to the corresponding nonlinear system were obtain via a trust-region-dogleg algorithm.

[9] Zero in the tranformed scale corresponds to 0.5 in the original scale. This observation goes along the lines of Cadenillas and Zapatero (1999) who treated a similar problem for the control of an exchange rate.

levels much closer to the target than when she/he intervenes in the strong side. This difference between the target and Merton's proportion also causes asymmetry between the left and right part of the no-transaction region: the distance between the target asset proportion and the left boundary is larger than the one between the target and the right boundary.

5.2 Sensitivity Analysis

To conduct sensitivity analysis, we perturb individual parameter values from their baseline values, thereby indicating how the optimal strategy is affected. Results are displayed at tables 1-5. Regarding the transaction costs parameters the intuition is clear: the investor rebalances more often with lower transaction costs. When fixed costs increase it is optimal to wait longer before intervening although the sizes of interventions will be larger. When proportional costs increase it is optimal to wait longer before intervening but unlike when fixed costs increase interventions tend to be smaller. As volatility increases the no-transaction regions become wider and the magnitude of interventions becomes larger. As κ in (2.9) increases, optimal interventions resulting from hitting the "weak side" of the target tend to bring asset holdings to a level located lower than the target. The higher the pressure on the "weak side" of the target the sooner the investor should intervene; the opposite holds true at the "strong side" of the target. Finally, as λ increases, the investor becomes more concerned about tracking error; thus, the width of the no transaction region becomes narrower.

The above findings hold for both (ergodic and discounted) problems. Tables 1-5 indicate that the outer boundaries of the ergodic control bands are always within the outer boundaries of the control band that corresponds to the discounted problem. Thus an investor that adopts an ergodic criterion intervenes earlier than one that adopts a discounted criterion with discount rate equal to 0.05. Sensitivity analysis of the discounted problem with respect to the discount rate indicates that control bands tend to shrink for decreasing discount rate (table 6) but the ergodic control band lies always within the discounted one for all the discount rates examined. Figure 3 displays that control boundaries depend linearly to the discount rate.

Table 1. Control bands for the ergodic and discounted problems for different levels of *K*.

K	L1	l1	u1	U1	L2	l2	u2	U2
0.003	0.4616	0.4714	0.5023	0.5389	0.4428	0.4735	0.5070	0.5395
0.004	0.4551	0.4720	0.5021	0.5413	0.4392	0.4741	0.5051	0.5420
0.005	0.4519	0.4731	0.5019	0.5446	0.4338	0.4746	0.5023	0.5456

Note. κ=0.1, σ=0.2, λ=1, β=0.05, $\widetilde{y}$=0.5, k=0.05.

Table 2. Control bands for the ergodic and discounted problems for different levels of *k*.

k	L1	l1	u1	U1	L2	l2	u2	U2
0.03	0.4553	0.4763	0.5008	0.5438	0.4385	0.4808	0.4998	0.5444
0.04	0.4521	0.4736	0.5018	0.5442	0.4360	0.4776	0.5011	0.5451
0.05	0.4519	0.4731	0.5019	0.5446	0.4338	0.4746	0.5023	0.5456

Note. κ=0.1, σ=0.2, λ=1, β=0.05, $\tilde{y}$=0.5, K=0.005.

Table 3. Control bands for the ergodic and discounted problems for for different levels of σ.

σ	L1	l1	u1	U1	L2	l2	u2	U2
0.1	0.4651	0.4767	0.4962	0.5281	0.4470	0.4746	0.4919	0.5290
0.2	0.4519	0.4731	0.5019	0.5446	0.4338	0.4746	0.5023	0.5456
0.3	0.4461	0.4681	0.5082	0.5581	0.4198	0.4708	0.5088	0.5584

Note. κ=0.1, λ=1, β=0.05, $\tilde{y}$=0.5, k=0.05, K=0.005.

Table 4. Control bands for the ergodic and discounted problems for for different levels of κ.

κ	L1	l1	u1	U1	L2	l2	u2	U2
0.09	0.4521	0.4730	0.5020	0.5448	0.4346	0.4757	0.5031	0.5461
0.10	0.4519	0.4721	0.5019	0.5446	0.4338	0.4746	0.5023	0.5456
0.15	0.4431	0.4681	0.4924	0.5440	0.4294	0.4694	0.4981	0.5436

Note. σ=0.2, λ=1, β=0.05, $\tilde{y}$=0.5, k=0.05, K=0.005.

Table 5. Control bands for the ergodic and discounted problems for for different levels of λ.

λ	L1	l1	u1	U1	L2	l2	u2	U2
0.5	0.4398	0.4609	0.5008	0.5531	0.4153	0.4634	0.4985	0.5534
1	0.4519	0.4731	0.5019	0.5446	0.4338	0.4746	0.5023	0.5456
1.5	0.4598	0.4804	0.5023	0.5416	0.4423	0.4795	0.5025	0.5416

Note. κ=0.1, σ=0.2, β=0.05, $\tilde{y}$=0.5, k=0.05, K=0.005.

Table 6. Sensitivity of the discounted problem with respect to β.

β	**L2**	**l2**	**u2**	**U2**
0.02	0.43396	0.47475	0.50220	0.54552
0.025	0.43394	0.47475	0.50222	0.54555
0.03	0.43391	0.47475	0.50225	0.54557
0.035	0.43389	0.47472	0.50227	0.54560
0.04	0.43386	0.47472	0.50227	0.54562
0.045	0.43384	0.47472	0.50230	0.54567
0.05	0.43382	0.47470	0.50232	0.54570
0.055	0.43379	0.47470	0.50232	0.54572
0.06	0.43377	0.47470	0.50235	0.54575
0.065	0.43374	0.47467	0.50237	0.54577
0.07	0.43372	0.47467	0.50237	0.54580
0.075	0.43369	0.47467	0.50240	0.54582
0.08	0.43367	0.47467	0.50242	0.54585

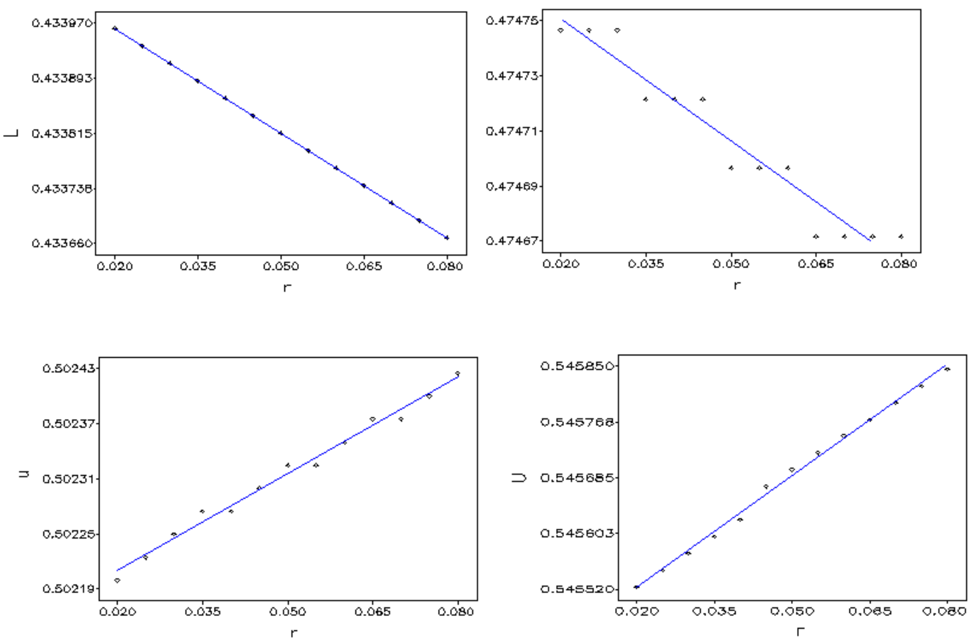

Figure 3. Regression fit for the control boundary-discount rate relationships.

6. Concluding Remarks

The vast majority of stochastic impulse control models presented in the literature so far examine discounted optimization objectives. However there are problems like index tracking or control of an exchange rate that an ergodic criterion is more suitable. This article presented a methodology for treating a tracking problem with an ergodic criterion and compared the resulting optimal policies with the ones derived from a discounted criterion. This approach can be readily modified to examine other impulse control problems like control of an exchange rate, cash management, price adjustment with menu costs, etc. An advantage of the methodology is that it may easily accommodate policy constraints via Lagrange multipliers or constrained nonlinear optimization algorithms.

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