The Shapley-Folkman Theorem and the Range of a Bounded Measure: An Elementary and Unified Treatment

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Abstract

We present proofs, based on the Shapley-Folkman theorem, of the convexity of the range of a strongly continuous, finitely additive measure, as well as that of an atomless, countably additive measure. We also present proofs, based on diagonalization and separation arguments respectively, of the closure of the range of a purely atomic or purely nonatomic countably additive measure. A combination of these results yields Lyapunov's celebrated theorem on the range of a countably additive measure. We also sketch, through a comprehensive bibliography, the pervasive diversity of the applications of the Shapley-Folkman theorem in mathematical economics. (95 words)

Key Words: Strongly continuous measure, atomless measure, range of a measure, diagonalization argument, Hahn decomposition, Shapley-Folkman theorem, Lyapuonov's theorem.

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1 Introduction

Lyapunov's convexity theorem [41, 42] has found substantial application in optimal control theory, mathematical economics, statistics and game theory, and its importance is reflected in the several proofs that have been offered since it was first announced in 1940; see [1, 23, 31] for exposition and references to applications, and [11, 27, 38, 49, 58, 64] for a selection of alternative proofs. However the argument that has retained its hold, and received textbook treatment, is the connection of the theorem to extreme point phenomenon in an infinite-dimensional space, as emphasized by Lindenstrauss [38]; in addition to [1, Section 13.9], see [40]. As such it has remained outside the reach of the undergraduate equipped only with a first course in real analysis. In this note, we offer a proof based on separation, Cantor's diagonalization argument and the Shapley-Folkman theorem, which, thanks to Cassell ([21, 22], also see [65, 55, 57, 60]), is itself a consequence of elementary probabilistic considerations and simple linear algebraic manipulation. In addition to its pedagogic value, the proof reported here testifies to the elementary nature, surprisingly missed so far, of Lyapunov's claim.

The fact that the proof presented here, and the alternative one presented in [58] (see Remarks 1 and 4 below for the difference in approach), rely on the Shapley-Folkman theorem is of some substantive consequence for mathematical economics: the subject can be seen (admittedly in hindsight) to have evolved by providing asymptotic implementations, and rates of convergence, of the fundamental results obtained through Lyapunov's theorem or its nonstandard analogues (as in [39, 34, 35, 36]), and thereby systematically replacing the tools for these idealized (continuum) objects by their counterparts for finite ones. For this trajectory, see [54, 10, 2] and their followers: in alphabetical order, [3, 4, 5, 6, 7, 8, 17, 18, 19, 20, 25, 29, 30, 44, 46, 47, 48, 53, 63]), as well as Remark 7 below. (An expression of this point of view is also available in [56, 37, 57].) For an application to matching and to probability theory, see [16] and [61, 32, 45] respectively.

After collecting preliminaries in Section 2, the main results are presented in Sections 3 and 4. We conclude the paper with Section 5 consisting of a series of remarks delineating our reliance on, and difference from, earlier ideas.

2 Preliminaries on Bounded Measures

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space. For any set $A \subseteq \mathbb{R}^m$ let co A denotes its convex hull. The Shapley-Folkman theorem, see [54, 2, 65, 57], states that: if S_i ($i = 1, \ldots, n$) are nonempty subsets of $\mathbb{R}^m$ and $x \in \text{co} \sum_{i=1}^{n} S_i$ then x has a representation $x = \sum_{i=1}^{n} x_i$ such that $x_i \in \text{co } S_i$ for all i and $x_i \in S_i$ for at least $(n - m)$ indices i.

Let T be a nonempty set and $\mathcal{T}$ a field of subsets of T, i.e., (i) $\emptyset, T \in \mathcal{T}$; (ii) $A, B \in \mathcal{T} \Rightarrow A \cup B \in \mathcal{T}$; and (iii) $A, B \in \mathcal{T} \Rightarrow A \setminus B \in \mathcal{T}$. $\mathcal{T}$ is a Boolean algebra with set theoretic unions, intersections and complementation. μ: $\mathcal{T} \longrightarrow \mathbb{R}^m$ is a finitely additive measure if $\mu(\emptyset) = 0$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{T}$ and $A \cap B = \emptyset$. We shall assume throughout that μ is bounded, i.e., sup $\{|\mu_i(A)|: A \in \mathcal{T}\} < \infty$, $1 \leq i \leq m$, and when $m = 1$, we shall be explicit in calling it a finitely additive scalar measure. For a finitely additive scalar measure μ, the positive, negative and total variations of μ are: $\mu^+(E) = \sup \{\mu(F) : F \subseteq E, F \in \mathcal{T}\}$, $E \in \mathcal{T}$; $\mu^-(E) = -\inf \{\mu(F) : F \subseteq E, F \in \mathcal{T}\}$, $E \in \mathcal{T}$ and $|\mu| = \mu^+ + \mu^-$. $E \in \mathcal{T}$ is an atom of μ if $\mu(E) \neq 0$ and for every $F \subseteq E$, $F \in \mathcal{T}$, $\mu(F) = 0$ or $\mu(F) = \mu(E)$. A finitely additive measure μ taking values in $\mathbb{R}^m$, is nonatomic or atomless if none of its coordinates has any atoms. For details regarding these concepts see [13, p. 53, p. 141].

A field $\mathcal{T}$ is an F-algebra if for any increasing sequence $\{A_n\}$, and any decreasing sequence $\{B_n\}$ with $A_n, B_n \in \mathcal{T}$, $A_n \subseteq B_n$ for all n there is a $C \in \mathcal{T}$ with $A_n \subseteq C \subseteq B_n$ for all n. This concept is introduced in [52], called an F-algebra in [9, p. 502] and a Boolean algebra with Seever property in [13, p. 210].

$\mathcal{T}$ is a σ-algebra if $\cup_{n=1}^{\infty} A_n \in \mathcal{T}$ whenever $A_n \in \mathcal{T}$ in (ii) above. When T is equipped with a σ-algebra $\mathcal{T}$, $(T, \mathcal{T})$ will be referred to as a measurable space. It is easy to see that every σ-algebra is an F-algebra: if $\mathcal{T}$ is a σ-algebra, $\{A_n\}$ an increasing sequence and $\{B_n\}$ a decreasing sequence both in $\mathcal{T}$ with $A_n \subseteq B_n$ for all n then either $\cup_{n=1}^{\infty} A_n$ or $\cap_{n=1}^{\infty} B_n$ can serve the role of C.

Let $\mathcal{T}$ be a σ-algebra. μ: $\mathcal{T} \longrightarrow \mathbb{R}^m$ is a countably additive measure (or a measure) if $\mu(\emptyset) = 0$ and $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $\{A_n\}$ is a sequence of pairwise disjoint sets in $\mathcal{T}$. When $m = 1$, we shall be explicit in calling it a scalar measure. Again, we shall assume throughout that μ is bounded. The notions of variations, atom and atomlessness are analogous to those given above. If λ and μ are two scalar measures, λ is absolutely continuous with respect to μ, $\lambda \ll \mu$, if $\lambda(A) = 0$ for every measurable set A for which $|\mu|(A) = 0$. The measure μ is purely atomic if there is a sequence $\{E_k\}$ of pairwise disjoint measurable sets such that $T = \cup_{k=1}^{\infty} E_k$, and every E_k is an atom of each μ_i, $i = 1, \ldots, m$. For introduction to this material, the reader can see, for example, the textbooks [1, 15, 28, 33, 50].

For any finitely or countably additive measure μ on $\mathcal{T}$ and for any $A \in \mathcal{T}$, the range of μ on A is $R_\mu(A) = \{\mu(A \cap E) \in \mathbb{R}^m : E \in \mathcal{T}\}$. $R_\mu(T)$ is the range of μ.

3 The Theorem of Armstrong-Prikry

The notion of nonatomicity or atomlessness of a finitely additive measure is not adequate to yield convexity and closedness properties of its range. This necessitates a consideration of a stronger concept, that of a strongly continuous measure. In this section we consider the example from [13, pp. 143-144] to establish lack of convexity and closedness, introduce the notion of a strongly continuous finitely additive measure and then present a result on the convexity of its range due to [9].

3.1 An Example

Let T = [0,1], $\mathcal{T}$ the Borel σ-algebra of T and λ the Lebesgue measure. Assume τ to be a 0–1 valued finitely additive measure on $\mathcal{T}$ such that $\tau(A) = 0$ if $\lambda(A) = 0$. Then $\mu = \lambda + 2\tau$ defines a nonnegative, finitely additive scalar measure on $\mathcal{T}$. We will show that μ is nonatomic on $\mathcal{T}$ and $R_\mu(T) = [0, 1) \cup (2, 3]$, a set which is not convex or closed.

Let $E \in \mathcal{T}$ and $\mu(E) > 0$. If $\lambda(E) = 0$ then $\tau(E) = 0$ and $\mu(E) = 0$. So, $\mu(E) > 0$ implies that $\lambda(E) > 0$. Since λ is nonatomic, there is $F \subseteq E$ such that $0 < \lambda(F) < \lambda(E)$. Since $\tau(F) \leq \tau(E)$, $0 < \mu(F) = \lambda(F) + \tau(F) < \lambda(E) + \tau(E) = \mu(E)$. This shows that μ is nonatomic.

Clearly, $\mu(\emptyset) = 0$ and $\mu(T) = 3$. Let $E \in \mathcal{T}$. If $\tau(E) = 0$ then $\mu(E) = \lambda(E) \leq 1$ and if $\tau(E) = 1$ then $\mu(E) = \lambda(E) + 2 \geq 2$. If $\mu(E) = 2$ for some $E \in \mathcal{T}$, then $\tau(E) = 1$ and $\lambda(E) = 0$. But $\lambda(E) = 0$ implies that $\tau(E) = 0$, a contradiction. Similarly, if $\mu(E) = 1$ for some $E \in \mathcal{T}$ then $\mu(E^c) = 2$, a contradiction. This shows that $R_\mu(T) \subseteq [0, 1) \cup (2, 3]$.

We will now establish the converse, that $[0, 1) \cup (2, 3] \subseteq R_\mu(T)$. Since $\mu(T) = 3$, it suffices to show that $(0, 1) \subseteq R_\mu(T)$. Let $x \in (0, 1)$. There is a positive integer n such that $x \leq \sum_{i=1}^{n}(1/2^i) < 1$. Suppose that for some $E \in \mathcal{T}$, $\mu(E) = \sum_{i=1}^{n}(1/2^i)$. Then $\tau(E) = 0$ and $\mu(E) = \lambda(E)$. So, $\exists\ F \subseteq E$ such that $\lambda(F) = x$. $\tau(E) = 0$ implies that $\tau(F) = 0$ and $\mu(F) = x$. Thus, it suffices to show that there is a sequence $\{E_i\}$ of pairwise disjoint intervals in $\mathcal{T}$ such that for each i, $\mu(E_i) = \lambda(E_i) = 1/2^i$.

Consider the two intervals (0,1/2) and (1/2,1). Since τ is 0–1 valued, the μ measure of exactly one of these is 1/2 and of the other is 5/2. Denote by E_1 the one with μ measure 1/2 and the other by F_1. F_1 is an interval, say $F_1 = (a_1, b_1)$. Now suppose that for some n, intervals $E_1, \ldots,\ E_n$ and $F_1, \ldots,\ F_n$ have been constructed such that: $\mu(E_i) = 1/2^i$, $\mu(F_i) = 2 + (1/2^i)$, $E_i \cap F_i = \emptyset$, $F_i = (a_i, b_i)$ for $i = 1, \ldots,\ n$ and $E_i \subseteq F_{i-1}$ for $i > 1$. Let $c_n = (b_n - a_n)/2$. By considering the two sets (a_n, c_n) and (c_n, b_n) we have $\mu(E_{n+1}) = \lambda(E_{n+1}) = 1/2^{n+1}$, $\mu(F_{n+1}) = 2 + (1/2^{n+1})$ and $F_{n+1} = (a_{n+1}, b_{n+1})$. This construction yields the desired sequence of sets $\{E_i\}$ and shows that $R_\mu(T) = [0, 1) \cup (2, 3]$.

We now turn to the question of existence of a 0–1 valued finitely additive measure τ on $\mathcal{T}$ with the property that $\tau(A) = 0$ if $\lambda(A) = 0$. $\mathcal{I} \subseteq \mathcal{T}$ is an ideal in $\mathcal{T}$ if (i) $T \notin \mathcal{I}$, (ii) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$ and (iii) $A \in \mathcal{I}$, $B \in \mathcal{T}$, $B \subseteq A \Rightarrow B \in \mathcal{I}$. An ideal $\mathcal{I}$ in $\mathcal{T}$ is a maximal ideal in $\mathcal{T}$ if there is no ideal in $\mathcal{T}$ properly containing $\mathcal{I}$. If $\mathcal{I}$ is an ideal in $\mathcal{T}$ then by Zorn's lemma, there is a maximal ideal $\mathcal{I}^*$ in $\mathcal{T}$ containing $\mathcal{I}$. If $\mathcal{I}^*$ is a maximal ideal in $\mathcal{T}$ then the function ν on $\mathcal{T}$ defined as $\nu(A) = 0$ if $A \in \mathcal{I}^*$ and $\nu(A) = 1$ if $A \in \mathcal{T} \setminus \mathcal{I}^*$ defines a 0–1 valued finitely additive measure on $\mathcal{T}$. The reader can refer to [13, pp. 10-11, 14, 38] for this material. In terms of the example, notice that $\mathcal{B} = \{A \in \mathcal{T} : \lambda(A) = 0\}$ is an ideal in $\mathcal{T}$. So, there is a τ with the required properties.

3.2 Strongly Continuous Measures

Let $\mathcal{T}$ be a field of subsets of a nonempty set T and μ a finitely additive scalar measure on $\mathcal{T}$. μ is strongly continuous if for every $\epsilon > 0$, there exists a partition $\{E_1, \ldots,\ E_n\}$ of T such that $|\mu|(E_i) < \epsilon$ for every i. If μ takes values in $\mathbb{R}^m$ then μ is strongly continuous if μ_i is strongly continuous for every coordinate i.

If μ is a strongly continuous finitely additive scalar measure then it is nonatomic. Since μ is strongly continuous or nonatomic according as $|\mu|$ is strongly continuous or nonatomic, we can assume without loss of generality that $\mu \geq 0$. Let $A \in \mathcal{T}$ and $\mu(A) > 0$. For $0 < \epsilon < \mu(A)$, there is a partition

$\{E_1, \ldots,\ E_n\}$ of T such that $\mu(E_i) < \epsilon$ for every i. There exist two distinct indices i and j such that $\mu(E_i \cap A) > 0$ and $\mu(E_j \cap A) > 0$. If $B = E_i \cap A$ then $0 < \mu(B) < \mu(A)$. So, μ is nonatomic.

A nonatomic, finitely additive scalar measure need not be strongly continuous. In the example above, μ is nonatomic. It is not strongly continuous however, for $0 < \epsilon < 2$ there is no partition with the required properties.

We now examine the case of a countably additive scalar measure defined on a σ-algebra. The definition of a strongly continuous measure is the same as above. A measure is strongly continuous or nonatomic according as its total variation is strongly continuous or nonatomic. The preceding arguments show that a strongly continuous measure is nonatomic. Conversely, a nonatomic scalar measure is strongly continuous.

Let μ be a nonatomic countably additive scalar measure defined on a σ-algebra $\mathcal{T}$. To show that μ is strongly continuous, assume without loss of generality that $\mu \geq 0$ and fix $\epsilon > 0$. If $A \in \mathcal{T}$ with $\mu(A) > 0$, then $\exists\, B \subseteq A$, $B \in \mathcal{T}$ with $0 < \mu(B) < \epsilon$, see [26, Lemma 1]. Theorem 1.12.9 in [15] yields a finite partition $\{E_1, \ldots,\ E_n\}$ of T with the property that either $\mu(E_i) \leq \epsilon$, or E_i is an atom of measure greater than ϵ. By assumption, E_i is not an atom for any i. If $\mu(E_i) = \epsilon$ for some i then E_i can be partitioned into two disjoint sets, each with measure less than ϵ. So, μ is strongly continuous.

3.3 A Convexity Theorem

The range of a finitely additive, strongly continuous measure defined on an F-algebra is convex, [9, Theorem 2-2]. A simple proof, using the Shapley-Folkman theorem, is given below. It is preceded by a Lemma which establishes the corresponding result for the nonnegative, scalar case, and explicates the argument in [9, Lemma 2-1]; also see [12, 43].

Lemma 1 *Let T be a nonempty set and $\mathcal{T}$ a field of subsets of T which is also an F-algebra. If μ is a bounded, nonnegative, finitely additive, strongly continuous scalar measure on $\mathcal{T}$ then its range $R_\mu(T)$ is convex.*

Proof Let $x \in (0, \mu(T))$. Since μ is strongly continuous, for $\epsilon = 1/2$, there is a partition $\{E_1, \ldots, E_k\}$ of T such that $E_i \in \mathcal{T}$ and $\mu(E_i) < 1/2$ for $i = 1, \ldots,\ k$. Let k_1 be the last integer, possibly 0, so that $\sum_{i=1}^{k_1} \mu(E_i) \leq x$. Let $A_1 = \cup_{i=1}^{k_1} E_i$ and $B_1 = \cup_{i=1}^{k_1+1} E_i$. Then $\mu(A_1) \leq x \leq \mu(B_1)$, $A_1 \subseteq B_1$ and $\mu(B_1 \setminus A_1) < 1/2$.

Now suppose that for some n the sets $A_1 \subseteq \ldots \subseteq A_n \subseteq B_n \subseteq \ldots \subseteq B_1$ have been constructed so that $\mu(A_n) \leq x \leq \mu(B_n)$ and $\mu(B_n \setminus A_n) < 1/2^n$. Find a partition $\{E_1, \ldots, E_k\}$ of $B_n \setminus A_n$ such that $E_i \in \mathcal{T}$ and $\mu(E_i) < 1/2^{n+1}$ for $i = 1, \ldots,\ k$. Let k_n be the last integer, possibly 0, so that $\mu(A_n) + \sum_{i=1}^{k_n} \mu(E_i) \leq x$. Let $A_{n+1} = A_n \cup \left(\cup_{i=1}^{k_n} E_i\right)$ and $B_{n+1} = A_n \cup \left(\cup_{i=1}^{k_n+1} E_i\right)$. Then $A_n \subseteq A_{n+1} \subseteq B_{n+1} \subseteq B_n$, $\mu(A_{n+1}) \leq x \leq \mu(B_{n+1})$ and $\mu(B_{n+1} \setminus A_{n+1}) < 1/2^{n+1}$.

Since $\mathcal{T}$ is an F-algebra, $\exists\, C \in \mathcal{T}$ such that $A_n \subseteq C \subseteq B_n$ for every n. Since $\mu(A_n) \leq x \leq \mu(B_n)$ and $\mu(B_n \setminus A_n) \to 0$, $\mu(C) = x$. This completes the proof. ∎

Theorem 1 *Let T be a nonempty set and $\mathcal{T}$ a field of subsets of T which is also an F-algebra. If μ is a bounded, $\mathbb{R}^m$-valued, finitely additive, strongly continuous measure on $\mathcal{T}$ then its range $R_\mu(T)$ is convex.*

Proof Suppose that for any positive integer k, the range of a bounded, nonnegative, $\mathbb{R}^k$-valued, finitely additive, strongly continuous measure is convex. For an $\mathbb{R}^m$-valued, finitely additive, strongly continuous measure μ, define the $\mathbb{R}^{2m}$-valued measure $\hat{\mu} = (\mu_1^+, \mu_1^-, \ldots, \mu_m^+, \mu_m^-)$. $\hat{\mu}$ is nonnegative and strongly continuous, hence its range is convex. Since $\mu_i = \mu_i^+ - \mu_i^-$ for every i, the range of μ is obtained from that of $\hat{\mu}$ by a linear transformation, and is therefore, convex. So, it suffices to assume that μ is nonnegative.

If μ is a nonnegative, strongly continuous scalar measure, its range is convex by Lemma 1. Assume that μ takes values in $\mathbb{R}^m$ and define $\mu^* = \mu_1 + \mu_2 + \ldots + \mu_m$. The range of μ^* is convex. Let $x \in \text{co } R_\mu(T) \subseteq \mathbb{R}^m$.

Since the range of μ^* is convex, there is a family $\{A_i\}$ ($i = 1, \ldots, 2m$) of elements of $\mathcal{T}$ such that $\cup_{i=1}^{2m} A_i = T$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\mu^*(A_i) = (2m)^{-1}\mu^*(T)$ for each i.

Since $R_\mu(T) = \sum_{i=1}^{2m} R_\mu(A_i)$, $x \in \text{co } \sum_{i=1}^{2m} R_\mu(A_i)$. By the Shapley–Folkman theorem, see [21, 57, 65], there is a set of indices I, $|I| \leq m$, such that

$$x \in \sum_{i \notin I} R_\mu(A_i) + \text{co} \sum_{i \in I} R_\mu(A_i)$$

Let $S_1 = \cup_{i \notin I} A_i$. Then $x \in R_\mu(S_1) + \text{co } R_\mu(T \setminus S_1)$. Thus, there exist two points x_1 and z_1 such that $x_1 \in R_\mu(S_1)$, $z_1 \in \text{co } R_\mu(T \setminus S_1)$ and $x = x_1 + z_1$. Furthermore, $\mu^*(S_1) \geq 2^{-1}\mu^*(T)$.

The same argument can be iteratively repeated replacing T by $T \setminus S_1$, so that, by induction, at the n^{th} step one obtains n measurable sets $\{S_i\}$ $(1 \leq i \leq n)$ and $n+1$ points $\{x_i\}$ $(1 \leq i \leq n)$ and z_n with $x_i \in R_\mu(S_i)$ and $z_n \in \text{co } R_\mu(T \setminus \cup_{i=1}^n S_i)$ such that

$$x = \sum_{i=1}^{n} x_i + z_n$$

Observe that $\mu^*(S_1) \geq 2^{-1}\mu^*(T)$ and for each $i > 1$, $\mu^*(S_i) \geq 2^{-1}\mu^*(T \setminus \cup_{j=1}^{i-1} S_j)$. This yields, $\mu^*(T \setminus \cup_{i=1}^n S_i) \leq (2^n)^{-1}\mu^*(T) \to 0$ as $n \to \infty$. This implies that $\mu(T \setminus \cup_{i=1}^n S_i) \to 0$ as $n \to \infty$, and $z_n \to 0$ as $n \to \infty$. Hence $x = \sum_{i=1}^{\infty} x_i$.

Since $x_i \in R_\mu(S_i)$ for each i, $\exists\, K_i \subseteq S_i$ such that $\mu(K_i) = x_i$. Let $E_1 = K_1$ and $E_{n+1} = E_n \cup K_{n+1}$. Let $F_1 = E_1 \cup (T \setminus S_1)$ and $F_{n+1} = E_{n+1} \cup (T \setminus \cup_{i=1}^{n+1} S_i)$. It is obvious that $E_n \subseteq E_{n+1}$ and $E_n \subseteq F_n$ for every n. We will show that $F_{n+1} \subseteq F_n$. $F_{n+1} = E_{n+1} \cup (T \setminus \cup_{i=1}^{n+1} S_i) = E_n \cup K_{n+1} \cup (T \setminus \cup_{i=1}^{n+1} S_i)$ $\subseteq E_n \cup (T \setminus \cup_{i=1}^n S_i) = F_n$. The inclusion follows because $K_{n+1} \subseteq T \setminus \cup_{i=1}^n S_i$.

Since $\mathcal{T}$ is an F-algebra, $\exists\, C \in \mathcal{T}$ such that $E_n \subseteq C \subseteq F_n$ for every n. So, $\mu(E_n) \leq \mu(C) \leq \mu(F_n)$ for every n. Since $F_n \setminus E_n = T \setminus \cup_{i=1}^n S_i$, $\mu(F_n \setminus E_n) \to 0$ as $n \to \infty$. Moreover, $\mu(E_n) \leq x \leq \mu(F_n)$ for every n. So, $\mu(C) = x$ and $R_\mu(T)$ is convex. This completes the proof. ∎

In [12] it is shown that the range of a finitely additive, strongly continuous measure defined on a σ-algebra is convex. This is implied by Theorem 1 since a σ-algebra is an F-algebra. The arguments used in [9, 12] to prove Theorem 1 are similar to those in [27]. In contrast, the above proof based on the Shapley-Folkman theorem is shorter and simpler. Examples in [9, 12] demonstrate that the range of a finitely additive, strongly continuous measure need not be closed.

4 The Theorem of Lyapunov

Throughout this section, $(T, \mathcal{T})$ is a measurable space and μ an $\mathbb{R}^m$-valued, countably additive measure on $\mathcal{T}$.

Theorem 2 *(Lyapunov) The range of a bounded, $\mathbb{R}^m$-valued measure is a closed subset of $\mathbb{R}^m$. If the measure is atomless then its range is a convex subset of $\mathbb{R}^m$.*

In terms of proof, suppose that the range of a bounded, nonnegative measure is closed (and hence compact) and whenever the measure is atomless, the range is convex. Consider any $\mathbb{R}^m$-valued measure μ and let $\hat{\mu} = (\mu_1^+, \mu_1^-, \ldots, \mu_m^+, \mu_m^-)$. Since $\mu_i = \mu_i^+ - \mu_i^-$ for every i, $R_\mu(T)$ is obtained from $R_{\hat{\mu}}(T)$ by a linear transformation. If $R_{\hat{\mu}}(T)$ is compact then so is $R_\mu(T)$ and if μ is atomless then so is $\hat{\mu}$ and the convexity of $R_{\hat{\mu}}(T)$ implies that of $R_\mu(T)$. Therefore, in the proofs it suffices to consider nonnegative measures only.

Lemma 2 below establishes the convexity of the range of an atomless measure. Lemmas 4 and 5 show that the range is closed if the measure is atomless or is purely atomic. Lemma 6 combines these two results to establish that the range of a measure is closed.

4.1 Convexity and Closedness of the Range in the Atomless Case

Let μ be a bounded, nonnegative, atomless measure taking values in $\mathbb{R}^m$ and define $\mu^* = \mu_1 + \mu_2 + \ldots + \mu_m$.

Lemma 2 *If μ is a bounded, nonnegative, atomless measure on $\mathcal{T}$ then $R_\mu(T)$ is a convex subset of $\mathbb{R}^m$.*

Proof Since a σ-algebra is an F-algebra, a countably additive measure is a finitely additive measure and a countably additive atomless measure is strongly continuous, Theorem 1 shows that $R_\mu(T)$ is a convex set and the proof is complete. ∎

Remark 1 Since the measure is countably additive in Lemma 2 and only finitely additive in Theorem 1, a somewhat simpler proof of Lemma 2 can be given and we outline it below.

By [26, Lemma 2] the range of a nonnegative, atomless, scalar measure is convex. This can also be deduced from Lemma 1 by taking $C = \cup_{n=1}^{\infty} A_n$ or $C = \cap_{n=1}^{\infty} B_n$. So, the range of μ^* (as defined above) is convex. Let $x \in \text{co}\, R_\mu(T) \subseteq \mathbb{R}^m$. The arguments in the proof of Theorem 1 can be repeated verbatim to show that there is a sequence $\{S_i\}$ of pairwise disjoint sets in $\mathcal{T}$ such that $x_i \in R_\mu(S_i)$ and $x = \sum_{i=1}^{\infty} x_i$. (The last two paragraphs in the proof of Theorem 1 can be omitted in the present context.) Since $x_i \in R_\mu(S_i)$ for all i, $x \in R_\mu(\cup_{i=1}^{\infty} S_i) \subseteq R_\mu(T)$. This shows that $R_\mu(T)$ is convex.

The next lemma is used in the proof of the closedness of the range of an atomless measure.

Lemma 3 *Let $p \in \mathbb{R}^m$, $p \neq 0$. Then $p \cdot \mu(B)$ for each $B \in \mathcal{T}$ defines a scalar measure on $\mathcal{T}$ and is absolutely continuous with respect to μ^*. Suppose that for some $P \in \mathcal{T}$, $p \cdot \mu(B) \geq 0$ for every measurable set $B \subseteq P$. Then there exist two measurable sets $E_1, E_2 \subseteq P$ such that: (i) $E_1 \cup E_2 = P$, $E_1 \cap E_2 = \emptyset$, (ii) if $B \subseteq E_1$ then $\mu^*(B) > 0 \Leftrightarrow p \cdot \mu(B) > 0$, i.e., each of μ^* and $p \cdot \mu$ is absolutely continuous with respect to the other on E_1 and (iii) $p \cdot \mu(B) = 0$ for every $B \subseteq E_2$.*

An analogous result holds if for some $P \in \mathcal{T}$, $p \cdot \mu(B) \leq 0$ for every $B \subseteq P$: if $B \subseteq E_1$ then $\mu^*(B) > 0 \Leftrightarrow p \cdot \mu(B) < 0$ in (ii) above.

Proof It is easy to verify that $p \cdot \mu(\cdot)$ is a scalar measure on $\mathcal{T}$. If $\mu^*(B) = 0$ then $\mu(B) = 0$ and $p \cdot \mu(B) = 0$, which verifies absolute continuity.

Suppose that for some $P \in \mathcal{T}$, $p \cdot \mu(B) \geq 0$ for every $B \subseteq P$. Let

$$\mathcal{A} = \{B \subseteq P : p \cdot \mu(B) = 0\} \text{ and } \alpha = \sup\{\mu^*(B) : B \in \mathcal{A}\}.$$

Since $\emptyset \in \mathcal{A}$, $\mathcal{A}$ is nonempty. α is finite because μ^* is bounded. Since $\mathcal{A}$ is closed under countable union of pairwise disjoint sets and set difference, it is closed under countable union of sets. Let $\{B_n\}$ be a sequence of sets in $\mathcal{A}$ such that $\{\mu^*(B_n)\} \to \alpha$. If $E_2 = \cup_{n=1}^{\infty} B_n$ then $E_2 \in \mathcal{A}$. For each n, $B_n \subseteq E_2$ implies that $\mu^*(B_n) \leq \mu^*(E_2)$ and $\mu^*(E_2) = \alpha$. Let $E_1 = P \setminus E_2$. If for some $B \subseteq E_1$, $p \cdot \mu(B) = 0$ and $\mu^*(B) > 0$, then $p \cdot \mu(E_2 \cup B) = 0$ and $\mu^*(E_2 \cup B) > \alpha$, a contradiction. So, $B \subseteq E_1$, $p \cdot \mu(B) = 0$ implies that $\mu^*(B) = 0$. ∎

Lemma 4 *If μ is a bounded, nonnegative, atomless measure on $\mathcal{T}$ then $R_\mu(T)$ is a closed subset of $\mathbb{R}^m$.*

Proof The proof is by induction. It is well known that the range of a nonnegative, atomless scalar measure is a closed interval, [26, Lemma 2] or Lemma 1 above. Before proceeding to the general case of higher dimension, we will deal with one special case, that the range of a signed, atomless scalar measure is closed. If λ is such a measure, then by Hahn decomposition, there is $A \in \mathcal{T}$ such that $\lambda(A \cap E) \geq 0$ and $\lambda(A^c \cap E) \leq 0$ for every $E \in \mathcal{T}$. Thus, the range of λ is the interval $[\lambda(A^c), \lambda(A)]$, which is closed. Assume that $R_\mu(T)$ is closed when dim $R_\mu(T) \leq m - 1$ and we will prove it for the case when dim $R_\mu(T) = m$.

Let $\{A_n\}$ be a sequence of measurable sets such that $\{\mu(A_n)\} \to y$. Assume that y belongs to the boundary of $R_\mu(T)$. By the separating hyperplane theorem, $\exists\, p \in \mathbb{R}^m, p \neq 0$ such that $p{\cdot}y = \sup\ p{\cdot}\mu(B)$, $B \in \mathcal{T}$. $p \cdot \mu$ is an atomless, signed, scalar measure and its range is closed. So, for some $P \in \mathcal{T}$, $p \cdot \mu(P) = p \cdot y$.

If for some $B \subseteq P$, $p \cdot \mu(B) < 0$ then $p \cdot \mu(P \setminus B) > p \cdot \mu(P)$, a contradiction. So, $p \cdot \mu(B) \geq 0$ for every $B \subseteq P$. Similarly, if for some $B \subseteq P^c$, $p \cdot \mu(B) > 0$ then $p \cdot \mu(P \cup B) > p \cdot \mu(P)$, again a contradiction. So, $p \cdot \mu(B) \leq 0$ for every $B \subseteq P^c$.

By Lemma 3, there exist E_1, $E_2 \subseteq P$ and F_1, $F_2 \subseteq P^c$ such that $E_1 \cup E_2 = P$, $E_1 \cap E_2 = \emptyset$, $F_1 \cup F_2 = P^c$, $F_1 \cap F_2 = \emptyset$, μ^* and $p \cdot \mu$ are absolutely continuous with respsect to each other on E_1 and on F_1 and $p \cdot \mu(B) = 0$ if $B \subseteq E_2$ or if $B \subseteq F_2$. If necessary, passing to a subsequence we can assume that all the limits appearing in the following equation exist.

$$y = \lim_{n\to\infty} \mu(E_1 \cap A_n) + \lim_{n\to\infty} \mu(E_2 \cap A_n) + \lim_{n\to\infty} \mu(F_1 \cap A_n) + \lim_{n\to\infty} \mu(F_2 \cap A_n)$$

Multiplying by p and interchanging limits we get

$$p \cdot y = \lim_{n\to\infty} p \cdot \mu(E_1 \cap A_n) + \lim_{n\to\infty} p \cdot \mu(E_2 \cap A_n) + \lim_{n\to\infty} p \cdot \mu(F_1 \cap A_n) + \lim_{n\to\infty} p \cdot \mu(F_2 \cap A_n)$$

The second and the fourth terms in the RHS are zero and the third term is nonpositive. So, $p \cdot y \leq \lim_{n\to\infty} p \cdot \mu(E_1 \cap A_n)$. Since $p \cdot y = \sup\ p \cdot \mu(B)$, $B \in \mathcal{T}$, $p \cdot y \geq \lim_{n\to\infty} p \cdot \mu(E_1 \cap A_n)$. Therefore, $\lim_{n\to\infty} p \cdot \mu(E_1 \cap A_n) = p \cdot y = p \cdot \mu(P) = p \cdot \mu(E_1)$ and $\lim_{n\to\infty} p \cdot \mu(F_1 \cap A_n) = 0$.

On F_1, μ^* is absolutely continuous with respect to $p \cdot \mu$. Since $p \cdot \mu(F_1 \cap A_n) \to 0$, $\mu^*(F_1 \cap A_n) \to 0$ and $\mu(F_1 \cap A_n) \to 0$. On E_1, μ^* is absolutely continuous with respect to $p \cdot \mu$. Since $p \cdot \mu(E_1 \cap A_n) \to p \cdot \mu(E_1)$, $p \cdot \mu(E_1 \setminus A_n) \to 0$. Therefore, $\mu^*(E_1 \setminus A_n) \to 0$, $\mu(E_1 \setminus A_n) \to 0$ and $\mu(E_1 \cap A_n) \to \mu(E_1)$.

Since $p \cdot \mu(B) = 0$ if $B \subseteq E_2$ or if $B \subseteq F_2$, dim $R_\mu(E_2) \leq m - 1$ and dim $R_\mu(F_2) \leq m - 1$. By the induction hypothesis $R_\mu(E_2)$ and $R_\mu(F_2)$ are closed.

So, $y = \mu(E_1) + x_1 + x_2$ where $x_1 \in R_\mu(E_2)$ and $x_2 \in R_\mu(F_2)$. Since $R_\mu(E_2) \subseteq R_\mu(T)$ and $R_\mu(F_2) \subseteq R_\mu(T)$, $y \in R_\mu(T)$ and $R_\mu(T)$ is closed. ∎

4.2 Closedness of the Range

4.2.1 The Purely Atomic Case

Lemma 5 *If μ is a bounded, purely atomic nonnegative measure and each of the measures μ_i is absolutely continuous with respect to every other measure μ_ℓ $(i, \ell = 1, \ldots, m)$, then its range is closed.*

Proof Let $S = \cup_{k=1}^{\infty} S_k$ where each S_k is an atom of every μ_i, $i = 1, \ldots, m$. Let $\{A_n\}$ be a sequence of measurable subsets of S and assume that $\{\mu(A_n)\} \to y$. We will show that for some measurable subset B of S, $\mu(B) = y$.

Consider $\mu(S_k \cap A_n)$; $k,\ n = 1, 2, \ldots$. By Cantor's diagonalization argument [51, Theorem 7.23], there is a subsequence $\{n_j\}$ such that $\mu(S_k \cap A_{n_j})$ converges for every k, say to y_k. We will show that $\sum_{k=1}^{\infty} y_k = y$.

$$\begin{aligned} y &= \lim_{j\to\infty} \mu(A_{n_j}) = \lim_{j\to\infty} \sum_{k=1}^{\infty} \mu(S_k \cap A_{n_j}) \\ &= \lim_{j\to\infty} \left[\mu(S_1 \cap A_{n_j}) + \sum_{k=2}^{\infty} \mu(S_k \cap A_{n_j}) \right] = y_1 + \lim_{j\to\infty} \sum_{k=2}^{\infty} \mu(S_k \cap A_{n_j}) \end{aligned}$$

Repetition of the argument gives, for any integer $K > 1$, $y = \sum_{k=1}^{K} y_k + \lim_{j\to\infty} \sum_{k=K+1}^{\infty} \mu(S_k \cap A_{n_j})$. Since $\mu(S) = \sum_{k=1}^{\infty} \mu(S_k) < \infty$, $\mu(S_k) \to 0$ as $k \to \infty$. This gives $y = \sum_{k=1}^{\infty} y_k$.

Fix any k. If $\mu_i(S_k \cap A_{n_j}) = 0$ for some i then $\mu_\ell(S_k \cap A_{n_j}) = 0$ for all ℓ because of absolute continuity. If $\mu_i(S_k \cap A_{n_j}) = \mu_i(S_k)$ for some i then $\mu_i(S_k \setminus A_{n_j}) = 0$, $\mu_\ell(S_k \setminus A_{n_j}) = 0$ and $\mu_\ell(S_k \cap A_{n_j}) = \mu_\ell(S_k)$. So, for any k, $\mu(S_k \cap A_{n_j}) = 0$ or $\mu(S_k \cap A_{n_j}) = \mu(S_k) \neq 0$, i.e., for any k, $y_k = 0$ or $y_k = \mu(S_k)$. Let $B_k = \emptyset$ or $B_k = S_k$ in these two cases respectively and $B = \cup_{k=1}^{\infty} B_k$. Then $\mu(B) = y$. ∎

4.2.2 The General Case

Lemma 6 *If μ is a bounded, nonnegative measure then $R_\mu(T)$ is a closed subset of $\mathbb{R}^m$.*

Proof Let T be a nonsingular $m \times m$ matrix with positive elements and write $\mu' = T\mu$. Each of the nonnegative measures $\mu'_1, \ldots, \mu'_m$ is absolutely continuous with respect to every other one. Since the range of μ is obtained from the range of μ' by applying T^{-1}, we can assume that μ itself has this absolute continuity property. It follows that an atom of any coordinate is an atom of all others. Let $\{S_k\}$, $k = 1, 2, \ldots$ be the common set of atoms and write $S = \cup_{k=1}^{\infty} S_k$. Then μ is purely atomic on S and atomless on $T \setminus S$. Lemma 5 shows that $R_\mu(S)$ is closed and Lemma 4 shows that $R_\mu(T \setminus S)$ is closed. Since $R_\mu(T)$ is a sum of these two sets and the sum of two compact sets is compact, $R_\mu(T)$ is compact. So, the range of a nonnegative measure is closed. ∎

5 Concluding Remarks

We conclude with a series of observations on the relationship of the proof reported here to earlier work.

1. Our motivation for the proof of convexity of the range using the Shapley-Folkman theorem comes from [58]. The proof of convexity there, involves two steps. First it is shown, using the Shapley-Folkman theorem, that the integral of a correspondence is a convex set. From this, in the next step, it is deduced that the range of a measure is convex. This step uses Radon-Nikodym derivatives. Our proof is more direct. Furthermore, slight variations in our arguments lead to convexity of the range for both finitely additive and countably additive measures. This is significant since Radon-Nikodym derivatives may not exist in the finitely additive setting.

2. The proof above of the convexity of the range using the Shapley-Folkman theorem is quite different from those in [27, 9, 12]. It is also shorter and simpler. In [49], the proof of convexity is based on the intermediate value theorem and the axiom of choice, and any reader comfortable with the axiom of choice can substitute the argument in [49] for that based here on the Shapley-Folkman theorem. The axiom of choice and a separation argument are also used in [11] to establish the convexity and closedness of the range. The analogue of Lyapunov's theorem, established in [39], is obtained as a consequence of Steinitz' theorem.

3. The statement and the proof of Lemma 6 is due to [27, Lemma 11]. There are substantial differences in the details, however. The separation argument used in Lemma 4 to establish the closedness of the range of an atomless measure is more transparent. In particular, it does not require the translation of the measure as in [27, Lemma 9]. In the purely atomic case, [26, 27] appeal to Tychonov's theorem, whereas we have used a diagonalization argument. In similar contexts, diagonalization argument has been used in [14, 24]. The separation argument used above is considerably simpler than that of [14].

4. In almost all the proofs discussed above either Radon-Nikodym derivatives and/or the Lebesgue decomposition theorem are used explicitly. The text book proofs of the latter make use of the former. Surprisingly, the present proof does not use Radon-Nikodym derivatives at all. Lemma 3, the key ingredient in the proof of Lemma 4, obtains a decomposition from elementary measure theoretic considerations.

5. Ironically, the Hahn decomposition theorem [50, Theorem 6.14] is used in the proof of the convexity part of the claim in [27, Lemma 7] rather than in the proof of the closedness claim as in the proof reported here. Whereas [50, Theorem 6.14] gives a functional-analytic proof of the Hahn decomposition theorem, note that an elementary purely measure-theoretic treatment, as is being emphasized in this note, is available in [33, Theorem 19.6]. Note that for the finitely additive case we have used the Jordan decomposition extensively in our proof of convexity to reduce the consideration of a signed measure to a non-negative one, and that an exact Hahn decomposition may not exist in this case; see [13, pp. 52-57] for ϵ-Hahn decompositions.

6. Note that one can dispense with the Hahn decomposition theorem in the case of a scalar signed measure, and simply decompose it into its non-positive and non-negative parts, and not even require the minimality of this decomposition, which is to say, require the existence of a Jordan decomposition, [50, Section 6.6]. Since the sum of the variation measure is not identical to the variation measure of a sum, this is not possible in the vector case. The fact that the Jordan decomposition does work for Lindenstrauss' proof is brought out in [40]. This again brings out the difference of the argument reported here.

7. It is worthy of notice that in the trajectory of applications of the Shapley-Folkman mentioned in the introduction, Anderson [2] represents a discontinuity in that only a weaker form of the Shapley-Folkman theorem, one bypassing explicit measures of non-convexity and drawing on elementary linear algebraic conditions, is used in the subsequent literature; see [21] for this distinction, and [46] for explicitly referring to it. We leave it to the reader to provide a categorization of the applications according to this "weak-strong" criteria, and in the interests of space, also leave it to her to track down subsequent work that draws on the principal result in [2] rather than on the Shapley-Folkman theorem itself.

8. The authors of [24, Section 4] extend their theorem, and therefore Lyapunov's theorem, to the case of an arbitrary atomless signed measure. The arguments reported there hinge on a prior claim regarding a finite atomless signed measure, and as such can be applied *verbatim* here to extend our proofs to cover the case of an unbounded measure.

9. Lyapunov [42] presents an example establishing the failure of both claims of the theorem for measures taking values in infinite-dimensional spaces rather than in $\mathbb{R}^m$. In keeping with the motivation behind this note, we avoid any reference to the substantial literature extending, under additional hypotheses, Lyapunov's theorem to such spaces; see [59] and his references for an up-to-date treatment.

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