# Interest Rate Risk of Banking Accounts: Measurement Using the VaR Framework

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*In order to measure the interest rate risk of banking accounts such as deposits and loans, this paper extends the value at risk (hereafter, VaR) analysis framework, which is useful for the risk evaluation of trading accounts.*

*In order to apply the VaR concept derived from trading accounts to banking accounts, we should take into account the following issues: (1) the longer risk evaluation period because of the inflexibility of adjustability of banking account positions; (2) the evaluation of risk included in the administered rates (the long-term prime rate and the short-term prime rate); and (3) the prepayment risk (associated with housing loans, etc.). Therefore, in this paper we first construct a VaR model including a term-structure model to express the stochastic process of market rate, the administered rate model, and the prepayment function model. Then, we perform a simulation using an imaginary portfolio to analyze the factors determining interest rate risk.*

*In conclusion, it has been proved that the factor of administered rates increases interest rate risk both in single products and in a portfolio. Taking into account the behavior of customers who want better interest rate conditions, the factor of prepayment decreases the present value, which is itself the basis of calculating risk. Finally, we perform a sensitivity analysis of model parameters to show the magnitude of model risk.*

Key words: Value at risk; Monte Carlo simulation; Holding period; Term-structure model; Heath-Jarrow-Morton model; Administered rate; Prepayment

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## I. Introduction

It is essential to measure the risk of financial transactions as accurately as possible in order to price individual transactions, to hedge for risk control, to calculate the required total capital prepared for the risk of the whole portfolio, and to make decisions on the best resource allocation, paying attention to the trade-off between profitability and risk. In this context, we can say that the quantitative method of risk measurement is the most important technique for the management of financial institutions.

Financial institutions confront various kinds of risks. Of those risks, market risk and credit risk seem most important. In this paper, we focus on market risk, especially the market risk of whole banking account (or non-trading account) portfolios, which account for the most typical portfolios of Japanese banks. As is widely recognized, value at risk (hereafter, abbreviated as VaR), a kind of stochastic risk evaluation method conducted on a present value (hereafter, PV) basis, has by now taken root for the interest rate risk evaluation of trading accounts. On the other hand, as to the interest rate risk evaluation of banking accounts including loans and deposits, there is no consensus on the definition of risk. In addition, in the process of risk measurement, technical skills are not yet mature enough. First, there is no consensus on whether we should measure risk on an earnings basis (so-called earning at risk [EaR]) or on a PV basis (VaR). Second, some features of banking account transactions (difficulties in closing positions, the possibility of prepayment, and the linking of products to administered rates) make risk calculation more difficult than in the case of trading accounts.

It is preferable to use the same rule to measure the aggregate risk of portfolios of financial institutions. From this point of view, we propose a theoretical framework to measure the interest rate risk of banking accounts within the VaR framework, which has become a popular method of risk measurement for trading accounts. Using this theoretical framework and an imaginary portfolio, we perform a simulation to analyze what kinds of factors determine the interest rate risk of banking accounts. Of course, it is difficult at this stage to judge which of EaR and VaR is superior. However, it seems very meaningful to study the possibilities of the extension of the risk measurement method with VaR, for the following reasons. (1) If credit risk can also be measured under the VaR framework, as shown by Oda and Muranaga (1997), we can expect to be able to manage market risk and credit risk in aggregate on a PV basis. (2) If credit liquidation and the disposal of deposit business by selling branches become more popular, as in the United States, the difference in liquidity between banking accounts and trading accounts will decrease in scale.

This paper consists of the following parts. First, in Section II, we give an overview of the differences between trading accounts and banking accounts. We list the points that we should consider when we apply the VaR framework, which has been developed as the risk management method for trading accounts, to banking accounts. In Section III, we show how to approach model-building when we extend the VaR model, and explain the outline of the model that we use in this paper. In Section IV, we perform a simulation under the extended VaR framework with the imaginary portfolio and analyze the various factors that change the interest risk of banking accounts. Finally, in Section V, we derive some implications for risk management.

## II. The Approach to Extending the VaR Model: Taking into Account Features of Banking Accounts

### A. Features of Banking Accounts: Comparison with Trading Accounts

When we calculate the interest rate risk of banking accounts within the VaR framework, we have to list sequentially the types of financial transactions that are actually included in banking accounts, and state what kinds of features those transactions have by comparison with trading accounts. There are two definitions of banking accounts: (1) the total of non-trading accounts; and (2) non-trading accounts excluding investment securities accounts. We use the second, narrower definition for banking accounts in this paper. Because we exclude investment securities such as bonds and stocks, we focus mainly on loans on the assets side and deposits and financial debentures on the liabilities side. When we calculate risk, we may deal with off-balance-sheet positions such as swaps, futures, and options held in banking accounts mainly for hedging purposes, and the exchange rate risk of foreign currency assets and liabilities. However, because we can deal with these transactions basically in the same way as the off-balance-sheet transactions of trading accounts and exchange rate risk, which have already been discussed actively in the literature, we do not deal with these in this paper.

Next, Appendix Table 1 shows the product lineup of standard city banks. We can derive the features of deposits and loans that are representative products of banks as follows.

- Most products do not possess liquidity.
- Most loans and deposits are on a fixed rate basis (the applied interest rate does not change until the contractual maturity).
- Some long-term loans and long-term deposits are on a floating rate basis; therefore, the applied rate is settled every renewal period under predetermined interest rate conditions. The interest rates are market rates in some cases and administered rates, such as the short-term prime rate and the long-term prime rate, in other cases.
- Even in the case of fixed-rate loans and deposits, when maturity arrives and products are rolled over, the applied rate is renewed according to the market rate, short-term prime rate, and long-term prime rate at that time. Especially in the case of non-maturity loans and non-maturity deposits, maturities arrive randomly and applied interest rates are renewed according to the change in the corresponding interest rate applicable at the time.
- In the case of corporate loans, there are few prepayments during the transaction term. On the other hand, in the case of individual loans (especially housing loans), there are frequent prepayments. Recently, it has become usual that the applied interest rate is fixed for the first few years or based on a floating rate such as the short-term prime rate. However, there remain some housing loans that started in the past and whose applied interest rate is floating based on the long-term prime

rate or on a fixed rate until maturity.[1]

• There are frequent prepayments in the case of individual term deposits.

As compared to trading accounts, the three main differences are as follows.

[1] It is difficult to close a position immediately.

[2] There are products such as housing loans and time deposits that allow customers to cancel before maturity (hereafter, we always call the cancellation before maturity "prepayment").

[3] There are not only products linked to market rates but also products linked to administered rates (e.g., the prime rate).

**B. Approach to Extension of VaR**

It is widely recognized that the VaR framework which measures portfolio risk in terms of the probability distribution of changes in PV is effective for the risk calculation of trading accounts. However, considering the features of banking account portfolios outlined above, what kind of modifications are required in methods for calculating current value and risk? The first change is due to the problem of liquidity ([1] in Section II.A). The liquidity of trading account products is high, and the banks usually hold such products only for a short term. Therefore, it is usual that the holding period of the portfolio for risk calculation (the required period for liquidation)—or risk calculation period[2]—is usually one day, or at the most two weeks. On the other hand, for banking accounts, the banks intend to hold a product for a long time and the liquidity of the products is low. Therefore, the risk calculation period should be assumed to be longer than that of a trading account. In calculating risk, it is necessary to assume that the interest rate fluctuation process lasts longer, and to explicitly account for the fluctuation pattern of the position itself (dynamic simulation). In addition, because of [2] in Section II.A, it is essential to consider prepayment (cancellation of loans as interest rates decrease and cancellation of deposits as interest rates rise) to be a kind of option and to recognize its value. Finally, because of [3] in Section II.A, it is necessary to make an explicit model of the relationship between market rates, which are risk factors for trading account VaR, and administered rates such as prime rates. In the following sections, we elaborate these points and discuss our approach to the extension of the VaR model to calculate the interest rate risk calculation of banking accounts.

**1. Lack of liquidity**

When we calculate the risk of trading accounts, we can assume relatively short holding periods such as one day or two weeks because of the specific characteristics of such accounts. Then, using the VaR model's variance-covariance method, we calculate the risk amount in a simple way, usually under the following assumptions.

[1] We assume that the value of the position changes linearly according to the change in the interest rate (that is, we omit convexity risks).

[2] We assume that market rates of various maturities fluctuate according to a

1. Financial institutions except for city banks still sell housing loans linked to long-term prime or to fixed interest rates.

2. In this paper, we use both terms "holding period" and "risk evaluation period" to mean the same thing, without distinction.

multivariate lognormal process with specified drift rates,[3] volatilities, and correlations.[4]

On the other hand, banking account products possess little liquidity, so their holding period is longer and it is necessary to address the following four problems.

[1] The longer the holding period, the larger the fluctuation of interest rates, so convexity risk may be non-negligible. We cannot capture this kind of nonlinear risk accurately under the ordinary variance-covariance method, so we have to introduce a simulation method.

[2] The VaR model assumes that market rates of various terms are random variables under a multivariate lognormal process. On the one hand, this assumption makes the calculation of risk easy. However, on the other hand, when considering yield curve fluctuations over a long time, such an assumption is likely to produce an unrealistic yield curve that contradicts the non-arbitrage conditions. Therefore, when making a simulation, it is necessary to use some term-structure model to solve this problem.

[3] It is necessary to take account explicitly of position fluctuation during the holding period.

[4] Because it is not always the case that loss is maximized at the end of the holding period, it is necessary to calculate the loss amounts at each time during the holding period.

**2. Prepayment**

In banking accounts, unlike trading accounts, there are products that allow customers to cancel before maturity, e.g., mortgage loans and time deposits. Though the interest rate to be applied and the maturity are decided when the contract is made, it is possible for borrowers and depositors to cancel at any time during the term of the loan or deposit. The right of prepayment belongs not to banks but to customers. Unless we take account of the value of prepayment, we cannot accurately evaluate the value of housing loans and time deposits, and cannot calculate their risk. Therefore, we need some valuation model of prepayments in the simulation.

**3. Administered rates**

The price of interest rate-sensitive products in trading accounts usually depends on the market rate. Therefore, we consider only the market interest rate as a risk factor. However, products of banking accounts require not only market rates but also administered rates as index rates. For example, a floating-rate housing loan changes its applied rate when an index rate such as the short-term prime rate or the long-term

3. The drift rate is often ignored.

4. The following equations specify this process.

$$dr_i = \mu_i r_i dt + \sigma_i r_i dW_i \quad (i = 1, 2, \ldots, n),$$
$$dW_i dWj = \rho_{ij} dt,$$

where

$r_i$: the market interest rates selected for the risk factor,

$W_i$: the standard Wiener process,

$\mu_i$: drift rate,

$\sigma_i$: volatility,

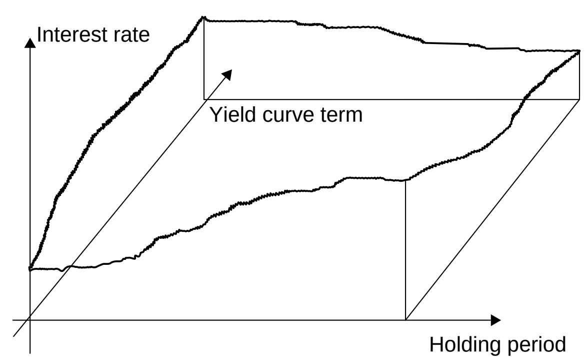
$\rho_{ij}$: correlation.

prime rate changes by more than a specific amount. In addition, the short-term prime rate is applied to short-term loans in many cases. When the loan is rolled over at maturity, the short-term prime rate at that time is applied to the renewed loan. In order to calculate the value and risk of these products, we have to make models of administered rates as well as of the market rate. Market rate-linked products do not owe interest rate risk to the cash flow after interest rate renewal in the future. On the other hand, because administered rate-linked products owe interest rate risk also after the interest rate novation, we have to take care of the cash flows after renewal.

## 4. Definition of interest rate risk

In this paper, as in the case of the VaR of trading accounts, we calculate the interest rate risk of banking accounts as the fluctuation of the discounted PV of the portfolio over a specific holding period. So we must use two time axes, one to represent time during the holding period (risk measurement period), and the other to describe the yield-curve term structure seen at each holding. Figure 1 shows conceptually how one simulation generates a series of yield curves during the holding period.[5] First, we generate the shape of the yield surface (interest rate environment) under the term-structure model. Next, we calculate the portfolio PV at each point during the holding period. To obtain the PV, we take the expected value of all cash flows. In other words, based on the yield curve of every point generated along the holding period axis by the simulation, we have to make a simulation to generate the yield curve along the yield construction period.[6] Using the future path of market rates generated in this way, we construct the future path of administered rates. The development of the yield curve determines future cash flows, which are discounted by the series of generated market rates to obtain the PV in one simulation. The simulation is repeated many times, and the average value is set to be the PV at the given time. We measure the risk by how much the PV changes as time passes.[7]

**Figure 1 Diagram of Yield Curves Simulated during the Evaluation of Interest Rate Risk Using a Term-Structure Model**




5. Technically speaking, when we adopt the Heath-Jarrow-Morton (HJM) model for the stochastic development of the yield curve, we should be careful that the yield curve construction period becomes shorter as time passes.
6. We can also use a lattice method.
7. For exact simulation, we should use the original probability measure of the stochastic process of the yield curve.

**5. Position change**

In considering a trading account, we assume that positions are closed out in the near future, and deal only with the positions which we actually hold. We usually measure risk statically, not taking account of any position changes in the future. In considering a banking account, because of the longer holding time involved, it seems more realistic to assume that some matured or prepaid products are not removed from the portfolio but keep rolling over with interest rate renewal (the rest become products linked to the market rates). We perform a dynamic simulation to calculate risk taking account of changes in portfolio composition.

What kinds of new assets and liabilities are accepted as present positions mature depends on the management strategy of a financial institution. Therefore, it may be appropriate for an institution to assume that its portfolio will develop following its own strategy and to calculate risk accordingly. When we make simulations, we assume several scenarios for position transition so that we can understand how such assumptions affect the size of the calculated interest rate risk. When we deal with position dynamics, the holding period should be considered to be time actually passing. Therefore, as with the rollover rule along the yield construction axis, some of the matured products rollover and the rest switch to the market rate-based products. Of course, the interest rate is renewed. However, from this point of view, because the portfolio whose risk we are going to calculate itself changes, we measure both "the PV change resulting from change of interest rate environment" and "the PV change resulting from position change" at the same time.[8]

## III. Details of the Model Used in the Simulation

In this section, we build the basic framework for calculation of interest rate risk of banking accounts, as analyzed in the last section. We precisely define the measurement of interest rate risk discussed in the last section. Then, we explain how to model each risk factor: the market rate, administered rates, and prepayments.

**A. Definition of Interest Rate Risk**

As described in the first section, in calculating the interest rate risk of banking accounts, we think of an account as a portfolio and analyze how much its current value is likely to decrease during the holding period due to interest rate fluctuations. This point of view means that we capture the risk of banking accounts by applying VaR, a risk methodology for trading accounts. Concretely speaking,

- We discount all cash flows that result from assets and liabilities during the specific period according to the interest rate environment in order to calculate the PV of banking accounts.

8. In this paper, we compare the PVs of the following seven years' cash flows from each time grid in the holding period. This means that we neglect the net cash flow of interest payments and receipts during the holding period. Although this problem relates to the appropriate definition of risk amount, we should analyze the transition of the portfolio value including the accumulated cash flows arising in the holding period in order to evaluate the future value of the existing portfolio.

- We calculate the minimum of the PVs determined by the change of interest rate term structure during a specific holding period.
- We calculate the $\alpha$ percentile point of the distribution of the difference between the above minimum and the initial PV, which we call the $\alpha$ percentile point "risk amount."[9]

**B. The Process of Market Rate Fluctuation**

There is much research, both theoretical and empirical, on time-series models of market interest rates. There are well-known models that include one risk factor and describe the dynamics of instantaneous spot rates, such as the Vasicek (1977) model or the Cox-Ingersoll-Ross (1985) (CIR) model.

These one-factor models have the advantage of simple mathematical form. On the other hand, they depend on the assumption that all interest rates in the term structure are perfectly correlated with each other. They are fine for the purpose of valuing interest rate derivative products with a short maturity, but it is possible that this kind of model does not satisfactorily describe the various fluctuations of yield curves over a long time period, as we attempt to do in this paper.

Therefore, it is necessary for us to devise an interest rate model that is consistent with the fluctuations of the term structure seen in the real world. Taking account of this requirement, we adopt the HJM model as our stochastic model of market rates.[10]

**1. Outline of the HJM model**

Heath, Jarrow, and Morton (1992) focused on the instantaneous forward rate $[f(t, T), 0 \leq t \leq T]$ at future time $T$ seen from time $t$, and developed a multi-factor model as follows:

$$df(t, T) = \mu_f(t, T)dt + \sum_{i=1}^{N}\sigma_{fi}(t, T)dW_i(t), 0 \leq t \leq T, \tag{1}$$

where $\mu_f(t, T)$ is a drift term, $\sigma_{fi}(t, T)$ are volatility terms, and each $[W_i(t), t \geq 0]$ is an independent standard Wiener process. The price of a discount bond is given under the equivalent Martingale measure as

$$P(t, T) = \exp[-\int_t^T f(t, s)ds]. \tag{2}$$

By assuming various forms of volatility function, this flexible model can express various kinds of yield curve fluctuations. Many famous one-factor models fit into this framework. In addition, the HJM scheme has the advantage that it does not require a market price of risk, which is usually essential for pricing derivative securities using no arbitrage arguments. The framework requires only that volatility structures be specified. However, especially in Japan, there has not been enough empirical research into how to set up volatility structures conforming to the actual term structure of interest rates, or how to estimate parameters.

9. Please refer to Appendix 1 for the mathematical expression of the interest rate risk defined in this way.
10. Please refer to Appendix 2 for other typical multi-factor models such as the Brennan-Schwartz model or the Cox-Ingersoll-Ross model.

### 2. The HJM model in this paper

We adopt the two-factor HJM model in order to describe the fluctuation of market rates because (1) the HJM model is superior to other multi-factor models in terms of completeness; (2) there is some degree of freedom about the identification of a volatility function, so that we can set up the model flexibly for specific purposes; and (3) empirical analyses such as the "principal component analysis" of interest rates show that two or three factors can explain more than 90 percent of all fluctuations.[11]

There exist various views about what kind of volatility functions we should take into account. In this paper, after considering the result of empirical analysis, we comply with Heath, Jarrow, and Morton (1992) and Ritchken and Sankarasubramanian (1995). We assume the following function for the volatility term $\sigma_{fi}(t,T)$ of equation (1).

$$\sigma_{f1}(t, T) = \sigma_1 e^{-\kappa(T-t)},\ \sigma_{f2}(t, T) = \sigma_2.\ (\sigma_1, \sigma_2, \kappa\text{: constant parameters}) \qquad (3)$$

The former term implies that the volatility of forward rates decreases exponentially with time, consistent with a change in yield curve slope. The latter term influences the volatility of every forward rate equally, in both the near and the distant future, describing parallel shifts.

In our simulation, we do not directly model the forward rates but model $P(t, T)$, the price of a zero-coupon bond at time $t$ with maturity $T$, as follows.[12]

$$dP(t, T) = \left[-\frac{\partial \log P(t, T)}{\partial T}\right]_{T=t} P(t, T)dt + \sum_{i=1}^{2} \sigma_{pi}(t, T)P(t, T)dW_i(t). \quad (4)$$

The volatility terms for the zero-coupon bond $\sigma_{pi}(t, T)$ are derived as

$$\sigma_{p1}(t, T) = -\frac{\sigma_1}{\kappa}[1 - e^{-\kappa(T-t)}],\ \sigma_{p2}(t, T) = -\sigma_2(T - t). \qquad (5)$$

### C. Stochastic Process Describing Administered Rates

In order to calculate the interest rate risk of banking accounts, it is necessary to discuss how to model not only market rates but also administered rates such as the short-term prime rate and the long-term prime rate. It is true that we have said less about modeling administered rates than about modeling the market rate. Our reasons are that (1) administered rates are used only in limited cases; and (2) the administered rates are to some extent linked to market rates, so we have not considered them as independent risk factors. However, as we discuss below, it is possible that risks due to changes in administered rates are bigger than market risk. If so, we cannot neglect this source of risk.

11. For example, please refer to Litterman and Scheinkman (1991).
12. Please refer to Heath, Jarrow, and Morton (1992), which shows how to transform the forward rate stochastic process described by equation (1) to the zero-coupon bond stochastic process described by equation (4).

**1. Short-term prime rate**

The short-term prime rate is an average of the funding rates of liquid deposits or from the interbank market or the open market, and the expense rate. However, it does not always respond immediately to changes in market rates. In this paper, we construct a model of the procedure for revising the short-term prime rate, as follows.[13]

- We generate the three-month interest rate in accordance with the market rate model (the HJM model).
- When the three-month interest rate changes by more than 0.25 percent since the last time the short-term prime rate was revised, we decide to revise the short-term prime rate.
- However, the revision does not take place immediately. It is revised after some time lag.
- The change in the short-term prime rate is equal to the difference between the three-month interest rate at the revision time (i.e., after the time lag) and the three-month interest rate at the previous revision time, and is always a multiple of 0.125 percent.

If we make such a model, we should consider what risk premium should be paid for the new risk factor (uncertainty due to the randomness of the time lag) introduced into the administered rate.

**2. Long-term prime rate**

The coupon rate of financial debentures is revised when the difference between the current coupon rate and the yield of financial debentures in the secondary market becomes more than 0.2 percent. The minimum unit of revision is 0.1 percent. The coupon rate is checked every month in this way. The long-term prime rate is determined as the coupon rate plus 0.9 percent. In this paper, we model this as follows.

- We assume that the spread between an index market rate (five-year swap rate) and the secondary market yield of a financial debenture obeys a normal distribution. We estimate its average and variance from historical data.
- We generate the five-year market rate in accordance with the stochastic model of market interest rates (i.e., the HJM model).
- We calculate the secondary market yield of financial debentures by adding a spread (which obeys a normal distribution, generated as above) to the five-year market rate.
- When the difference between the calculated secondary market yield and the current coupon rate of the financial debenture (the current long-term prime rate minus 0.9 percent) becomes more than 0.2 percent, we revise the coupon rate by a minimum unit of 0.1 percent.
- When the coupon rate is revised, we revise the long-term prime rate to the new coupon rate plus 0.9 percent.

13. We can devise another model of the short-term prime rate as follows. First, we divide the level taken by a market interest rate into several ranges. Each range has a corresponding short-term prime rate. The time lag between the time at which the market rate moves from one range to another and the time at which the prime rate actually changes to the corresponding level follows a probability distribution—for example, an exponential distribution. (Please refer to Appendix 3.)

**D. Evaluation of Prepayment Value**

In the field of mortgage securities, there has been discussion of how to make a model of prepayment that includes option value. On the other hand, because there are few publicly available data about prepayment behavior in time deposits, the discussion has been less active than in the case of mortgage securities. In this paper, we review the research on mortgage securities, and develop a prepayment model for banking accounts. The candidates for consideration are rational models, prepayment function models, discrete models, etc.[14]

In the case of banking accounts, we should consider prepayments of loans, especially housing loans, and time deposits. We can directly adapt a model of mortgage securities to housing loans. In addition, except for the difference between assets and liabilities and the problem of parameters, we can apply the analysis of mortgage securities to time deposits. For example, the rational prepayment model assumes that investors compare current applied interest rates with interest rates on other products, take account of future changes of interest rates before maturity, and exchange their assets or liabilities for other, more favorable products. In this paper, for compatibility with the Monte Carlo simulation we perform later, we do not adopt the rational model but instead use the prepayment function model.

**1. Overview of prepayment function model**

In the prepayment function model, we assume a function that expresses the prepayment and statistically estimate parameters. In this kind of model, we assume that when mortgagors think about prepayment, they take account of interest rates and the economic situation, but only that of the present and the past, not of the future, as is assumed in the rational prepayment model. Therefore, we can use this model for estimating value in a Monte Carlo simulation.

For example, Schwartz and Torous (1989) suggest the following function model. Let $v$ be a vector of explanatory variables that affect prepayment, and $\beta$ be a vector of coefficients, then the prepayment function $\pi(t)$ can be described as

$$\pi(t) = \pi_0(t)\exp(\beta' v), \tag{6}$$

where $\pi_0(t)$ is the baseline function that expresses the prepayment rate, independent of the change in interest rate environment, and the prime symbol means the transpose of a vector. We assume the following log-logistic function for the baseline function.

$$\pi_0(t) = \frac{\gamma p(\gamma t)^{p-1}}{1 + (\gamma t)^p}. \tag{7}$$

As one of the explanatory variables, for example, we use the difference between the current applied interest rate and the interest rate of other candidate products. In addition, we can use a dummy variable in order to express the seasonal fluctuation.

14. Please refer to Appendix 4 for the details of each model.

Finally, because the attributes of investors may significantly affect the function's form and its parameters, we can make a separate model for each attribute group, just as we can do in the rational prepayment model.

**2. Prepayment function model in this paper**

Under the prepayment function model in this paper, following Schwartz and Torous (1989), we use the following as explanatory variables.

- The difference between interest rates
  (the interest rate that has already been applied to the product) minus (the interest rate that will be applied to it if we make a new contract for it).
- The third power of the above difference
  This represents the fact that the larger the difference becomes, the faster prepayment activity accelerates.
- The ratio of the remaining amount
  (the remaining amount when we take account of prepayment) divided by (the remaining amount when we assume no prepayment).

However, because we have no data to help us evaluate these parameters, we set the value of the parameters to several levels and estimate the magnitude of the effect of prepayment.

When examining products that include the option to prepay, as we discussed in Section II.B.4, we should strictly calculate their value (including the time value of the option) by a Monte Carlo or lattice method.[15] However, if we attempt to value the option at the same time as we simulate the fluctuation of portfolio value with a Monte Carlo method, the calculation load becomes too heavy. In this paper, we do not carry out a so-called "simulation on the simulation." We first generate yield curves for each time. Then, the future prepayment schedule is determined based on the forward rates implied by the yield curve and the assumed prepayment function model. Finally, we discount the fixed future cash flows to obtain the PV of the product.

## IV. Detail of the Method and Result of the Simulation

In this section, taking account of the above theoretical considerations, we carry out a simulation using an imaginary portfolio, and explore how the features of banking accounts actually affect the PV and risk due to interest rates.

### A. Composition of an Imaginary Portfolio

In this subsection, we explain the content of the basic form of the imaginary portfolio that we used in our simulation (see Appendix Table 2). The numbers in the table show the share of each asset and liability in total assets and liabilities. When we made the imaginary portfolio, we referred to the balance sheets of several city banks as of March 1995. We assumed that both total assets and total liabilities were ¥30 trillion. In the actual simulation, we varied the composition of the portfolio so that we could explore how PV and risk were affected.

15. If we adopt the rational model, we must use the lattice method, just as we do when we calculate prices of American-type options.

First, the portfolio has products divided into three categories determined by the method for setting the interest rate (linking the products to the market rate, the short-term prime rate, or the long-term prime rate). The products linked to the market rate are further divided into two categories: one has a constant spread to the market rate (perfect linkage), the other has a weaker correlation with the market rate (imperfect linkage). There are four categories in total.

Next, in order to specify which interest rate is applied to the initial term and to each term after rollovers, we assigned the composition ratio of interest rate renewal time-span. In such cases, we should classify not on the basis of the product's remaining term but of the term of the applied rate.

In addition, maturity information is also important. For example, we compare a three-year floating rate loan with interest rate reset every six months to a six-month fixed rate loan. After six months have passed, the interest rate on the floating rate note should be reset. However, after the same six months, at maturity of the fixed rate loan, there is a possibility that the product is replaced by other products. In Appendix Table 2, we classify the term of a product according to the term of the applied interest rate. However, when the term of the applied interest rate and final maturity differ from each other (i.e., in the case that interest payments are made before maturity), we show the final maturity in the remarks column.

In addition, to be strictly accurate, the PVs of the differences between the three kinds of index interest rates described above and the actual applied interest rates fluctuate as the yield curves change. However, for simplicity, we assume no spread irrespective of the market rate, the short-term prime rate, and the long-term prime rate. We also neglect products in which the customer can choose which index rate to use.

**B. Simulation Method**

**1. Setting the yield curve span and the length of holding period**

In this paper, we generate yield curves during the holding period by means of the HJM model, with the initial yield curve derived from the London interbank offered rate (LIBOR) and swap rates. In Japan, only swap rates of fewer than 10 years have enough liquidity to be used as index rates. Therefore, the span of the yield curve is limited to 10 years. In addition, if we assume the maximum holding period is three years,[16] the maximum interest rate term that we can use at the end of the holding period is seven years. When we calculate the PV, we discount the cash flows that occur within seven years of each point of the holding period.[17]

**2. Assumption about position change**

First, among financial products that have no explicit maturity, there are products with no interest rates, such as current deposits and cash. If we assume that the balances of these products do not change in the future, which means that there is

16. It is a provisional assumption that the holding period is not longer than three years, so that the assumption is not based on the specific theory.

17. Because we recognize the PV of a portfolio as the value of the discounted cash flows that arise during the "limited" yield curve span (seven years in this paper), the PV is not exact. Moreover, in this simulation, we assume the products are rolled over only in the case that the renewed maturity after rollover is not beyond seven years from the current grid point. Regarding the products without specific maturity—for example, a liquid deposit—we always evaluate just the cash flows occurring within seven years.

no fluctuation of PV (that is, risk) associated with interest rate fluctuations, we can dispense with these products in this simulation. Second, as for financial products with an interest rate, such as liquid deposits (partially linked to the market rate) and overdrafts (linked to the short-term prime rate), we assume for convenience that the interest rate renewal term is one month and that the balance does not change.

A rollover ratio of 100 percent means that the amount that comes to maturity is fully rolled over to the same product. We vary the rollover ratio up to 100 percent and observe the effect on the PV and risk.

As for prepayment, if it occurs along the axis of the yield curve term, we have to invest or fund with the simulated market rate applying immediately afterward. For convenience, prepayment does not occur along the axis of the holding period.

### 3. Configuration of parameters

#### *a)* Parameters of the HJM model

As for the standard parameters of the HJM model, we assumed that the parameters $\sigma_1$ and $\kappa$, which determine the volatility term relating to the difference between the long-term and short-term interest rates, were $4.56 \times 10^{-5}$ and $6.32 \times 10^{-2}$, and that the volatility term for parallel shifts, $\sigma_2$, which affects the whole term until maturity, was $4.04 \times 10^{-5}$.[18]

#### *b)* Parameters that prescribe the stochastic model of the prime rate

#### *1)* Short-term prime rate

We assume that the time lag between the fluctuation of the short-term prime rate and the fluctuation of the market rate obeys an exponential distribution.[19] When we estimated the parameter $\lambda$ of the exponential distribution, we used the data of the end day of every month from January 1989 (when the "new short-term prime rate" was introduced) to February 1996. For each of the 23 times of short-term prime rate renovation, we specified how many months of time lag the short-term prime rate was revised afterward (see the table below). We can calculate the average time lag of 1.065 with this histogram and obtain the parameter $\lambda$ of 0.939, which is equal to the reciprocal of the expected value of the exponential distribution.

| Time lag | Frequency |
|---|---|
| Zero months | 14 |
| One month | 6 |
| Two months | 2 |
| Three months | 1 |
| Total | 23 |

#### *2)* Long-term prime rate

The secondary market rate of financial debentures consists of the five-year market rate (five-year swap rate) and the spread, which obeys a normal distribution with mean $\mu$ and standard deviation $\sigma$. In order to determine $\mu$ and $\sigma$, we used the data of the spread

18. We used the parameters estimated from weekly data for government bonds from April 1988 to July 1995. We are grateful to Katsuya Komoribayashi of the Dai-Ichi Mutual Life Insurance Company, who gave us much assistance with this estimation.
19. The assumption that the random variable $X$ which represents the time lag follows an exponential distribution means that the probability density function of $X$ [$f(x)$] can be described as follows with parameter $\lambda$.

$$f(x) = \lambda \exp(-\lambda x), \text{ for } x \geq 0.$$

between the secondary market rate of financial debentures and the five-year swap rate from October 1987 to February 1996, so that we acquired $\mu$ of –0.360 and $\sigma$ of 0.161.

***c)* Parameters of prepayment function**

We adopted in equation (6) as explanatory variables the difference between interest rates, the third power of the difference, and the ratio of the remaining amount. Because we have no data to estimate the parameters $\beta_1$, $\beta_2$, and $\beta_3$, we quote the result of U.S. mortgage market analysis by Schwartz and Torous (1989).

$$\beta_1 = 0.39678,\ \beta_2 = 0.00356,\ \beta_3 = 3.74351. \tag{8}$$

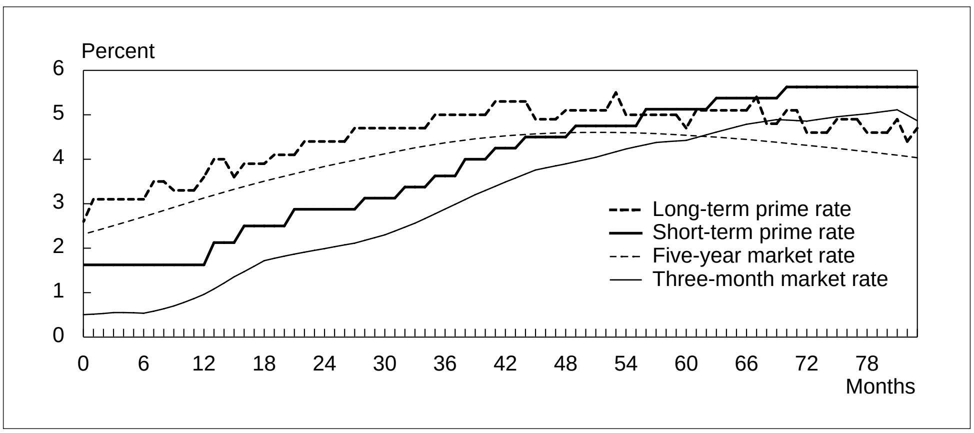
Parameters $\beta_1$ and $\beta_2$ show how much the prepayment rate fluctuates relative to the baseline function as the interest rate difference fluctuates. In the above case, if there is 1 percent of interest rate difference, the prepayment rate increases 1.5 times as before. Parameter $\beta_3$ shows that as the ratio of remaining amount decreases, prepayment becomes unlikely to occur. For example, if 10 percent of the balance has already been prepaid, the prepayment ratio decreases as much as 30 percent compared to the initial level. We set up the parameters of baseline functions according to the maturity.[20]

## C. Simulation Result

### 1. Trial calculation of yield curve and risk amount

Here, we confirm the yield curve form of the market rate and the administered rate used in this simulation. For example, the yield curve of the forward rate that we use in order to evaluate the PV of the initial time (PV0) is shown in Figure 2. Although

**Figure 2 Implied Market Interest Rates on the Initial Time (End of 1995) and Future Paths of Administered Rates**




20. We set up the value of parameters $p$ and $\gamma$ as the following table. With these parameters, the simulation shows the unrealistic result that almost all the product should be prepaid before maturity. Therefore, we introduce another parameter, $a$, so that we use $\tilde{\pi}(t) = a\pi(t)$ instead of $\pi$ as our prepayment function.

| Maturity | One year | Three years | Five years | Seven years |
|---|---|---|---|---|
| $p$ | 3 | 3 | 3 | 3 |
| $\gamma$ | 0.30 | 0.10 | 0.05 | 0.03 |
| $a$ | 0.05 | 0.10 | 0.15 | 0.20 |

in Section II.B.4 we described that it is necessary to carry out a "simulation on the simulation" in order to calculate the precise PV, we do not carry it out because it requires a huge calculation load. Therefore, we regard the forward interest rate implied by the yield curves at each time that is generated along the direction of the holding period as the future path (along the construction term of the yield curve) of the market rate. Similarly, we generate the future path of administered rate by including factors such as the time lag.

Next, we confirm what kind of yield curves are formed at the end of the holding period (three years later) if we generate many paths along the direction of the holding period. Figure 3 shows the result when we generated 10 paths.[21]

**Figure 3 Yield Curve Movement Three Years Later (10 Paths)**

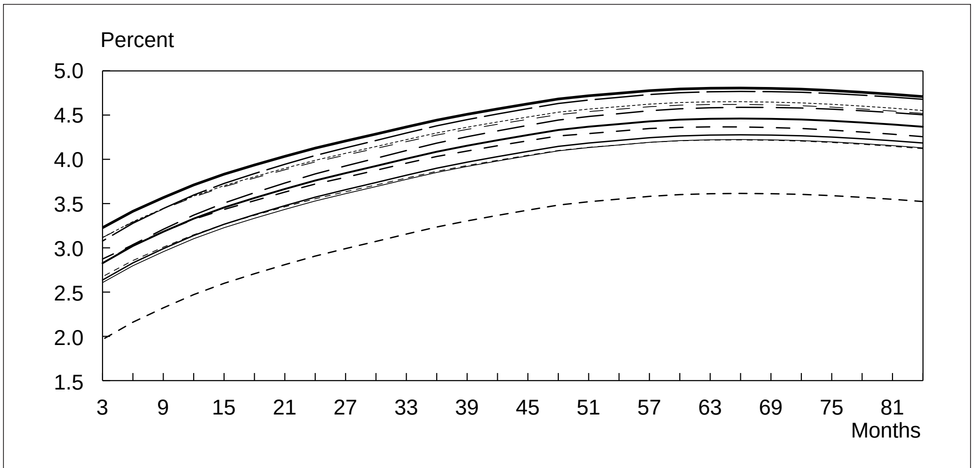
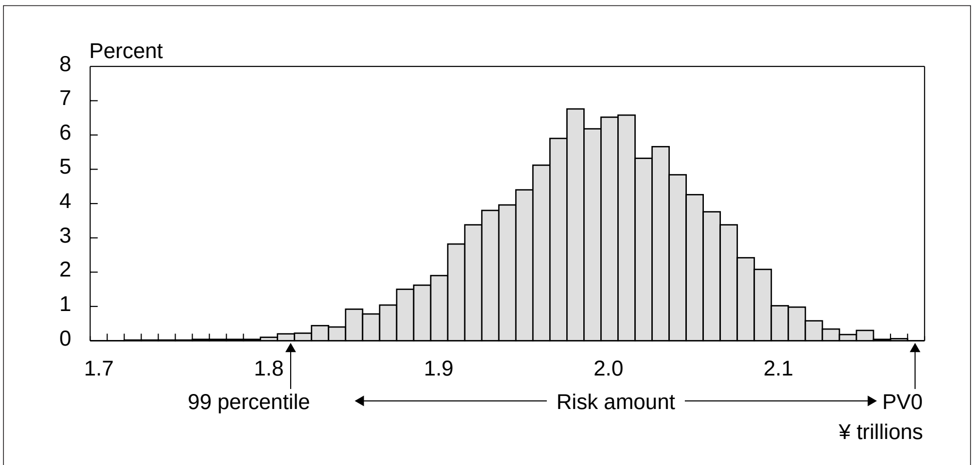



**Figure 4 Distribution of minPV (Basic Portfolio, Perfect Rollover, No Prepayment)**




21. Figure 3 shows a yield curve derived from the HJM model with the initial parameters. In this figure, we cannot clearly recognize the transition of the yield curve slope. However, changing the parameters of the HJM model, we can generate various shapes of yield curves as shown in Appendix 5.

In our simulation, we distribute the minimum PVs (minPVs) of 36 timings on each path, and define the risk amount as the difference between the 99 percentile point and the PV0. Figure 4 shows how the minPVs are distributed. We carried out 5,000 simulations, using a basic portfolio and assuming perfect rollover, and drew a histogram of the result. The distribution has an almost smooth shape with the limitation on the right-hand side, PV0.

**2. The effect on the PV of individual products and risk according to the features of banking accounts**

First, we carry out a simulation with the individual products in order to capture the effect of administered rates and prepayment, which we consider to be the representative risk factors of banking accounts. In the simulations below, we always carry out 500 simulations (500 paths). Because one simulation generates 36 PVs, in fact, we calculate the risk amount based on 18,000 PVs.

***a)* The effect of the administered rate on the risk amount**

We assume the three kinds of interest rates (the market rate, the short-term prime rate, and the long-term prime rate) to be the loan rate, and verify the effect that the administered rate has on the risk amount (Table 1). The object product is ¥5 trillion of a six-month loan. We assume that the balance will be constant and will be rolled over for seven years.[22]

**Table 1 Effect of Administered Rates (Risk of Individual Products)**

¥ billions

| | Market rate-linked | Short-term prime rate-linked | Long-term prime rate-linked |
|---|---|---|---|
| One-year holding period | 1.3 | 88.5 | 160.5 |
| Three-year holding period | 1.4 | 97.0 | 371.3 |

The risk of short-term prime rate-linked and long-term prime rate-linked products is larger than that of market rate-linked products. Because the transition of the short-term prime rate and the long-term prime rate is a kind of stepwise function, it does not link perfectly to the transition of the market rate so that it leads to an increase in risk. In addition, as a factor specific to the short-term prime rate, it has the effect on the risk that the short-term prime rate is revised with some time lag as the market rate fluctuates. On the other hand, as for the long-term prime rate-linked products, because the basis between the secondary market rate of financial debentures and the market rate (five-year swap rate) fluctuates, the risk is larger than that of market rate-linked products. In addition, in our simulation we discount the interest cash flow of the long-term prime rate, which is a kind of five-year interest rate, with the six-month interest rate. In this case, the initial yield curve becomes upward-sloping. Therefore, the spread effect diminishes as time passes so that the PV decreases. For this reason, it seems that the risk of a three-year holding period is much larger than that of a one-year holding period.

22. In the following section, we show the risk amount of just a few kinds of holding periods because of the limited space of this paper. However, actually, we calculated the risk amounts of holding periods up to three years with one-month intervals, i.e., 36 holding periods (please refer to Appendix Table 3).

***b)* The effect of prepayment on the PV**

With a five-year housing loan (¥1.5 trillion, market rate-linked [2.8 percent fixed]), we verify the effect that prepayment has on the PV0 and on risk (Table 2). As Table 2 shows, when we compare the prepayment by baseline function (independent of the interest rate difference)[23] with the one including the customers' action based on the interest rate difference, we can verify that the PV decreases (¥8.5 billion ⇒ ¥4.7 billion) because of the option (etc.) held by customers.

**Table 2 Effect of Prepayment (Five-Year Housing Loan)**

¥ billions

| | Prepayment | | | |
|---|---|---|---|---|
| | Baseline only | Plus interest rate factor | All factors | None |
| PV0 | 8.5 | 4.7 | 3.4 | 0 |
| One-year holding period risk | 43.3 | 46.1 | 46.0 | 47.5 |
| Three-year holding period risk | 64.1 | 65.7 | 64.7 | 64.5 |

**3. The effect that the features of banking accounts have on the PV of the portfolio and the risk amount**

In this subsection, we proceed further, calculating the PV0 and risk of a portfolio that consists of various individual products.

***a)* The effect of the administered rate on the risk amount**

In order to capture clearly the effect of the administered rate, just for this simulation we increase the linkage ratio to the market rate of liquid deposits in the basic portfolio from 20 percent to 100 percent. In this case, the risk of the basic portfolio is ¥666.6 billion under the conditions of no prepayment and a three-year holding period (Table 3). In addition, by exchanging all the administered rate-linked products into market rate-linked products (see Appendix Table 4), the risk decreases greatly, so that the risk of a six-month holding period is ¥24.7 billion and that of a three-year holding period is ¥133.6 billion. Therefore, we can say that administered rate-linked products have much influence on the risk not only of individual products but also of portfolios.

**Table 3 Effect of the Products Linked to Administered Rate**

¥ billions

| Portfolio | Basic portfolio | Market rate-linked products only |
|---|---|---|
| Six-month holding period | 338.0 | 24.7 |
| Three-year holding period | 666.6 | 133.6 |

***b)* The effect of prepayment**

When we take account of prepayment in the basic portfolio only by baseline function, the PV0 is ¥2,215.7 billion (Table 4). Moreover, if we include the prepayment based on the interest rate difference, the PV0 decreases by as much as ¥5.8 billion to ¥2,209.9 billion, and the risk of a three-year holding period is ¥326.1 billion. It is true

23. In this case, the prepayment by the baseline function is a less-favored form of prepayment for customers because of the normal slope yield curve (i.e., the longer-term interest rate is larger than the shorter-term interest rate). Therefore, the PV with prepayment of the baseline function is larger than that without prepayment (0 ⇒ ¥8.5 billion).

that the level of risk itself decreases compared with that without the factor of interest rate difference (¥359.1 billion). The more a position is reduced by prepayment, the less the probability of PV fluctuation becomes. In this sense, prepayment has the effect of decreasing the risk. On the other hand, the possibility of prepayment immediately changes the PV. Consequently, prepayment has the effect of increasing the risk. It seems that the former effect surpasses the latter effect, so that the risk decreases in this case.

**Table 4  Effect of Prepayment (Three-Year Holding Period)**

¥ billions

| PV0 (baseline) | PV0 (plus interest rate factor) | Risk amount |
|---|---|---|
| 2,215.7 | 2,209.9 | 326.1 |

## 4. Comparison with the usual VaR

In the most popular VaR calculation method of trading account so far, we set up several term interest rates as risk factors, assume the multivariate lognormal process among them, and then calculate VaR using a variance-covariance matrix. Under this method, as the holding period $t$ become longer, the risk simply increases $\sqrt{t}$ times. On the other hand, under the HJM method adopted in this simulation, we actually generate the fluctuation of the yield curve for three years. Comparing the risk from both methods,[24] using the portfolio that consists of market rate-linked products (see Appendix Table 4), we obtained the result shown in Table 5.

**Table 5  Comparison with the Variance-Covariance Method**

¥ billions

| Method | Variance-covariance | HJM | HJM |
|---|---|---|---|
| Rollover | — | None | Perfect |
| Prepayment | None | None | None |
| One-month holding period | 29.1 | 3.7 | 3.7 |
| Six-month holding period | 72.6 | 24.7 | 24.7 |
| Three-year holding period | 178.0 | 120.9 | 133.6 |

## 5. The effect of model parameters on risk amount

### *a)* Parameters of the HJM model

Here, we study the sensitivity of risk to the parameters of the HJM model in this paper. Using the basic portfolio, we increase the amount of $\sigma_1$, $\kappa$, and $\sigma_2$ as much as five times so that we can compare the resulting risk with that before the change is made. In order to facilitate comparison, we assume no prepayment and assume perfect rollover. While $\sigma_1$ affects mainly the short-term interest rate, $\sigma_2$ affects interest rates of all terms (Table 6). In other words, because $\sigma_1$ and $\sigma_2$ express the fluctuation of the yield curve slope and the fluctuation of parallel shift for each, as the amount of these increases, the risk increases. On the other hand, $\kappa$ expresses the extent to which the influence of $\sigma_1$ on the far-future forward rate is smaller than that

24. In order to analyze the difference of Monte Carlo simulation results derived by varying the stochastic process of the yield curve, we should compare the case with the multivariate lognormal distribution process of yield-curve fluctuation and the case with the HJM model. However, in this paper, we compare the HJM model with the variance-covariance method, which is generally the most popular.

on the near-future forward rate. Therefore, as the $\kappa$ increases, the far-future forward rate tends not to fluctuate, so that the risk decreases.

**Table 6 Effect of Parameter Change of the HJM Model**

¥ billions

| | Standard | $\sigma_1$ | $\kappa$ | $\sigma_2$ |
|---|---|---|---|---|
| Six-month holding period | 256.0 | 332.4 | 255.9 | 351.7 |
| One-year holding period | 326.9 | 352.3 | 303.7 | 459.7 |
| Three-year holding period | 359.1 | 409.3 | 342.7 | 650.5 |

### *b)* Parameters of prepayment

When we take account of interest rate differences as described above, prepayment decreases the PV0, which is itself the basis of risk. Then, using a basic portfolio assuming perfect rollover, we increase the parameters $\beta_1$ and $\beta_2$ twice so that we get the result that the PV0 decreases by a further ¥9.3 billion, as Table 7 shows. (In this comparison, we take account not of the ratio of the remaining amount but of the baseline function.)

**Table 7 Effect of Parameter Change of Prepayment on PV**

¥ billions

| | $\beta_1 = 0$<br>$\beta_2 = 0$ | $\beta_1 = 0.39678$<br>$\beta_2 = 0.00356$ | $\beta_1 = 0.39678 \times 2$<br>$\beta_2 = 0.00356 \times 2$ |
|---|---|---|---|
| PV0 | 2,215.8 | 2,209.9 | 2,200.6 |

## V. Conclusion: Implication for Risk Management

In this paper, we have tried to suggest the basic framework with which we calculate the interest rate risk of banking accounts. By using this framework to calculate the risk of an imaginary portfolio, we could capture an approximation of the interest rate risk of banking accounts, and the relation between the administered rate and the risk amount or prepayment and the risk amount, etc., which are features of banking accounts. It is true that the simulation result in this paper is no more than a calculation example that is derived under various assumptions. However, the result proves that the interest rate risk of a banking account is affected by various factors.

It is true that the models of market rates, prepayment, and administered rates in this paper are no more than examples. There can be various types of models in the case of risk management of actual financial institutions. In addition, in this paper, we have made many assumptions in order to simplify the simulation as much as possible. Nevertheless, we faced the problem that the calculation workload becomes too heavy after the process of including the various features of banking accounts. In actual banking operations, we should take account of the trade-off between the accuracy of risk measurement and the calculation workload in order to choose the most suitable model. When doing this, we should first carefully consider each building block such as the administered rate factor or the prepayment factor. Then, we should continue to simplify each building block.

We neglect the following points from the standpoint of simplification in this model: (1) measuring the risk of non-maturity products; (2) including the interest rate spread and interest rate cash flow when we calculate (the discounted) cash flow; and (3) integrated management of the interest rate risk of banking accounts together with that of trading accounts or credit risk. These are important problems that we should study from now on. In addition, in order to estimate the prepayment function realistically, it is essential to include data on prepayments or cancellations. It is very important to arrange the transaction data from the viewpoint of actual risk management business. We hope that more theoretical and quantitative studies are cultivated in this field, in which little work has been conducted in Japan.

## APPENDICES

In these appendices, we show (1) measure of risk; (2) major multi-factor term-structure models; (3) modeling of administered rates; and (4) mathematical expression of prepayment models.

## APPENDIX 1: MEASURE OF RISK

Let $[f(s, u), 0 \leq s \leq u]$ be the instantaneous forward rate at future time $u$ evaluated at time $s$. And let $CF_i[s, u, f(s, u)]$ be the future cash flow at time $u$ foreseen at time $s$ of each asset or liability $i$. We set the risk evaluation period as $[t, \tau]$. We consider sample paths ($\omega$) of change of the term structure of interest rates $\{f(s, u), s \in [t, \tau], u \in [s, s+T]\}$ and let $R(t, \tau, \omega)$ be the set of term structures on the paths $\{f(s, u), s \in [t, \tau]\}$. Then, given the term structure of interest rate $f(s, u) \in R(t, \tau, \omega)$ at each time $s$,[25] $PV_f(s, T)$, the PV of the banking account evaluated in the period $u \in [s, s+T]$, can be obtained as follows with the information available at time $s$ $\mathcal{F}_s$,

$$PV_f(s, T) = \sum \tilde{E}\{\int_s^{s+T} \exp[-\int_s^u r(u)du]\, CF_i[s, u, f(s, u)]du \,|\, \mathcal{F}_s\}, \qquad \text{(A.1)}$$

$$t \leq s \leq u \leq s+T,\ s \in [t, \tau],$$

$$r(t) = f(t, t),$$

where $\tilde{E}[\bullet]$ means the expected value under the risk-neutral probability measure (the equivalent Martingale measure). Then, we define $VaR_\alpha(t)$ as

$$\Pr\{\min_{f \in R(t, \tau, \omega)} [PV_f(s, T) - PV_0(t, T)] \leq -VaR_\alpha(t)\} = \alpha, \qquad \text{(A.2)}$$

the potential loss, with probability $\alpha$, due to changes in the whole term structure of interest rates at the time $t$.

## APPENDIX 2: MODELING OF MARKET RATES

### A. The Brennan-Schwartz Model

The model of Brennan and Schwartz (1979) is a representative two-factor model. The model assumes stochastic processes for two interest rates: the instantaneous spot rate and the long-term bond rate (consol rate). Because these two factors are directly connected with short-term and long-term interest rates, it is easy to understand the model intuitively. However, when it comes to pricing interest rate derivative securities, the model can only derive a partial differential equation that should be satisfied given a market price of risk. Therefore, we must solve this partial differential equation numerically in order to calculate a price.

Let $[r(t), t \geq 0]$ be the instantaneous spot rate, $[l(t), t \geq 0]$ be the interest rate of a long-term bond (consol bond), and $[W_1(t), t \geq 0]$, $[W_2(t), t \geq 0]$ be two standard

25. As for products with fixed cash flow, we can calculate the discounted value using this term structure of interest rates. On the other hand, as for products with option value, we calculate the expected value by generating some paths on the basis of this initial term structure.

Wiener processes defined on some probability space $(\Omega, F, P)$. The Brennan and Schwartz (1979) model assumes that the interest rates follow the processes below.

$$\begin{aligned} dr(t) &= \beta_1(r, l, t)dt + \eta_1(r, l, t)dW_1(t), \\ dl(t) &= \beta_2(r, l, t)dt + \eta_2(r, l, t)dW_2(t), \\ dW_1(t)dW_2(t) &= \rho dt, \end{aligned} \tag{A.3}$$

where $\beta_i(\bullet)$, $\eta_i(\bullet)$, $i = 1, 2$ represent functions that depend on the short-term interest rate, the long-term interest rate, and time. $\rho$ is a constant number that represents the correlation between the two Wiener processes. Under the assumption that there is an active market for consol bonds, a discussion about non-arbitrage conditions determines that the market price of risk for long-term interest rate $[\lambda_2(r, l, t), t \geq 0]$ must satisfy the following.

$$\lambda_2(r, l, t) = -\eta_2(r, l, t)/l + [\beta_2(r, l, t) - l^2 + rl]/\eta_2(r, l, t). \tag{A.4}$$

In addition, if we assume

$$\begin{aligned} \eta_1(r, l, t) &= r\sigma_1, \\ \eta_2(r, l, t) &= l\sigma_2, \\ \beta_1(r, l, t) &= r[\alpha \ln(\frac{l}{pr}) + \frac{1}{2}\sigma_1^2], \end{aligned} \tag{A.5}$$

given the market price of risk for short-term interest rate $[\lambda_1(r, l, t), t \geq 0]$, the price of a discount bond $B(r, l, t)$ satisfies the following partial differential equation.

$$\begin{aligned} &\frac{1}{2}B_{rr}r^2\sigma_1^2 + B_{rl}rl\rho\sigma_1\sigma_2 + \frac{1}{2}B_l l^2\sigma_2^2 + B_r r[\alpha\ln(l/pr) + \frac{1}{2}\sigma_2^2 - \lambda_1\sigma_1] \\ &\quad + B_l l(\sigma_2^2 + l - r) - B_t - Br = 0, \\ &\qquad B(r, l, 0) = 1, \end{aligned} \tag{A.6}$$

where $B_x$ represents the first partial derivative of $B(r, l, t)$ with respect to $x$ and $B_{xx}$ represents the second partial derivative.

**B. The Multi-Factor CIR Model**

The multi-factor model within the framework of Cox, Ingersoll, and Ross (1985) is also well known. In this model, we assume that we can describe the instantaneous spot rate $[r(t), t \geq 0]$, as the sum of $k$ state variables $[y_j(t), t \geq 0, i = 1, \ldots, K]$ and that each state variable fluctuates according to a square root process with mean reversion property. In other words, we can describe $[r(t), t \geq 0]$ as follows,

$$r(t) = \sum_{j=1}^{K} y_j(t), \tag{A.7}$$

and we can assume that the stochastic process for each state variable follows the square root process below.

$$dy_j(t) = \kappa_j[\theta_j - y_j(t)]dt + \sigma_j\sqrt{y_j(t)}dW_j(t), \;\; j = 1, \ldots, K, \tag{A.8}$$

where each $[W_j(t),\, t \geq 0]$ is an independent standard Wiener process. It is known that in this model, we can calculate the price of a discount bond, $P(t, \tau)$, at time $t$ with maturity $\tau$ as a natural extension of the pricing formula of the one-factor CIR model.

$$P(t, \tau) = \prod_{j=1}^{K} A_j(t, \tau) \exp\,[-\sum_{j=1}^{K} B_j(t, \tau) y_j(t)], \tag{A.9}$$

$$A_j(t, \tau) = \left[\frac{2\gamma_j \exp\,[1/2(\kappa_j + \lambda_j + \gamma_j)(\tau - t)]}{2\gamma_j + (\kappa_j + \lambda_j + \gamma_j)(e^{\gamma_j(\tau - t)} - 1)}\right]^{\frac{2\kappa_j\theta_j}{\sigma_j^2}}, \tag{A.10}$$

$$B_j(t, \tau) = \frac{2(e^{\gamma_j(\tau - t)} - 1)}{2\gamma_j + (\kappa_j + \lambda_j + \gamma_j)(e^{\gamma_j(\tau - t)} - 1)}, \tag{A.11}$$

$$\gamma_j = \sqrt{(\kappa_j + \lambda_j)^2 + 2\sigma_j^2}, \tag{A.12}$$

where $\lambda_j y_j$ is the risk premium for each state variable, and $\lambda_j$ is constant.

Longstaff and Schwartz (1992) paid attention to the point that, within the two-factor CIR framework, we can describe both the instantaneous spot rate $[r(t),\, t \geq 0]$ and its volatility $[V(t),\, t \geq 0]$ by linear equations in the same state variables. They suggested estimation of the parameters with a generalized autoregressive conditional heteroskedasticity (GARCH) model. Formally, when state variables are assumed to satisfy the following partial differential equation,

$$dy_j(t) = \kappa_j[\theta_j - y_j(t)]dt + \sigma_j\sqrt{y_j(t)}dW_j(t), \;\; j = 1, 2, \tag{A.13}$$

we can describe the instantaneous spot rate $[r(t),\, t \geq 0]$ and its volatility $[V(t),\, t \geq 0]$ as follows:

$$\begin{aligned} r(t) &= \alpha y_1(t) + \beta y_2(t), \\ V^2(t) &= \alpha^2 y_1(t) + \beta^2 y_2(t). \end{aligned} \tag{A.14}$$

Alternatively, Chen and Scott (1995) suggest, assuming $K$ is equal to two or three, the estimation of parameters with a Kalman filter, and discuss its features. They confirm that there are two principal variables which are estimated from actual interest rate changes: (1) a factor that affects the whole term structure of interest rates equally; and (2) a factor that mainly affects the short-term interest rate and to a lesser extent affects the long-term interest rate.

The disadvantage of this model is that the state variables are not directly observable and are hard to interpret, so an intuitively clear picture of the changes in interest rates is not available. Moreover, in practice, it is difficult to estimate the parameters.

**C. The HJM Model**

We assume the following process describing the instantaneous forward rate $[f(t, T), 0 \leq t \leq T]$ at future time $T$ viewed from time $t$,

$$df(t, T) = \alpha(t, T)dt + \sum_{n=1}^{N} \sigma_n(t, T)dW_n(t), \ 0 \leq t \leq T, \tag{A.15}$$

where each $[W_n(t), t \geq 0]$ is an independent standard Wiener process, $[\alpha(t, T), 0 \leq t \leq T]$ and $[\sigma_n(t, T), 0 \leq t \leq T]$, $n = 1, \ldots, N$ are a drift term and a volatility term that satisfy some regularity conditions. Now, letting $\beta_n(t)$, $n = 1, \ldots, N$ be the market prices of risk, then under the equivalent Martingale measure,

$$\begin{aligned} df(t, T) &= \alpha(t, T)dt + \sum_{n=1}^{N} \sigma_n(t, T)d\tilde{W}_n(t) \\ &+ \sum_{n=1}^{N} \beta_n(t)\sigma_n(t, T)dt, \ t \in [0, T]. \end{aligned} \tag{A.16}$$

We can derive the following relation based on a consideration of the absence of arbitrage:

$$\alpha(t, T) = -\sum_{i=1}^{N} \sigma_n(t, T)[\beta(t) - \textstyle\int_t^T \sigma_n(t, v)dv], \ 0 \leq t \leq T. \tag{A.17}$$

Using this equation, we can obtain a formula for bond prices that does not include the market price of risk but contains volatility functions under the equivalent Martingale measure. We calculate the discount bond price $P(t, T)$ on the equivalent Martingale measure as follows:

$$P(t, T) = \exp[-\textstyle\int_t^T f(t, s)ds]. \tag{A.18}$$

Therefore, we can describe the process by which the discount bond price fluctuates as follows:

$$dP(t, T) = r(t)P(t, T)dt - \sum_{n=1}^{N} [\textstyle\int_t^T \sigma_n(t, s)ds]P(t, T)d\tilde{W}_n(t), \tag{A.19}$$

$$r(t) = f(t, t).$$

## APPENDIX 3: MODELING ADMINISTERED INTEREST RATES

### A. The Model That Fixes the Level of Administered Rate Corresponding to the Range of Market Rates

Let $[r_s(t), t \geq 0]$ be the short-term prime rate, and $[r(t, \tau), \tau > t \geq 0]$ be the market rate (actually, the three-month CD rate) at time $t$ with fixed maturity of $\tau$,

$$r_s(t+\Delta t) = \begin{cases} r_s^1, & 0 \le r(t,\tau) \le r^1, \\ \vdots & \vdots \\ r_s^i, & r^{i-1} \le r(t,\tau) \le r^i, \\ \vdots & \vdots \\ r_s^m, & r^{m-1} \le r(t,\tau), \end{cases} \tag{A.20}$$

where $\Delta t$ is the time lag since the time when $r(t)$ crossed a rate boundary most recently and obeys the exponential distribution EXP($\lambda$).

**B. The Model That Describes the Revision of the Short-Term Prime Rate**

Assume

$r_{3CD}(t)$: three-month CD rate at time $t$,

$\bar{t}$: the time at which the short-term prime rate was last revised,

$\bar{\bar{t}}$: the time, $> \bar{t}$, at which the condition abs $[r_{3CD}(t) - r_{3CD}(\bar{t})] \ge 0.25$ first becomes true,

abs($x$): the absolute value of $x$.

Then,

$$r_s(\bar{\bar{t}} + \Delta t) = r_s(\bar{t}) + \text{int}\left[\frac{r_{3CD}(\bar{\bar{t}} + \Delta t) - r_{3CD}(\bar{t})}{0.125} + 0.5\right] \times 0.125, \tag{A.21}$$

$\Delta t$: a random variable that represents a time lag with exponential distribution EXP($\lambda$),

int[$X$]: the largest integer that does not exceed $X$.

**C. Modeling of the Long-Term Prime Rate**

Assume

$r_l(t)$: the long-term prime rate at time $t$,

$r_c(t)$: the coupon rate of a five-year financial debenture at time $t$,

$\bar{r}_c(t)$: the secondary market rate of a five-year financial debenture at time $t$,

$\bar{t}$: the time at which the condition abs $[\bar{r}_c(t) - r_c(t)] \ge 0.20$ first becomes true.

Then,

$$r_c(\bar{t} + \Delta t) = r_c(\bar{t}) + \text{int}\left[\frac{\bar{r}_c(\bar{t} + \Delta t) - r_c(\bar{t} + \Delta t)}{0.1} + 0.5\right] \times 0.1, \tag{A.22}$$

$\Delta t$: a random variable that obeys EXP($\lambda$)[26],

int[$X$]: the largest integer that does not exceed $X$.

Here, we assume that the secondary market rate of a debenture can be obtained by adding some spread to the market interest rate of the same maturity (i.e., a five-year swap rate),

$$\bar{r}_c(t) = r(t) + r_{spread}(t), \tag{A.23}$$

$r_{spread}(t) \sim N(\mu, \sigma^2)$, the normal distribution with mean $\mu$ and variance $\sigma^2$.

26. In our simulation, we do not take account of this time lag in the movement of the long-term prime rate.

## APPENDIX 4: PREPAYMENT MODELS

### A. Rational Models

We can trace back the research that tries to explain the prepayment behavior of a mortgagor in terms of rational decision-making at least to Dunn and McConnell (1981). This class of models assumes that the mortgagor takes account of the value of the remaining liabilities along the path of future interest rates and prepays only when this value exceeds the sum of the remaining principal amount and the refinancing cost. Johnston and Van Drunen (1988), Stanton (1993a,b), and McConnell and Singh (1994) represent this class of model.

"Rational" does not always mean that everyone acts according to a rational standard without exception. Some models assume a time lag (i.e., see Dunn and Spatt [1986]). Some models divide the mortgagors into several groups according to the level of refinancing cost at prepayment in order to reflect borrowers' attitudes to the fluctuation of interest rates. Other models also take into account prepayments that are generated independently of economic conditions (Stanton [1993a,b]).

In this kind of model, we have to compare the values mentioned above during the construction of the yield curve, working backward from maturity, just as we do when we evaluate an American-type option. It is difficult for the Monte Carlo simulation in this paper to operate this kind of calculation, which is better suited to a lattice method.

McConnell and Singh (1994) first calculate with a lattice method the interest rate boundary at which prepayment happens. Then, they suggest a model that accelerates the prepayment at the boundary rate during Monte Carlo simulation. Let $V[r(t), t]$ be the value of the mortgagor's liabilities, and assume the refinancing cost to be equal to $F(t)$, the remaining principal at prepayment times the constant ratio $RF$. In this situation, the mortgagor always compares $V[r(t), t]$ and $(1+RF)F(t)$. He prepays just after the value of his liabilities exceeds the sum of the remaining principal and the refinancing cost. Under a model with continuous evaluation, the interest rate boundary for prepayment $r_c(t)$ satisfies

$$V[r_c(t), t] = (1+RF)F(t). \tag{A.24}$$

In addition, it is not a realistic assumption that all the borrowers decide to prepay simultaneously. So, using a time lag $\alpha$, we assume that the prepayment ratio $\pi(t)$ changes according to

$$\pi(t) = \begin{cases} \pi_0(t), & r(t) > r_c(t), \\ \pi_0(t) + \alpha, & r(t) < r_c(t). \end{cases} \tag{A.25}$$

Here, $\pi_0(t)$ is the prepayment ratio that is generated independently of the interest rate (the so-called background prepayment). In addition, if we classify the mortgagors into several groups on the basis of their levels of "rationality," we allocate a different refinancing cost to each group so that we can vary the probability of rational prepayment.

**B. The Prepayment Function Model**

Schwartz and Torous (1989) suggest the following model. Let $v$ be an explanatory vector variable that affects the prepayment, and $\beta$ be the coefficient vector. Then, the prepayment function is described as follows.

$$\pi(t) = \pi_0(t) \exp(\beta' v), \tag{A.26}$$

where $\pi_0(t)$ is the prepayment ratio that is independent of environmental change. We assume that $\pi_0(t)$ is the log-logistic function as follows.

$$\pi_0(t) = \frac{\gamma p (\gamma t)^{p-1}}{1 + (\gamma t)^p}. \tag{A.27}$$

As explanatory variables, we take account of the difference between the mortgage contract rate $c$ and the refinancing rate $l(t)$ (replacing this with the long-term interest rate).

$$v_1(t) = c - l(t - s),\ s \geq 0, \tag{A.28}$$

where $s$ is a parameter that represents the time lag between the decision to prepay and actual prepayment.

Furthermore, we introduce a variable that represents the acceleration of prepayment due to the widening of the interest rate differential, a variable that represents the ratio of the balance assuming prepayment to the balance assuming no prepayment, and a variable that represents the seasonal fluctuation of prepayment, and so on.

**C. The Discrete Model**

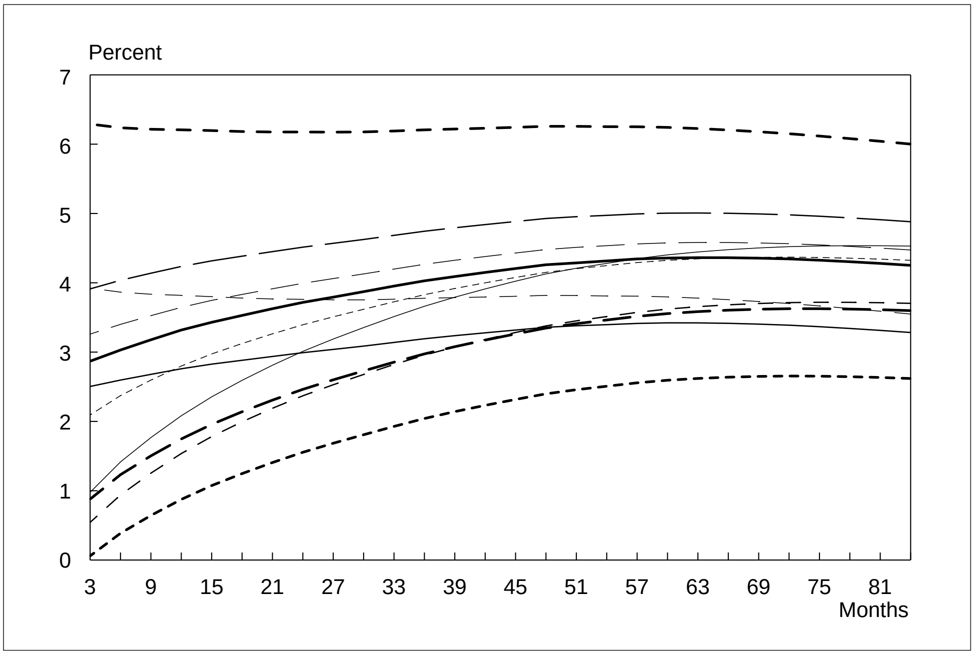
The simplest example is the Zipkin (1993) model. When Zipkin formulated the prepayment model of mortgage securities, he assumed the discrete interest rate model based on a Markov chain and proposed a model with a prepayment ratio that is a deterministic function of the interest rate. When we assume a continuous model for the interest rate, we divide interest rates into several ranges, and assume a different prepayment function for each range.

$$\pi(t) = \begin{cases} \pi_1, & r(t) \leq r_1, \\ \quad\vdots & \\ \pi_i, & r_{i-1} \leq r(t) \leq r_i, \\ \quad\vdots & \\ \pi_n, & r_{n-1} \leq r(t). \end{cases} \tag{A.29}$$

In the case of time deposits, $r(t)$ should be the difference between the interest rate of existing time deposits and the current rate of a new time deposit.

## APPENDIX 5: GENERATING YIELD CURVES USING THE HJM MODEL

By changing the parameter of the HJM model, we can generate various kinds of yield curves. For example, if we set $\sigma_1 = 0.000365$, $\kappa = 0.0632$, and $\sigma_2 = 0.000202$, we get the following curves after three years.



**Appendix Table 1 The Features of the Main Products in Banking Accounts**

| Assets | Loans | | |
|---|---|---|---|
| | Overdrafts | Short-term loans | Long-term loans |
| Holding period | — | Until maturity (several months) | Until maturity (mid- and long-term) |
| Liquidity | No | No | No |
| Interest rate type | Linked to base interest rate | Fixed until maturity | Fixed until maturity or linked to base interest rate |
| Cancellation of contract | Withdrawal at any time | Principally, no | Principally no, but frequent cancellation as for consumer loans (in particular, housing loans) |
| Characteristics of cash flow | Principal: withdrawal at any time; interest: periodical | One-time repayment at maturity | One-time repayment at maturity or by installments |

| Liabilities | Deposits | | | | |
|---|---|---|---|---|---|
| | Liquid deposits | | | | Time deposits |
| | Current | Ordinary | Saving | Notice | Time |
| Holding period | Until withdrawal | Until withdrawal | Until withdrawal | Until withdrawal | Until maturity (several years) |
| Liquidity | No | No | No | No | No |
| Interest rate type | No interest | Fixed until withdrawal | Fixed until withdrawal | Fixed until withdrawal | Fixed, partly variable |
| Cancellation of contract | Withdrawal at any time | Withdrawal at any time | Withdrawal at any time | Withdrawal with prior notice | Yes |
| Characteristics of cash flow | Principal: withdrawal at any time | Principal: withdrawal at any time; interest: periodical | Principal: withdrawal at any time; interest: periodical | Repayment of principal and interest at maturity | Repayment of principal and interest at maturity and mid-term interest payment |

Note: 1. The holding period alluded to in this table indicates the banks' actual holding period, which is a different concept from the "holding period" as a measurement period in the main text.

## Appendix Table 2 Imaginary Portfolio of Banking Accounts (Basic Form)

| Liabilities | Variable/fixed | Interest rate | Linkage | Non-maturity | 1M | 3M | 6M | 12M | 2Y | 3Y | 4Y | 5Y | 6Y | 7Y | Total | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Current deposits | No interest | — | | 5 | | | | | | | | | | | 5 | |
| Other demand deposits (ordinary, saving, notice) | Non-maturity | Market | Imperfect | — | 20 | | | | | | | | | | 20 | |
| Time deposits | Fixed rate | Market | Perfect | | — | — | 25 | 35 | — | 10 | — | — | — | — | 70 | |
| Time deposits | Variable rate | Market | Perfect | | | — | 5 | — | | | | | | | 5 | Maturity is 3 years |
| | | | | | | | | | | | | | | | 100 | |

| Assets | Variable/fixed | Interest rate | Linkage | Non-maturity | 1M | 3M | 6M | 12M | 2Y | 3Y | 4Y | 5Y | 6Y | 7Y | Total | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Cash and deposits | No interest | — | | 5 | | | | | | | | | | | 5 | |
| Short-term loans (bills discounted, loans on bills) | Fixed rate | Short-term prime | — | | — | — | 20 | — | | | | | | | 20 | |
| Long-term loans (loans on deeds) | Fixed rate | Market | Perfect | | — | — | 5 | — | | | | | | | 5 | |
| Long-term loans (loans on deeds) | Variable rate | Short-term prime | — | | — | — | 25 | — | | | | | | | 25 | Maturity is 3 years |
| Long-term loans (loans on deeds) | Variable rate | Long-term prime | — | | — | — | 5 | — | | | | | | | 5 | Maturity is 3 years |
| Long-term loans (loans on deeds) | Fixed rate | Market | Perfect | | | | | | 5 | — | — | 5 | — | — | 10 | |
| Housing loans | Variable rate | Short-term prime | — | | — | — | 5 | — | | | | | | | 5 | Maturity is 7 years |
| Housing loans | Fixed rate | Market | Perfect | | | | | — | — | — | — | 5 | — | — | 5 | |
| Overdrafts (corporate loans, card loans) | Non-maturity | Short-term prime | — | — | 20 | | | | | | | | | | 20 | |
| | | | | | | | | | | | | | | | 100 | |

Note: 1. The table shows the share of each product assuming that the total sums of liabilities and assets are 100, respectively.

**Appendix Table 3 Simulation Results**

Target products: individual loans
Perfect rollover

| Holding period (months) | Market rate-linked (risk, ¥ billions) | Short-term prime rate-linked (risk, ¥ billions) | Long-term prime rate-linked (risk, ¥ billions) |
|---|---|---|---|
| 1 | 1.0 | 50.1 | 27.2 |
| 2 | 1.0 | 51.1 | 29.1 |
| 3 | 1.0 | 51.1 | 33.8 |
| 4 | 1.0 | 51.1 | 33.8 |
| 5 | 1.0 | 51.1 | 33.8 |
| 6 | 1.0 | 71.1 | 94.3 |
| 7 | 1.3 | 71.5 | 94.9 |
| 8 | 1.3 | 71.5 | 94.9 |
| 9 | 1.3 | 71.5 | 94.9 |
| 10 | 1.3 | 71.5 | 94.9 |
| 11 | 1.3 | 71.5 | 94.9 |
| 12 | 1.3 | 88.5 | 160.5 |
| 13 | 1.4 | 88.5 | 161.4 |
| 14 | 1.4 | 88.5 | 164.3 |
| 15 | 1.4 | 88.5 | 164.3 |
| 16 | 1.4 | 88.5 | 164.3 |
| 17 | 1.4 | 88.5 | 164.3 |
| 18 | 1.4 | 89.8 | 224.3 |
| 19 | 1.4 | 89.8 | 225.9 |
| 20 | 1.4 | 89.8 | 225.9 |
| 21 | 1.4 | 89.8 | 225.9 |
| 22 | 1.4 | 89.8 | 225.9 |
| 23 | 1.4 | 89.8 | 225.9 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 33 | 1.4 | 91.4 | 331.5 |
| 34 | 1.4 | 91.4 | 331.5 |
| 35 | 1.4 | 91.4 | 331.5 |
| 36 | 1.4 | 97.0 | 371.3 |
| PV0 | 0 | 269.2 | 467.5 |

## Appendix Table 4 Imaginary Portfolio of Banking Accounts (No Administered Rate-Linked Products)

| Liabilities | Variable/fixed | Interest rate | Linkage | Non-maturity | 1M | 3M | 6M | 12M | 2Y | 3Y | 4Y | 5Y | 6Y | 7Y | Total | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Current deposits | No interest | — | | 5 | | | | | | | | | | | 5 | |
| Other demand deposits (ordinary, saving, notice) | Non-maturity | Market | Imperfect | — | 0 | | | | | | | | | | 0 | |
| Time deposits | Fixed rate | Market | Perfect | | — | — | 45 | 35 | — | 10 | — | — | — | — | 90 | |
| Time deposits | Variable rate | Market | Perfect | | | — | 5 | — | | | | | | | 5 | Maturity is 3 years |
| | | | | | | | | | | | | | | | 100 | |

| Assets | Variable/fixed | Interest rate | Linkage | Non-maturity | 1M | 3M | 6M | 12M | 2Y | 3Y | 4Y | 5Y | 6Y | 7Y | Total | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Cash and deposits | No interest | — | | 5 | | | | | | | | | | | 5 | |
| Short-term loans (bills discounted, loans on bills) | Fixed rate | Short-term prime | — | | — | — | 0 | — | | | | | | | 0 | |
| Long-term loans (loans on deeds) | Fixed rate | Market | Perfect | | — | — | 80 | — | | | | | | | 80 | |
| Long-term loans (loans on deeds) | Variable rate | Short-term prime | — | | — | — | 0 | — | | | | | | | 0 | Maturity is 3 years |
| Long-term loans (loans on deeds) | Variable rate | Long-term prime | — | | — | — | 0 | — | | | | | | | 0 | Maturity is 3 years |
| Long-term loans (loans on deeds) | Fixed rate | Market | Perfect | | | | | | 5 | — | — | 5 | — | — | 10 | |
| Housing loans | Variable rate | Short-term prime | — | | — | — | 0 | — | | | | | | | 0 | Maturity is 7 years |
| Housing loans | Fixed rate | Market | Perfect | | | | | — | — | — | — | 5 | — | — | 5 | |
| Overdrafts (corporate loans, card loans) | Non-maturity | Short-term prime | — | — | 0 | | | | | | | | | | 0 | |
| | | | | | | | | | | | | | | | 100 | |

## References


Brennan, M. J., and E. S. Schwartz, "A Continuous Time Approach to the Pricing of Bonds," *The Journal of Banking and Finance*, 3, 1979, pp. 133–155.

Chen, R. R., and L. O. Scott, "Multi-Factor Cox-Ingersoll-Ross Model of the Term Structure: Estimates and Tests from a Kalman Filter Model," working paper, University of Georgia, 1995.

Cox, J. C., J. E. Ingersoll, and S. A. Ross, "An Intertemporal Equilibrium Model of Asset Prices," *Econometrica*, 53, 1985, pp. 363–384.

Dunn, K. B., and J. J. McConnell, "Valuation of GNMA Mortgage-Backed Securities," *The Journal of Finance*, 36, 1981, pp. 599–617.

———, and C. S. Spatt, "The Effect of Refinancing Costs and Market Imperfections on the Optimal Call Strategy and the Pricing of Debt Contracts," working paper, Carnegie-Mellon University, 1986.

Heath, D., R. A. Jarrow, and A. Morton, "Contingent Claim Valuation with a Random Evolution of Interest Rates," *The Review of Futures Markets*, 9, 1990, pp. 54–76.

———, ———, and ———, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation," *Econometrica*, 60, 1992, pp. 77–105.

Johnston, E. T., and L. D. Van Drunen, "Pricing Mortgage Pools with Heterogeneous Mortgagors: Empirical Evidence," working paper, University of Utah, 1988.

Litterman, R., and J. Scheinkman, "Common Factors Affecting Bond Returns," *Journal of Fixed Income*, 1, 1991, pp. 54–61.

Longstaff, F. A., and E. S. Schwartz, "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model," *The Journal of Finance*, 47, 1992, pp. 1259–1282.

McConnell, J. J., and M. Singh, "Rational Prepayments and the Valuation of Collateralized Mortgage Obligations," *The Journal of Finance*, 59, 1994, pp. 891–921.

Mori, A., M. Ohsawa, and T. Shimizu, "Calculation of Value at Risk and Risk/Return Simulation," *IMES Discussion Paper Series* 96-E-8, Bank of Japan, 1996.

Oda, N., and J. Muranaga, "A New Framework for Measuring the Credit Risk of a Portfolio: The 'ExVaR' Model," *Monetary and Economic Studies*, Institute for Monetary and Economic Studies, Bank of Japan, 15 (2), 1997, pp. 27–62.

Ritchken, P., and L. Sankarasubramanian, "A Multifactor Model of the Quality Option in Treasury Futures Contracts," *The Journal of Financial Research*, 18, 1995, pp. 261–279.

Schwartz, E. S., and W. N. Torous, "Prepayment and the Valuation of Mortgage-Backed Securities," *The Journal of Finance*, 55, 1989, pp. 375–392.

Stanton, R., "Rational Prepayment and the Valuation of Mortgage-Backed Securities," working paper, University of California, Berkeley, 1993a.

———, "Pool Heterogeneity and Rational Learning: The Valuation of Mortgage-Backed Securities," working paper, University of California, Berkeley, 1993b.

Vasicek, O. A., "An Equilibrium Characterization of the Term Structure," *The Journal of Financial Economics*, 5, 1977, pp. 177–188.

Wright, D., and J. Houpt, "An Analysis of Commercial Bank Exposure to Interest Rate Risk," *Federal Reserve Bulletin*, 82, 1996, pp. 115–128.

Zipkin, P., "Mortgages and Markov Chains: A Simplified Evaluation Model," *Management Science*, 39, 1993, pp. 683–691.