

DEGENERATE FOLIATIONS IN SASAKIAN SEMI-RIEMANNIAN MANIFOLDS

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Abstract: *In the Semi-Riemannian case we do not have the liability of the existence of such a metric being a difference from the Riemannian case. A Semi-Riemannian manifold provided with a normal contact metric structure is called Sasakian manifold.*

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In the Semi-Riemannian case we do not have the liability of the existence of such a metric being a difference from the Riemannian case.

Definition Let $M=M^{2n+1}$ a connected Semi-Riemannian manifold and φ, ξ, η tensorial fields on M of types $(1,1)$, $(1,0)$ and $(0,1)$ respectively. We say that (φ, ξ, η) define an almost contact structure on M if:

$$(1) \quad \varphi^2 Y = -Y + \eta(Y)\xi \quad \forall Y \in X(M)$$

$$(2) \quad \eta(\xi) = 1$$

In these conditions we have the relations:

$$(3) \quad \eta(\varphi Y) = 0 \quad \forall Y \in X(M)$$

$$(4) \quad \varphi \xi = 0$$

We define the contact distribution like $D = \{Y \in X(M) \mid \eta(Y) = 0\}$.

Definition A metric g on M for which:

$$(5) \quad g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y) \quad \forall X, Y \in X(M)$$

where $\varepsilon \in \{-1, 1\}$ is called compatible metric with the almost contact structure.

In the Semi-Riemannian case we do not have the liability of the existence of such a metric being a difference from the Riemannian case.

Definition If on a Semi-Riemannian manifold M^{2n+1} we have an almost contact structure (φ, ξ, η) and a compatible metric with the structure we shall say that (φ, ξ, η, g) define an almost contact metric structure.

For $Y=\xi$ in (1) we have with (2) and (4):

$$(6) \quad g(X, \xi) = \varepsilon \eta(X) \quad \forall X \in X(M)$$

and if in (6) $Y=\xi$:

$$(7) \quad g(\xi, \xi) = \varepsilon$$

Remark If we shall consider in the upper $g'=-g$, $\xi'=-\xi$, $\eta'=-\eta$ and $\varphi'=\varphi$ we have together with $\varepsilon'=-\varepsilon$ also an almost contact metric structure. We can further suppose that $\varepsilon=1$.

Definition: Let $M=M^{2n+1}$ a connected Semi-Riemannian manifold provided with an almost contact metric structure (φ, ξ, η, g) . We shall say that the structure is contact metric if:

$$(8) \quad (d\eta)(X, Y) = g(\varphi X, Y) \quad \forall X, Y \in X(M)$$

Definition: An almost contact metric structure (φ, ξ, η, g) is normal if:

$$(9) \quad (\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \quad \forall X, Y \in X(M)$$

where ∇ is the Levi-Civita connection for the metric g .

Definition A Semi-Riemannian manifold provided with a normal contact metric structure is called Sasakian manifold.

On a Sasakian manifold the following relations hold:

$$(10) \quad \nabla_X \xi = -\varphi X \quad \forall X \in X(M)$$

$$(11) \quad N^1(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0 \quad \forall X, Y \in X(M)$$

$$(12) \quad N^2(X, Y) = (L_{\varphi X} \eta)Y - (L_{\varphi Y} \eta)X = 0 \quad \forall X, Y \in X(M)$$

$$(13) \quad N^3(X) = (L_\xi \varphi)X = 0 \quad \forall X \in X(M)$$

$$(14) \quad N^4(X) = (L_\xi \eta)X = 0 \quad \forall X \in X(M)$$

where $N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y]$.

Let now a φ -basis of TM that is an orthonormal basis of the form: $\{\xi, Y_1, \dots, Y_n, \varphi Y_1, \dots, \varphi Y_n\}$ where $Y_i \in D$, $i=1, \dots, n$. We denote by

$$\varepsilon_i = g(Y_i, Y_i) = g(\varphi Y_i, \varphi Y_i), \quad i=1, \dots, n.$$

We define now the 1-differential forms:

$$(15) \quad \omega_i(Z) = g(Y_i, Z) \quad \forall Z \in X(M) \quad \forall i=1, \dots, n$$

and from (15) we have:

$$(16) \quad g(\varphi Y_i, Z) = -\omega_i(\varphi Z) \quad \forall Z \in X(M) \quad \forall i=1, \dots, n$$

Let now two arbitrary vector fields:

$$(17) \quad X = \eta(X)\xi + \sum_{i=1}^n \varepsilon_i \omega_i(X) Y_i - \sum_{i=1}^n \varepsilon_i \omega_i(\varphi X) \varphi Y_i$$

$$(18) \quad Z = \eta(Z)\xi + \sum_{j=1}^n \varepsilon_j \omega_j(Z) Y_j - \sum_{j=1}^n \varepsilon_j \omega_j(\varphi Z) \varphi Y_j$$

We have now:

$$(19) \quad [X, Z] = \{ \eta(X) \nabla_{\xi} \eta(Z) - \eta(Z) \nabla_{\xi} \eta(X) + \sum_{j=1}^n \varepsilon_j \omega_j(X) \nabla_{Y_j} \eta(Z) - \sum_{j=1}^n \varepsilon_j \omega_j(Z) \nabla_{Y_j} \eta(X) + \sum_{j=1}^n \varepsilon_j \omega_j(\varphi Z) \nabla_{\varphi Y_j} \eta(X) - \sum_{j=1}^n \varepsilon_j \omega_j(\varphi X) \nabla_{\varphi Y_j} \eta(Z) \} \xi +$$

$$\sum_{j=1}^n \varepsilon_j \{ \eta(X) \nabla_{\xi} \omega_j(Z) - \eta(Z) \nabla_{\xi} \omega_j(X) + \sum_{i=1}^n \varepsilon_i \omega_i(X) \nabla_{Y_i} \omega_j(Z) - \sum_{i=1}^n \varepsilon_i \omega_i(Z) \nabla_{Y_i} \omega_j(X) + \sum_{i=1}^n \varepsilon_i \omega_i(\varphi Z) \nabla_{\varphi Y_i} \omega_j(X) - \sum_{i=1}^n \varepsilon_i \omega_i(\varphi X) \nabla_{\varphi Y_i} \omega_j(Z) \} Y_j +$$

$$\sum_{j=1}^n \varepsilon_j \{ \eta(Z) \nabla_{\xi} \omega_j(\varphi Z) - \eta(X) \nabla_{\xi} \omega_j(\varphi Z) + \sum_{i=1}^n \varepsilon_i \omega_i(Z) \nabla_{Y_i} \omega_j(\varphi X) - \sum_{i=1}^n \varepsilon_i \omega_i(X) \nabla_{Y_i} \omega_j(\varphi Z) + \sum_{i=1}^n \varepsilon_i \omega_i(\varphi X) \nabla_{\varphi Y_i} \omega_j(\varphi Z) - \sum_{i=1}^n \varepsilon_i \omega_i(\varphi Z) \nabla_{\varphi Y_i} \omega_j(\varphi X) \} \varphi Y_j +$$

$$\sum_{j=1}^n \varepsilon_j \{ \eta(X) \omega_j(Z) - \eta(Z) \omega_j(X) \} [\xi, Y_j] + \sum_{j=1}^n \varepsilon_j \{ \eta(Z) \omega_j(\varphi X) - \eta(X) \omega_j(\varphi Z) \} [\xi, \varphi Y_j] -$$

$$\sum_{i,j=1}^n \varepsilon_i \varepsilon_j \omega_i(Z) \omega_j(X) [Y_i, Y_j] + \sum_{i,j=1}^n \varepsilon_i \varepsilon_j \{ \omega_i(Z) \omega_j(\varphi X) - \omega_i(X) \omega_j(\varphi Z) \} [Y_i, \varphi Y_j] -$$

$$\sum_{i,j=1}^n \varepsilon_i \varepsilon_j \omega_i(\varphi Z) \omega_j(\varphi X) [\varphi Y_i, \varphi Y_j].$$

If we define now:

$$(20) \quad Z_j = \omega_j(Z) X - \omega_j(X) Z \quad \forall j=1, \dots, n$$

$$(21) \quad W_j = \omega_j(\varphi Z) X - \omega_j(\varphi X) Z \quad \forall j=1, \dots, n$$

$$(22) \quad T = \eta(X) Z - \eta(Z) X$$

we obtain:

$$(23) \quad [X, Z] = (\operatorname{div} Z) X - (\operatorname{div} X) Z - \{ \operatorname{div} T - 2 \sum_{j=1}^n \varepsilon_j \omega_j(W_j) \} \xi + \sum_{j=1}^n \varepsilon_j (\operatorname{div} Z_j) Y_j -$$

$$\sum_{j=1}^n \varepsilon_j (\operatorname{div} W_j) \varphi Y_j + \sum_{j=1}^n \varepsilon_j \eta(Z_j) [\xi, Y_j] - \sum_{j=1}^n \varepsilon_j \eta(W_j) \varphi [\xi, Y_j] - \sum_{i,j=1}^n \varepsilon_i \varepsilon_j \omega_j(W_i) [\varphi Y_i, Y_j] -$$

$$\sum_{\substack{i,j=1 \\ i < j}}^n \varepsilon_i \varepsilon_j \omega_j(\varphi W_i) \varphi [\varphi Y_i, Y_j] + \sum_{\substack{i,j=1 \\ i < j}}^n \varepsilon_i \varepsilon_j \omega_i(\varphi W_j) \varphi [\varphi Y_i, Y_j] + \sum_{\substack{i,j=1 \\ i < j}}^n \varepsilon_i \varepsilon_j \{ \omega_i(Z_j) - \omega_j(\varphi W_i) \} [Y_i, Y_j].$$

In order that the two vector fields define a foliation we must have:

$$(24) \quad -\{ \operatorname{div} T - 2 \sum_{j=1}^n \varepsilon_j \omega_j(W_j) \} \xi + \sum_{j=1}^n \varepsilon_j (\operatorname{div} Z_j) Y_j - \sum_{j=1}^n \varepsilon_j (\operatorname{div} W_j) \varphi Y_j +$$

$$\sum_{j=1}^n \varepsilon_j \eta(Z_j) [\xi, Y_j] - \sum_{j=1}^n \varepsilon_j \eta(W_j) \varphi[\xi, Y_j] - \sum_{i,j=1}^n \varepsilon_i \varepsilon_j \omega_j(W_i) [\varphi Y_i, Y_j] - \sum_{\substack{i,j=1 \\ i < j}}^n \varepsilon_i \varepsilon_j \omega_j(\varphi W_i) \varphi[\varphi Y_i, Y_j] +$$

$$\sum_{\substack{i,j=1 \\ i > j}}^n \varepsilon_i \varepsilon_j \omega_i(\varphi W_j) \varphi[\varphi Y_i, Y_j] + \sum_{\substack{i,j=1 \\ i < j}}^n \varepsilon_i \varepsilon_j \{ \omega_i(Z_j) - \omega_j(\varphi W_i) \} [Y_i, Y_j] \in \text{Span}(X, Z).$$

Example

Let on \mathbf{R}_v^{2n+1} with the coordinates $(x^1, \dots, x^n, y^1, \dots, y^n, z)$ the usual contact structure defined by:

(25)
$$\eta = dz - \sum_{i=1}^n y^i dx^i$$

(26)
$$\xi = \frac{\partial}{\partial z}$$

(27)
$$\varphi = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & y^t & 0 \end{pmatrix}$$

where $y^t = (y^1, \dots, y^n)$ and I the identity.

The contact distribution D is generated by $\left\{ \frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z} \right\}_{i=1, \dots, n}$.

The compatible metric is:

(28)
$$g = \sum_{i,j=1}^n (\varepsilon_i \delta_{ij} + y^i y^j) dx^i dx^j - 2 \sum_{i=1}^n y^i dx^i dz + \sum_{i=1}^n \varepsilon_i (dy^i)^2 + dz^2$$

Let

(29)
$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} + \sum_{i=1}^n b_i \frac{\partial}{\partial y^i} + c \frac{\partial}{\partial z}$$

(30)
$$Z = \sum_{i=1}^n d_i \frac{\partial}{\partial x^i} + \sum_{i=1}^n e_i \frac{\partial}{\partial y^i} + f \frac{\partial}{\partial z}$$

where $a_i, b_i, c, d_i, e_i, f \in F(M)$.

We have: $\omega_i = \varepsilon_i dy^i \forall i=1, \dots, n$. It is easy to see that:

(6.31)
$$\omega_i(X) = \varepsilon_i b_i, \omega_i(\varphi X) = -\varepsilon_i a_i, \omega_i(Z) = \varepsilon_i e_i, \omega_i(\varphi Z) = -\varepsilon_i d_i, \eta(X) = c - \sum_{i=1}^n a_i y^i,$$

$$\eta(Z) = f - \sum_{i=1}^n d_i y^i$$

From (20)-(22) we have:

(32)
$$Z_j = \varepsilon_j \left[\sum_{i=1}^n (a_i e_j - d_i b_j) \frac{\partial}{\partial x^i} + \sum_{i=1}^n (b_i e_j - e_i b_j) \frac{\partial}{\partial y^i} + (c e_j - f b_j) \frac{\partial}{\partial z} \right]$$

$$(33) \quad W_j = \varepsilon_j \left[\sum_{i=1}^n (d_i a_j - a_i d_j) \frac{\partial}{\partial x^i} + \sum_{i=1}^n (e_i a_j - b_i d_j) \frac{\partial}{\partial y^i} + (f a_j - c d_j) \frac{\partial}{\partial z} \right]$$

$$(34) \quad T = \sum_{i=1}^n \left[(d_i c - a_i f) + \sum_{j=1}^n (a_i d_j - a_j d_i) y^j \right] \frac{\partial}{\partial x^i} + \sum_{i=1}^n \left[(e_i c - b_i f) + \sum_{j=1}^n (b_i d_j - a_j e_i) y^j \right] \frac{\partial}{\partial y^i} + \sum_{i=1}^n (c d_i - f a_i) \frac{\partial}{\partial z}$$

For a field $U = \sum_{i=1}^n U^i \frac{\partial}{\partial x^i} + \sum_{i=1}^n U^{n+i} \frac{\partial}{\partial y^i} + U^{2n+1} \frac{\partial}{\partial z}$ we have

$$\text{div } U = \sum_{i=1}^n \frac{\partial U^i}{\partial x^i} + \sum_{i=1}^n \frac{\partial U^{n+i}}{\partial y^i} + \frac{\partial U^{2n+1}}{\partial z}$$

therefore:

$$(35) \quad \text{div } Z_j = \varepsilon_j \left[\sum_{i=1}^n \frac{\partial (a_i e_j - d_i b_j)}{\partial x^i} + \sum_{i=1}^n \frac{\partial (b_i e_j - e_i b_j)}{\partial y^i} + \frac{\partial (c e_j - f b_j)}{\partial z} \right]$$

$$(36) \quad \text{div } W_j = \varepsilon_j \left[\sum_{i=1}^n \frac{\partial (d_i a_j - a_i d_j)}{\partial x^i} + \sum_{i=1}^n \frac{\partial (e_i a_j - b_i d_j)}{\partial y^i} + \frac{\partial (f a_j - c d_j)}{\partial z} \right]$$

$$(6.37) \quad \text{div } T = \sum_{i=1}^n \left[\frac{\partial (d_i c - a_i f)}{\partial x^i} + \sum_{j=1}^n \frac{\partial (a_i d_j - a_j d_i)}{\partial x^i} y^j \right] + \sum_{i=1}^n \left[\frac{\partial (e_i c - b_i f)}{\partial y^i} + \sum_{j=1}^n \frac{\partial (b_i d_j - a_j e_i)}{\partial y^i} y^j + (b_i d_i - a_i e_i) \right] + \sum_{i=1}^n \frac{\partial (c d_i - f a_i)}{\partial z} y^i$$

$$(38) \quad \omega_j(W_j) = a_j e_j - b_j d_j$$

The condition of integrability is:

$$(39) \quad \left[\sum_{j=1}^n (2\varepsilon_j + 1) \omega_j(W_j) - \text{div } T \right] \xi + \sum_{j=1}^n \varepsilon_j (\text{div } Z_j) Y_j - \sum_{j=1}^n \varepsilon_j (\text{div } W_j) \phi Y_j \in \text{Span}(X, Z)$$

and with (35)-(38) becomes:

$$(6.40) \quad - \sum_{j=1}^n \left[\sum_{i=1}^n \frac{\partial (a_j d_i - a_i d_j)}{\partial x^i} + \sum_{i=1}^n \frac{\partial (a_j e_i - b_i d_j)}{\partial y^i} + \frac{\partial (f a_j - c d_j)}{\partial z} \right] \frac{\partial}{\partial x^j} + \sum_{j=1}^n \left[\sum_{i=1}^n \frac{\partial (a_i e_j - d_i b_j)}{\partial x^i} + \sum_{i=1}^n \frac{\partial (b_i e_j - e_i b_j)}{\partial y^i} + \frac{\partial (c e_j - f b_j)}{\partial z} \right] \frac{\partial}{\partial y^j} + \left[\sum_{j=1}^n 2(\varepsilon_j + 1)(e_j a_j - b_j d_j) + \sum_{j=1}^n \frac{\partial (f a_j - c d_j)}{\partial x^j} - \sum_{j=1}^n \frac{\partial (c e_j - f b_j)}{\partial y^j} \right] \frac{\partial}{\partial z} \in \text{Span}(X, Z)$$

The conditions that $g(X, X) = 0$, $g(Z, Z) \neq 0$, $g(X, Z) = 0$ become:

$$(41) \begin{cases} \left(\sum_{i=1}^n a_i y^i - c \right)^2 + \sum_{i=1}^n \varepsilon_i a_i^2 + \sum_{i=1}^n \varepsilon_i b_i^2 = 0 \\ \left(\sum_{i=1}^n d_i y^i - f \right)^2 + \sum_{i=1}^n \varepsilon_i d_i^2 + \sum_{i=1}^n \varepsilon_i e_i^2 \neq 0 \\ \left(\sum_{i=1}^n a_i y^i - c \right) \left(\sum_{j=1}^n d_j y^j - f \right) + \sum_{i=1}^n \varepsilon_i (a_i d_i + b_i e_i) = 0 \end{cases}$$

From (40) we have:

$$\text{rank} \begin{pmatrix} a_j & d_j & - \left[\sum_{i=1}^n \frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + \sum_{i=1}^n \frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} + \frac{\partial(f a_j - c d_j)}{\partial z} \right] \\ b_k & e_k & \left[\sum_{i=1}^n \frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} + \sum_{i=1}^n \frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + \frac{\partial(c e_k - f b_k)}{\partial z} \right] \\ c & f & \left[\sum_{p=1}^n 2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + \sum_{p=1}^n \frac{\partial(f a_p - c d_p)}{\partial x^p} - \sum_{p=1}^n \frac{\partial(c e_p - f b_p)}{\partial y^p} \right] \end{pmatrix} = 2$$

where $j, k=1, \dots, n$.

We have now:

$$\begin{pmatrix} a_j & d_j & - \left[\sum_{i=1}^n \frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + \sum_{i=1}^n \frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} + \frac{\partial(f a_j - c d_j)}{\partial z} \right] \\ b_k & e_k & \left[\sum_{i=1}^n \frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} + \sum_{i=1}^n \frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + \frac{\partial(c e_k - f b_k)}{\partial z} \right] \\ c & f & \left[\sum_{p=1}^n 2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + \sum_{p=1}^n \frac{\partial(f a_p - c d_p)}{\partial x^p} - \sum_{p=1}^n \frac{\partial(c e_p - f b_p)}{\partial y^p} \right] \end{pmatrix} = 0$$

therefore:

$$\begin{aligned} & \left[\sum_{i=1}^n \frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + \sum_{i=1}^n \frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} + \frac{\partial(f a_j - c d_j)}{\partial z} \right] (c e_k - f b_k) + \\ & \left[\sum_{i=1}^n \frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} + \sum_{i=1}^n \frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + \frac{\partial(c e_k - f b_k)}{\partial z} \right] (c d_j - f a_j) + \\ & \left[\sum_{p=1}^n 2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + \sum_{p=1}^n \frac{\partial(f a_p - c d_p)}{\partial x^p} - \sum_{p=1}^n \frac{\partial(c e_p - f b_p)}{\partial y^p} \right] (a_j e_k - b_k d_j) = 0 \end{aligned}$$

$$\begin{aligned}
 & (ce_k - fb_k) \sum_{i=1}^n \frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + (ce_k - fb_k) \sum_{i=1}^n \frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} + \\
 & (ce_k - fb_k) \frac{\partial(fa_j - cd_j)}{\partial z} + (cd_j - fa_j) \sum_{i=1}^n \frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} + \\
 & (cd_j - fa_j) \sum_{i=1}^n \frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + (cd_j - fa_j) \frac{\partial(ce_k - fb_k)}{\partial z} + \\
 & (a_j e_k - b_k d_j) \sum_{p=1}^n 2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + (a_j e_k - b_k d_j) \sum_{p=1}^n \frac{\partial(fa_p - cd_p)}{\partial x^p} - \\
 & (a_j e_k - b_k d_j) \sum_{p=1}^n \frac{\partial(ce_p - fb_p)}{\partial y^p} = 0
 \end{aligned}$$

$$\begin{aligned}
 & (ce_k - fb_k) \sum_{i=1}^n \frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + (ce_k - fb_k) \sum_{i=1}^n \frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} + \\
 & + (ce_k - fb_k)^2 \frac{\partial \frac{fa_j - cd_j}{ce_k - fb_k}}{\partial z} - (fa_j - cd_j) \sum_{i=1}^n \frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} + \\
 & (cd_j - fa_j) \sum_{i=1}^n \frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + \\
 & (a_j e_k - b_k d_j) \sum_{p=1}^n 2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + (a_j e_k - b_k d_j) \sum_{p=1}^n \frac{\partial(fa_p - cd_p)}{\partial x^p} - \\
 & (a_j e_k - b_k d_j) \sum_{p=1}^n \frac{\partial(ce_p - fb_p)}{\partial y^p} = 0
 \end{aligned}$$

$$\begin{aligned}
 & (ce_k - fb_k) \sum_{i=1}^n \frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + (ce_k - fb_k)^2 \frac{\partial \frac{fa_j - cd_j}{ce_k - fb_k}}{\partial z} + (cd_j - fa_j) \sum_{i=1}^n \frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + \\
 & (a_j e_k - b_k d_j) \sum_{p=1}^n 2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + \\
 & \sum_{i,p=1}^n \delta_{ij} (a_i e_k - b_k d_i) \frac{\partial(fa_p - cd_p)}{\partial x^p} - \delta_{jp} (fa_p - cd_p) \frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} \\
 & + \sum_{i,p=1}^n \delta_{kp} (ce_p - fb_p) \frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} - \delta_{ij} (a_i e_k - b_k d_i) \frac{\partial(ce_p - fb_p)}{\partial y^p} = 0
 \end{aligned}$$

From (38) we have:

$$(39) \quad \frac{\partial cd - af}{\partial x^1} (bd - ae)^2 + \frac{\partial bd - ae}{\partial y^1} (bf - ce)^2 + \frac{\partial cd - af}{\partial z} (bf - ce)^2 = 0$$

From (39), (40) we have:

$$(41) \frac{bd - ae}{bf - ce} = \frac{ay^1 - c}{(ay^1 - c)y^1 - a}$$

$$(42) \frac{cd - af}{bd - ae} = -\frac{1}{ay^1 - c}$$

$$(43) \frac{cd - af}{bf - ce} = \frac{-1}{(ay^1 - c)y^1 - a}$$

If we replace (41)-(43) in (39) we have:

$$(44) \left[\frac{\partial a}{\partial z} - a^2 \right] (y^1)^2 + \left[\frac{\partial a}{\partial x^1} - \frac{\partial c}{\partial z} + 2ac \right] y^1 + \left[-\frac{\partial c}{\partial x^1} + a \frac{\partial c}{\partial y^1} - c \frac{\partial a}{\partial y^1} - \frac{\partial a}{\partial z} - a^2 - c^2 \right] = 0$$

If we note $A = ay^1 - c$ we have:

$$(45) \frac{\partial A}{\partial x^1} + \left(A \frac{\partial a}{\partial y^1} - a \frac{\partial A}{\partial y^1} - A^2 \right) + \left(y^1 \frac{\partial A}{\partial z} - \frac{\partial a}{\partial z} \right) = 0$$

We have.

$$\operatorname{div} X = \sum_{i=1}^n \left(\frac{\partial X^i}{\partial x^i} + \sum_{j=1}^{2n+1} \left| \frac{i}{i} \right| X^j \right) + \sum_{a=1}^n \left(\frac{\partial X^{n+a}}{\partial y^a} + \sum_{j=1}^{2n+1} \left| \frac{n+a}{n+a} \right| X^j \right) + \left(\frac{\partial X^{2n+1}}{\partial z} + \sum_{j=1}^{2n+1} \left| \frac{2n+1}{2n+1} \right| X^j \right)$$

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