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DIPARTIMENTO DI ECONOMIA

CONTINGENT CLAIM PRICING IN A DUAL EXPECTED UTILITY THEORY FRAMEWORK

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Working Paper n° 82, 2007

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Abstract

This paper investigates the price for contingent claims in a dual expected utility theory framework, the dual price, considering complete arbitrage-free financial markets. In this framework this dual price is obtained, for the first time in the literature, without any comonotonicity hypothesis and for contingent claims written on n underlying assets following generic Itô processes. An application is also considered assuming geometric brownian motion for the underlying assets and the Wang transform as distortion function.

KEYWORDS

Contingent Claims Pricing, Dual Utility Theory, Wang Transform.

1 Introduction

After Wang [11] and Wang [12], a unified approach to evaluate insurance and financial risks has emerged in the literature. The common mathematical tool is that of the Choquet integral or, from an economics perspective, that of the dual expected utility (DEU) theory, a particular class of the non expected utility theory, presented in Yaari [13]. As shown in Quiggin [8], an important advantage in using the DEU theory is the absence of paradoxes such as Allais [1] and Ellsberg [4], originating from the interpretation of the choices by the classical expected utility (EU) theory presented by von Neumann and Morgenstern [9].

In the DEU framework "attitudes toward risks are characterized by a distortion applied to probability distribution functions, in contrast to expected utility in which attitudes toward risks are characterized by a utility function of wealth" (Wang and Young [10]).

In Wang [11] and Wang [12] an expression for the risk-adjusted premium for a risk R, $H[R, \alpha]$, is given in terms of a Choquet integral. The assumption that the assets can be priced applying such risk-adjusted premium to the present value of the future asset price and that the European option pay-off is comonotone[1] with the terminal underlying stock price, allows the derivation of an implied value of the parameter α. Such an approach has been adopted for European call option pricing and the standard Black and Scholes [2] and Merton [7] formula has been recovered.

In Hamada and Sherris [5], the approach developed in Wang [11] and Wang [12] is formally considered in a complete arbitrage-free financial market with one risk asset. In particular a pricing formula for a contingent claim whose pay-off is comonotone with the terminal value of the underlying asset is obtained and is consistent with the Black-Scholes-Merton option pricing formula.

In Cenci *et al.* [3] a general dynamic framework for optimal portfolio selection is set in the context of Yaari's DEU theory. In particular, using the Wang transform as distortion function, the problem has been explicitly solved in complete arbitrage-free markets with a risk-free and two risky assets.

In this paper the price in the DEU theory framework for a contingent claim, named in this paper as "dual price", is obtained in complete arbitrage-free financial markets. Such pricing formula holds also for a claim contingent on n underlying assets following generic Itô processes, without any comonotonicity hypothesis between its pay-off and the underlying assets.

An application is also considered using the Wang transform as distortion function and a geometric brownian motion for the underlying assets dynamics. In this case the dual price of a contingent claim is equivalent to the standard Black-Scholes-Merton option pricing formula in a complete unconstrained arbitrage-free market.

The remainder of the paper is structured as follows. Outlines of EU and DEU theories are given in section 2. In section 3 the financial market model is described. In section 4 the concept of dual price is introduced and a pricing formula for a contingent claim is obtained. In section 5 the Wang transform is used as distortion operator and the corresponding pricing formula is obtained. In the final section the conclusions are drawn.

2 DEU Decision Theory

The EU theory by von Neumann and Morgenstern [9] is the most frequently used approach to solve problems of choice under uncertainty. The main disadvantage in using such an approach is that, as shown by several authors since the sixties, actual decisions are not fully consistent with all EU theory axioms.

Denoting with χ the opportunity set and with capital letters the opportunities which can be degenerate or not degenerate random variables, the EU theory axioms are:

A.1) Completeness – $\forall X, Y \in \chi$ it is either $X \succeq Y$ or $X \preceq Y$

A.2) Transitivity – if $X \succeq Y$ and $Y \succeq Z$ then $X \succeq Z$

A.3) Continuity – if $X \succeq Y \succeq Z \Rightarrow \exists p \in [0,1] \ : \ Y \sim pX + (1-p)Z$

[1] The random variables X and Y are comonotone if there exist a random variable Z and two not decreasing real functions f and h such that $X = f(Z)$ and $Y = h(Z)$.

A.4) Independence – if $X \succeq Y \Rightarrow \forall p \in [0,1]\ \ pX+(1-p)Z \succeq pY+(1-p)Z$,

where $X \succeq Y$ is short for Y not preferred to X and $X \sim Y$ is short for X indifferent to Y.

As shown, for example, in Allais [1] and Ellsberg [4], the independence axiom is violated in several empirical tests. In order to avoid this problem, theories of choice alternative to the EU theory, called non-expected utility theories, have been presented in the literature. In particular the DEU theory is a non-expected utility theory whose axioms are A.1, A.2, A.3 and axiom A.4 is replaced with

A.4*) Comonotonicity – if X, Y, Z are pairwise comonotonic and $X \succeq Y$

$$\Rightarrow \forall p \in [0,1]\ pX + (1-p)Z \succeq pY + (1-p)Z.$$

As shown in Yaari [13], under axioms A.1, A.2, A.3 and A.4*, there exists a non-decreasing function $g : [0,1] \to [0,1]$, with $g(0) = 0$ and $g(1) = 1$, such that

- $X \succeq Y \Leftrightarrow \mathbb{E}_g[X] \geq \mathbb{E}_g[Y]$;
- $X \preceq Y \Leftrightarrow \mathbb{E}_g[X] \leq \mathbb{E}_g[Y]$;
- $X \sim Y \Leftrightarrow \mathbb{E}_g[X] = \mathbb{E}_g[Y]$,

with

$$\mathbb{E}_g[X] = \int_{-\infty}^{+\infty} x dg\left(F_X(x)\right),$$

where $F_X(x)$ is the probability distribution functions of the random variables X.

The analytical form of the so called distortion function g, embeds the degree of aversion towards risk of the decision maker. In particular, if g is increasing and concave, as shown in Quiggin [8], the decision maker is risk-averse and the resulting ordering of preferences is consistent with the first and second order stochastic dominance principles. It can be shown that the following properties hold[2]:

P.1) if $g(F(x)) = F(x) \Rightarrow \mathbb{E}_g[X] = \mathbb{E}[X]$

P.2) $\mathbb{E}_g[aX+b] = a\mathbb{E}_g[X] + b\ \ \forall a > 0, b \geq 0$

P.3) if X and Y are comonotonic $\Rightarrow \mathbb{E}_g[X+Y] = \mathbb{E}_g[X] + \mathbb{E}_g[Y]$

P.4) if g is concave $\Rightarrow \mathbb{E}_g[X] \leq E[X]$ and $\mathbb{E}_g[X+Y] \geq \mathbb{E}_g[X] + \mathbb{E}_g[Y]$

3 Financial Market Model

3.1 The Assets

In this paper a complete and arbitrage-free financial market with a risk-free and n risky traded assets is considered. The price at time $s \in [t,T]$ of the risky assets and that of the risk-free asset are respectively denoted by $\{P_i(s),\ i = 1,...,n\}$ and $P_0(s)$.

[2] See, for instance, Wang and Young [10].

The n independent risk sources are represented through a standard Wiener vector process $W = (W_1, ..., W_n)^*$ in $\mathbb{R}^n$, while the dynamics of the risky assets is described by the following Itô stochastic differential equations:

$$dP_i(s) = P_i(s)\left(\mu_i(s, P(s))ds + \sum_{j=1}^{n} \sigma_{ij}(s, P(s))dW_j(s)\right), \; i = 1, ..., n, s \in [t, T], \tag{1}$$

where $\mu_i(s, P_i(s))$ is the instantaneous return rate of the i-th risky asset and $\{\sigma_{ij}(s, P(s)),\ s \in [t, T]\}$ is the volatility matrix.

The deterministic evolution of the risk-free asset price is:

$$dP_0(s) = P_0(s)r(s)ds$$

where $r(s)$ is the risk-free rate and $P_0(0) = 1$. In the remainder of the paper it is obviously assumed that $\mu_i(s, P(s)) > r(s)$, $\forall s \in [t, T], \forall i = 1, ...n$.

In order to obtain the existence of a unique solution of Eq. (1) it is also assumed that, $\forall s \in [t, T]$,

H.1) The processes $r(s)$, $\mu(s, P(s)) = (\mu_1(s, P(s)), ..., \mu_n(s, P(s)))^*$ and the matrix $\sigma(s, P_i(s)) = \{\sigma_{ij}(s, P(s))\}$ are adapted to the filtration $\mathcal{F}_s = \sigma(W(u),\ u \in [t, s])$

H.2) $\sigma(s, P(s))$ is not degenerate in strong form so that if

$$D(s, P(s)) = \sigma^*(s, P(s))\sigma(s, P(s)) \Rightarrow$$

$$\forall \varepsilon > 0 : \xi^* D(s, P(s))\xi \geq \epsilon \left\|\xi\right\|^2 a.s. \forall \, (s, \xi) \in [t, T] \times \mathbb{R}^n$$

H.3) $r(s) \geq -\eta, \; \eta > 0$

3.2 The Portfolio

At time t a price-taker agent $\mathcal{I}$ with an initial wealth $X(t) = x > 0$ is considered. At time $s \in [t, T]$ the agent $\mathcal{I}$ selects the quantity of each risky asset $(\phi_1(s), ..., \phi_n(s))$ and the quantity of the risk-free asset $\phi_0(s) = (X(s) - \sum_{i=1}^{n} \phi_i(s)P_i(s))/P_0(s)$ to hold over the infinitesimal time interval $[s, s + ds)$.

The trading strategy of $\mathcal{I}$ is representable by the process $(\phi_0(s), ..., \phi_n(s))^*$ which is assumed to be adapted to the current information $\mathcal{F}_s$ and such that $\int_t^T [\phi_i(s)]^2 \, ds < +\infty \; \forall i = 1, ..., n \; a.s.$. In order to have a self-financing trading strategy the following relation must hold:

$$\sum_{i=0}^{n} \phi_i(s)P_i(s) = X(t) + \sum_{i=0}^{n} \int_t^s \phi_i(u)dP_i(u) \quad , \forall s \in [t, T]$$

or, in differential form,

$$d\sum_{i=0}^{n} \phi_i(s)P_i(s) = \sum_{i=0}^{n} \phi_i(s)dP_i(s) \quad , \forall s \in [t, T].$$

Under these hypotheses the wealth of the agent $\mathcal{I}$ at time s is:

$$X(s) = \sum_{i=0}^{n} \phi_i(s) P_i(s).$$

where

$$\phi_i(s) = \begin{cases} \frac{X(s)\pi_i(s)}{P_i(s)} & i = 1, ..., n \\ \frac{X(s)[1-\sum_{i=1}^{n} \pi_i(s)]}{P_0(s)} & i = 0 \end{cases},$$

having defined the portfolio process

$$\pi(s) = (\pi_1(s), ..., \pi_n(s))^*.$$

The resulting wealth evolution can be expressed through the following relation[3]:

$$dX(s) = X(s)[r(s) + \hat{\mu}(s, P(s)) \cdot \pi(s)]ds + X(s)\pi(s) \cdot \sigma(s, P(s))dW(s) \quad (2)$$

with $X(t) = x$ and $\hat{\mu}_i = \mu_i - r$, $\forall i = 1, ...n$.

In the remainder of the paper the solution of Eq. (2) relative to the portfolio process π and initial condition $X(t) = x$ is denoted with $X_s^{t,x,\pi}$, $s \in [t, T]$, and the associated conditional probability function of the final wealth $Prob[X_T^{t,x,\pi} \leq y | X(t) = x]$ with $F_{t,x}^{\pi}(y)$.

4 Dual Price

The optimal portfolio choice requires, within the DEU theory, the solution of the following optimal stochastic control problem:

$$\begin{cases} v(t,x) = \sup_{\pi \in \mathcal{K}} \mathbb{E}_g[X_T^{t,x,\pi}] \\ v(T,x) = x \end{cases}, \quad (3)$$

where $\mathcal{K}$ is the feasible set, the vector $\pi(s) = (\pi_1(s), \pi_2(s), ..., \pi_n(s))^*$, $s \in [t, T]$, represents the control variables and the final wealth $X_T^{t,x,\pi}$ is determined by the solution of Eq. (2).

In an arbitrage-free market the price homogeneity is valid[4]:

$$X_T^{t,\lambda x,\pi} = \lambda X_T^{t,x,\pi}, \; \lambda > 0, \quad (4)$$

and the following proposition holds:

Proposition 4.1 In a complete arbitrage-free market the dual price $\hat{p}_t$ at time t of a contingent claim with pay-off $Y_T = Y(P_1(T), P_2(T), ..., P_n(T))$ at maturity time T is

[3]For $a, b \in \mathbb{R}^n$ the inner product between the vector a and b is denoted with $a \cdot b \equiv \sum_{i=1}^{n} a_i b_i$.

[4]See, for instance, Karatzas and Kou [6].

$$\hat{p}_t = \frac{x}{v(t,x)}\mathbb{E}_g[Y_T],$$

where $v(t,x)$ is the solution of Eq. (3).

Proof

In a complete market any contingent claim with pay-off Y_T at maturity time T, can be replicated by a (hedging) self-financing portfolio $X_T^{t,\tilde{x},\tilde{\pi}}$ for some initial wealth $\tilde{x}$ and portfolio process $\tilde{\pi}$:

$$Y_T = X_T^{t,\tilde{x},\tilde{\pi}}\, a.s. \Rightarrow Y_T \sim X_T^{t,\tilde{x},\tilde{\pi}}.$$

The law of one price implies that $\tilde{x} = \hat{p}_t$, where $\hat{p}_t$ is supposed to be the contingent claim dual price at time t for the agent $\mathcal{I}$:

$$Y_T \sim X_T^{t,\hat{p}_t,\tilde{\pi}}.$$

Supposing that $\tilde{\pi} \neq \overline{\pi}$, where $\overline{\pi}$ is the optimal control:

$$\mathbb{E}_g[Y_T] = \mathbb{E}_g[X_T^{t,\hat{p}_t,\tilde{\pi}}] < \mathbb{E}_g[X_T^{t,\hat{p}_t,\overline{\pi}}],$$

therefore the opportunity $X_T^{t,\hat{p}_t\overline{\pi}}$ is preferred to the opportunity Y_T. Hence the agent $\mathcal{I}$ always invests his initial wealth $\hat{p}_t$ in the optimal portfolio $X_T^{t,\hat{p}_t,\overline{\pi}}$ rather than in the contingent claim which, therefore, should be valued less than $\hat{p}_t$, contrarily to the hypothesis that $\hat{p}_t$ is the contingent claim dual price for the agent $\mathcal{I}$. Hence $\tilde{\pi} = \overline{\pi}$ and

$$Y_T \sim X_T^{t,\hat{p}_t,\overline{\pi}} \Leftrightarrow \mathbb{E}_g[Y_T] = \mathbb{E}_g[X_T^{t,\hat{p}_t,\overline{\pi}}]. \tag{5}$$

Using (4), $X_T^{t,\hat{p}_t,\overline{\pi}} = X_T^{t,\frac{\hat{p}_t}{x}x,\overline{\pi}} = \frac{\hat{p}_t}{x}X_T^{t,x,\overline{\pi}}$, the relation (5) becomes, using property P.2:

$$\mathbb{E}_g[Y_T] = \mathbb{E}_g[\frac{\hat{p}_t}{x}X_T^{t,x,\overline{\pi}}] = \frac{\hat{p}_t}{x}\mathbb{E}_g[X_T^{t,x,\overline{\pi}}] = \frac{\hat{p}_t}{x}v(t,x) \Rightarrow$$

$$\hat{p}_t = \frac{x}{v(t,x)}\mathbb{E}_g[Y_T]. \tag{6}$$

It is interesting to note that the pricing formula above holds also for contingent claims whose pay-off is not comonotone with the underlying assets. ■

Eq. (6) resembles the one obtained in the EU theory. In fact, whitin the EU framework, the optimization problem is

$$v(t,x) = \sup_{\pi\in\mathcal{K}} \mathbb{E}[\mathcal{U}(X_T^{t,x,\pi})],$$

where $\mathcal{U}(\cdot)$ is the utility function, and the price of a contingent claim Y is[5]

$$\hat{p}_t = \frac{1}{\partial v(t,x)/\partial x}\mathbb{E}[\mathcal{U}'(X_T^{t,x,\overline{\pi}})Y_T].$$

[5]See, for instance, Karatzas and Kou [6].

5 Dual Price and the Wang Transform: an Application

In Wang [11] and Wang [12] a general framework for pricing insurance and financial risks has been introduced. The methodology consists of using a particular distortion function, known now in the literature as the Wang transform. If $F_X(x)$ denotes the probability distribution function of the random variable X, the Wang transform is

$$g(F_X(x)) = \Phi(\Phi^{-1}(F_X(x)) + \alpha),\ \alpha > 0, \tag{7}$$

where Φ is the normal cumulative distribution function,

$$\Phi(x) = \int_{-\infty}^{x} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}},$$

and α is a parameter. The effect of the Wang transform is a horizontal translation of the probability distribution $F_X(x)$ with an overweighting of the left tail and an underweighting of the right one. Since the Wang function $g(\cdot)$ is increasing and concave, it characterizes a risk-averse agent and the resulting order preferences are compatible with first and second order stochastic dominance principles.

In the remainder of this section it is assumed that the distortion function is given by Eq. (7) and that the risky asset prices follow a geometric brownian motion with constant coefficients. The portfolio stochastic evolution is therefore described by Eq. (2) with constant coefficients:

$$dX_s = X_s(r + \hat{\mu} \cdot \pi)ds + X_s\pi \cdot \sigma dW_s,\ \ s \in [t, T], \tag{8}$$

with $X_t = x$.

In Cenci *et al.* [3][6] it is shown that in this case $v(t, x) \propto x$, thus Eq. (3) leads to a non standard Bellman equation, being $\frac{\partial v/\partial x}{\partial^2 v/\partial x^2}$ ill defined. Hence Eq. (3) must be solved using a non standard Bellman equation for any given distortion function $g(\cdot)$. It has been shown also that using the Wang transform as distortion function and a price dynamics described by a geometric brownian motion with constant coefficients, the optimal control is determined solving the static non linear programming problem

$$\sup_{\pi \in K} f(\pi) = \hat{\mu} \cdot \pi - \beta\sqrt{\pi \cdot D\pi}, \tag{9}$$

where $K = \mathbb{R}^n$ in the unconstrained case, $D = \sigma^*\sigma$ and the parameter β, the Wang parameter, is defined as $\beta = \alpha/\sqrt{T - t}$. The solution of Eq. (3) is

$$v(t, x) = xe^{r(T-t)}e^{f(\overline{\pi})(T-t)}, \tag{10}$$

where $\overline{\pi}$ is the optimal control.

[6] For more details on Eqs. (9)-(**??**), see [3].

In the unconstrained case it has been shown that the solution of Eq. (9) is

$$f(\overline{\pi}) = \begin{cases} 0 & if \quad \beta > \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}} & \rightarrow \overline{\pi} = 0 \\ 0 & if \quad \beta = \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}} & \rightarrow \overline{\pi} = cD^{-1}\hat{\mu}, \ c \geq 0 \\ \nexists & if \quad 0 < \beta < \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}} & \end{cases} . \tag{11}$$

From Eq. (11) it is clear that in order to obtain a well diversified optimal portfolio the Wang parameter must be $\beta = \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}}$, which implies

$$\overline{\pi} = cD^{-1}\hat{\mu} \tag{12}$$

with $c \geq 0$ and

$$f(\overline{\pi}) = 0.$$

Previous analysis shows that in order to have a fully diversified optimal portfolio solution, the Wang parameter must be chosen as $\beta = \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}}$. The corresponding optimal control is given by Eq. (12), $f(\overline{\pi}) = 0$ and, from Eq. (10),

$$v(t, x) = xe^{r(T-t)}. \tag{13}$$

Using the market price of risk vector q defined as

$$\mu_i - r = \sum_{j=1}^{n} \sigma_{ij} q_j, \tag{14}$$

it can be easily shown that the Wang parameter coincides with the norm of the market price of risk vector given in Eq. (14), $\beta = ||q||$.

The following proposition holds:

Proposition 5.1 The Wang transform with parameter $\beta = \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}}$ is equivalent to a Girsanov transformation with kernel $\theta = q$.

Proof

The portfolio dynamics is given by Eq. (8) and after a Wang transform with parameter $\beta = \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}}$, as shown in appendix A, the dynamics of the optimal portfolio is determined by the following stochastic differential equations:

$$d\overline{X}_s = r\overline{X}_s ds + \overline{X}_s \overline{\pi} \cdot \sigma d\overline{W}_s, \quad s \in [t, T], \tag{15}$$

where $d\overline{W}_s$ is the Wiener processes associated with the transformed measure, hereafter named as the Wang measure. Under the measure $dW_s^{'}$ generated by a Girsanov transformation with kernel θ,

$$dW_s \rightarrow dW_s^{'} = dW_s + \theta_s ds, \ s \in [t, T],$$

the optimal portfolio dynamics is

$$dX_s^{'} = X_s^{'}(r + \hat{\mu} \cdot \overline{\pi})ds + X_s^{'}\overline{\pi} \cdot \sigma(dW_s^{'} - \theta_s ds) \Rightarrow$$

$$dX_s^{'} = X_s^{'}(r + \hat{\mu} \cdot \overline{\pi} - \overline{\pi} \cdot \sigma\theta_s)ds + X_s^{'}\overline{\pi} \cdot \sigma dW_s^{'}. \quad (16)$$

Comparing Eq. (15) with Eq. (16) it can be argued that the previous Girsanov transformation will coincide with the Wang transform if and only if

$$\hat{\mu} \cdot \overline{\pi} - \overline{\pi} \cdot \sigma\theta_s = 0.$$

The thesis follows from the definition of the market price of risk given in Eq. (10),

$$\hat{\mu} = \sigma q \Leftrightarrow q = \sigma^{-1}\hat{\mu},$$

and taking $\theta_s = q$.

■

From proposition 5.1 the effect of the Wang transform with $\beta = \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}}$ over the risky assets price dynamics can be inferred:

Proposition 5.2 After the Wang transform with parameter $\beta = \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}}$ the dynamics of the n risky assets is described by the stochastic differential equations

$$d\overline{P}_i(s) = r\overline{P}_i(s)ds + \overline{P}_i(s)\sum_{j=1}^{n} \sigma_{ij} d\overline{W}_j,\ i = 1, ..., n,\ s \in [t, T],$$

where $d\overline{W}_s$ is the Wiener processes associated with the Wang measure. In other words the measure after the Wang transform with parameter $\beta = \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}}$ is the well known risk-neutral measure $\mathbb{Q}$.

Proof

The dynamics of the n risky assets is described in the "real world" by the stochastic differential equations

$$dP_i(s) = \mu_i P_i(s)ds + P_i(s)\sum_{j=1}^{n} \sigma_{ij} dW_j,\ i = 1, ..., n,\ s \in [t, T].$$

From proposition 5.1. it can be argued that the Wang transform on the n assets dynamics is a Girsanov transformation with kernel given by the market price of risk q: for $i = 1, ..., n$, $s \in [t, T]$

$$dP_i(s) \to d\overline{P}_i(s),$$

where

$$d\overline{P}_i(s) = \mu_i\overline{P}_i(s)ds + \overline{P}_i(s)\sum_{j=1}^{n} \sigma_{ij}(d\overline{W}_j - q_j ds) \Rightarrow$$

$$d\overline{P}_i(s) = (\mu_i - \sum_j \sigma_{ij} q_j)\overline{P}_i(s)ds + \overline{P}_i(s)\sum_{j=1}^{n} \sigma_{ij} d\overline{W}_j$$

or, using the definition of the market price of risk given in Eq. (14),

$$d\overline{P}_i(s) = r\overline{P}_i(s)ds + \overline{P}_i(s)\sum_{j=1}^{n}\sigma_{ij}d\overline{W}_j.$$

■

From the latter proposition the dual price of a generic contingent claim can be explicitely deduced. Such a result is expressed in the following proposition:

Proposition 5.3 The dual price at time t of a contingent claim with pay-off $Y_T = Y(P_1(T), P_2(T), ..., P_n(T))$ at maturity time T is

$$\hat{p}_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[Y_T],$$

where $\mathbb{Q}$ is the risk-neutral measure.

Proof

Eq. (6) gives the dual price in the context of the DEU theory:

$$\hat{p}_t = \frac{x}{v(t,x)}\mathbb{E}_g[Y_T].$$

Using the Wang transform with parameter $\beta = \sqrt{\hat{\mu}\cdot D^{-1}\hat{\mu}}$ as distortion function, the solution of Eq. (3) is given by Eq. (13),

$$v(t,x) = xe^{r(T-t)}.$$

The thesis follows from proposition 5.2: the transformed measure is the risk-neutral one,

$$\mathbb{E}_g[Y_T] = \mathbb{E}^{\mathbb{Q}}[Y_T].$$

■

6 Final Remarks

In this paper the price for contingent claims consistent with the DEU theory, the dual price, has been investigated for the first time in a complete arbitrage-free market for n underlying assets following generic Itô processes and without any comonotonicity hypothesis. Using the results of a recent paper on dynamic portfolio selection in a DEU theory framework the dual price for contingent claims has been deduced. In particular it has been shown that if the n underlying follows geometrical brownian motion, using the Wang transform as distortion operator, the standard Black-Scholes-Merton valuation formula, based on the discounted contingent claim expected pay-off under the risk-neutral measure, is obtained.

A Portfolio Dynamics After the Wang Transform

If the price process follows a geometric brownian motion with constant coefficients, the "real world" portfolio dynamics is given by Eq. (8),

$$dX_s = X_s(r + \hat{\mu} \cdot \pi)ds + X_s \pi \cdot \sigma dW_s,$$

with $s \in [t, T]$ and $X_t = x$. Hence $\ln X_T^{t,x,\pi} \sim Normal(m, \Sigma^2)$, where

$$m = \ln x + [r + \hat{\mu} \cdot \pi - \frac{1}{2}\pi \cdot D\pi](T - t), \tag{17}$$

$$\Sigma^2 = \pi \cdot D\pi(T - t), \tag{18}$$

with $D = \sigma^* \sigma$.

The conditional probability function of the final wealth, $F_{t,x}^{\pi}(y) = Prob(X_T^{t,x,\pi} \leq y | X_t^{t,x,\pi} = x)$ is

$$F_{t,x}^{\pi}(y) = \Phi(\frac{\ln y - m}{\Sigma}),$$

where $\Phi(\cdot)$ is the normal cumulative distribution function. After the Wang transform the conditional probability function of the final wealth is distorted:

$$g(F_{t,x}^{\pi}(y)) = \Phi[\Phi^{-1}(F_{t,x}^{\pi}(y)) + \alpha] = \Phi(\frac{\ln y - m + \alpha\Sigma}{\Sigma}). \tag{19}$$

Using expressions (17) and (18) for m and Σ it can be easily shown that

$$\frac{\ln y - m + \alpha\Sigma}{\Sigma} = \frac{\ln y - \ln x - \{r + [\hat{\mu} \cdot \pi - \beta\sqrt{\pi \cdot D\pi}] - \frac{1}{2}\pi \cdot D\pi\}(T - t)}{\Sigma}. \tag{20}$$

For $\beta = \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}}$ and $\overline{\pi} = cD^{-1}\hat{\mu},\ c \geq 0$, results

$$\hat{\mu} \cdot \overline{\pi} - \beta\sqrt{\overline{\pi} \cdot D\overline{\pi}} =$$

$$\hat{\mu} \cdot cD^{-1}\hat{\mu} - \sqrt{\hat{\mu} \cdot D^{-1}\hat{\mu}}\sqrt{cD^{-1}\hat{\mu} \cdot DcD^{-1}\hat{\mu}} =$$

$$c\hat{\mu} \cdot D^{-1}\hat{\mu} - c\hat{\mu} \cdot D^{-1}\hat{\mu} = 0$$

hence Eq. (20) becomes

$$\frac{\ln y - m + \alpha\Sigma}{\Sigma} = \frac{\ln y - \ln x - \{r - \frac{1}{2}\pi \cdot D\pi\}(T - t)}{\Sigma}. \tag{21}$$

Substituting the right hand side of Eq. (21) into Eq. (19) gives

$$g(F_{t,x}^{\pi}(y)) = \Phi[\Phi^{-1}(F_{t,x}^{\pi}(y)) + \alpha] = \Phi(\frac{\ln y - m^W + \alpha\Sigma}{\Sigma}),$$

where

$$m^W = \ln x + [r - \frac{1}{2}\pi \cdot D\pi](T - t). \tag{22}$$

The comparison of Eq. (22) and Eq. (17) shows that after the Wang transform the portfolio dynamics at the optimum value $\overline{\pi} = cD^{-1}\hat{\mu},\ c \geq 0$ is

$$d\overline{X}_s = r\overline{X}_s ds + \overline{X}_s \overline{\pi} \cdot \sigma d\overline{W}_s,$$

where $d\overline{W}_s$ is the Wiener process associated with the Wang measure.

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