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Existence and Uniqueness of a Fixed-Point
for Local Contractions

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Existence and Uniqueness of a Fixed-Point for Local Contractions

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EXISTENCE AND UNIQUENESS OF A FIXED-POINT FOR LOCAL CONTRACTIONS

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This paper proves the existence and uniqueness of a fixed-point for local contractions without assuming the family of contraction coefficients to be uniformly bounded away from 1. More importantly it shows how this fixed-point result can apply to study existence and uniqueness of solutions to some recursive equations that arise in economic dynamics.

KEYWORDS: Fixed-point theorem, Local contraction, Bellman operator, Koopmans operator, Thompson aggregator, Recursive utility.

1. INTRODUCTION

Fixed-point results for local contractions turned out to be useful to solve recursive equations in economic dynamics. Many applications in dynamic programming are presented in Rincón-Zapatero and Rodríguez-Palmero (2003) for the deterministic case and in Matkowski and Nowak (2008) for the stochastic case. Applications to recursive utility problems can be found in Rincón-Zapatero and Rodríguez-Palmero (2007). Previous fixed-point results for local contractions rely on a metric approach.[1] The idea underlying this approach is based on the construction of a metric that makes the local contraction a global contraction in a specific subspace. The construction of an appropriate metric is achieved at the cost of restricting the family of contraction coefficients to be uniformly bounded away from 1. Contrary to the previous literature, we prove a fixed-point result using direct arguments that do not require the application of the Banach Contraction Theorem for a specific metric. The advantage of following this strategy of proof is that it allows us to deal with a family of contraction coefficients that has a supremum equal to 1. In that respect, the proposed fixed-point result generalizes the fixed-point results for local contractions stated in the literature. An additional benefit is that the stated fixed-point theorem applies to operators that are local contractions with respect to an uncountable family of semi-distances.

We exhibit two applications to illustrate that, from an economic perspective, it is important to have a fixed-point result that encompasses local contractions associated with a family of contraction coefficients that are arbitrarily close to 1. The first application deals with the existence and uniqueness of solutions to the Bellman equation in the unbounded case, while the second one addresses the existence and uniqueness of a recursive utility function derived from Thompson aggregators.[2] We also present two applications to illustrate that, in some circumstances, it is relevant not to restrict the cardinality of the family of semi-distances.

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[1]See Rincón-Zapatero and Rodríguez-Palmero (2003), Matkowski and Nowak (2008) and Rincón-Zapatero and Rodríguez-Palmero (2009).

[2]Contrary to Blackwell aggregators, Thompson aggregators may not satisfy a uniform contraction property. See Marinacci and Montrucchio (2007) for details.

The paper is organized as follows: Section 2 defines local contractions and states a fixed-point theorem. Sections 3 and 4 show how the fixed-point result can apply to the issue of existence and uniqueness of solutions to the Bellman and Koopmans equations respectively. In section 5 we present two applications that give rise to local contractions associated with an uncountable family of semi-distances. The proofs of all theorems and propositions are postponed to Appendices.

2. AN ABSTRACT FIXED-POINT THEOREM

In the spirit of Rincón-Zapatero and Rodríguez-Palmero (2007), we state a fixed-point theorem for operators that are local contractions in an abstract space.[3] Let F be a set and $\mathscr{D} = (d_j)_{j \in J}$ be a family of semi-distances defined on F. We let σ be the weak topology on F defined by the family $\mathscr{D}$. A sequence $(f_n)_{n \in \mathbb{N}}$ is said σ-Cauchy if it is d_j-Cauchy for each $j \in J$. A subset A of F is said sequentially σ-complete if every σ-Cauchy sequence in A converges in A for the σ-topology. A subset $A \subset F$ is said σ-bounded if $\mathrm{diam}_j(A) \equiv \sup\{d_j(f, g) : f, g \in A\}$ is finite for every $j \in J$.

DEFINITION 2.1 Let r be a function from J to J. An operator $T : F \to F$ is a *local contraction* with respect to $(\mathscr{D}, r)$ if for every j there exists $\beta_j \in [0, 1)$ such that

$$\forall f, g \in F, \quad d_j(Tf, Tg) \leqslant \beta_j d_{r(j)}(f, g).$$

The main technical contribution of this paper is the following existence and uniqueness result of a fixed-point for local contractions.

THEOREM 2.1 *Assume that the space F is σ-Hausdorff.[4] Consider a function $r : J \to J$ and let $T : F \to F$ be a local contraction with respect to $(\mathscr{D}, r)$. Consider a non-empty, σ-bounded, sequentially σ-complete and T-invariant subset $A \subset F$. If the following condition is satisfied*

$$(2.1) \qquad \forall j \in J, \quad \lim_{n \to \infty} \beta_j \beta_{r(j)} \ldots \beta_{r^n(j)} \mathrm{diam}_{r^{n+1}(j)}(A) = 0$$

then the operator T admits a fixed-point $f^\star$ in A. Moreover, if $h \in F$ satisfies

$$(2.2) \qquad \forall j \in J, \quad \lim_{n \to \infty} \beta_j \beta_{r(j)} \ldots \beta_{r^n(j)} d_{r^{n+1}(j)}(h, A) = 0$$

then the sequence $(T^n h)_{n \in \mathbb{N}}$ is σ-convergent to $f^\star$.[5]

The arguments of the proof of Theorem 2.1 are very simple and straightforward. The details are postponed to Appendix A.

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[3] From now on we should write RZ-RP (2003) for Rincón-Zapatero and Rodríguez-Palmero (2003), RZ-RP (2007) for Rincón-Zapatero and Rodríguez-Palmero (2007) and RZ-RP (2009) for Rincón-Zapatero and Rodríguez-Palmero (2009).

[4] That is, for each pair $f, g \in F$, if $f \neq g$ then there exists $j \in J$ such that $d_j(f, g) > 0$.

[5] If A is a non-empty subset of F then for each h in F, we let $d_j(h, A) \equiv \inf\{d_j(h, g) : g \in A\}$.

REMARK 2.1   Theorem 2.1 generalizes a fixed-point existence result proposed in Hadžić (1979). [6] To be precise, Hadžić (1979) imposed the additional requirement that each semi-distance d_j is the restriction of a semi-norm defined on a vector space E containing F such that E is a locally convex topological vector space. Under such conditions the existence result cannot be used for the two applications proposed in Section 3 and Section 4. Moreover, Hadžić (1979) does not provide any criteria of stability similar to condition (2.2). A detailed comparison of Theorem 2.1 with the result established in Hadžić (1979) is presented in Appendix B.

REMARK 2.2   If h is a function in A then condition (2.2) is automatically satisfied, implying that the fixed-point $f^\star$ is unique in A. Actually $f^\star$ is the unique fixed-point on the set $B \subset F$ defined by

$$B \equiv \left\{ h \in F \,:\, \forall j \in J, \quad \lim_{n \to \infty} \beta_j \beta_{r(j)} \ldots \beta_{r^n(j)} d_{r^{n+1}(j)}(h,A) = 0 \right\}.$$

REMARK 2.3   If the function r is the identity, i.e., $r(j) = j$ then the operator T is said to be a 0-local contraction and, in that case, conditions (2.1) and (2.2) are automatically satisfied. In particular, if a fixed-point exists it is unique on the whole space F.

REMARK 2.4   Assume that the space F is sequentially σ-complete and choose an arbitrary $f \in F$. As in RZ-RP (2007), we can show that the set $F(f)$ defined by

$$F(f) \equiv \left\{ g \in F \,:\, \forall j \in J, \quad d_j(g,f) \leqslant [1/(1-\beta_j)]d_j(Tf,f) \right\}$$

is non-empty, σ-bounded, σ-closed and T-invariant. Applying Theorem 2.1 by choosing $A \equiv F(f)$ we obtain the following corollary.

COROLLARY 2.1   *Let $T : F \to F$ be a 0-local contraction with respect to a family $\mathscr{D} = (d_j)_{j \in J}$ of semi-distances. Assume that the space F is sequentially σ-complete. Then the operator T admits a unique fixed-point $f^\star$ in F. Moreover, for any arbitrary $f \in F$ the sequence $(T^n f)_{n \in \mathbb{N}}$ is σ-convergent to $f^\star$.*

Corollary 2.1 is a generalization of a result first stated in RZ-RP (2003) (see Theorem 1).[7] Unfortunately, the proposed proof in RZ-RP (2003) is not correct. As Matkowski and Nowak (2008) have shown, an intermediate step (Proposition 1b) used in their method of proof is false. RZ-RP (2009) have provided a corrigendum of their fixed-point result but at the cost of assuming that the family $(\beta_j)_{j \in J}$ of contraction coefficients is uniformly bounded away from 1, i.e., $\sup_{j \in J} \beta_j < 1$.[8]

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[6]We are grateful to a referee for pointing out this reference.

[7]If the family J is assumed to be countable then Corollary 2.1 coincides with Theorem 1 in RZ-RP (2007).

[8]Matkowski and Nowak (2008) also prove a similar fixed-point result under this additional assumption.

From an economic perspective, the main contribution of this paper is to show that it is important to establish a fixed-point theorem that allows the contraction coefficients to be arbitrarily closed to 1. The economic applications presented in Section 3 and Section 4 aim to illustrate this fact.

An additional difference of Theorem 2.1 with respect to the fixed-point results of Matkowski and Nowak (2008) and RZ-RP (2009) is that the family J is not assumed to be countable. Although in many applications it is sufficient to consider a countable family of semi-distances, in some circumstances, it may be helpful not to restrict the cardinality of the family of semi-distances. Two applications are presented in Section 5.

REMARK 2.5   An interesting observation about Theorem 2.1 is that its proof only requires each β_j to be non-negative. The requirement that β_j belongs to $[0, 1)$ is used only in the proof of Corollary 2.1.

3. DYNAMIC PROGRAMMING: UNBOUNDED BELOW CASE

We propose to consider the framework of Section 3.3 in RZ-RP (2003). The state space is $X \equiv \mathbb{R}_+^\ell$, there is a technological correspondence $\Gamma : X \to X$, a return function $U : \operatorname{gph} \Gamma \to Z \equiv [-\infty, \infty)$ where $\operatorname{gph} \Gamma$ is the graph of Γ and $\beta \in (0, 1)$ is the discounting factor. Given $x_0 \in X$, we denote by $\Pi(x_0)$ the set of all admissible paths $\widetilde{x} = (x_t)_{t \geqslant 0}$ defined by

$$\Pi(x_0) \equiv \{\widetilde{x} = (x_t)_{t \geqslant 0} : \ \forall t \geqslant 0, \quad x_{t+1} \in \Gamma(x_t)\}.$$

The dynamic optimization problem consists of solving the following maximization problem:

$$v^\star(x_0) \equiv \sup\{S(\widetilde{x}) : \ \widetilde{x} \in \Pi(x_0)\} \quad \text{where} \quad S(\widetilde{x}) \equiv \sum_{t \geqslant 0} \beta^t U(x_t, x_{t+1}).$$

We denote by $C(X, Z)$ the space of continuous functions from X to Z, and we let $C^\star(X)$ be the space of functions f in $C(X, Z)$ such that the restriction of f to $X^\star \equiv X \setminus \{0\}$ takes values in $\mathbb{R}$. Among others, we make the following assumptions.

DP1. The correspondence Γ is continuous with nonempty and compact values.

DP2. The function $U : \operatorname{gph}(\Gamma) \to [-\infty, \infty)$ is continuous on $\operatorname{gph}(\Gamma)$.

DP3. There is a continuous function $q : X^\star \to X^\star$ with $(x, q(x)) \in \operatorname{gph} \Gamma$ and $U(x, q(x)) > -\infty$ for all $x \in X^\star$.

We denote by $\mathscr{B}$ the Bellman operator defined on $C(X, Z)$ as follows:

$$\mathscr{B}f(x) \equiv \sup\{U(x, y) + \beta f(y) : \ y \in \Gamma(x)\}.$$

Under the previous assumptions, the function $\mathscr{B}f$ belongs to $C(X, Y)$. Moreover, for every $f \in C^\star(X)$, we have $\mathscr{B}f(x) \geqslant U(x, q(x)) + \beta f(q(x)) > -\infty$ for all $x \in X^\star$. This implies that $\mathscr{B}$ maps $C^\star(X)$ into $C^\star(X)$. Under suitable conditions, the value

function $v^\star$ coincides with the fixed-point of the Bellman operator $\mathscr{B}$. To establish this relationship, we introduce the following assumptions.[9]

DP4. There exist three functions w_-, w_+, and w in $C^\star(X)$ such that

$$w_- \leqslant w_+ < w \quad \text{and} \quad \frac{w_- - w}{w_+ - w} = O(1) \quad \text{at } 0$$

together with

(a) $\mathscr{B}w < w$, $\mathscr{B}w_- \geqslant w_-$, $\mathscr{B}w_+ \leqslant w_+$

(b) $(w_+ - w)/(\mathscr{B}w - w) = O(1)$ at 0

(c) for any $x_0 \in X^\star$, the set $\Pi^0(x_0)$ is non-empty[10] and for each admissible path $\widetilde{x} = (x_t)_{t \geqslant 0}$ in $\Pi^0(x_0)$ it follows that

$$\lim_{t \to \infty} \beta^t w_-(x_t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \beta^t w_+(x_t) = 0.$$

DP5. There exists a countable increasing family $(K_j)_{j \in \mathbb{N}}$ of non-empty and compact subsets of X such that for any compact subset K of X, there exists j with $K \subset K_j$ and such that $\Gamma(K_j) \subset K_j$ for all $j \in \mathbb{N}$.

We denote by $[w_-, w_+]$ the order interval in $C^\star(X)$, i.e., the space of all functions $f \in C^\star(X)$ satisfying $w_- \leqslant f \leqslant w_+$. The following theorem is analogous to the main result of Section 3.3 (see Theorem 6) in RZ-RP (2003).[11]

THEOREM 3.1 *Assume (DP1)–(DP5). Then the following statements hold:*
(a) *The Bellman equation has a unique solution f in $[w_-, w_+] \subset C^\star(X)$.*
(b) *The value function $v^\star$ is continuous in $X^\star$ and coincides with the fixed-point f.*
(c) *For any function g in $[w_-, w_+]$, the sequence $(\mathscr{B}^n g)_{n \in \mathbb{N}}$ converges to $v^\star$ for the topology associated with the family $(d_j)_{j \in \mathbb{N}}$ of semi-distances defined on the space $[w_-, w_+]$ by*

$$d_j(f, g) \equiv \sup_{x \in K_j^\star} \left| \ln\left(\frac{f - w}{w_+ - w}(x) \right) - \ln\left(\frac{g - w}{w_+ - w}(x) \right) \right|$$

where $K_j^\star = K_j \setminus \{0\}$.

Using the convexity property of the Bellman operator, RZ-RP (2003) proved (refer to page 1553) that the operator $\mathscr{B}$ is a 0-local contraction with respect to the family $(d_j)_{j \in \mathbb{N}}$ where the contraction coefficient β_j is defined by

$$\beta_j \equiv 1 - \exp\{-\mu_j\} \quad \text{with} \quad \mu_j \equiv \sup\{d_j(f, \mathscr{B}w) : f \in [w_-, w_+]\}.$$

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[9]Given two functions f and g in $C^\star(X)$ with $g(x) \neq 0$ in a neighborhood of 0, we say that $f/g = O(1)$ at $x = 0$ if there exists a neighborhood V of 0 in X such that f/g is bounded in $V \setminus \{0\}$.

[10]$\Pi^0(x_0)$ is the subset of $\Pi(x_0)$ of all admissible paths $\widetilde{x}$ in $\Pi(x_0)$ such that $S(\widetilde{x})$ exists and satisfies $S(\widetilde{x}) > -\infty$.

[11]Our set of assumptions is slightly different from the one used by RZ-RP (2003). In particular condition DP4(b) is not imposed in RZ-RP (2003). We make this assumption to ensure that the distance $d_j(f, \mathscr{B}w)$ is well-defined. See Appendix C for details.

Observe that for each j and each pair of functions f, g in $[w_-, w_+]$ we have

$$d_j(f, g) = \sup_{x \in K_j^*} \left| \ln \left(\frac{f - w}{g - w}(x) \right) \right|$$

implying that

$$\mu_j = \max \left\{ \left\| \ln \theta_+ \right\|_{K_j^*}, \left\| \ln \theta_- \right\|_{K_j^*} \right\}$$

where $\theta_+ \equiv (w - w_+)/(w - \mathscr{B}w)$ and $\theta_- \equiv (w - w_-)/(w - \mathscr{B}w)$.[12] Since the family $(K_j)_{j \in \mathbb{N}}$ covers the space X, we get

$$\sup_{j \in J} \mu_j = \max \left\{ \left\| \ln \theta_+ \right\|_{X^*}, \left\| \ln \theta_- \right\|_{X^*} \right\}.$$

If either the function $\ln \theta_+$ or the function $\ln \theta_-$ is unbounded, then the supremum $\sup_{j \in J} \beta_j$ of the contraction coefficients is 1. In this case, the fixed-point results of Matkowski and Nowak (2008) and RZ-RP (2009) cannot apply to prove Theorem 3.1. In contrast, Theorem 2.1 makes possible to provide a straightforward proof of Theorem 3.1. To illustrate that it is possible to exhibit economic applications that give rise to an unbounded sequence $(\mu_j)_{j \in J}$ we borrow two examples from RZ-RP (2003).

3.1. *Logarithmic utility function and technology with decreasing returns*

We consider Example 10 in RZ-RP (2003). Fix a function $F : [0, \infty) \to \mathbb{R}$ continuous and strictly increasing with $F(0) = 0$. Moreover, assume that there exists $\overline{x} > 0$ with $F(\overline{x}) = \overline{x}$, $F(x) > x$ for all $x < \overline{x}$ and $F(x) < x$ for all $x > \overline{x}$. We consider the Bellman operator where the action space X is $\mathbb{R}_+$; the correspondence Γ is defined by $\Gamma(x) = [0, F(x)]$ for all $x \in X$; and the utility function U is defined by $U(x, y) = \ln(F(x) - y)$ for all $(x, y) \in \text{gph}\,\Gamma$.

We follow RZ-RP (2003) and pose

$$w_-(x) = \begin{cases} \dfrac{1}{(1-\beta)^2} \ln \dfrac{1}{2} + \dfrac{1}{1-\beta} \ln x & \text{if} \quad x \leqslant \overline{x} \\[3mm] \dfrac{1}{(1-\beta)^2} \ln \dfrac{1}{2} + \dfrac{1}{1-\beta} \ln \overline{x} & \text{if} \quad x > \overline{x}. \end{cases}$$

There exists $\sigma > 0$ small enough such that $\overline{x}^{1-\sigma} x^\sigma \geqslant F(x)$ for every x in $[0, \overline{x}]$. We then pose

$$w_+(x) = \begin{cases} \dfrac{\sigma}{1-\beta\sigma} \ln x + \dfrac{1-\sigma}{(1-\beta)(1-\beta\sigma)} \ln \overline{x} & \text{if} \quad x \leqslant \overline{x} \\[3mm] \dfrac{1}{1-\beta} \ln x & \text{if} \quad x > \overline{x}. \end{cases}$$

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[12]If f is a function in $C(X, Y)$ and K is a subset of X, we let $\left\| f \right\|_K \equiv \sup\{|f(x)| : x \in K\}$.

Finally, the function w is defined by $w(x) = a + (1 - \beta)^{-1} \ln \overline{x}$ if $x \leqslant \overline{x}$ and $w(x) = a + w_+(x)$ if $x \geqslant \overline{x}$ where $a > 0$. We claim that the sequence $(\mu_j)_{j \in J}$ is not bounded.

PROPOSITION 3.1   We have

$$\liminf_{x \to \infty} \theta_+(x) = 0 \quad \text{or} \quad \limsup_{x \to \infty} \theta_-(x) = \infty.$$

PROOF OF PROPOSITION 3.1:   We denote by ξ the function in $C(X^\star)$ defined by $\xi \equiv w - \mathscr{B}w$. For each $x \in X^\star$ we have $\xi(x) > 0$. We split the proof in two cases. First we assume that there exists $M \in (0, \infty)$ such that $\xi(x) \leqslant M$ for x large enough. Observe that for every $x \geqslant \overline{x}$ we have

$$\theta_-(x) = \frac{1}{\xi(x)} \left[a + \frac{1}{1 - \beta} \ln x - w_-(\overline{x}) \right].$$

This implies that $\lim_{x \to \infty} \theta_-(x) = \infty$.

Assume now that $\limsup_{x \to \infty} \xi(x) = \infty$. For every $x \geqslant \overline{x}$ we have $\theta_+(x) = a/\xi(x)$ implying that $\liminf_{x \to \infty} \theta_+(x) = 0$.   Q.E.D.

3.2. *Homogenous utility function and technology with decreasing returns*

We consider Example 11 in RZ-RP (2003). There is a function $F : [0, \infty) \to \mathbb{R}$ strictly increasing and continuously differentiable on $(0, \infty)$ with $F(0) = 0$ and $F'(0^+) > 1$. Moreover, there exists $\overline{x} > 0$ with $F(\overline{x}) = \overline{x}$ and $F(x) < x$ for all $x > \overline{x}$. We consider the Bellman operator where the action space X is $\mathbb{R}_+$; the correspondence Γ is defined by $\Gamma(x) = [0, F(x)]$ for all $x \in X$; the utility function U is defined by $U(x, y) = (F(x) - y)^\theta / \theta$ for all $(x, y) \in \text{gph } \Gamma$ where $\theta < 0$.

We follow RZ-RP (2003) and pose $w \equiv 0$ and $w_+ \equiv \psi$ where $\psi \equiv \mathscr{B}0$. In this example we have $\psi(x) = F(x)^\theta / \theta$. There exists $x_1 \in (0, \overline{x})$ such that $x < F(x)$ for every $x \leqslant x_1$. Then we pose $w_-(x) \equiv (1 - \beta)^{-1}(F(x) - x)^\theta / \theta$ if $x \leqslant x_1$ and $w_-(x) \equiv w_-(x_1)$ if $x \geqslant x_1$. We claim that the sequence $(\mu_j)_{j \in J}$ is not bounded when F is unbounded.

PROPOSITION 3.2   If F is unbounded then $\lim_{x \to \infty} \theta_-(x) = \infty$.

PROOF OF PROPOSITION 3.2:   Observe that for this example we have $\theta_- = w_-/\psi$. It follows that for all $x \geqslant x_1$

$$\theta_-(x) = \frac{(F(x_1) - x_1)^\theta}{1 - \beta} F(x)^{-\theta}.$$

Since F is not bounded, we must have $\lim_{x \to \infty} F(x) = \infty$ and we get the desired result.   Q.E.D.

4. RECURSIVE PREFERENCES FOR THOMPSON AGGREGATORS

Consider a model where an agent chooses consumption streams in the space ℓ_+^∞ of non-negative and bounded sequences $x = (x_t)_{t \in \mathbb{N}}$ with $x_t \geqslant 0$. The space ℓ^∞ is endowed with the sup-norm $\|x\|_\infty \equiv \sup\{|x_t| : t \in \mathbb{N}\}$. We propose to investigate whether it is possible to represent the agent's preference relation on ℓ_+^∞ by a recursive utility function derived from an aggregator

$$W : X \times Y \to Y$$

where $X = \mathbb{R}_+$ and $Y = \mathbb{R}_+$. The answer obviously depends on the assumed properties of the aggregator function W.

After the seminal contribution of Lucas and Stokey (1984), there has been a wide literature dealing with the issue of existence and uniqueness of a recursive utility function derived from aggregators that satisfy a uniform contraction property (Blackwell aggregators). We refer to Becker and Boyd III (1997) for an excellent exposition of this literature.[13] In what follows we explore whether a unique recursive utility function can be derived from Thompson aggregators.

Throughout this section, we assume that W satisfies the following conditions:

ASSUMPTION 4.1 W is a Thompson aggregator as defined by Marinacci and Montrucchio (2007), i.e., the following conditions are satisfied:
W1. The function W is continuous, non-negative, non-decreasing and satisfies the condition $W(0,0) = 0$.
W2. There exists a continuous function $f : X \to Y$ such that $W(x, f(x)) \leqslant f(x)$.[14]
W3. The function W is concave in the second variable at 0.[15]
W4. For every $x > 0$ we have $W(x, 0) > 0$.

REMARK 4.1 We can find in Marinacci and Montrucchio (2007) a list of examples of Thompson aggregators that do not satisfy a uniform contraction property. For instance, one may consider $W(x, y) = (x^\eta + \beta y^\sigma)^{1/\rho}$ where $\eta, \sigma, \rho, \beta > 0$ together with the following conditions: $\sigma < 1$ and either $\sigma < \rho$ or $\sigma = \rho$ and $\beta < 1$. Another example is the aggregator introduced by Koopmans, Diamond, and Williamson (1964): $W(x, y) = (1/\theta) \ln(1 + \eta x^\delta + \beta y)$ with $\theta, \beta, \delta, \eta > 0$. This aggregator is always Thompson but it is Blackwell only if $\beta < \theta$.

In order to define formally the concept of a recursive utility function we need to introduce some notations. We denote by π the linear functional from ℓ^∞ to $\mathbb{R}$

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[13]See also: Epstein and Zin (1989), Boyd III (1990), Duran (2000), Duran (2003), Le Van and Vailakis (2005) and Rincón-Zapatero and Rodríguez-Palmero (2007).

[14]Marinacci and Montrucchio (2007) assume that there is a sequence $(x^n, y^n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+^2$ with $(x^n)_{n \in \mathbb{N}}$ increasing to infinite and $W(x^n, y^n) \leqslant y^n$ for each n. This assumption, together with the others, implies that for each $x \in X$, there exists $y_x \in Y$ such that $W(x, y_x) \leqslant y_x$. We require that we can choose $x \mapsto y_x$ continuous.

[15]In the sense that $W(x, \alpha y) \geqslant \alpha W(x, y) + (1 - \alpha) W(x, 0)$ for each $\alpha \in [0, 1]$ and each $x, y \in \mathbb{R}_+$.

defined by $\pi x = x_0$ for every $x = (x_t)_{t \in \mathbb{N}}$ in ℓ^∞. We denote by σ the operator of ℓ^∞ defined by $\sigma x = (x_{t+1})_{t \in \mathbb{N}}$.

DEFINITION 4.1   Let $\mathbb{X}$ be a subset of ℓ^∞ stable under the shift operator σ.[16] A function $u : \mathbb{X} \to \mathbb{R}$ is a *recursive utility function* on $\mathbb{X}$ if

$$\forall x \in \mathbb{X}, \quad u(x) = W(\pi x, u(\sigma x)).$$

We propose to show that we can use the Thompson metric introduced by Thompson (1963) to prove the existence of a continuous recursive utility function when the space $\mathbb{X}$ is the subset of all sequences in ℓ^∞_+ which are uniformly bounded away from 0, i.e., $\mathbb{X} \equiv \{x \in \ell^\infty : \inf_{t \in \mathbb{N}} x_t > 0\}$.[17] The topology on $\mathbb{X}$ derived from the sup-norm is denoted by τ. This space of feasible consumption patterns also appears in Boyd III (1990).

4.1. *The operator*

In the spirit of Marinacci and Montrucchio (2007) we introduce the following operator. First, denote by $\mathcal{V}$ the space of sequences $V = (v_t)_{t \in \mathbb{N}}$ where v_t is a τ-continuous function from $\mathbb{X}$ to $\mathbb{R}_+$. The real number $v_t(x)$ is interpreted as the utility at time t derived from the consumption stream $x \in \mathbb{X}$. For each sequence of functions $V = (v_s)_{s \in \mathbb{N}}$ and each period t, we denote by $[TV]_t$ the function from $\mathbb{X}$ to $\mathbb{R}_+$ defined by

$$\forall x \in \mathbb{X}, \quad [TV]_t(x) \equiv W(x_t, v_{t+1}(x)).$$

Since W and v_{t+1} are continuous the function $[TV]_t$ is continuous. In particular, the mapping T is an operator on $\mathcal{V}$, i.e., $T(\mathcal{V}) \subset \mathcal{V}$.

We denote by $\mathcal{K}$ the family of all sets $\mathbb{K} = [a\mathbf{1}, b\mathbf{1}]$ with $0 < a < b < \infty$.[18] We consider the subspace F of $\mathcal{V}$ composed of all sequences V such that on every set $\mathbb{K} = [a\mathbf{1}, b\mathbf{1}] \in \mathcal{K}$ the family $V = (v_t)_{t \in \mathbb{N}}$ is uniformly bounded from above and away from 0, i.e., $V = (v_t)_{t \in \mathbb{N}}$ belongs to F if for every $0 < a < b < \infty$ there exist $\underline{v}$ and $\overline{v}$ such that

$$\forall t \in \mathbb{N}, \quad \forall x \in [a\mathbf{1}, b\mathbf{1}], \quad 0 < \underline{v} \leqslant v_t(x) \leqslant \overline{v} < \infty.$$

Observe that T maps F into F since W is monotone with respect to both variables.[19] The objective is to show that T admits a unique fixed-point $V^\star$ in F. The reason is that if $V^\star = (v_t^\star)_{t \in \mathbb{N}}$ is a fixed-point of T then the function $v_0^\star$ is a recursive utility

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[16]I.e., for every $x \in \mathbb{X}$ we have σx still belongs to $\mathbb{X}$.

[17]See also Montrucchio (1998) for another reference where the Thompson metric is used.

[18]We denote by $\mathbf{1}$ the sequence $x = (x_t)_{t \in \mathbb{N}}$ in ℓ^∞ defined by $x_t = 1$ for every t. The order interval $[a\mathbf{1}, b\mathbf{1}]$ is the set $\{x \in \ell^\infty_+ : a \leqslant x_t \leqslant b, \ \forall t \in \mathbb{N}\}$.

[19]We can easily check that for every $V = (v_t)_{t \in \mathbb{N}}$ in F and for every $\mathbb{K} \equiv [a\mathbf{1}, b\mathbf{1}]$, we have $W(a, \underline{v}) \leqslant [TV]_t(x) \leqslant W(b, \overline{v})$.

function. Indeed, we will show that for each consumption stream $x \in \mathbb{X}$ and every time t, we have $\lim_{n\to\infty}[T^n 0]_t(x) = v_t^\star(x)$. Since $[T^n 0]_t(\sigma x) = [T^n 0]_{t+1}(x)$ and

$$[T^n 0]_t(x) = W(x_t, W(x_{t+1}, \ldots, W(x_{t+n}, 0)\ldots)),$$

passing to the limit we get that $v_t^\star(\sigma x) = v_{t+1}^\star(x)$. This property is crucial in order to prove that $v_0^\star$ is a recursive utility on $\mathbb{X}$. Indeed, we have $v_0^\star(x) = [TV^\star]_0(x) = W(x_0, v_1^\star(x)) = W(x_0, v_0^\star(\sigma x))$.[20]

4.2. *The Thompson metric*

Fix a set $\mathbb{K}$ in $\mathcal{K}$. We propose to introduce the semi-distance $d_\mathbb{K}$ on F defined as follows:[21]

$$d_\mathbb{K}(V, V') \equiv \max\{\ln M_\mathbb{K}(V|V'), \ln M_\mathbb{K}(V'|V)\}$$

where

$$M_\mathbb{K}(V|V') \equiv \inf\{\alpha > 0 : \forall x \in \mathbb{K}, \quad \forall t \in \mathbb{N}, \quad v_t(x) \leqslant \alpha v_t'(x)\}.$$

Let $V^\infty \in \mathcal{V}$ be the sequence of functions $(v_t^\infty)_{t\in\mathbb{N}}$ defined by $v_t^\infty(x) \equiv f(\|x\|_\infty)$. Observe that $[TV^\infty]_t(x) \leqslant v_t^\infty(x)$ for every $t \in \mathbb{N}$ and every x in $\mathbb{X}$. We denote by V^0 the sequence of functions $T0 = ([T0]_t)_{t\in\mathbb{N}}$, i.e., $V^0 = (v_t^0)_{t\in\mathbb{N}}$ with $v_t^0(x) = W(x_t, 0)$. The monotonicity of T then implies that T maps the order interval $[V^0, V^\infty]$ into $[V^0, V^\infty]$. Moreover, both V^0 and V^∞ belong to F.[22] We can then adapt the arguments of Theorem 9 in (Marinacci and Montrucchio, 2007, Appendix B) to show that T is a 0-local contraction on $[V^0, V^\infty]$ with respect to the family $\mathcal{D} = (d_\mathbb{K})_{\mathbb{K}\in\mathcal{K}}$. More precisely, we can prove that

$$d_\mathbb{K}(TV, TV') \leqslant \beta_\mathbb{K} d_\mathbb{K}(V, V')$$

where $\beta_\mathbb{K} \equiv 1 - [\mu_\mathbb{K}]^{-1}$ and $\mu_\mathbb{K} \equiv M_\mathbb{K}(V^\infty|V^0)$. Recall that

$$M_\mathbb{K}(V^\infty|V^0) \equiv \inf\{\alpha > 0 : \forall x \in \mathbb{K}, \quad \forall t \in \mathbb{N}, \quad f(\|x\|_\infty) \leqslant \alpha W(x_t, 0)\}$$

implying that

$$\mu_\mathbb{K} = \sup_{x\in\mathbb{K}} \sup_{t\in\mathbb{N}} \frac{f(\|x\|_\infty)}{W(x_t, 0)} = \sup_{x\in\mathbb{K}} \frac{f(\|x\|_\infty)}{\inf_{t\in\mathbb{N}} W(x_t, 0)} = \frac{f(b)}{W(a, 0)}.$$

The set $[V^0, V^\infty]$ is sequentially complete with respect to the family $\mathcal{D}$.[23] Therefore, we can apply Corollary 2.1 to get the existence of a unique fixed-point $V^\star = (v_t^\star)_{t\in\mathbb{N}}$

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[20]Observe that the time t utility $v_t^\star(x)$ of the consumption stream x does not depend on the past consumption since $v_t^\star(x) = v_{t-1}^\star(\sigma x) = \ldots = v_0^\star(\sigma^t x)$.

[21]The function $d_\mathbb{K}$ is well-defined, we refer to Appendix D for details.

[22]See Appendix D for details.

[23]See Appendix D for details.

of T in $[V^0, V^\infty]$.[24] The function $u^\star \equiv v_0^\star : \mathbb{X} \to \mathbb{R}_+$ is then a recursive utility function associated with the aggregator W and continuous for the sup-norm topology.[25] We have thus provided a sketch of the proof of the following result.[26]

THEOREM 4.1 *Given a Thompson aggregator W, there exists a recursive utility function $u^\star : \mathbb{X} \to \mathbb{R}$ which is continuous on $\mathbb{X}$ for the sup-norm. Moreover, this function is unique among all continuous functions which are bounded on every order interval of $\mathscr{K}$.*

REMARK 4.2 In the spirit of Kreps and Porteus (1978), Epstein and Zin (1989), Ma (1998), Marinacci and Montrucchio (2007) and Klibanoff, Marinacci, and Mukerji (2009), we can adapt the arguments above in order to deal with uncertainty.

REMARK 4.3 Consider the KDW aggregator

$$W(x, y) = (1/\theta)\ln(1 + \eta x^\delta + \beta y)$$

for any $\theta, \beta, \delta, \eta > 0$. Applying Theorem 4.1 we get the existence of a recursive utility function defined on $\mathbb{X}$ and continuous for the sup-norm. When $\beta < \theta$ the aggregator W is Blackwell and the existence of a continuous recursive utility function can be established by applying the Continuous Existence Theorem in Boyd III (1990) or Becker and Boyd III (1997). We propose to show that the case $\beta \geqslant \theta$ is not covered by the Continuous Existence Theorem. Observe first that the lowest $\alpha > 0$ satisfying the uniform Lipschitz condition

$$|W(x, y) - W(x, y')| \leqslant \alpha|y - y'|$$

for all $x > 0$ and $y, y' \geqslant 0$, is $\alpha = \beta/\theta$. Assume by way of contradiction that the conditions of the Continuous Existence Theorem are met. Then there exists a positive continuous function $\varphi : \mathbb{X} \to (0, \infty)$ such that

$$M \equiv \sup_{\boldsymbol{x} \in \mathbb{X}} \frac{W(\pi\boldsymbol{x}, 0)}{\varphi(\boldsymbol{x})} < \infty \quad \text{and} \quad \chi \equiv \sup_{\boldsymbol{x} \in \mathbb{X}} \alpha \frac{\varphi(\sigma\boldsymbol{x})}{\varphi(\boldsymbol{x})} < 1.$$

For every $\boldsymbol{x} \in \mathbb{X}$ and every $n \geqslant 1$, we obtain

$$\begin{aligned}
\alpha^n W(x_n, 0) &\leqslant M\alpha^n \varphi(\sigma^n \boldsymbol{x}) \\
&\leqslant M\left[\alpha\frac{\varphi(\sigma^n\boldsymbol{x})}{\varphi(\sigma^{n-1}\boldsymbol{x})} \times \ldots \times \alpha\frac{\varphi(\sigma\boldsymbol{x})}{\varphi(\boldsymbol{x})}\right]\varphi(\boldsymbol{x}) \\
&\leqslant M\chi^n \varphi(\boldsymbol{x}).
\end{aligned}$$

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[24]Observe that the family of contraction coefficients is such that $\sup_{\mathbb{K} \in \mathscr{K}} \beta_\mathbb{K} = 1$. We will show that uniqueness is obtained on the whole set F.

[25]Observe that $u^\star$ is non-decreasing. This follows from the fact that

$$u^\star(\boldsymbol{x}) = \lim_{n \to \infty} W(x_0, W(x_1, \ldots, W(x_n, 0)\ldots)).$$

[26]See Appendix D for details.

Choosing $x = a\mathbf{1}$ for any $a > 0$, we get

$$\forall n \geqslant 1, \quad \alpha^n W(a, 0) \leqslant M \chi^n \varphi(a\mathbf{1}).$$

Since $\alpha \geqslant 1$ and $\chi < 1$, it follows that $W(a, 0) = 0$ for every $a > 0$: contradiction.

5. RECURSIVE PREFERENCES FOR BLACKWELL AGGREGATORS

We borrow the notations of Section 4 and consider a model where an agent chooses consumption streams in the space ℓ_+^∞. We propose to investigate if it is possible to represent the agent's preference relation on ℓ_+^∞ by a recursive utility function derived from an aggregator $W : X \times Y \to Y$ where $X = \mathbb{R}_+$ and Y is a subset of $[-\infty, \infty)$ containing 0. The answer will obviously depend on the properties the aggregator W satisfies. Throughout this section, we will assume that W is a Blackwell aggregator, i.e., W is continuous on $X \times Y$, non-decreasing on $X \times Y$ and satisfies a Lipschitz condition with respect to its second argument, i.e., there exists $\delta \in (0, 1)$ such that

$$|W(x, y) - W(x, y')| \leqslant \delta |y - y'|, \quad \forall x \in X, \quad \forall y, y' \in Y.$$

The objective is to find a subspace $\mathbb{X} \subset \ell_+^\infty$ stable under σ such that W admits a recursive utility function from $\mathbb{X}$ to $\mathbb{R}$.

Taking ℓ^∞ as the commodity space is a choice that is made in many intertemporal models.[27] The advantage of ℓ^∞ with respect to other spaces (for instance ℓ^p with $1 \leqslant p < \infty$) is that it does not impose severe restrictions on the kind of dynamics that can be considered (see Chapter 15 in Stockey, Lucas, and Prescott (1989) for a discussion). In addition, the existence of a non-empty interior for ℓ_+^∞ simplifies considerably the application of a separation theorem that underlies the theorems of welfare economics in an intertemporal setting (see Lucas and Prescott (1971)).

The choice of ℓ^∞ as a commodity space introduces some complications on the choice of the appropriate topology. One may consider several topologies on ℓ^∞. There is the topology derived from the sup-norm and the product topology. There are also the weak topology $\sigma(\ell^\infty, \ell^1)$, the Mackey topology $\tau(\ell^\infty, \ell^1)$ and the absolute weak topology $|\sigma|(\ell^\infty, \ell^1)$ which is defined as the smallest locally convex-solid topology on ℓ^∞ consistent with the duality $\langle \ell^\infty, \ell^1 \rangle$.[28] In particular we have (see Page 292 in Aliprantis and Border (1999))

$$\sigma(\ell^\infty, \ell^1) \subset |\sigma|(\ell^\infty, \ell^1) \subset \tau(\ell^\infty, \ell^1).$$

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[27] See among others Lucas and Prescott (1971), Bewley (1972), Kehoe, Levine, and Romer (1990), Magill and Quinzii (1994), Levine and Zame (1996) and Alipranits, Border, and Burkinshaw (1997). In some models this choice is imposed directly while in some others it is implied by the assumptions made on the production activity.

[28] This family is the weak topology generated by the family of semi-norms $\{\eta_{\boldsymbol{q}} : \boldsymbol{q} \in \ell^1\}$ where

$$\forall \boldsymbol{x} \in \ell^\infty, \quad \eta_{\boldsymbol{q}} = \langle |\boldsymbol{x}|, |\boldsymbol{q}| \rangle = \sum_{t \in \mathbb{N}} |x_t q_t|.$$

Assuming continuity of preference orderings with respect to one of the afore-mentioned topologies plays a crucial role in establishing existence of equilibrium in intertemporal models. As shown by Brown and Lewis (1981), assigning to ℓ^∞ one of these topologies is an abstract way of formalizing the idea that agents are impatient. In particular continuity of preference orders with respect to the Mackey topology permits equilibria of finite horizon economies to approximate the equilibria of infinite horizon economies since it implies that consumption in the very distant future is unimportant.

In what follows we show how our fixed-point result can apply to prove existence of recursive utility functions, defined on subsets of ℓ^∞_+ endowed with a specific topology, in two particular frameworks.

5.1. *Unbounded from below*

In this subsection we allow for aggregators that are unbounded from below. More precisely, we assume that $Y = [-\infty, \infty)$ and that $W(x, y) \in \mathbb{R}$ for every $x \neq 0$ and $y \in \mathbb{R}$. We let $\mathscr{A}$ be the space of sequences $a \in \ell^\infty_+$ such that

$$\sum_{t \in \mathbb{N}} \delta^t |W(a_t, 0)| < \infty$$

and we let $\mathbb{X}$ be the union of all intervals $[a, b\mathbf{1}]$ where $a \in \mathscr{A}$ and $b > \|a\|_\infty$. It is straightforward to see that the set $\mathbb{X}$ is a subset of ℓ^∞_+ stable under σ.[29] We let $\mathscr{K}$ be the set of all order intervals $\mathbb{K} \equiv [a, b\mathbf{1}]$ where $a \in \mathscr{A}$ and $b > \|a\|_\infty$. A direct consequence of Theorem 2.1 is the following existence result.

PROPOSITION 5.1 There exists a recursive utility function $U : \mathbb{X} \to \mathbb{R}$ continuous for the product topology on every order interval $\mathbb{K}$ in $\mathscr{K}$. Moreover, for any function $V : \mathbb{X} \to \mathbb{R}$ continuous for the product topology on every order interval $\mathbb{K}$ in $\mathscr{K}$ satisfying

(5.1) $\qquad \forall \mathbb{K} \in \mathscr{K}, \quad \lim_{t \to \infty} \delta^t \sup_{x \in \mathbb{K}} |V(\sigma^t x)| = 0$

we have[30]

$$\lim_{s \to \infty} \sup_{x \in \mathbb{K}} |U(x) - W(x_0, W(x_1, \ldots, W(x_s, V(\sigma^{s+1} x)) \ldots))| = 0.$$

The proof of Proposition 5.1 is based on an application of Theorem 2.1 with an uncountable family of semi-distances. We refer to Appendix E for the details.

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[29]When $W(0, y) = -\infty$ for any $y \in Y$, a feasible consumption stream in $\mathbb{X}$ must be strictly positive.

[30]This implies that U is the unique fixed-point of T on the set of all functions $V : \mathbb{X} \to \mathbb{R}$ bounded and continuous for the product topology on every order interval $\mathbb{K}$ in $\mathscr{K}$ satisfying (5.1).

5.2. *Weak absolute continuity*

In this subsection we restrict our attention to aggregators that are bounded from below. More precisely, we assume that $Y = [0, \infty)$ and for simplicity we impose $W(0,0) = 0$. We will also assume that for any $y \in Y$, the function $x \mapsto W(x,y)$ is concave. We show that under our assumptions, there exists a recursive utility function defined on ℓ_+^∞ and continuous for the absolute weak topology, and in particular for the Mackey topology.[31]

PROPOSITION 5.2   There exists a recursive utility function $U : \ell_+^\infty \to \mathbb{R}$ which is continuous for the absolute weak topology. Moreover, the function U is the unique recursive utility function among all functions $V : \ell_+^\infty \to \mathbb{R}$ continuous for the absolute weak topology and satisfying

$$(5.2) \qquad \lim_{t \to \infty} \delta^t \sup_{\boldsymbol{x} \in \mathbb{K}} |V(\sigma^t \boldsymbol{x})| = 0$$

for every non-empty set $\mathbb{K} \subset \ell_+^\infty$ compact for the absolute weak topology.

The proof of Proposition 5.2 is based on an application of Theorem 2.1 with an uncountable family of semi-distances. We refer to Appendix F for the details.

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APPENDIX A: PROOF OF THEOREM 2.1

Fix an element g in A. Since T is a local contraction, for every pair of integers $q > n > 0$, we have

$$d_j(T^q g, T^n g) \leqslant \beta_j d_{r(j)}(T^{q-1} g, T^{n-1} g) \leqslant \ldots \leqslant \beta_j \beta_{r(j)} \ldots \beta_{r^{n-1}(j)} d_{r^n(j)}(T^{q-n} g, g).$$

Since A is T-invariant, $T^{q-n} g$ belongs to A and we get

$$d_j(T^q g, T^n g) \leqslant \beta_j \beta_{r(j)} \ldots \beta_{r^{n-1}(j)} \operatorname{diam}_{r^n(j)}(A).$$

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[31]Stroyan (1983) also proves existence and uniqueness of a Mackey continuous recursive utility function for aggregators studied by Koopmans, Diamond, and Williamson (1964). However, the arguments of his proof rely on non-standard analysis.

It follows from condition (2.1) that the sequence $(T^n g)_{n \in \mathbb{N}}$ is d_j-Cauchy for each j. Since A is assumed to be sequentially σ-complete, there exists $f^\star$ in A such that $(T^n g)_{n \in \mathbb{N}}$ is σ-convergent to $f^\star$. We claim that $f^\star$ satisfies all properties of Theorem 2.1.

CLAIM A.1   The function $f^\star$ is a fixed-point of T.

PROOF OF CLAIM A.1:   Since the sequence $(T^n g)_{n \in \mathbb{N}}$ converges for the topology σ to $f^\star$, we have

$$\forall j \in J, \quad d_j(Tf^\star, f^\star) = \lim_{n \to \infty} d_j(Tf^\star, T^{n+1}g).$$

Recall that the operator T is a local contraction with respect to $(\mathscr{D}, r)$, this implies that

$$\forall j \in J, \quad d_j(Tf^\star, f^\star) \leqslant \beta_j \lim_{n \to \infty} d_{r(j)}(f^\star, T^n g).$$

Since convergence for the σ-topology implies convergence for the semi-distance $d_{r(j)}$, we get that $d_j(Tf^\star, f^\star) = 0$ for every $j \in J$. This in turn implies that $Tf^\star = f^\star$ since σ is Hausdorff.   *Q.E.D.*

CLAIM A.2   For every $h \in F$ satisfying (2.2), the sequence $(T^n h)_{n \in \mathbb{N}}$ is σ-convergent to $f^\star$.

PROOF OF CLAIM A.2:   Fix an arbitrary $h \in F$. For each $j \in J$ and every $n \geqslant 1$, we have

$$
\begin{aligned}
d_j(T^{n+1}h, T^{n+1}f^\star) \quad &\leqslant \quad \beta_j d_{r(j)}(T^n h, T^n f^\star) \\
&\leqslant \quad \beta_j \beta_{r(j)} \ldots \beta_{r^n(j)} d_{r^{n+1}(j)}(h, f^\star) \\
&\leqslant \quad \beta_j \beta_{r(j)} \ldots \beta_{r^n(j)} \left[d_{r^{n+1}(j)}(h, A) + \text{diam}_{r^{n+1}(j)}(A) \right].
\end{aligned}
$$

Since $Tf^\star = f^\star$, it follows from conditions (2.1) and (2.2) that $(T^n h)_{n \in \mathbb{N}}$ is d_j-convergent to $f^\star$. Since this is true for every j we have thus proved that $(T^n h)_{n \in \mathbb{N}}$ is σ-convergent to $f^\star$.   *Q.E.D.*

The proof of Theorem 2.1 follows from Claims A.1 and A.2.

APPENDIX B: RELATION TO THE LITERATURE

Consider a set F and a family $\mathscr{D} = (d_j)_{j \in J}$ of semi-distances on F such that F is σ-Hausdorff where we recall that σ is the weak topology defined by the family $\mathscr{D}$. Fix $r : J \to J$ and let $T : F \to F$ be a local Lipschitz function with respect to $(\mathscr{D}, r)$ in the sense that for every j there exists $\beta_j \geqslant 0$ such that[32]

$$\forall f, g \in F, \quad d_j(Tf, Tg) \leqslant \beta_j d_{r(j)}(f, g).$$

Assume that F is sequentially σ-complete. We propose to apply Theorem 2.1 for a specific set A. Assume that there exists f in F such that the series

$$(B.1) \qquad \sum_{n=0}^{\infty} \beta_j \beta_{r(j)} \ldots \beta_{r^n(j)} d_{r^{n+1}(j)}(f, Tf)$$

is convergent for every $j \in J$. Denote by $\mathscr{O}(f)$ the orbit of f and let A be the σ-closure of $\mathscr{O}(f)$.[33] The set A is T-invariant and sequentially σ-complete. We first prove that A is σ-bounded. Fix $j \in J$ and observe that

$$\text{diam}_j(A) \equiv \sup\{d_j(f, g) : f, g \in A\} = \text{diam}_j(\mathscr{O}(f)) \leqslant 2 \sup_{n \in \mathbb{N}} d_j(T^{n+1}f, f).$$

Since T is a local Lipschitz function with respect to $(\mathscr{D}, r)$, we get that for every $n \geqslant 1$

$$d_j(T^{n+1}f, f) \leqslant d_j(Tf, f) + \beta_j d_{r(j)}(Tf, f) + \ldots + \beta_j \beta_{r(j)} \ldots \beta_{r^{n-1}(j)} d_{r^n(j)}(Tf, f).$$

---

[32]If $\beta_j \in [0, 1)$ for each j then F is a local contraction. The concept of a local Lipschitz function was first introduce by Hadžić (1979) in a more specific framework.

[33]The orbit of f is the set $\mathscr{O}(f) \equiv \{T^n f : n \in \mathbb{N}\}$.

This implies that

$$(B.2) \qquad \mathrm{diam}_j(A) \leqslant 2\left[d_j(f, Tf) + \sum_{n=0}^{\infty} \beta_j \beta_{r(j)} \dots \beta_{r^n(j)} d_{r^{n+1}(j)}(f, Tf) \right] < \infty$$

and the set A is σ-bounded. From (B.2) we have that for each $n \geqslant 1$,

$$\beta_j \beta_{r(j)} \dots \dots \beta_{r^n(j)} \, \mathrm{diam}_{r^{n+1}(j)}(A) \leqslant 2 \sum_{k=0}^{\infty} \beta_j \beta_{r(j)} \dots \beta_{r^{k+n}(j)} d_{r^{k+n+1}(j)}(f, Tf)$$

implying that (2.1) follows from (B.1). We can thus apply Theorem 2.1 to get the following corollary which generalizes Lemma 2 in Hadžić (1979).[34]

COROLLARY B.1 *Consider a family $\mathscr{D} = (d_j)_{j \in J}$ of semi-distances defined on a set F such that F is Hausdorff and sequentially complete with respect to the associated topology σ. Let $T : F \to F$ be a locally Lipschitz operator with respect to $(\mathscr{D}, r)$ for some $r : J \to J$. Assume that there exists f in F satisfying (B.1). Then T admits a unique fixed point in the closure of the orbit of f.*

REMARK B.1   In Hadžić (1979) it is assumed that each semi-distance d_j is the restriction of a semi-norm defined on a vector space E containing F such that E is a locally convex topological vector space. We have proved that this assumption is superfluous. Moreover, Hadžić (1979) does not provide any criteria of stability similar to condition (2.2).

APPENDIX C: PROOF OF THEOREM 3.1

For any subset A of X, we denote by $A^{\star}$ the set $A \setminus \{0\}$. Recall that $C^{\star}(X)$ is the space of all continuous functions from $X = \mathbb{R}_+^{\ell}$ to $Z = [-\infty, \infty)$ such that $f(x) > -\infty$ for every $x \neq 0$. Let $F = [w_-, w_+]$ be the order interval in $C^{\star}(X)$, i.e., the space of all functions $f \in C^{\star}(X)$ satisfying $w_- \leqslant f \leqslant w_+$.

REMARK C.1   If $w_-(0) > -\infty$ then every function f in F takes values in $\mathbb{R}$. We claim that if $w_-(0) = -\infty$ then every function f in F satisfies $f(0) = -\infty$.[35]

Observe that for every function f in F, we can construct a function $\Psi(f) : X^{\star} \to \mathbb{R}$ by posing

$$\forall x \in X^{\star}, \quad \Psi(f)(x) \equiv \ln\left(\frac{f - w}{w_+ - w}(x) \right).$$

The function $\Psi(f)$ is continuous on $X^{\star}$. Moreover, for any compact set K of X, the function $\Psi(f)$ is

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[34]Hadžić (1979) allows the operator T to be multi-valued. The arguments of the proof of Theorem 2.1 can easily be adapted to deal with multi-valued operators.

[35]Indeed, if $w_+(0) = -\infty$ the result is trivial. We claim that we always have $w_+(0) = -\infty$. Assume by way of contradiction that $w_+(0) \in \mathbb{R}$. This implies that $w(0)$ also belongs to $\mathbb{R}$. It follows from (DP4) that there exists $M > 1$ and an open neighborhood V of 0 in X such that $w(x) - w_-(x) \leqslant M(w(x) - w_+(x))$ for every $x \in V^{\star}$, implying that

$$\forall x \in V^{\star}, \quad -w_-(x) \leqslant (M-1)w(x) - Mw_+(x).$$

Passing to the limit when x tends to 0, we obtain a contradiction.

bounded on $K^\star$.[36] We recall that for each j the function d_j is defined on F by

$$d_j(f, g) \equiv \sup_{x \in K_j^\star} \left| \Psi(f)(x) - \Psi(g)(x) \right|.$$

Given a function f in F, we denote by f_{K_j} the restriction of f to K_j. Denote by F_j the space of all functions f_{K_j} when f belongs to F. Since K_j is a compact subset of X, the space $\Psi(F_j)$ composed of all functions $\Psi(f_{K_j})$ with f_{K_j} in F_j, is a subset of $C_b(K_j^\star)$ the space of continuous and bounded functions defined on $K_j^\star$. It is straightforward to check that d_j is a semi-distance on F. We denote by σ the topology on F defined by the family $\mathscr{D} = (d_j)_{j \in \mathbb{N}}$.

CLAIM C.1 The topology σ is Hausdorff.

PROOF OF CLAIM C.1: Indeed, let f and g two functions in F with $f \neq g$. Assume there exists $x \neq 0$ such that $f(x) \neq g(x)$. Then there exists j large enough such that $x \in K_j$, implying that $d_j(f, g) > 0$. Now assume that $f(x) = g(x)$ for every $x \neq 0$. By continuity at 0, we must have $f(0) = g(0)$ which contradicts the fact that $f \neq g$. *Q.E.D.*

CLAIM C.2 The space F is sequentially σ-complete.

PROOF OF CLAIM C.2: Indeed, let $(f_n)_{n \in \mathbb{N}}$ be a sequence in F which is σ-Cauchy. For every j the sequence is d_j-Cauchy. Recall that for f and g in F we have

$$d_j(f, g) \equiv \sup_{x \in K_j^\star} \left| \Psi(f)(x) - \Psi(g)(x) \right|.$$

We split the analysis in two parts. We first study the interesting case where $w_-(0) = -\infty$.[37] We denote by $C_b(X^\star)$ the space of continuous functions defined on $X^\star$ and bounded in a neighborhood of 0. Since the sequence $(f_n)_{n \in \mathbb{N}}$ is d_j-Cauchy it follows that the sequence $(g_n)_{n \in \mathbb{N}}$ is δ_j-Cauchy where $g_n \equiv \Psi(f_n)$ belongs to $C_b(X^\star)$ and δ_j is the semi-distance defined on $C_b(X^\star)$ by[38]

$$\forall \xi, \varphi \in C_b(X^\star), \quad \delta_j(\xi, \varphi) \equiv \sup\{|\xi(x) - \varphi(x)| : x \in K_j^\star\}.$$

Fix $j \in J$ and let g_n^j be the restriction of g_n to $K_j^\star$. We already proved that (DP4) implies that g_n^j belongs to $C_b(K_j^\star)$. Since the $C_b(K_j^\star)$ endowed with the sup-norm δ_j is a Banach space, there exists a continuous and bounded function $g^j : K_j^\star \to \mathbb{R}$ such that

$$\lim_{n \to \infty} \sup_{x \in K_j^\star} |g_n^j(x) - g^j(x)| = 0.$$

We denote by $g^\star$ the function defined on $X^\star$ by $g^\star(x) = g^j(x)$ where j is such that K_j contains x. Since for every compact set K of X, there exists $j \in J$ such that $K \subset K_j$, we can follow standard arguments

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[36]Indeed, for every x in $X^\star$ we have

$$0 \leqslant \Psi(f)(x) \leqslant \ln \frac{w_- - w}{w_+ - w}(x).$$

It follows from (DP4) that there exists an open neighborhood V of 0 in X such that $(w_- - w)/(w_+ - w)$ is bounded on $V^\star$ by $M > 1$. Now, take f in F and K a compact subset of X. The set $K \setminus V$ is a compact subset of $X^\star$ implying that there exists $M(K, f) > 0$ such that $\Psi(f)(x) \leqslant M(K, f)$ for every $x \in K \setminus V$. Therefore, we have proved that $\Psi(f)$ is bounded on $K^\star$ by $\max\{\ln(M), M(K, f)\}$.

[37]We already proved that (DP4) implies that $w_+(0) = -\infty$. In particular, every function f in F also satisfies $f(0) = -\infty$.

[38]If ξ belongs to $C_b(X^\star)$ then it is bounded on $K_j^\star$. Indeed, there exists V an open neighborhood of 0 such that ξ is bounded on $V^\star$. Since ξ is continuous, it is bounded on the compact set $K_j \setminus V \subset X^\star$.

to show that $g^\star$ is well defined and is continuous on $X^\star$. Observe that the sequence $(g_n)_{n \in \mathbb{N}}$ converges uniformly to $g^\star$ on every set $K_j^\star$. Let us define the function $f^\star : X^\star \to \mathbb{R}$ by

$$\forall x \in X^\star, \quad f^\star(x) = (w_+(x) - w(x))e^{g^\star(x)} + w(x).$$

The function $f^\star$ is continuous on $X^\star$ and the sequence $(d_j(f_n, f^\star))_{n \in \mathbb{N}}$ converges to 0 for every j. Moreover, for every $x \in X^\star$, we have $w_-(x) \leqslant f^\star(x) \leqslant w_+(x)$.[39]

We propose to define the function $f : X \to Y$ by posing $f(x) = f^\star(x)$ if $x \in X^\star$ and $f(0) = -\infty$. To prove that F is sequentially σ-complete, it is sufficient to show that f is continuous on X. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $X^\star$ converging to 0. Observe that

$$f(x_k) = e^{g^\star(x_k) - g_n(x_k)}(w_+(x_k) - w(x_k))e^{g_n(x_k)} + w(x_k).$$

For $\kappa \in \mathbb{N}$ large enough, there exists j such that $\{x_k : k \geqslant \kappa\} \subset K_j$. It follows that for every $k \geqslant \kappa$ we have

$$f(x_k) \leqslant e^{-\delta_j(g^\star, g_n)}(w_+(x_k) - w(x_k))e^{g_n(x_k)} + w(x_k).$$

Since the sequence $(\delta_j(g^\star, g_n))_{n \in \mathbb{N}}$ converges to 0, there exists $m \in \mathbb{N}$ such that $\delta_j(g^\star, g_m) \leqslant 1/2$. Therefore, we have

$$f(x_k) \leqslant e^{-1/2}(w_+(x_k) - w(x_k))e^{g_m(x_k)} + w(x_k).$$

Since $g_m(x) \geqslant 0$, for all $x \in X^\star$, we have

$$f(x_k) \leqslant e^{-1/2}w_+(x_k) + (1 - e^{-1/2})w(x_k).$$

Since $w_+(0) = -\infty$ and we cannot have $w(0) = +\infty$, passing to the limit, we obtain that

$$\limsup_{k \to \infty} f(x_k) = -\infty.$$

If $w_-(0) > -\infty$ then we can replace $K_j^\star$ in the definition of d_j by K_j. In that case, we can follow standard arguments to prove that F is sequentially σ-complete.   *Q.E.D.*

CLAIM C.3   For every $f \in F$ the function $\mathscr{B}f$ also belongs to F.

PROOF OF CLAIM C.3:   It follows from Assumptions (DP1)–(DP3) that $\mathscr{B}$ maps functions in $C^\star(X)$ into functions in $C^\star(X)$. Let f be a function in $F = [w_-, w_+]$. By monotonicity of $\mathscr{B}$ and Assumption (DP4) we get the desired result.   *Q.E.D.*

For every j, the semi-distance d_j is well defined on the set $\mathscr{C}$ of all functions $f \in C^\star(X)$ for which there exist $0 < m < M < \infty$ and a neighborhood V of 0 in X satisfying

$$\forall x \in V^\star, \quad m \leqslant \frac{w - f}{w - w_+}(x) \leqslant M.$$

In other words, $\mathscr{C}$ is the set of all functions $f \in C^\star(X)$ such that at 0 we have

$$\frac{w - f}{w - w_+} = O(1) \quad \text{and} \quad \frac{w - w_+}{w - f} = O(1).$$

Observe that the order interval $[w_-, w_+]$ is a subset of $\mathscr{C}$. Indeed, for every f in $[w_-, w_+]$ and all $x \in X^\star$ we have

$$1 = \frac{w - w_+}{w - w_+}(x) \leqslant \frac{w - f}{w - w_+}(x) \leqslant \frac{w - w_-}{w - w_+}(x).$$

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[39]This follows from the fact that for every n we have

$$\forall x \in X^\star, \quad 0 \leqslant g_n(x) \leqslant \ln[(w(x) - w_-(x))/(w(x) - w_+(x))].$$

Since at 0 we have

$$\frac{w - w_-}{w - w_+} = O(1)$$

there exists a neighborhood V of 0 in X and $0 < M < \infty$ such that

$$\forall x \in V^\star, \quad \frac{w - w_-}{w - w_+}(x) \leqslant M.$$

We claim that the function $\mathscr{B}w$ also belongs to $\mathscr{C}$. Indeed, we know from Assumption (DP4) that $w > w_-$ implying by monotonicity of $\mathscr{B}$ that $\mathscr{B}w \geqslant \mathscr{B}w_- \geqslant w_-$. In particular, we have

$$\forall x \in X^\star, \quad \frac{w - \mathscr{B}w}{w - w_+}(x) \leqslant \frac{w - w_-}{w - w_+}(x).$$

We have proved that $(w - \mathscr{B}w)/(w - w_+) = O(1)$ at 0. The fact that $\mathscr{B}w$ belongs to $\mathscr{C}$ follows from Assumption (DP4.b).

We can now follow the arguments in (Rincón-Zapatero and Rodríguez-Palmero, 2003, p.1553) to prove that for every j we have

$$\forall f, g \in F, \quad d_j(\mathscr{B}f, \mathscr{B}g) \leqslant (1 - \exp\{-\mu_j\}) d_j(f, g)$$

where

$$\mu_j \equiv \sup_{f \in F} d_j(f, \mathscr{B}w).$$

APPENDIX D: PROOF OF THEOREM 4.1

Recall that

- $\mathscr{V}$ is the space of sequences $V = (v_t)_{t \in \mathbb{N}}$ where v_t is a τ-continuous function from $\mathbb{X}$ to $\mathbb{R}_+$;

- $\mathscr{K}$ is the family of all sets $[a\mathbf{1}, b\mathbf{1}]$ with $0 < a < b < \infty$;

- F is the subset of $\mathscr{V}$ composed of all sequences V such that on every set $\mathbb{K}$ the family $V = (v_t)_{t \in \mathbb{N}}$ is uniformly bounded from above and away from 0, i.e., $V = (v_t)_{t \in \mathbb{N}}$ belongs to F if for every $0 < a < b < \infty$ there exist $\underline{v}$ and $\overline{v}$ such that

$$(D.1) \qquad \forall t \in \mathbb{N}, \quad \forall \boldsymbol{x} \in [a\mathbf{1}, b\mathbf{1}], \quad 0 < \underline{v} \leqslant v_t(\boldsymbol{x}) \leqslant \overline{v} < \infty.$$

Consider two functions V and V' in $\mathscr{V}$ and recall that for each set $\mathbb{K} \in \mathscr{K}$, the number $M_{\mathbb{K}}(V|V')$ is defined by

$$M_{\mathbb{K}}(V|V') \equiv \inf\{\alpha > 0 : \forall \boldsymbol{x} \in \mathbb{K}, \quad \forall t \in \mathbb{N}, \quad v_t(\boldsymbol{x}) \leqslant \alpha v_t'(\boldsymbol{x})\}.$$

The functions V and V' in $\mathscr{V}$ are said to be comparable if $M_{\mathbb{K}}(V|V') \in (0, \infty)$ and $M_{\mathbb{K}}(V'|V) \in (0, \infty)$ for every $\mathbb{K}$ in $\mathscr{K}$. This defines an equivalence relation on $\mathscr{V}$ and the set of all functions V' in $\mathscr{V}$ comparable to a function V is called the component of V and is denoted by $\mathscr{C}_V$.

Observe that any pair of functions in F are comparable. Indeed, assume that V and V' belong to F and fix $\mathbb{K}$ in $\mathscr{K}$. We let $(\underline{v}, \overline{v})$ and $(\underline{v}', \overline{v}')$ be the real numbers satisfying (D.1) for V and V' respectively. It is straightforward to check that

$$0 < \underline{v}/\overline{v}' \leqslant M_{\mathbb{K}}(V|V') \leqslant \overline{v}/\underline{v}' < \infty \quad \text{and} \quad 0 < \underline{v}'/\overline{v} \leqslant M_{\mathbb{K}}(V'|V) \leqslant \overline{v}'/\underline{v} < \infty.$$

Moreover, if $M_{\mathbb{K}}(V|V') < 1$ then for every $t \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{K}$ we have

$$v_t(\boldsymbol{x}) < v_t'(\boldsymbol{x}) \leqslant M_{\mathbb{K}}(V'|V) v_t(\boldsymbol{x})$$

implying that $M_{\mathbb{K}}(V'|V) > 1$. It follows that

$$d_{\mathbb{K}}(V, V') \equiv \max\{\ln M_{\mathbb{K}}(V|V'), \ln M_{\mathbb{K}}(V'|V)\} \geqslant 0.$$

As a consequence, fixing an arbitrary V in F, the set F is a subset of the component $\mathscr{C}_V$. Actually the set F coincides with the component $\mathscr{C}_V$, i.e., if V' is a function in $\mathscr{V}$ comparable to V then V' belongs to F.

LEMMA D.1 *For each $\mathbb{K} \in \mathscr{K}$, the function $d_{\mathbb{K}}$ is a semi-distance on F. The topology σ on F defined by the family $\mathscr{D} = (d_{\mathbb{K}})_{\mathbb{K} \in \mathscr{K}}$ is Hausdorff.*

PROOF OF LEMMA D.1:   Consider a sequence $V = (v_t)_{t \in \mathbb{N}}$ in F. Given a set $\mathbb{K}$ in $\mathscr{K}$, we denote by $V^{\mathbb{K}} = (v_t^{\mathbb{K}})_{t \in \mathbb{N}}$ the sequence of functions from $\mathbb{K}$ to $\mathbb{R}_+$ where $v_t^{\mathbb{K}}$ is the restriction of v_t to $\mathbb{K}$. The space of all sequences $V^{\mathbb{K}}$ when V belongs to F is denoted by $F^{\mathbb{K}}$. It is straightforward to adapt the arguments in Thompson (1963) to show that $d_{\mathbb{K}}$ is a distance on $F^{\mathbb{K}}$, implying that $d_{\mathbb{K}}$ is a semi-distance on F. Since the family $\mathscr{K}$ covers $\mathbb{X}$ the topology σ on F defined by the family $\mathscr{D} = (d_{\mathbb{K}})_{\mathbb{K} \in \mathscr{K}}$ is Hausdorff.[40]   *Q.E.D.*

We recall that T is the operator on $\mathscr{V}$ defined by

$$\forall t \in \mathbb{N}, \quad \forall \boldsymbol{x} \in \mathbb{X}, \quad [TV]_t(\boldsymbol{x}) \equiv W(x_t, v_{t+1}(\boldsymbol{x})).$$

The mapping T maps F into F. Indeed, fixing $\mathbb{K} = [a\mathbf{1}, b\mathbf{1}]$ in $\mathscr{K}$ we have

$$\forall t \in \mathbb{N}, \quad \forall \boldsymbol{x} \in \mathbb{K}, \quad 0 < W(a, \underline{v}) \leqslant [TV]_t(\boldsymbol{x}) \leqslant W(b, \overline{v}) < \infty.$$

Recall that $V^0 = (v_t^0)_{t \in \mathbb{N}}$ is the element in $\mathscr{V}$ defined by $V^0 \equiv T0$. Observe that

$$\forall t \in \mathbb{N}, \quad \forall \boldsymbol{x} \in \mathbb{X}, \quad v_t^0(\boldsymbol{x}) = W(x_t, 0).$$

If $\mathbb{K} = [a\mathbf{1}, b\mathbf{1}]$ belongs to $\mathscr{K}$ then we have

$$\forall t \in \mathbb{N}, \quad \forall \boldsymbol{x} \in \mathbb{K}, \quad 0 < W(a, 0) \leqslant v_t^0(\boldsymbol{x}) \leqslant W(b, 0) < \infty$$

implying that V^0 belongs to F.

Recall that $V^\infty \in \mathscr{V}$ is the sequence of functions $(v_t^\infty)_{t \in \mathbb{N}}$ defined by $v_t^\infty(\boldsymbol{x}) \equiv f(\|\boldsymbol{x}\|_\infty)$. Fix $\mathbb{K} = [a\mathbf{1}, b\mathbf{1}] \in \mathscr{K}$. Since f is continuous there exist $\underline{z}$ and $\overline{z}$ in $[a, b]$ such that

$$\forall \boldsymbol{x} \in \mathbb{K}, \quad f(\underline{z}) \leqslant f(\|\boldsymbol{x}\|_\infty) \leqslant f(\overline{z}).$$

Since $f(\underline{z}) \geqslant W(\underline{z}, f(\underline{z}))$ we get $f(\underline{z}) > 0$ implying that the sequence V^∞ belongs to F. Moreover, for every $t \in \mathbb{N}$ and every $\boldsymbol{x} \in \mathbb{X}$ we have

$$[TV^\infty]_t(\boldsymbol{x}) = W(x_t, f(\|\boldsymbol{x}\|_\infty)) \leqslant W(\|x\|_\infty, f(\|\boldsymbol{x}\|_\infty)) \leqslant f(\|\boldsymbol{x}\|_\infty)$$

implying that $TV^\infty \leqslant V^\infty$.

The monotonicity of T implies that T maps the order interval $[V^0, V^\infty]$ into $[V^0, V^\infty]$. We can then adapt the arguments of Theorem 9 in (Marinacci and Montrucchio, 2007, Appendix B) to show that T is a 0-local contraction on $[V^0, V^\infty]$ with respect to the family $\mathscr{D} = (d_{\mathbb{K}})_{\mathbb{K} \in \mathscr{K}}$. Recall that we denote by σ the topology defined by $\mathscr{D}$.

LEMMA D.2 *The set $[V^0, V^\infty]$ is sequentially σ-complete.*

PROOF OF LEMMA D.2:   We first relate the distance $d_{\mathbb{K}}$ with the semi-norm $\|\cdot\|_{\mathbb{K}}$ defined by

$$\forall V \in [V^0, V^\infty], \quad \|V\|_{\mathbb{K}} \equiv \sup_{t \in \mathbb{N}} \sup_{\boldsymbol{x} \in \mathbb{K}} |v_t(\boldsymbol{x})|.$$

Fix two sequences $V = (v_t)_{t \in N}$ and $V' = (v_t')_{t \in \mathbb{N}}$ in $[V^0, V^\infty]$ and fix $\mathbb{K} = [a\mathbf{1}, b\mathbf{1}]$ in $\mathscr{K}$. Since

$$\forall \boldsymbol{x} \in \mathbb{K}, \quad \forall t \in \mathbb{N}, \quad v_t(\boldsymbol{x}) \leqslant M_{\mathbb{K}}(V|V') v_t'(\boldsymbol{x})$$

we get that

$$\forall \boldsymbol{x} \in \mathbb{K}, \quad \forall t \in \mathbb{N}, \quad v_t(\boldsymbol{x}) - v_t'(\boldsymbol{x}) \leqslant \left[M_{\mathbb{K}}(V|V') - 1 \right] v_t'(\boldsymbol{x}).$$

Permuting V and V', and using the fact that $\max\{v_t(\boldsymbol{x}), v_t'(\boldsymbol{x})\} \leqslant f(b)$ we obtain that

(D.2) $\left\| V - V' \right\|_{\mathbb{K}} \leqslant \left[\exp\{d_{\mathbb{K}}(V, V')\} - 1 \right] f(b).$

---

[40]The family $\mathscr{K}$ covers $\mathbb{X}$ if for each $\boldsymbol{x} \in \mathbb{X}$ there exists $\mathbb{K} \in \mathscr{K}$ containing $\boldsymbol{x}$.

Now let $(V_n)_{n\in\mathbb{N}}$ be a σ-Cauchy sequence in $[V^0, V^\infty]$. Fix $\mathbb{K} \in \mathcal{K}$ and denote by $V_n^{\mathbb{K}}$ the restriction of V_n to the set $\mathbb{K}$. It follows from (D.2) that the sequence $(V_n^{\mathbb{K}})_{n\in\mathbb{N}}$ is a Cauchy sequence for the sup-norm $\|\cdot\|_{\mathbb{K}}$. Observe that $V_n^{\mathbb{K}}$ belongs to the space of bounded and continuous functions from $\mathbb{N} \times \mathbb{K}$ to $\mathbb{R}$. This space is a Banach space when endowed with the sup-norm $\|\cdot\|_{\mathbb{K}}$. Therefore there exists $V^{\mathbb{K}} = (v_t^{\mathbb{K}})_{t\in\mathbb{N}}$ a sequence of continuous functions $v_t^{\mathbb{K}} : \mathbb{K} \to \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \left\| V_n^{\mathbb{K}} - V^{\mathbb{K}} \right\|_{\mathbb{K}} = 0.$$

Since V_n belongs to $[V^0, V^\infty]$, passing to the limit we get that $V^0(\boldsymbol{x}) \leqslant V^{\mathbb{K}}(\boldsymbol{x}) \leqslant V^\infty(\boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{K}$. Observe that if $\mathbb{K}$ and $\mathbb{K}'$ are two sets in $\mathcal{K}$ satisfying $\mathbb{K} \subset \mathbb{K}'$ then $V^{\mathbb{K}}(\boldsymbol{x}) = V^{\mathbb{K}'}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{K}$. Therefore, we can define without ambiguity the function $V : \mathbb{X} \to \mathbb{R}_+^{\mathbb{N}}$ as follows:

$$\forall \boldsymbol{x} \in \mathbb{X}, \quad V(\boldsymbol{x}) = V^{\mathbb{K}}(\boldsymbol{x})$$

where $\mathbb{K}$ is any set in $\mathcal{K}$ containing $\boldsymbol{x}$.[41] Since $V^0(\boldsymbol{x}) \leqslant V^{\mathbb{K}}(\boldsymbol{x}) \leqslant V^\infty(\boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{K}$, we get that $V^0(\boldsymbol{x}) \leqslant V(\boldsymbol{x}) \leqslant V^\infty(\boldsymbol{x})$. To conclude the proof of Lemma D.2 we only have to show that V is τ-continuous on $\mathbb{X}$. Fix $\boldsymbol{x}$ in $\mathbb{X}$ and let $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{X}$ τ-converging to $\boldsymbol{x}$. Fix $a > 0$ and $b < \infty$ such that

$$a < \inf_{t\in\mathbb{N}} x_t \quad \text{and} \quad \sup_{t\in\mathbb{N}} x_t < b.$$

Observe that $\boldsymbol{x}$ belongs to $\mathbb{K} \equiv [a\mathbf{1}, b\mathbf{1}]$. Since $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges for the sup-norm to $\boldsymbol{x}$, for n large enough $\boldsymbol{x}_n$ also belongs to $\mathbb{K}$. Since $V^{\mathbb{K}}$ is τ-continuous on $\mathbb{K}$, it follows that the sequence $(V(\boldsymbol{x}_n))_{n\in\mathbb{N}}$ converges to $V(\boldsymbol{x})$. *Q.E.D.*

We can apply Corollary 2.1 to get the existence of a unique fixed-point $V^\star = (v_t^\star)_{t\in\mathbb{N}}$ of T in $[V^0, V^\infty]$. Actually uniqueness is obtained in a much larger set.

LEMMA D.3 *Let $V = (v_t)_{t\in\mathbb{N}}$ be a sequence in $\mathcal{V}$ which is a fixed-point of T. If the sequence $(v_t(\boldsymbol{x}))_{t\in\mathbb{N}}$ is bounded from above for every $\boldsymbol{x}$ in $\mathbb{X}$ then V belongs to $[V^0, V^\infty]$. In particular, $V^\star$ is the unique fixed-point of T on F.*

PROOF OF LEMMA D.3: We let $V = (v_t)_{t\in\mathbb{N}}$ be a sequence in $\mathcal{V}$ which is a fixed-point of T. Assume that for every $\boldsymbol{x}$, the sequence $V(\boldsymbol{x})$ is bounded from above, i.e., there exists $\overline{v}(\boldsymbol{x}) \in \mathbb{R}$ such that

$$\forall t \in \mathbb{N}, \quad 0 \leqslant v_t(\boldsymbol{x}) \leqslant \overline{v}(\boldsymbol{x}).$$

Fix $\boldsymbol{x} \in \mathbb{X}$. Since W is non-decreasing we have

$$\forall t \in \mathbb{N}, \quad v_t(\boldsymbol{x}) = W(x_t, v_{t+1}(\boldsymbol{x})) \geqslant W(x_t, 0) = v_t^0(\boldsymbol{x}).$$

We have thus proved that $V \geqslant V^0$. We claim that we also have $V \leqslant V^\infty$. Let $\overline{x} \equiv \|x\|_\infty$. We should prove that for every $t \in \mathbb{N}$ we have $v_t(\boldsymbol{x}) \leqslant f(\overline{x})$. We split the analysis in two cases.

First, assume that for every $t \in \mathbb{N}$ there exists $T \geqslant t$ such that $v_T(\boldsymbol{x}) \leqslant f(\overline{x})$. The monotonicity of W and the definition of f imply

$$v_{T-1}(\boldsymbol{x}) = W(x_{T-1}, v_T(\boldsymbol{x})) \leqslant W(\overline{x}, f(\overline{x})) \leqslant f(\overline{x}).$$

If $t < T - 1$ we reproduce the above argument recursively to show that $v_t(\boldsymbol{x}) \leqslant f(\overline{x})$.

Now, assume that there exists $\tau \in \mathbb{N}$ such that $v_t(\boldsymbol{x}) > f(\overline{x})$ for every $t \geqslant \tau$. Following the arguments in (Marinacci and Montrucchio, 2007, Lemma 4) we can prove that the function $y \mapsto W(x, y)/y$ is strictly decreasing for any $x > 0$. In particular, if $y > f(x)$ then $W(x, y)/y < W(x, f(x))/f(x) \leqslant 1$. It follows that

$$\forall t \geqslant \tau, \quad v_t(\boldsymbol{x}) = W(x_t, v_{t+1}(\boldsymbol{x})) \leqslant W(\overline{x}, v_{t+1}(\boldsymbol{x})) < v_{t+1}(\boldsymbol{x}).$$

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[41]We always have $\boldsymbol{x} \in [\underline{x}\mathbf{1}, \overline{x}\mathbf{1}]$ where $\underline{x} \equiv \inf_{t\in\mathbb{N}} x_t$ and $\overline{x} \equiv \sup_{t\in\mathbb{N}} x_t$. By definition of $\mathbb{X}$, we have $\underline{x} > 0$ and $\overline{x} < \infty$ for each $\boldsymbol{x} \in \mathbb{X}$, implying that $[\underline{x}\mathbf{1}, \overline{x}\mathbf{1}]$ belongs to $\mathcal{K}$.

The sequence $(v_t(\boldsymbol{x}))_{t \geqslant \tau}$ is strictly increasing and bounded by $\overline{v}(\boldsymbol{x})$. Let us denote its limit by $\ell(\boldsymbol{x})$. Since $v_t(\boldsymbol{x}) > f(\overline{x})$ for every $t \geqslant \tau$, we get that $\ell(\boldsymbol{x}) \geqslant f(\overline{x})$. Since the mapping $y \mapsto W(\overline{x}, y)/y$ is strictly decreasing we obtain

$$(\text{D.3}) \qquad \frac{W(\overline{x}, \ell(\boldsymbol{x}))}{\ell(\boldsymbol{x})} < \frac{W(\overline{x}, f(\overline{x}))}{f(\overline{x})} \leqslant 1.$$

Moreover, since

$$\forall t \geqslant \tau, \quad v_t(\boldsymbol{x}) = W(x_t, v_{t+1}(\boldsymbol{x})) \leqslant W(\overline{x}, v_{t+1}(\boldsymbol{x}))$$

passing to the limit we get that

$$(\text{D.4}) \qquad \ell(\boldsymbol{x}) \leqslant W(\overline{x}, \ell(\boldsymbol{x})).$$

Combining (D.3) and (D.4) we get a contradiction.

We have thus proved that $V \leqslant V^\infty$, implying that V belongs to the order interval $[V^0, V^\infty]$.   *Q.E.D.*

The function $u^\star \equiv v_0^\star : \mathbb{X} \to \mathbb{R}_+$ is a recursive utility function associated to the aggregator W and continuous for the sup-norm topology.[42] Actually, we have

$$\forall \mathbb{K} \in \mathscr{K}, \quad \lim_{n \in \mathbb{N}} d_{\mathbb{K}}(T^n 0, V^\star) = 0.$$

This implies that for each $\boldsymbol{x} \in \mathbb{X}$ we have

$$u^\star(\boldsymbol{x}) = \lim_{n \to \infty} W(x_0, W(x_1, \ldots, W(x_n, 0) \ldots)).$$

Since W is non-decreasing with respect to both variables, we get that $u^\star$ is also non-decreasing. This in turn implies that the function $u^\star$ is bounded on every set $\mathbb{K}$ in $\mathscr{K}$.

Now, let $u : \mathbb{X} \to \mathbb{R}_+$ be a continuous function which is bounded on every set $\mathbb{K}$ in $\mathscr{K}$. Assume that u is a recursive utility function. We propose to prove that u coincides with $u^\star$. We let $V = (v_t)_{t \in \mathbb{N}}$ be the sequence in $\mathscr{V}$ defined by $v_t(\boldsymbol{x}) = u(\sigma^t \boldsymbol{x})$. Since u is a recursive utility function then V is a fixed-point of T. We claim that V coincides with $V^\star$. In order to apply Lemma D.3 we only have to show that for every $\boldsymbol{x} \in \mathbb{X}$, the sequence $(v_t(\boldsymbol{x}))_{t \in \mathbb{N}}$ is bounded from above. Fix $\boldsymbol{x} \in \mathbb{X}$. We propose to show that there exists $\overline{v}(\boldsymbol{x}) \in \mathbb{R}$ such that $v_t(\boldsymbol{x}) \leqslant \overline{v}(\boldsymbol{x})$ for every $t \in \mathbb{N}$. We always have $\boldsymbol{x} \in [\underline{x}\mathbf{1}, \overline{x}\mathbf{1}]$ where $\underline{x} \equiv \inf_{t \in \mathbb{N}} x_t$ and $\overline{x} \equiv \sup_{t \in \mathbb{N}} x_t$. Since $\boldsymbol{x}$ belongs to $\mathbb{X}$, the set $\mathbb{K}_{\boldsymbol{x}} \equiv [\underline{x}\mathbf{1}, \overline{x}\mathbf{1}]$ belongs to $\mathscr{K}$. By assumption, the function u is bounded on $\mathbb{K}_{\boldsymbol{x}}$ by some $\overline{u}(\boldsymbol{x}) \in \mathbb{R}$. Observe that for every $t \in \mathbb{N}$ we have $\sigma^t \boldsymbol{x}$ belongs to $\mathbb{K}_{\boldsymbol{x}}$, implying that

$$\forall t \in \mathbb{N}, \quad v_t(\boldsymbol{x}) = u(\sigma^t \boldsymbol{x}) \leqslant \overline{u}(\boldsymbol{x}).$$

We can thus choose $\overline{v}(\boldsymbol{x}) \equiv \overline{u}(\boldsymbol{x})$ and we have proved that the sequence $(v_t(\boldsymbol{x}))_{t \in \mathbb{N}}$ is bounded from above.

Applying Lemma D.3, we get that $V = V^\star$ implying that $u = u^\star$.

APPENDIX E: PROOF OF PROPOSITION 5.1

Let F be the space of functions $V : \mathbb{X} \to \mathbb{R}$ continuous for the product topology on every $\mathbb{K} \in \mathscr{K}$.[43] For every set $\mathbb{K} \in \mathscr{K}$ we let $d_{\mathbb{K}}$ be the semi-distance on F defined by

$$d_{\mathbb{K}}(U, V) \equiv \sup\{|U(\boldsymbol{x}) - V(\boldsymbol{x})| : \boldsymbol{x} \in \mathbb{K}\} = \|U - V\|_{\mathbb{K}}.$$

---

[42]Modifying the definition of the set $\mathscr{V}$ one can prove that $u^\star$ is continuous for the product topology on every order interval $[a\mathbf{1}, b\mathbf{1}]$ in $\mathscr{K}$.

[43]Recall that $\mathscr{K}$ is the set of all order intervals $\mathbb{K} \equiv [\boldsymbol{a}, b\mathbf{1}]$ where $b > \|\boldsymbol{a}\|_\infty$ and $\boldsymbol{a}$ is a sequence in ℓ_+^∞ satisfying

$$\sum_{t \in \mathbb{N}} \delta^t |W(a_t, 0)| < \infty.$$

The space F is sequentially complete with respect to the topology defined by the family $\mathscr{D} \equiv (d_{\mathbb{K}})_{\mathbb{K} \in \mathscr{K}}$. Observe that if $\mathbb{K}$ belongs to $\mathscr{K}$ then $\sigma\mathbb{K} = \{\sigma\boldsymbol{x}\colon \boldsymbol{x} \in \mathbb{K}\}$ also belongs to $\mathscr{K}$. We let $r : \mathscr{K} \to \mathscr{K}$ be the mapping defined by $r(\mathbb{K}) = \sigma\mathbb{K}$. Given $U \in F$ we let $TU : \mathbb{X} \to \mathbb{R}$ be the function defined by $[TU](\boldsymbol{x}) = W(\pi\boldsymbol{x}, U(\sigma\boldsymbol{x}))$. Since W is continuous and non-decreasing, the mapping T is an operator on F, i.e., it maps F into F. Since W satisfies a Lipschitz contraction property, we get that T is a local contraction with respect to $(\mathscr{D}, r)$. More precisely, we have

$$d_{\mathbb{K}}(U, V) \leqslant \delta d_{r(\mathbb{K})}(U, V).$$

For each $s \geqslant 1$ we have

$$\|T0\|_{r^s(\mathbb{K})} = \sup_{\boldsymbol{x} \in \mathbb{K}} |W(x_s, 0)|.$$

Since $\mathbb{K}$ belongs to $\mathscr{K}$, it follows that the series

$$\sum_{s=0}^{\infty} \delta^s \|T0\|_{r^s(\mathbb{K})}$$

is convergent. We can then apply Corollary B.1 (see Appendix B) to get the existence of a fixed-point U of the operator T which is unique in A the closure of the orbit $\mathscr{O}(0)$ of 0. Now, fix a function $V : \mathbb{X} \to \mathbb{R}$ continuous for the product topology on every order interval $\mathbb{K}$ in $\mathscr{K}$ satisfying (5.1). We have to prove that for every $\mathbb{K} \in \mathscr{K}$,

(E.1) $$\lim_{s \to \infty} \sup_{\boldsymbol{x} \in \mathbb{K}} |U(\boldsymbol{x}) - W(x_0, W(x_1, \ldots, W(x_s, V(\sigma^{s+1}\boldsymbol{x}))\ldots))| = 0.$$

In other words, we should prove that

$$\forall \mathbb{K} \in \mathscr{K}, \quad \lim_{s \to \infty} d_{\mathbb{K}}(T^s V, U) = 0.$$

According to Theorem 2.1, it is sufficient to prove that

$$\forall \mathbb{K} \in \mathscr{K}, \quad \lim_{s \to \infty} \delta^s d_{r^s(\mathbb{K})}(V, A) = 0.$$

Since 0 belongs to A, we have

$$d_{r^s(\mathbb{K})}(V, A) \leqslant d_{r^s(\mathbb{K})}(V, 0) = \|V\|_{\sigma^s \mathbb{K}}$$

and the desired result follows from (5.1).

APPENDIX F: PROOF OF PROPOSITION 5.2

We denote by $\mathscr{K}$ the set of all subsets $\mathbb{K}$ of ℓ_+^∞ such that

$$\sum_{t \in \mathbb{N}} \delta^t \sup\{|W(x_t, 0)| : \boldsymbol{x} \in \mathbb{K}\} < \infty.$$

Let $\boldsymbol{x}$ be any element in ℓ_+^∞. Observe that $0 \leqslant x_t \leqslant \|\boldsymbol{x}\|_\infty$ for all $t \in \mathbb{N}$. Since W is non-decreasing we get $0 \leqslant W(x_t, 0) \leqslant W(\|\boldsymbol{x}\|_\infty, 0)$ for all $t \in \mathbb{N}$, implying that

$$\sum_{t \in \mathbb{N}} \delta^t W(x_t, 0) < \infty.$$

In particular, for every $\boldsymbol{x} \in \mathbb{X}$, the set $\{\boldsymbol{x}\}$ belongs to $\mathscr{K}$. Choose $\eta > 0$ such that

(F.1) $$\sum_{t \in \mathbb{N}} \delta^t W(\eta, 0) < 1.$$

We denote by $\mathscr{D}$ the family of all non-empty sets $\mathbb{K} \subset \ell_+^\infty$ such that there exists $\boldsymbol{x} \in \mathbb{K}$ satisfying

$$\sup_{\boldsymbol{z} \in \mathbb{K}} \sum_{t \in \mathbb{N}} \delta^t W(x_t, 0)|z_t - x_t| < \infty \quad \text{and} \quad \sup_{\boldsymbol{z} \in \mathbb{K}} \sum_{t \in \mathbb{N}} \delta^t W(\eta, 0)|z_t - x_t| < \infty.$$

CLAIM F.1 The family $\mathscr{D}$ is stable under σ, contains all non-empty subsets of $\mathbb{X}$ that are compact for the absolute weak topology, and covers $\mathbb{X}$.

PROOF OF CLAIM F.1: Let $\boldsymbol{x}$ be any consumption stream in ℓ_+^∞. The set $\{\boldsymbol{x}\}$ belongs to $\mathscr{D}$. This implies that $\mathscr{D}$ is non-empty and covers $\mathbb{X}$. The stability of $\mathscr{D}$ is obvious. Now, let $\mathbb{K}$ be a non-empty set of ℓ_+^∞ that is compact for the absolute weak topology. Since $\mathbb{K}$ is non-empty, we let $\boldsymbol{x}$ be any element of $\mathbb{K}$. We already proved that

$$\sum_{t \in \mathbb{N}} \delta^t W(x_t, 0) < \infty.$$

Therefore the sequence $\boldsymbol{q}$ belongs to ℓ_+^1 where $q_t = \delta^t W(x_t, 0)$ for every $t \in \mathbb{N}$. Observe that the sequence $\boldsymbol{r} = (r_t)_{t \in \mathbb{N}}$ defined by $r_t = \delta^t W(\eta, 0)$ also belongs to ℓ_+^1. Since $\mathbb{K} - \{\boldsymbol{x}\}$ is compact for $|\sigma|(\ell^\infty, \ell^1)$, there exists $M > 0$ such that

$$\sup_{\boldsymbol{z} \in \mathbb{K}} \sum_{t \in \mathbb{N}} \delta^t W(x_t, 0)|z_t - x_t| = \sup_{\boldsymbol{z} \in \mathbb{K}} \langle |\boldsymbol{z} - \boldsymbol{x}|, q \rangle < M$$

and

$$\sup_{\boldsymbol{z} \in \mathbb{K}} \sum_{t \in \mathbb{N}} \delta^t W(\eta, 0)|z_t - x_t| = \sup_{\boldsymbol{r} \in \mathbb{K}} \langle |\boldsymbol{z} - \boldsymbol{x}|, q \rangle < M.$$

This implies that $\mathbb{K}$ belongs to $\mathscr{D}$. *Q.E.D.*

CLAIM F.2 The family $\mathscr{D}$ is a subset of $\mathscr{K}$.

PROOF OF CLAIM F.2: Let $\mathbb{K}$ be a set in $\mathscr{D}$ and let $\boldsymbol{x}$ be any element of $\mathbb{K}$ and $M > 0$ such that

$$\sup_{\boldsymbol{z} \in \mathbb{K}} \sum_{t \in \mathbb{N}} \delta^t W(x_t, 0)|z_t - x_t| < M \quad \text{and} \quad \sup_{\boldsymbol{z} \in \mathbb{K}} \sum_{t \in \mathbb{N}} \delta^t W(\eta, 0)|z_t - x_t| < M.$$

We denote by $\mathbb{N}_\eta$ the subset of all $t \in \mathbb{N}$ such that $x_t \leqslant \eta$. Now let $t \in \mathbb{N}_\eta$, i.e., $x_t < \eta$. If $z_t \geqslant \eta$ then by concavity of $W(\cdot, 0)$ we have

$$|W(z_t, 0) - W(x_t, 0)| \leqslant \frac{W(\eta, 0)}{\eta}|z_t - x_t|.$$

If $z_t < \eta$ then

$$|W(z_t, 0) - W(x_t, 0)| \leqslant 2W(\eta, 0).$$

It follows that for every $\boldsymbol{z} \in \mathbb{K}$ we have

$$\sum_{t \in \mathbb{N}_\eta} \delta^t |W(z_t, 0) - W(x_t, 0)| \leqslant \sum_{t \in \mathbb{N}_\eta} \delta^t \left[\frac{W(\eta, 0)}{\eta}|z_t - x_t| + 2W(\eta, 0) \right]$$
$$\leqslant M/\eta + 2.$$

Now if $t \notin \mathbb{N}_\eta$ then $x_t \geqslant \eta > 0$ and by concavity of $W(\cdot, 0)$ we have for every $\boldsymbol{z} \in \mathbb{K}$

$$|W(z_t, 0) - W(x_t, 0)| \leqslant \frac{W(x_t, 0)}{x_t}|z_t - x_t| \leqslant \frac{W(x_t, 0)}{\eta}|z_t - x_t|.$$

This implies that for every $\boldsymbol{z} \in \mathbb{K}$

$$\sum_{t \in \mathbb{N}} \delta^t |W(z_t, 0) - W(x_t, 0)| \leqslant \sum_{t \in \mathbb{N}_\eta} \delta^t |W(z_t, 0) - W(x_t, 0)|$$
$$+ \sum_{t \notin \mathbb{N}_\eta} \delta^t |W(z_t, 0) - W(x_t, 0)|$$
$$\leqslant \sum_{t \in \mathbb{N}_\eta} \delta^t |W(z_t, 0) - W(x_t, 0)|$$
$$+ \sum_{t \notin \mathbb{N}_\eta} \delta^t \frac{W(x_t, 0)}{\eta}|z_t - x_t|$$
$$\leqslant (M/\eta + 2) + M/\eta = 2(M/\eta + 1).$$

We have shown that

$$
\begin{aligned}
\sum_{t\in\mathbb{N}}\delta^t\sup\{|W(z_t,0)| \,:\, \boldsymbol{z}\in\mathbb{K}\} \;\leqslant\;& \sum_{t\in\mathbb{N}}\delta^t W(x_t,0)\\
+\;& \sup_{\boldsymbol{z}\in\mathbb{K}}\sum_{t\in\mathbb{N}}\delta^t|W(z_t,0)-W(x_t,0)|\\
\leqslant\;& \sum_{t\in\mathbb{N}}\delta^t W(x_t,0)+2(M/\eta+1)<\infty.
\end{aligned}
$$

This implies that the set $\mathbb{K}$ belongs to $\mathscr{K}$.   Q.E.D.

We let H be the space of functions $U:\ell_+^\infty\to\mathbb{R}$ which are continuous on ℓ_+^∞ for the absolute weak topology and we let F be the space of functions $U:\ell_+^\infty\to\mathbb{R}$ which are bounded and continuous for the product topology on every set $\mathbb{K}$ of $\mathscr{D}$.

CLAIM F.3   Any function in F is also continuous on ℓ_+^∞ for the absolute weak topology, i.e., F is a subset of H.

PROOF OF CLAIM F.3:   Let $V:\mathbb{X}\to\mathbb{R}$ be function in F. Let $(\boldsymbol{x}^\alpha)_{\alpha\in A}$ be a net in ℓ_+^∞ converging to $\boldsymbol{x}$ in ℓ_+^∞ for the absolute weak topology. Recall that we have

$$
\sum_{t\in\mathbb{N}}\delta^t W(x_t,0)<\infty
$$

implying that the sequence $\boldsymbol{q}$ belongs to ℓ_+^1 where $q_t=\delta^t W(x_t,0)$ for every $t\in\mathbb{N}$. Observe that the sequence $\boldsymbol{r}=(r_t)_{t\in\mathbb{N}}$ defined by $r_t=\delta^t W(\eta,0)$ also belongs to ℓ_+^1. The convergence of $(\boldsymbol{x}^\alpha)_{\alpha\in A}$ to $\boldsymbol{x}$ for the absolute weak topology implies that

$$
\lim_{\alpha\in A}\langle\boldsymbol{q},|\boldsymbol{x}^\alpha-\boldsymbol{x}|\rangle=0 \quad\text{and}\quad \lim_{\alpha\in A}\langle\boldsymbol{r},|\boldsymbol{x}^\alpha-\boldsymbol{x}|\rangle=0.
$$

Therefore, there exists $\alpha_0\in A$ such that for all $\alpha\geqslant\alpha_0$ we have

$$
\sum_{t\in\mathbb{N}}\delta^t W(x_t,0)|x_t^\alpha-x_t|\leqslant 1 \quad\text{and}\quad \sum_{t\in\mathbb{N}}\delta^t W(\eta,0)|x_t^\alpha-x_t|\leqslant 1.
$$

It follows that the set

$$
\mathbb{K}\equiv\{\boldsymbol{x}\}\cup\{\boldsymbol{x}^\alpha \,:\, \alpha\geqslant\alpha_0\}
$$

belongs to $\mathscr{D}$.[44] Since $(\boldsymbol{x}^\alpha)_{\alpha\geqslant\alpha_0}$ converges for the absolute weak topology, it also converges for the product topology.[45] Since the restriction of V to $\mathbb{K}$ is continuous for the product topology, we get that

$$
\lim_{\alpha\geqslant\alpha_0}V(\boldsymbol{x}^\alpha)=V(\boldsymbol{x}).
$$

Q.E.D.

For each $\mathbb{K}\in\mathscr{D}$ we let $d_\mathbb{K}$ be the semi-distance on F defined by

$$
d_\mathbb{K}(U,V)\equiv\sup_{\boldsymbol{x}\in\mathbb{K}}|U(\boldsymbol{x})-V(\boldsymbol{x})|.
$$

The space F is sequentially complete for the topology defined by the family $\mathscr{D}$. For any function U in F, we let TU be the function defined on ℓ_+^∞ by $[TU](\boldsymbol{x})=W(\pi\boldsymbol{x},U(\sigma\boldsymbol{x}))$. As in the proof of Proposition 5.1 we can show that T maps F into F and is a local contraction contraction with respect to $(\mathscr{D},r)$ where $r(\mathbb{K})=\sigma\mathbb{K}$. We let A be the closure of the orbit $\mathscr{O}(0)$ of the null function. Since $\mathscr{D}$ is a subset of $\mathscr{K}$,

---

[44]The family $\mathscr{D}$ was introduced because we do not know if the set $\{\boldsymbol{x}\}\cup\{\boldsymbol{x}^\alpha \,:\, \alpha\geqslant\alpha_0\}$ is compact for the absolute weak topology.

[45]Fix any $s\in\mathbb{N}$ and let $\boldsymbol{q}$ be defined by $q_t=0$ if $t\neq s$ and $q_s=1$. The sequence $\boldsymbol{q}$ belongs to ℓ_+^1.

we can apply Theorem 2.1 to conclude that there exists a function U in A which is a fixed-point of T. Claim F.3 implies that the fixed-point U of T is continuous on ℓ_+^∞ for the absolute weak topology.

Denote by $C(|\sigma|)$ the set of all non-empty subset of ℓ_+^∞ which are compact for the absolute weak topology. We already proved that $C(|\sigma|)$ is a subset of $\mathcal{D}$. If $\mathbb{K}$ belongs to $C(|\sigma|)$ then we can extend the definition of $d_\mathbb{K}$ to the larger space H. Indeed, every function in H is continuous for the absolute weak topology and therefore must be bounded on $\mathbb{K}$. Moreover, the mapping T can be extended to H and satisfies $T(H) \subset H$.

Now fix a function $V : \ell_+^\infty \to \mathbb{R}$ continuous for the absolute weak topology, i.e., $V \in H$ and satisfying

$$\lim_{t \to \infty} \delta^t \sup_{\boldsymbol{x} \in \mathbb{K}} |V(\sigma^t \boldsymbol{x})| = 0$$

for every non-empty set $\mathbb{K} \in C(|\sigma|)$. We can adapt the arguments of the proof of Proposition 5.1 to show that

$$\forall \mathbb{K} \in C(|\sigma|), \quad \lim_{n \to \infty} \delta^{n+1} d_{\sigma^{n+1}\mathbb{K}}(V, A) = 0.$$

We can also adapt the arguments of the proof of Theorem 2.1 to show that the condition above implies

$$\forall \mathbb{K} \in C(|\sigma|), \quad \lim_{n \to \infty} d_\mathbb{K}(T^n V, T^n U) = 0.$$

If V is a fixed-point of T then V must coincide with U.

REFERENCES

ALIPRANITS, C. D., K. C. BORDER, AND O. BURKINSHAW (1997): "New proof of the existence of equilibrium in a single-sector growth model," *Macroeconomic Dynamics*, 1(4), 669–679.

ALIPRANTIS, C. D., AND K. C. BORDER (1999): *Infinite Dimensional Analysis*. Springer-Verlag, Berlin, second edn.

BECKER, R. A., AND J. H. BOYD III (1997): *Capital Theory, Equilibrium Analysis and Recursive Utility*. Basil Blackwell Publisher.

BEWLEY, T. (1972): "Existence of equilibria in economies with infinitely many commodities," *Journal of Economic Theory*, 4(3), 8514–540.

BOYD III, J. H. (1990): "Recursive utility and the Ramsey problem," *Journal of Economic Theory*, 50(2), 326–345.

BROWN, D., AND L. LEWIS (1981): "Myopic economic agents," *Econometrica*, 49(2), 359–368.

DURAN, J. (2000): "On dynamic programming with unbounded returns," *Economic Theory*, 15(2), 339–352.

——— (2003): "Discounting long run average growth in stochastic dynamic programs," *Economic Theory*, 22(2), 395–413.

EPSTEIN, L. G., AND S. E. ZIN (1989): "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework," *Econometrica*, 57(4), 937–969.

HADŽIĆ, O. (1979): "Some theorems on the fixed points of multivalued mappings in locally convex spaces," *Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques*, 27(3–4), 277–285.

KEHOE, T. J., D. K. LEVINE, AND P. M. ROMER (1990): "Determinacy of equilibria in dynamic models with finitely many consumers," *Journal of Economic Theory*, 50(1), 1–21.

KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI (2009): "Revursive smooth ambiguity preferences," *Journal Economic Theory*, 144(3), 930–976.

KOOPMANS, T. C., P. A. DIAMOND, AND R. E. WILLIAMSON (1964): "Stationary utility and time perspective," *Econometrica*, 32(1), 82–100.

KREPS, D. M., AND E. L. PORTEUS (1978): "Temporal resolution of uncertainty and dynamic choice theory," *Econometrica*, 46(1), 185–200.

LE VAN, C., AND Y. VAILAKIS (2005): "Recursive utility and optimal growth with bounded or unbounded returns," *Journal of Economic Theory*, 123(2), 187–209.

LEVINE, D. K., AND W. R. ZAME (1996): "Debt constraints and equilibrium in infinite horizon economies with incomplete markets," *Journal of Mathematical Economics*, 26(1), 103–131.

Lucas, R. E., and E. C. Prescott (1971): "Investment under uncertainty," *Econometrica*, 39(5), 659–681.

Lucas, R. E. J., and N. L. Stokey (1984): "Optimal growth with many consumers," *Journal of Economic Theory*, 32(1), 139–171.

Ma, C. (1998): "A discrete-time intertemporal asset pricing model: GE approach with recursive utility," *Mathematical Finance*, 8(3), 249–275.

Magill, M., and M. Quinzii (1994): "Infinite horizon incomplete markets," *Econometrica*, 62(4), 853–880.

Marinacci, M., and L. Montrucchio (2007): "Unique solutions of some recursive equations in economic dynamics," Collegio Carlo Alberto, Working Paper #46.

Matkowski, J., and A. Nowak (2008): "On discounted dynamic programming with unbounded returns," Working Paper, University of Zielona Góra.

Montrucchio, L. (1998): "Thompson metric, contraction property and differentiability of policy functions," *Journal of Economic Behavior and Organization*, 33(3–4), 449–466.

Rincón-Zapatero, J. P., and C. Rodríguez-Palmero (2003): "Existence and uniqueness of solutions to the Bellman equation in the unbounded case," *Econometrica*, 71(5), 1519–1555.

——— (2007): "Recursive utility with unbounded aggregators," *Economic Theory*, 33(2), 381–391.

——— (2009): "Corrigendum to "Existence and uniqueness of solutions to the Bellman equation in the unbounded case" Econometrica, Vol. 71, No. 5 (September, 2003), 1519-1555," *Econometrica*, 77(1), 317–318.

Stockey, N. L., J. Lucas, and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, MA.

Stroyan, K. D. (1983): "Myopic utility functions on sequential economies," *Journal of Mathematical Economics*, 11(3), 267–276.

Thompson, A. C. (1963): "On certain contraction mappings in a partially ordered vector space," *Proceeding of the American Mathematical Society*, 14(3), 438–443.