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**Uncertainty in Spatial Duopoly with Possibly
Asymmetric Distributions: a State Space Approach***

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Abstract

In spatial competition firms are likely to be uncertain about consumer locations when launching products either because of shifting demographics or of asymmetric information about preferences. Realistically distributions of consumer locations should be allowed to vary over states and need not be uniform. However, the existing literature models location uncertainty as an additive shock to a uniform consumer distribution. The additive shock restricts uncertainty to the mean of the consumers locations. We generalize this approach to a state space model in which a vector of parameters gives rise to different distributions of consumer tastes in different states, allowing other moments (besides the mean) of the consumer distribution to be uncertain. We illustrate our model with an asymmetric consumer distribution and obtain a unique subgame perfect equilibrium with an explicit, closed-form solution. An equilibrium existence result is then given for the general case. For symmetric distributions, the unique subgame perfect equilibrium in the general case can be described by a simple closed-form solution.

JEL Codes: C72, D43, D81, L10, L13, R30, R39

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1 Introduction

The Hotelling model of spatial competition can be thought of as one of the earliest problems in information economics: firms make strategic choices (locations/characteristics and prices) without knowing an individual consumer's type or location. However, in typical formulations, common knowledge of the normal or uniform distribution of consumer types gives rise to the familiar mill price equilibrium. Since individual consumer types are not observed in the standard model the assumption that the distribution of types is observable seems internally inconsistent. Furthermore casual empiricism indicates that firms are often less than perfectly informed about consumer tastes and frequently exert considerable effort to generate market research data.¹

Uncertainty about the population of consumers and their tastes arise naturally due to rural-urban drift, migration (both local and international), births and deaths, imperfect information dissemination and the obvious swings of fashion. In order to more accurately reflect market reality, the recent demand location uncertainty literature recasts the Hotelling model in a more general setting in which firms are uncertain not only about the type of an individual but are also uncertain about the distribution of consumer types.

The existing approaches to preference/demand location uncertainty focus on the restrictive case of an additive shock to consumer types. Early approaches to characteristic preference uncertainty, such as De Palma et al. (1985), used the law of large numbers to generate certain demand functions from individual preferences with idiosyncratic shocks. More recently the demand location uncertainty literature (Jovanovic (1981), Harter (1996), Casado-Izaga (2000) and Meagher and Zauner (2004, 2005)) has analyzed situations with perfectly correlated shocks which give rise to residual aggregate uncertainty.

Thus, the existing approaches generate no uncertainty at the distributional level; intuitively, the uncertainty is only about the shift or the mean of consumer

¹Data may be generated externally by market research survey, stated choice experiments or internally from customer behavior databases or from staff interaction with customers.

Table I: Share of urban settlements whose footprints intersect the Low Elevation Coastal Zone (LECZ) by urban settlement size, 2000

Region	<100K	100-500K	500K-1M	1-5M	5M+
Africa	9	23	39	50	40
Asia	12	24	37	45	70
Europe	17	22	37	41	58
Latin America	11	25	43	38	50
Australia and New Zealand	44	77	100	100	NA
North America	9	19	29	25	80
Small Island States	51	61	67	100	NA
World	13	24	38	44	65

Source: McGranahan, Balk, and Anderson (2007, Table 5, p.30).

locations/preferences. Furthermore, if there is aggregate uncertainty, consumers are typically uniformly distributed. The contributions of this paper are twofold: First, we show, by way of a *coastal city* example, how even simple situations fall beyond the scope of the existing ‘additive-shock-uniform’ approach. Second, we show how a judicious state space formulation gives an existence theorem for very general forms of uncertainty. For example the distribution of tastes could change shape across states and need not be from a fixed distributional type such as the uniform or normal. In the case of symmetric distributions, the state space approach gives simple closed form equations for equilibrium prices and locations.

Our coastal city example, in Section 3 is a Hotelling style linear city but with the population distributed according to a linear distribution instead of a uniform. To the best of our knowledge, this is the first use of an closed-form asymmetric distribution for consumers in the Hotelling model. In addition to the standard characteristic space interpretation, this representation also allows a geographic interpretation as a coastal city² such as Chicago, New York or Sydney. Two firms supplying the coastal city, when deciding where to locate, are certain where the coastal boundary is but are unsure how spread out the city will become by the time they build their facilities. In the coastal city example, both the mean and the dispersion of the consumer distribution are uncertain.

²Coastal cities include cities on oceans as well as on large lakes.

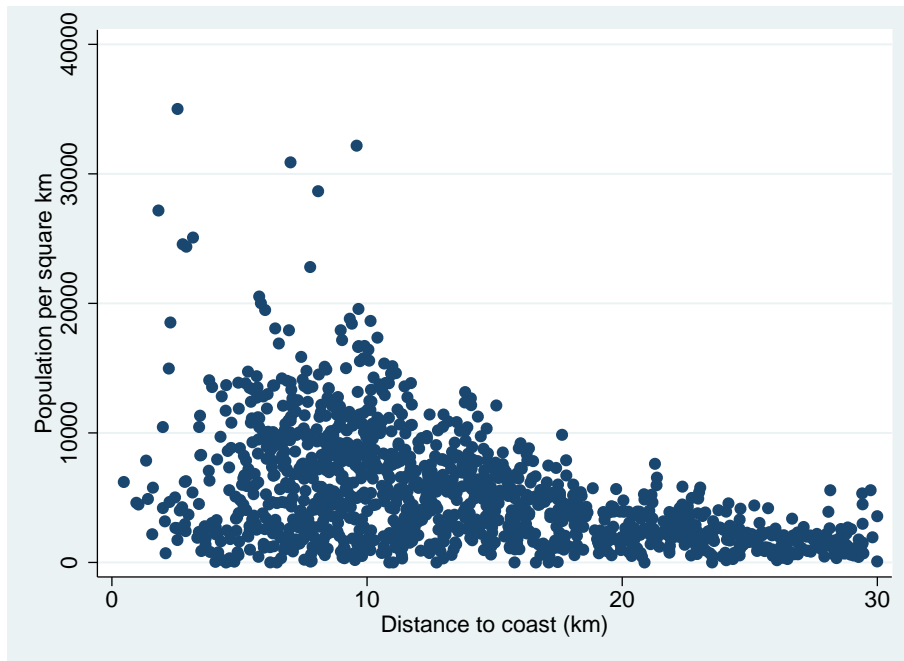


Figure 1: Population Density Per Square Kilometer by Distance to Coast for Chicago 2000. Source: US Census 2000, Gazetted Census Tracts Data Set

This issue of uncertainty beyond the mean has not previously been considered in the literature.

As Table I shows coastal cities are of considerable economic significance. Sixty five percent of the worlds large cities (over five million inhabitants) are coastal, with the rate even higher, eighty percent, in North America.

As the Chicago population density data in Figure 1 shows, average population density falls moving away from the coast. Many coastal cities share this feature. This feature can be approximated by a linear distribution and is suggestive of our coastal city example, as we will see in a later section.

We briefly lay out a model below to show how this class of problem can be solved by extending location theory. We solve the more general case in Section 4 following the coastal city example in Section 3.

2 Common Features of the Model

We present a model of aggregate location/taste uncertainty in the spirit of Meagher and Zauner (2004). Both firms and consumers are located at points on the real line \mathbf{R} . Each consumer demands either one or zero units of the good and has sufficient income to buy one unit of the good. For a consumer with “ideal” point/location x , the indirect utility function for consuming firm i 's product (located at x_i) at price p_i is given by

$$V(x, x_i, p_i) = A - p_i - \tau(x_i - x)^2, \quad \tau > 0. \quad (1)$$

There are two firms, $i = 1, 2$. The marginal cost of production of each firm is constant and normalized to 0. Firms are uncertain about the distribution of consumers but make decisions on the basis of a common prior, which we describe below. Firms choose locations $x_i \in \mathbf{R}$ ($i = 1, 2$ and, without loss of generality, $x_1 \leq x_2$) simultaneously, observe the location of their competitor, then choose prices simultaneously and finally the uncertainty is resolved.

This timing implies that firms's product flexibility and price flexibility are on a similar time scale. The major strategic thrust of modern manufacturing techniques which have emerged in recent decades is to allow product flexibility of small production runs so that firms can adjust the characteristics of their output to what consumers are buying (see Roberts, 2004 and Milgrom, Qian and Roberts, 1991). Restaurants have always been able to change menus from day to day (think of the specials board) and manufacturers, and others, can also be responsive in short time periods (see Thomke and von Hippel (2002) for a discussion of Dell and other custom manufacturing by firms). While fixed product characteristics may be an unreasonably pessimistic view of firms, instantaneous adjustment has been shown to be an overly optimistic assumption about price discovery. Under imperfect information firms have to experiment in order to discover demand conditions and this experimentation is more complex in a differentiated products setting (see Aghion, Espinosa and Jullien, 1993 and

Harrington, 1995). Indeed, if the consumer population do not remain constant in their behavior pricing cycles can emerge as in Keller and Rady (2003).

Consumers buy from the firm that gives them the highest (net) utility, hence there exists a unique point ξ , satisfying $V(\xi, x_1, p_1) = V(\xi, x_2, p_2)$, where consumers are indifferent between buying from firm 1 or firm 2. We will take this as the definition of ξ , which will be referred to as the *indifferent consumer location*. States of the world are indexed by S , which might be a vector, with density $f(S)$. In each state of the world the distribution of consumer locations, x , is given by $g_S(x)$.

3 An Asymmetric Coastal City Under Uncertainty

Consider a variant of the ice cream sellers story popularly used to motivate the Hotelling model. Two ice cream sellers must choose locations on a beach of length 1, represented by the unit interval, and post prices before their customers arrive for the day. Furthermore, assume this is a beach with a car park at the left end so all consumers enter from the same end and then walk a random distance to the right, determined jointly by their dislike of walking and crowds. This scenario could plausibly yield an asymmetric density of consumers on the interval $[0, \alpha]$, $0 < \alpha \leq 1$.

As Figure 1 shows asymmetric densities of consumer locations are not just confined to stories about ice cream sellers. Coastal cities are, by definition, bounded (on at least one side) by water. Typically access to shipping through a port played a strong role in the foundation of these coastal cities leading to a concentration of economic activity close to the coast. As the Chicago data shows over time the coast continues to exert a strong pull on the population. It is of course not immediately obvious if the higher population density near the coast is due directly to the attraction of the coast or if other agglomeration forces have taken over.

Table II: OLS Regression Results: Population per Square Kilometer by Distance to Coast for Chicago 2000.

Variable	Coefficient	Std. Error	p-value
Distance to coast	-283.46	16.19	0.000
Constant	9274.35	246.31	0.000
N	1254		
Adjusted R^2	19.6 %		

Source: US Census 2000, Gazetted Census Tracts Data Set. Distances are calculated using the Haversin Formula for Great Circle Distances as the minimum distance from the census tract centroids to a set of points on the coast.

Our focus is not to explain coastal population density but to observe that the standard Hotelling style uniform assumption is not accurate for this common situation, nor is the more recent triangle distribution model of Tabuchi and Thisse (1995). Rather than approximate a real density with a step function, as in Hotelling, we take the more flexible approach of linear approximation. For example approximating the Chicago data in Figure 1 by linear regression gives the estimates in Table II. The regression shows that population density in Chicago decreases on average at a rate of 283.46 people/ km^2 per kilometer from the coast.³

The following distribution is reminiscent of the coastal city discussed. For simplicity we assume consumers are distributed according to the linear density:

$$g_\alpha(x) = \begin{cases} \frac{2}{\alpha} - \frac{2x}{\alpha^2} & \text{if } 0 \leq x < \alpha \\ 0 & \text{otherwise} \end{cases}$$

Cities, and especially coastal cities in developing countries, are experiencing rapid population growth, for example coastal urban areas in China grew on average 3.39% per year from 1990 to 2000 (see McGranahan et al., 2007). We believe these large population changes are not deterministic from the point of view of firms, but they should appear as uncertainty over the distribution of consumers in the duopoly model. To clarify what we mean by uncertainty over

³Obviously population density is also likely to be affected by other geographic features like industrial zoning, rivers, hills, parks, urban decay etc. Indeed looking at the Chicago map shows industrial corridors close to the coast and the Chicago river. Nonetheless the adjusted- R^2 of 19.6% is comparable to cross sectional results for many economic variables.

the distribution reconsider the simple ice sellers story. Since the ice cream sellers are uncertain who will come to the beach they are therefore uncertain of which (linear) density of consumers will occur. This uncertainty about the realization of the consumer distribution is represented here by a common prior over α , denoted $f(\alpha)$ on $[0, 1]$. For simplicity and to yield explicit solutions we choose f to be the power density of order 2, i.e. $f(\alpha) = 3\alpha^2$ (which is also asymmetric).⁴

Assuming firms are risk neutral the expected profits for firm 1 are

$$E[\Pi_1(p_1, p_2, x_1, x_2)] = \int_0^1 \int_0^\xi p_1 g_\alpha(x) f(\alpha) dx d\alpha \quad (2)$$

$$= \int_0^\xi \left(p_1 \int_0^x 0 d\alpha + p_1 \int_x^1 \left(\frac{2}{\alpha} - \frac{2x}{\alpha^2} \right) 3\alpha^2 d\alpha \right) dx \quad (3)$$

$$= p_1 [x^3 - 3x^2 + 3x]_0^\xi \quad (4)$$

$$= p_1 (\xi^3 - 3\xi^2 + 3\xi) \quad (5)$$

and similarly for firm 2

$$E[\Pi_2(p_1, p_2, x_1, x_2)] = p_2 [x^3 - 3x^2 + 3x]_\xi^1 \quad (6)$$

$$= p_2 (1 - (\xi^3 - 3\xi^2 + 3\xi)) \quad (7)$$

This problem is quite different from those considered previously in the literature. Clearly, it is not just the mean of the consumer distribution that varies with the uncertainty.

As the following proposition shows, this problem does in fact give rise to a unique equilibrium.

⁴Choosing distributions which yield closed form solutions is extremely difficult for this problem. First, forming an expectation about demand is similar to the conjugate priors problem of Bayesian statistics which has only a few known solutions. In addition, the posterior from the conjugate priors must satisfy the Anderson et al. (1997) log-concavity type conditions for a location-price equilibrium which also has a relatively small number of closed formed solutions (see Meagher et al., 2008).

Proposition 1 *The coastal city problem under uncertainty has a unique sub-game perfect location-then-price equilibrium given by*

$$\begin{aligned}
x_1^* &= 1 - \frac{17}{24} \sqrt[3]{2} \approx 0.1076 \\
x_2^* &= 1 + \frac{3}{8} \sqrt[3]{2} \approx 1.4725 \\
p_1^* &= \tau \frac{13}{12} \sqrt[3]{2^2} \approx 1.7197\tau \\
p_2^* &= \frac{\tau}{3} \frac{13}{12} \sqrt[3]{2^2} \approx 0.5732\tau \\
\xi^* &= -\frac{1}{2} \sqrt[3]{2} + 1 \approx 0.3700
\end{aligned}$$

Proof: See appendix. \square

The obvious way to establish this proposition is by working directly with the specified payoff functions and their derivatives. Naturally one wonders if there are more general economic forces at work and if a more general result is possible. As the next section shows the appropriate formulation of the problem yields a very general existence result.

As one might expect the equilibrium locations and prices are asymmetric for the coastal cities model due to the asymmetric distributions. Specifically, firm 1 has expected demand of 0.7400, while firm 2 only has 0.2500, resulting in equilibrium expected profits of 1.2897 and 0.1433 for firms 1 and 2 respectively. Thus firm 2, the firm on the periphery, has prices one third the level of firm 1. These low prices plus low consumer density lead to profit for the coastal firm 1 being nine times higher than for the peripheral firm. These results are in strong contrast with the symmetric results produced from most models, in particular those with uniform distributions. In contrast to the asymmetric equilibrium of Tabuchi and Thisse (1995), the equilibrium here is unique.

Now, we turn to a more general version of this problem.

4 General State Space Results

Consider the more general setting in which the distribution of consumers over locations, x , conditional on the value of a vector of parameters M is given by $g(x|M)$. The marginal density of M , that is, the uncertainty distribution, is given by $f(M)$ with support \mathcal{S} . Without loss of generality assume $E[M] = 0$.

We analyze the pure-strategy subgame perfect Nash equilibria of this game. Given the above assumptions the consumer-cum-uncertainty distribution of consumers, $h(x)$, is given by:

$$h(x) = \int_{\mathcal{S}} g(x|M)f(M)dM. \quad (8)$$

For the following proposition it is useful to define

$$J(x) \equiv \frac{H(x)(1 - H(x))}{h(x)}, \quad (9)$$

where $H(\cdot)$ is the distribution function associated with the density function $h(\cdot)$. The following proposition shows the existence and uniqueness of the equilibrium in the general case.

Proposition 2 *Assume that the consumer-cum-uncertainty distribution $h(\cdot)$ is log-concave with mean 0 and support $[a, b]$. If $J(x)$ is strictly pseudo-concave and $\lim_{x \rightarrow a} J(x) = \lim_{x \rightarrow b} J(x)$ then there exists a unique subgame perfect equilibrium in the location-then-price game with uncertainty.*

Proof: The proof establishes the equivalence of our game to another spatial game with a known solution.

Prices are state independent thus for a fixed M firm i 's profit, π_i , is $\pi_i = p_i Q_i$, $i = 1, 2$, where Q_i is the demand for firm i . Since firms are risk neutral their payoffs are given by the expectation over M of the state contingent profits. Locations are also state independent implying ξ is independent of M . Hence, we have

$$E_M[Q_1(p_1, p_2, x_1, x_2, M)] = \int_{\mathcal{S}} \int_{-\infty}^{\xi(p_1, p_2, x_1, x_2)} g(z|M)f(M)dzdM \quad (10)$$

$$= \int_{-\infty}^{\xi(p_1, p_2, x_1, x_2)} \int_{\mathcal{S}} g(z|M)f(M)dMd z \quad (11)$$

$$= \int_{-\infty}^{\xi(p_1, p_2, x_1, x_2)} h(z) dz, \quad (12)$$

and a similar expression for firm 2. Thus $E_M[\pi_1] = p_1 H(\xi)$ and $E_M[\pi_2] = p_2(1 - H(\xi))$.

Thus, the expected payoffs and strategies for each firm are the same as in a standard certainty location game with a consumer distribution given by h . Since the two games are equivalent they will have the same equilibria which is seen to be unique by a direct application of Anderson et al. (1997, Proposition 2). \square

Corollary 1 *If, in addition to the conditions of Proposition 2, $h(\cdot)$ is symmetric then the unique sub-game perfect location-then-price equilibrium is:*

$$-x_1^* = x_2^* = \frac{3}{4h(0)}, \quad (13)$$

$$p_1^* = p_2^* = \frac{3\tau}{2h(0)^2}. \quad (14)$$

Proof: Proposition 2 and Anderson et al. (1997, Corollary 1). \square

If the density of consumers $h(x)$ is symmetric and neither too concave nor too convex, there is a unique equilibrium in which the prices and the locations depend only upon the density of the distribution at its mean. The standard spatial competition model uses a continuum of consumers to aggregate out the individual level uncertainty about consumer locations from firm profit functions. Under our state space approach firms know less: firms must still make their strategic decisions without knowing the locations of individual consumers but in addition they are uncertain about the distribution of consumers as well. However, because all decisions are made prior to the resolution of either source of uncertainty all that matters from the perspective of a firm is the combined effect of both forms of uncertainty on the density of consumers (at the marginal location).

5 Conclusion

We have argued that spatial competition theory should be extended to include uncertainty over the distribution of consumers, that the theory should allow for non-uniform consumer distributions and that these distributions should be allowed to vary in shape (mean, dispersion, support etc) over states of the world. We tackled all these issues in a setting where the adjustment of prices and product characteristics/locations occurs over a similar time frame to changes in the underlying distribution of consumer locations/tastes.

The coastal city model describes an asymmetric setting in which a linear consumer distribution is random, that is, the realized uncertainty determines which linear consumer distribution occurs. We showed that this situation has a unique duopoly equilibrium which, as one would expect, is quite different from the results of the standard uniform models.

For the general case we identified conditions on the joint density which yield a unique duopoly equilibrium. These conditions are relatively easy to test and could be employed in the numerical analysis of real data. For the case of symmetric densities we identified a closed form solution in which firms locate symmetrically. In the symmetric case equilibrium prices and locations depend on the density of the joint distribution at the mean, thus an increase in uncertainty, as expressed by a mean preserving spread of the joint distribution, would yield higher prices, greater differentiation and higher profits.

Appendix

Proof of Proposition 1

From Proposition 2 for existence and uniqueness of the equilibrium it suffices to show that h is log concave and that J is strictly pseudo concave with symmetric limits for its tails. First calculating h :

$$h(x) = \int_x^1 \left(\frac{2}{\alpha} - \frac{2x}{\alpha^2} \right) 3\alpha^2 d\alpha \quad (15)$$

$$= 3x^2 - 6x + 3 \quad (16)$$

and

$$H(x) = x^3 - 3x^2 + 3x \quad (17)$$

Hence

$$\frac{\partial^2 \ln(3x^2 - 6x + 3)}{\partial x^2} = -\frac{2}{(x-1)^2}. \quad (18)$$

Which establishes log concavity.

Now for the coastal city problem it can be shown that

$$J(x) = \frac{H(x)(1-H(x))}{h(x)} \quad (19)$$

$$= \frac{(x^3 - 3x^2 + 3x)(1 - (x^3 - 3x^2 + 3x))}{3x^2 - 6x + 3} \quad (20)$$

$$= -x(x-1)(x^2 - 3x + 3)/3 \quad (21)$$

Since this function is continuous the limits can be found by simple substitution

$$J(0) = 0 = J(1). \quad (22)$$

Finally strict pseudo concavity is easily established since the J function in this case is in fact concave:

$$\frac{\partial^2 (-x(x-1)(x^2 - 3x + 3)/3)}{\partial x^2} = -4(x-1)^2 < 0 \quad (23)$$

Having established existence and uniqueness we need only solve the system of first order conditions to find the equilibrium values for locations and prices:

$$h'(\xi^*) = \left[\frac{1}{H(\xi^*)} - \frac{1}{1-H(\xi^*)} \right] h(\xi^*)^2 \quad (24)$$

$$x_1^* = \xi^* - \frac{1-H(\xi^*)}{h(\xi^*)} (2-H(\xi^*)) \quad (25)$$

$$x_2^* = \xi^* + \frac{H(\xi^*)}{h(\xi^*)} (1+H(\xi^*)) \quad (26)$$

$$p_1^* = 4\tau \frac{H(\xi^*)}{h(\xi^*)^2} (1 - H(\xi^*) + H(\xi^*)^2) \quad (27)$$

$$p_2^* = 4\tau \frac{1 - H(\xi^*)}{h(\xi^*)^2} (1 - H(\xi^*) + H(\xi^*)^2). \quad (28)$$

Solving for ξ^* gives the cubic equation:

$$4(\xi^*)^3 - 12(\xi^*)^2 + 12(\xi^*) - 3 = 0. \quad (29)$$

This yields only one real root (which is the value of interest)

$$\xi^* = -\frac{1}{2} \sqrt[3]{2} + 1.$$

The other results follow immediately by substitution. QED

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