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An Analytical Approach to Merton's Rational Option Pricing Theory

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Abstract

In the early 70s Merton developed a theory based on economic arguments to study the properties of option and warrant prices. The main tool in his proofs was the portfolio dominance principle. In the context where the price of a contingent claim satisfies a partial differential equation we provide analytical proofs of Merton's rational option pricing theory. We use several versions of the maximum principle as well as the sliding and the moving planes methods to prove our results. Our approach enables us to extend the theory to nonlinear models.

Keywords: Merton, rational theory, option pricing, Black-Scholes, maximum principle.
JEL Classification: C02, G10, G11.

Resumen

En los años 70, Merton desarrolló una teoría basada en argumentos económicos y financieros para estudiar las propiedades del precio de una opción. La herramienta principal en sus demostraciones fue el principio de dominancia de portafolios. En el contexto donde el precio de una reclamación contingente satisface una ecuación diferencial parcial, nosotros damos demostraciones analíticas de la teoría racional de opciones de Merton para el caso de una opción. Para demostrar estos resultados, utilizamos diferentes versiones del principio del máximo, así como los métodos de planos móviles y de deslizamiento. Nuestra aproximación facilita extender la teoría a modelos no lineales.

Palabras Clave: Merton, teoría racional, estimación de opciones, Black-Scholes, principio del máximo.

1 Introduction

The rational theory of option pricing, RTOP, is an attempt to derive properties of option prices based on assumptions sufficiently weak to gain universal support [14]. To the extent that this attempt is successful, the resulting theorems become necessary conditions to be satisfied by any rational option pricing theory. The assumptions are both of economic and financial nature.

In this article, we develop a similar analysis for the rational option pricing theory of Merton [14], but we use partial differential equations techniques. In particular, the maximum principle ([8], [18] and [5]), moving planes and sliding methods [2] are used to obtain monotonicity properties of option prices as well as to establish comparison principles. The existing proofs, in contrast, relied on economic and financial arguments (dominance of portfolios).

Our results are not only of theoretical relevance, by providing analytical proofs to the theory, but are important in practice, e.g., when implementing numerical schemes (see [10]). Besides, they allow extending Merton's theory to some nonlinear models. These have become important also in practice ([6] and [16]). In particular, nonlinear models arise when pricing some bonds and in generalizing standard approaches, i.e., the Black-Scholes-Merton model. In these cases, representation formulas or Green's function techniques are not available anymore. Nevertheless, it would be important to be able to establish the same or similar qualitative results also in these cases.

In what follows, we will recall the essential features of Merton's RTOP as presented in [14].

In order to establish the restrictions on prices, we need the notion of dominance: security (portfolio) A dominates security (portfolio) B if, on some known date in the future, the return on A will exceed the return on B for some possible states, and will be at least as large as on B in all possible states.

We will also assume the following:

a) In perfect markets with no transactions costs and the ability to borrow and short-sell without restriction, the existence of a dominated security would be equivalent to the existence of an arbitrage opportunity.

b) If one assumes something like "symmetric market rationality" [13] and further that investors prefer more wealth to less, then any investor willing to purchase security B would prefer to purchase A.

c) The rate at which one is granted a loan is named lending rate and the return of the lender is the saving return.

We consider four kinds of financial instruments: an "American" or "European" call or put option.
We denote by

S: the underlying asset.

E: exercise price.

T: exercise time.

r: risk free interest rate, i.e., we assume lending and saving rates are equal.

σ: the volatility of underlying asset.

The basic principle considered by Merton is a necessary condition for a RTOP, namely, that the option be priced such that it is neither a dominant nor a dominated security.
The main results that Merton presents are the following:

1) Early exercise for an "American"-type put might be optimal.

2) The value of an "American"-type option is larger than or equal to the value of a "European"-type option, in other words, the "American"-type option dominates the "European".

3) If we consider two "European" or two"American"-type call options that differ only in the expiration time then the option with the largest expiration time has a larger value than the other.

4) If we consider two "European" or two "American"-type call options that differ only in the exercise price then the option with the largest exercise price has a smaller value than the other.

5) It is not optimal to exercise an "American" call non-dividend-paying asset before expiring. In this case, the value of an "American"-type warrant is the same as its "European" counterpart.

This paper is structured as follows: in section two, we recall some standard tools related to the maximum principle. In section three we prove

Merton's results using the maximum principle, the Fokker-Planck equation, the sliding and the moving planes methods instead of economic and financial arguments. In section four we present the conclusions and some issues for future research.

2   Preliminaries

For the sake of completeness, in this section we state the precise versions of the maximum principle that we will use in proving our results. Most of them are standard and we refer to [8] for details.

2.1   Maximum Principles for Parabolic Operators

2.1.1   Weak Maximum Principle

We assume that L is an operator of the form

$$Lu = -\Sigma_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \Sigma_{i=1}^{n} b^i u_{x_i} + cu, \tag{1}$$

where coefficients a^{ij}, b^i, and c are continuous. We will always assume the uniform parabolicity (see [8]) condition and also that $a^{ij} = a^{ji}$ $(i,j = 1,\ldots,n)$. We assume U to be an open, bounded subset of R^n and, set $U_T = (0,T] \times U$ for some fixed time $T > 0$. Recall also that the parabolic boundary of U_T is $\Gamma_T = \overline{U}_T - U_T$.

Theorem 1 *(Weak maximum principle for $c \equiv 0$)*
  Assume $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ and $c \equiv 0$ in U_T.[1]

i) *If*

$$u_t + Lu \leq 0 \qquad in \quad U_T,$$

  then

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u.$$

---

[1] C_1^2 means that the first derivative with respect to the time variable, t, is continuous and the second derivatives with respect to the space variables are also continuous.

ii) *Likewise, if*

$$u_t + Lu \geq 0 \qquad in \quad U_T,$$

then

$$\min_{\overline{U}_T} u = \min_{\Gamma_T} u.$$

Theorem 2 *(Weak maximum principle for $c \geq 0$)*
Assume $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ and $c \geq 0$ in U_T.

i) *If*

$$u_t + Lu \leq 0 \qquad in \quad U_T,$$

then

$$\max_{\overline{U}_T} u \leq \max_{\Gamma_T} u^+.$$

ii) *If*

$$u_t + Lu \geq 0 \qquad in \quad U_T,$$

then

$$\min_{\overline{U}_T} u \geq \min_{\Gamma_T} u^-.$$

Remark 1 *In particular, if $u_t + Lu = 0$ in U_T, then*

$$\max_{\overline{U}_T} \mid u \mid = \max_{\Gamma_T} \mid u \mid.$$

The following comparison lemma is immediate from the maximum principles already stated.

Lemma 1 *We consider u^1 and u^2, two solutions of the equation*

$$u_t + Lu = 0 \quad in \quad U_T,$$

and satisfying the condition

$$u^1 \leq u^2 \quad in \quad \Gamma_T,$$

then

$$u^1 \leq u^2 \quad in \quad U_T.$$

The corresponding results can be easily adapted for backward equations.

3 Main Results for European and American Calls

The Black-Scholes partial differential equation for a European call is (see [20])

$$\frac{\partial C_{Eu}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{Eu}}{\partial S^2} + rS\frac{\partial C_{Eu}}{\partial S} - rC_{Eu} = 0, \tag{2}$$

where C_{Eu} denotes the price of the European call, with payoff $C_{Eu}(T,S) = \max(S - E, 0)$, and boundary conditions
$C_{Eu}(t,0) = 0$ and $C_{Eu}(t,S) \to S$ if $S \to \infty$.

Similarly, the corresponding partial differential inequality for an American call is

$$\frac{\partial C_A}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_A}{\partial S^2} + rS\frac{\partial C_A}{\partial S} - rC_A \leq 0, \tag{3}$$

where C_A denotes the price of the American call, with payoff $C_A(t,S) = \max(S - E, 0)$, boundary conditions $C_A(t,0) = 0$, and $C_A(t,S) \to S$ if $S \to \infty$, and $C_A(t,S) \geq max(S - E, 0)$ at the free boundary.

Analogously, we will denote with P_{Eu} and P_A the corresponding prices for a European and an American put option.

Remark 2 *For the price of a European or an American call option, $V(t,S) \to S$ when $S \to \infty$, which is the standard boundary condition. For simplicity and without loss of generality, we will consider the boundary condition $V(t,\tilde{S}) = \tilde{S}$, for an $\tilde{S}$ fixed and sufficiently large. We could apply the maximum principle in unbounded domains, but imposing further technical assumptions. This is also necessary in order to accurately implement numerical methods.*

A first trivial statement, consequence of the maximum principle, is that

$C_{Eu}(T,S) \geq 0$ and $C_A(t,S) \geq 0$, which ensures the positivity of prices (see [7]).

Recall that if we consider the Black-Scholes equation for a European call on a dividend-paying asset, we have

$$\frac{\partial C_{ED}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{ED}}{\partial S^2} + (r - D_0)S\frac{\partial C_{ED}}{\partial S} - rC_{ED} = 0, \qquad (4)$$

with payoff $C_{ED}(T, S) = \max(S - E, 0)$.

For an American call on a dividend-paying asset, equality in equation (4) becomes an inequality. In this case, we have the following condition $C_{AD}(t, S) \geq \max(S - E, 0)$ at the free boundary.

Lemma 2 *An American call in absence of arbitrage opportunities must satisfy*

$$C_A(t, S) \geq \max(S - E, 0). \qquad (5)$$

Proof

This inequality has always been proved using economic and financial arguments as follows:

The put-call parity is given by

$C_{Eu} = P_{Eu} + S - PV(E)$, where PV stands for present value. Then

$$
\begin{aligned}
C_{Eu} &= P_{Eu} + S - VP(E) + E - E \\
&= (S - E) + \{E - VP(E)\} + P_{Eu} \\
&\geq S - E. \quad (since\ the\ terms\ left\ out\ are\ positive)
\end{aligned}
$$

Now we want to prove the same lemma using the maximum principle. For this example we give a detailed proof. Later, we will use similar arguments and, thus, omit some of the details.

We want to show that $C_A(t, S) \geq \max(S - E, 0)$, and we already know that $C_A(t, S) \geq 0$. Thus, we only have to verify that $C_A(t, S) \geq S - E$.

We consider the Black-Scholes inequality for an American call (3).

By standard regularity arguments [9], $C_A(t, S) \in C_1^2(U_t) \cap C(\bar{U}_t)$ and $\Gamma_t = \bar{U}_t - U_t$.

We make the next change of variable $\tau = T - t$. Then (3) is transformed into

$$\frac{\partial C_A}{\partial \tau} - L_{BS}(C_A)$$

$$= \frac{\partial C_A}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_A}{\partial S^2} - rS\frac{\partial C_A}{\partial S} + rC_A \geq 0, \qquad (6)$$

or,

$$\frac{\partial(-C_A)}{\partial \tau} - L_{BS}(-C_A)$$

$$= \frac{\partial(-C_A)}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2(-C_A)}{\partial S^2} - rS\frac{\partial(-C_A)}{\partial S} + r(-C_A) \leq 0, \qquad (7)$$

since $r \geq 0$, the inequality satisfies the conditions to apply the weak maximum principle with $c = r \geq 0$ (equation (2)). Then from equation (7), we want to verify, applying the maximum principle, that

$$(S - E) - C_A \leq 0.$$

Let $u = (S - E) - C_A$, then we have to show that

$$u_\tau - L_{BS}(u) \leq 0.$$

Substituting the value of u, we get

$$(S - E)_\tau + (-C_A)_\tau - L_{BS}(S - E) - L_{BS}(-C_A) \leq -rS + rS - rE$$
$$\leq -rE \leq 0,$$

since $r \geq 0$, and $E \geq 0$.

If we apply the maximum principle, we have:

$$\max_{\overline{U}_\tau}((S - E) - C_A) \leq \max_{\Gamma_\tau}((S - E) - C_A)^+ = 0,$$

given that we have in the boundary the following conditions:

i) If $S > E$

$$\Rightarrow \quad C_A = S - E \quad \Rightarrow \quad (S - E) - C_A = 0 \quad \Rightarrow \quad \max_{\Gamma_\tau}((S - E) - C_A)^+ = 0.$$

ii) If $S < E$

$$\Rightarrow \; C_A = 0 \;\; \Rightarrow \;\; (S - E) - C_A = S - E \le 0 \;\; \Rightarrow \;\; \max_{\Gamma_\tau}(S - E)^+ = 0.$$

iii) If $S = 0$

$$\Rightarrow \; C_A = 0 \;\; \Rightarrow \;\; (S - E) - C_A = -E \le 0 \;\; \Rightarrow \;\; \max_{\Gamma_\tau}(-E)^+ = 0.$$

iv) If S is large

$$\Rightarrow \; C_A \sim S \;\; \Rightarrow \;\; (S - E) - C_A \sim -E \le 0 \;\; \Rightarrow \;\; \max_{\Gamma_\tau}(-E)^+ = 0.$$

The last analysis reveals that

$$\max_{\bar{U}_\tau}((S - E) - C_A) \le 0$$

$$\Rightarrow \;\; (S - E) - C_A \le 0 \;\; in \;\; \bar{U}_\tau, \quad so \quad C_A \ge (S - E).$$

This completes our proof. $\square$

This financial interpretation leads to an important result: an American call on a non-dividend-paying asset will never be exercised prior to expiration, hence, it has the same value as a European call.

If we consider now an American call on a dividend-paying asset, we also recover and extend some important results.

Let us consider the Black-Scholes inequality for an American call on a dividend-paying asset

$$\frac{\partial C_{AD}}{\partial t} + \frac{\partial C_{AD}}{\partial S^2} + (r - D_0)S\frac{\partial C_{AD}}{\partial S} - rC_{AD} \le 0. \tag{8}$$

Similarly to the last proof, we have the change of variable $\tau = T - t$, and taking $u = (S - E) - C_{AD}$ we get

$$u_\tau - L_{BSD}(u) \le -rS + SD_0 + rS - rE \le SD_0 - rE.$$

In this case, we cannot apply the maximum principle because we do not know the sign of $SD_0 - rE$. For this reason, the inequality $C_{AD} \ge S - E$ is not always true, i.e., it tells us that the early exercise could occur.

The main idea of using the maximum principle is that we can say accurately when the early exercise is possible.

If $SD_0 < rE$ implies that $u_\tau + L_{BSD}(u) \leq 0$ and if we use the maximum principle, we have $C_{AD} \geq S - E$. The last proof shows that the early exercise does not occur.

For the sake of consistency, we will verify that the early exercise for an American put, P_A, cannot be ruled out with our methodology. Proceeding by contradiction, we try to show that $P_A - (E - S) \geq 0$.

Let $u = P_A - (E - S)$, and $\tau = T - t$

$$u_\tau - L_{BS}(u) \leq -rS - rE + rS \leq -rE \leq 0.$$

If we apply the maximum principle, we obtain the inequality

$$\max_{\bar{U}_\tau}(P_A - (E - S)) \leq \max_{\Gamma_\tau}(P_A - (E - S))^+.$$

We verifying the boundary

i)

$$If \quad S > E \quad \Rightarrow \quad \max_{\Gamma_\tau}(S - E)^+ = S - E.$$

ii)

$$If \quad S < E \quad \Rightarrow \quad \max_{\Gamma_\tau}(P_A - (E - S))^+ = 0.$$

iii)

$$If \quad S = 0 \quad \Rightarrow \quad \max_{\Gamma_\tau}(S)^+ = S.$$

iv)

$$If \quad S \to \infty \quad \Rightarrow \quad \max_{\Gamma_\tau}(S)^+ = S.$$

Since the boundary term does not have a definite sign, we do not obtain the inequality $P_A \geq \max(E - S, 0)$. This reveals that an American put option can, in principle, be exercised any time before expiring.

We would like to prove Lemma 2 in a different way. To do that, we prove the following theorem.

Theorem 3 *If the underlying asset does not give dividends, the value of a European call is the same as that of an American call, i.e., the early exercise for an American call does not occur.*

Before providing proof, some facts need to be presented. We need to refer to [7].

We have already proved this result, by showing that the American call satisfies the condition $C_A \geq \max(S - E)$, but now we want to establish this theorem using other tools that will also be useful in other contexts. It is well known that for a European option V, the solution to the Black-Scholes equation can be expressed using the Feynman-Kac formula [17]:

$$V(t, S) = E \left[g \left(X_T^{t,S} \right) exp \left(- \int_t^T r(\tau, X_\tau^{t,S} d\tau) \right) \right],\qquad(9)$$

where V can be a put or a call value, g is the payoff and for $\tau \in [0, t]$ the process $X_\tau^{t,S}$ is defined by considering $X_\tau^{t,S} \equiv S$ and where for $\tau \in [t, T]$ the process $X_\tau^{t,S}$ is defined to be the solution of the stochastic differential equation

$$dX_\tau^{t,S} = r \left(\tau, X_\tau^{t,S} \right) X_\tau^{t,S} dt + \sigma \left(\tau, X_\tau^{t,S} \right) dB_\tau \qquad and \qquad X_\tau^{t,S} = S. \quad (10)$$

The expectation in (9) is considered with respect to the risk neutral measure.

This is the fundamental theorem of asset pricing, which states that the arbitrage free price can be obtained as an expected value with respect to the risk neutral measure.

In turn, this can be written as an expected value, not with respect to the risk neutral measure, but with respect to the physical measure, with an appropriate discount factor (the so-called Samuelson formula [3]). Moreover, the density of the physical measure satisfies a Fokker-Planck equation, which can be written once the dynamics of S is given by a SDE (see [15]).

In fact, an analogous formulation can be given for an American call (for more details see [7] and [19]). Recall that the equation for a European call is (2).

According to Samuelson's formula the price of a European call at time t and exercise time T is

$$C_{Eu}(t, S, T) = e^{-r(T-t)} E_p^{t,S} \left(g \left(\frac{S_{(T-t)}}{E_p(S_{(T-t)})} S_0 e^{r(T-t)} \right) \right),\qquad(11)$$

where $g(\cdot)$ is the payoff; $E_p(S_t) = e^{\mu(T-t)}S_0$, μ is the return interest rate and E_p is the expected value at time t, with respect to the physical measure.

If the price of an American call is written in terms of a European *call*, we get the following formula

$$C_A(t, S, T) = E_p^{t,S}(C_A(\tau, S, t)/\tau) = \int_0^t \rho(\tau, S)C_{Eu}(\tau, S, t)d\tau, \qquad (12)$$

with $C_{Eu}(t, S, T)$ as before a European call with exercise time T and time before to expire t, and $\rho(t, S)$ is the probability that early exercise occurs at time t, when the underlying asset is S^2 and where we have used the fact that a European and an American call non-dividend-paying stock are identical.

We know that $\rho(t, S)$ satisfies the following Fokker-Planck equation(FP)

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 (\sigma^2 S^2 \rho)}{\partial S^2} - \frac{\partial (\mu S \rho)}{\partial S} \quad (\text{see also } [7]), \qquad (13)$$

with boundary conditions: $\rho(t, 0) = 0$ $\forall t \in [0, T]$, $\rho(S, t) \to 0$ as $S \to \infty$, and the initial condition $\rho(0, S) = \delta(S - S_0)$ with S_0 the initial underlying asset.

Proof of Theorem 3

Using this approach we can easily see that $C_A(t, S) \le C_{Eu}(t, S)$.

$$
\begin{aligned}
C_A(t, S) &= \int_0^t \rho(\tau, S)C_{Eu}(\tau, S)d\tau \\
&\le \int_0^t \rho(\tau, S)C_{Eu}(t, S)d\tau \quad (\textit{by Theorem 4, see below}) \\
&= C_{Eu}(t, S)\int_0^t \rho(\tau, S)d\tau \quad (\textit{since } C_{Eu}(t, S) \textit{ does not depend on } \tau), \\
&\le C_{Eu}(t, S). \qquad\qquad (\textit{since } \rho \textit{ is a probability distribution} \\
&\qquad\qquad\qquad\qquad\qquad \textit{function and integrates to one in } [0, T])
\end{aligned}
$$

The opposite inequality, $C_A \ge C_{Eu}$, is a general fact (see [14]) and follows easily from the maximum principle (see [7]). The proof is analogous to that of Lemma 2.

---

[2]In geometric terms, $\rho(t, S)$ is the probability that the process exists in the domain through the free boundary.

We can give an alternative proof by applying the maximum principle to ρ.

We consider the Fokker-Planck equation (13) and, according to the maximum principle [Theorem 1], we have

a)

$$\max_{\overline{U}_t}(-\rho) \leq \max_{\Gamma U_t}(-\rho)^+ = 0,$$

or

$$-\rho \leq 0 \quad \Rightarrow \quad \rho \geq 0.$$

The result of this inequality is that $\rho \geq 0$, because ρ is a probability density function.

b) On the other hand, it also satisfies

$$\min_{\overline{U}_t}(-\rho) \geq \min_{\Gamma U_t}(-\rho)^-.$$

Therefore

1) if $\rho = 1$

$$\min_{\overline{U}_t}(-\rho) \geq -1 \quad \Rightarrow \quad -\max_{\overline{U}_t}(\rho) \geq -1 \quad \Rightarrow \quad \max_{\overline{U}_t}(\rho) \leq 1.$$

2) if $\rho = 0$

$$\min_{\overline{U}_t}(-\rho) \geq 0 \quad \Rightarrow \quad -\max_{\overline{U}_t}(\rho) \geq 0 \quad \Rightarrow \quad \max_{\overline{U}_t}(\rho) \leq 0.$$

This means that in U_T the function is identically zero, except perhaps at the top when $t = T$.

We have proved before that $C_A(t, S) - (S - E) \geq 0$, i.e., the boundary in the right hand side is not reached because the early exercise does not occur (Lemma 2).

Since the only portion of the boundary where ρ can be different from zero is at $\{t = T\}$ and $\int_{U_T} \rho = 1$, we see that this measure is a Dirac delta concentrated in $\{t = T\}$.

In this way, we get the price of an American call:

$$
\begin{aligned}
C_A(t, S) &= \int_0^t \rho(\tau, S) C_{Eu}(\tau, S, t) d\tau \\
&= \int_0^t \delta(\tau - t, S) C_{Eu}(\tau, S, t) d\tau \\
&= C_{Eu}(t, S).
\end{aligned}
$$

We can say that the value of an American call is the same as that of a European call, since the early exercise does not occur.□

3.1   Others Results about American Call Options

We mention other results about American and European calls. Their proofs rely not only on the maximum principle, but on the so-called sliding and moving planes methods [2].

We consider the heat equation

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} + cu,
$$

in the rectangle $[0, T] \times [0, L]$, with initial and boundary conditions $u(0, x) = u_0(x)$ convex and $u(t, 0) = u(t, L) = 0$, $0 \le t \le T$, respectively.

Theorem 4 *Let u be a classical solution, then u is monotone in t, i.e.,*

$$
u(t, x) \ge u(t', x) \quad if \quad t \ge t' \quad \forall x \in [0, L]
$$

Proof

By contradiction. Assume that $u(t, x) < u(t', x)$ with $t \ge t'$ and let

$$
t^* = sup\{t \mid u(t, x) \ge u(t', x), \ x \in [0, L]\}.
$$

If $t^* > 0$ then

$$
u(t, x) \ge u(t', x) \quad \forall x \in [0, L] \quad and \quad 0 \le t'^*.
$$

Notice that this is true since the initial condition is convex as it can be easily verified [7].

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Define

$$u^\tau = u(t + \tau, x) \qquad for \qquad t \in [t^* - \epsilon, t^* + \epsilon],$$

and ϵ sufficiently small so that the measure of $[t^* - \epsilon, t^* + \epsilon] \times [0, L]$ is small and the maximum principle holds [5]. τ is considered to be also small.

Then $u^\tau(t^* - \epsilon, x) \geq u(t^* - \epsilon, x)$, by the definition of t^*.

From the maximum principle it follows that the previous inequality is valid in $[t^* - \epsilon, t^* + \epsilon] \times [0, L]$.

This in turn implies that $u(t, x) \geq u(t', x)$ for $t'^* + \epsilon$, and this is a contradiction.□

As a particular case, we have the next lemma.

Lemma 3 *Given two call options on the same stock S, with the same expiring time and with the same exercise price E, but evaluated at different times before expiring (real times), $t_2 > t_1$, they satisfy*

$$C(t_2, S, E) \leq C(t_1, S, E). \tag{14}$$

Notice that by Theorem 3, C can be either American or European.

Proof

Let $C(t_2, S, E)$ and $C(t_1, S, E)$ be two options with $t_2 > t_1$.
We put
$S = Ee^x$, $\quad t_i = T - \frac{\tau_i}{\frac{1}{2}\sigma^2}$ and $\quad C(t, S, E) = Ev(x, \tau_i)$, for $i = 1, 2$.

According to this change of variables equation (2) is transformed in the following heat equation and here the τ_i's correspond to the time of expiration

$$\frac{\partial v}{\partial \tau_i} = \frac{\partial^2 v}{\partial x^2} + (k_1 - 1)\frac{\partial v}{\partial x} - k_1 v, \quad k_1 = \frac{r}{\frac{1}{2}\sigma^2}$$

and the final condition is transformed in an initial condition of the following form

$$v(0, x) = \max(e^x - 1, 0),$$

which is convex and its boundary condition is

$$v(t, 0) = v(t, L) = 0, \qquad 0 \leq t \leq T.$$

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In the previous theorem we consider $t = \tau_1$, $t' = \tau_2$, $u(t, x) = v(\tau_1, x)$ and $u(t', x) = v(\tau_2, x).\square$

Remark 3 *Using the maximum principle, it is easy to show that for two given call options on the same stock S, with the same exercise price E, but with different expiring times, $T_2 > T_1$, they satisfy, $C(T_2, S) \geq C(T_1, S)$.*

Lemma 4 *Given two identical calls options, American or European, with the exception that one has an exercise price larger than the other, i.e., $E_2 > E_1$, they must satisfy the following inequality*

$$C(t, S, E_2) \quad \leq \quad C(t, S, E_1). \tag{15}$$

Proof

We are going to prove the lemma only for a European call. The case of an American call is similar.

Let two European call options be $C_{Eu}(t, S, E_1)$ and $C_{Eu}(t, S, E_2)$, respectively. They satisfy equation (2) with boundary conditions:

- $C_{Eu}(t, 0, E_i) = 0$, for $i = 1, 2$

- $C_{Eu}(t, S, E_i) \to S$ if $S \to \infty$, for $i = 1, 2$, and

- $C_{Eu}(T, S, E_i) = \max(S - E_i, 0)$, for $i = 1, 2$

We are going to use Lemma 1. We make a change of variables so that (2) becomes a parabolic equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x_i^2} \quad -\infty < x_i < \infty \quad y \quad \tau > 0,$$

with $x_i = log(S/E_i)$ for $i = 1, 2$ and initial condition

$$u(0, x_i) = \max(e^{\frac{1}{2}(k_1+1)x_i} - e^{(k_1-1)x_i}, 0).$$

If we apply Lemma 1, we have to prove that for $x_2 < x_1$,

$$u(0, x_1) > u(0, x_2).$$

First, we consider two cases

1) if $\;S \geq E_i \;\;\Rightarrow\;\; \frac{S}{E_i} \geq 1 \;\;\Rightarrow\;\; ln(\frac{S}{E_i}) \geq 0.$

2) if $\;S \leq E_i \;\;\Rightarrow\;\; \frac{S}{E_i} \leq 1 \;\;\Rightarrow\;\; ln(\frac{S}{E_i}) \leq 0.$

Without loss of generality, we only consider the first case $(S \geq E_i)$, because for the second case, we have $max(S - E_i, 0) = 0$.

Let $\;E_1 < E_2, \;\;\; S \geq 0 \;$ and $\;S > E_i \;$ for $\;i = 1, 2$, then

$$x_1 \geq x_2.$$

Since $k_1 = \frac{2r}{\sigma^2} > 0$

$$e^{\frac{1}{2}(k_1+1)x_1} \geq e^{\frac{1}{2}(k_1+1)x_2}, \tag{16}$$

and using direct calculus, we can prove

$$-e^{\frac{1}{2}(k_1-1)x_1} \leq -e^{\frac{1}{2}(k_1-1)x_2}. \tag{17}$$

From inequalities (16) and (17), we get

$$e^{\frac{1}{2}(k_1+1)x_1} - e^{\frac{1}{2}(k_1-1)x_1} \geq e^{\frac{1}{2}(k_1+1)x_2} - e^{\frac{1}{2}(k_1-1)x_2}.$$

We have

$$\max(e^{\frac{1}{2}(k_1+1)x_1} - e^{\frac{1}{2}(k_1-1)x_1}, 0) \geq \max(e^{\frac{1}{2}(k_1+1)x_2} - e^{\frac{1}{2}(k_1-1)x_2}, 0),$$

therefore $\;u(0, x_1) \geq u(0, x_2)\;$ for $\;x_2 < x_1.$

Then, we can apply Lemma 1 and have

$$u(t, x_1) \geq u(t, x_1).$$

Thus, we have proved that

$$C_{Eu}(t, S, E_2) \leq C_{Eu}(t, S, E_1) \;\; if \;\; E_1 < E_2. \square$$

Finally, according to Merton, now we want to establish that a European option is equivalent to a long position in the common stock levered by a limited-liability discount loan, where the borrower promises to pay E dollars at the end of τ periods, but in the event of default is only liable to the extent

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of the value of the common stock at that time. If the present value of such a loan is a decreasing function of the interest rate, then, for a given stock price, the option price will be an increasing function of the interest rate.

Let $P(\tau)$ be the price of a riskless (in terms of default) discounted loan (or bond), which pays one dollar τ years from now. If it is assumed that current and future interest rates are positive, then

$$1 = P(0) > P(t_1) > P(t_2) > \ldots > P(t_n), \quad for \quad 0 < t_1 < t_2 < \ldots < t_n.$$

Theorem 5 *If the exercise price of a European call option is E, the underlying asset does not pay dividends and we build the common stock over the option life , then*

$$C_{Eu}(t, S) \geq \max(0, S - EP(t)). \tag{18}$$

Proof

The proof is analogous to that of Lemma 2.

4 Conclusions

As we mentioned in the introduction, and from the techniques used in the proofs, it is clear that basically all results can be extended to semilinear models and to fully nonlinear equations (see [6] and [16]).

A relevant direction of future research is to consider models that allow for small arbitrage opportunities, in which case our approach would enable us to consider equations of the form (2) with variable interest rates. In this context, an arbitrage opportunity would be generated if $r(t)$ becomes negative for a short period.

However, the tools we employed would still be applicable provided these time "windows" are not very large. Along the same lines, using analytical techniques provides us with a more quantitative version of Merton's RTOP (see equation (8)).

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