SELECTING A UNIQUE COMPETITIVE EQUILIBRIUM WITH DEFAULT PENALTIES

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Selecting a Unique Competitive Equilibrium with Default Penalties*

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Abstract

The enlargement of the general-equilibrium structure to allow default subject to penalties results in a construction of a simple mechanism for selecting a unique competitive equilibrium. We consider economies for which a common credit money can be applied to uniquely select any competitive equilibrium with suitable default penalties. We identify two classes of such economies.

Keywords: Competitive equilibrium, credit mechanism, marginal utility of income, welfare economics

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1 Introduction

The problem of finding the most general conditions required to guarantee a unique competitive equilibrium (CE in short) for a general-equilibrium system is complex and challenging mathematically. By enlarging the problem, an approach is proposed that both offers a solution and facilitates an interesting selection.[1]

There appear to be two basic ways to obtain the uniqueness of equilibrium in economic models. The first is to seek reasonable conditions within the mathematical structure of a given economic model. A good example of this is the explorations of the conditions for an exchange economy that guarantee gross substitutability. The second is to extend or modify the actual model. In this paper we are concerned with the latter approach. Morris and Shin (2000) have approached the obtaining of the uniqueness via uncertainty of the information conditions, utilizing the global games formulation of Carlsson and van Damme (1993). Our approach is based on a different fact of economic life. If, instead of viewing the general equilibrium economy non-strategically, we must accept default as a strategic possibility, then every society should have rules to handle such a possibility. We introduce these default conditions and argue the uniqueness and other properties from them. Economic dynamics require institutions to carry process. As such, it appears to be quite plausible that there are many contributing factors that can lead to the same phenomenon. We believe that there may be several different economic factors that contribute to the uniqueness of equilibrium. However, institutionally, the presence of default conditions is certainly of importance.

To investigate the possibility of obtaining the uniqueness of CE from default conditions, we consider a simple mechanism for an economy that requires trade be in a credit money

[1] The problem of the uniqueness of competitive equilibrium is not a mere mathematical curiosity. It raises basic questions in macroeconomics in the employment of any dynamic model. It can be avoided by limiting oneself to sufficiently low dimensional models as has been done by those using dynamic programming models of macro-micro economic phenomena, where a single good is put up for sale and bought for cash (see Shubik and Whitt 1973, Lucas 1978, and Karatzas, Shubik and Sudderth 1994, for examples). However, the difficulties remain for higher dimensional problems.

issued by a government bank. Before trading begins, traders exchange personal IOUs for the credit money with the bank charging them an interest rate of zero. Each trader may exchange personal IOUs for the credit money without limit. But, after he has bought and received income from selling, he goes to the bank to settle up all outstanding credit.[2] Ending up with net credit is worthless for him, while a default penalty is levied against him for ending as a net debtor.[3]

By including an option to default, the mechanism enlarges the traders' budget sets. As a consequence, the mechanism puts stronger conditions on CEs. We say that a CE for an economy is a selection by the credit mechanism if the CE allocation remains to be a CE allocation in the presence of the option to default subject to penalties. We refer to CEs for the economy that can be selected by the mechanism as default-qualified competitive equilibria (DCEs in short).

We begin investigating the credit mechanism with a useful property of general-equilibrium analysis. Namely, under some mild conditions, a CE corresponds to a saddle-point of a Lagrangian function for each trader. This saddle-point characterization of a CE has useful applications to the design of default penalties towards the selection of a unique CE, as well as to the study of the already familiar welfare properties of CE allocations. We then consider economies for which a uniform credit money can be applied, such that any CE of the economy can be a unique DCE with suitable default penalties.[4] We identify two classes of such economies. Our analysis is carried out for pure exchange economies. However, we discuss extensions to production economies via Rader's equivalence principle.

[2] One way to play the model in a classroom is at the beginning to give each student a large stack of banknotes and inform him that at the end of the game, after he has bought and received income from selling, he has to return exactly the amount he started with initially or he will have to pay a default penalty.

For discussions on various credit mechanisms for the competitive model, the reader is referred to Shubik (1999).

[3] For example, default penalties may be in the form of asset confiscation from the debtors or jail sentences or other societal punishments.

[4] This will include CEs with the minimal cash flow property. Such CEs are special for the reason that they minimize the need for a substitute-for-trust in trade.

The rest of the paper is organized as follows. The next section briefly discusses saddle-point characterization of a CE and its applications. Section 3 presents results for pure exchange economies. Section 4 discusses extensions to economies with production and section 5 concludes the paper.

2 Saddle-Point Characterization of Competitive Equilibria

Consider an exchange economy $\mathcal{E} = \{X^i, u^i, a^i\}_{i=1}^n$ with trader i's consumption set X^i, utility function u^i, and endowment a^i. We assume *A1:* $X^i = \Re^m_+$; *A2:* u^i is non-satiated, continuous, and concave; *A3:* $a^i \in \Re^m_+$ with $a^i \neq 0$ and $u^i(a^i) > 0$; and *A4:* For each commodity $1 \leq h \leq m$, there is a trader i such that $u^i(x^i + \delta e^h) > u^i(x^i)$ for all $x^i \in \Re^m_+$ and for all $\delta > 0$, where $e^h \in \Re^m$ is the bundle with $e^h_h = 1$ and $e^h_k = 0$ for all $k \neq h$.[5] These are familiar assumptions in general-equilibrium analysis. A CE for economy $\mathcal{E}$ is a pair $(\bar{x}, \bar{p})$ with allocation $\bar{x} = (\bar{x}^1, \cdots, \bar{x}^n)$ and price vector $\bar{p}$ such that for all i, $\bar{x}^i$ solves

$$\begin{array}{lll} \max u^i(x^i) & & \\ \text{subject to:} & \bar{p} \cdot (a^i - x^i) \geq 0, & \text{(Utility Maximization)} \\ & x^i \in \Re^m_+. & \end{array} \tag{1}$$

and

$$\sum_{i=1}^n \bar{x}^i = \sum_{i=1}^n a^i. \qquad \text{(Market Clearance)} \tag{2}$$

2.1 Saddle Point Characterization

Due to *A4*, all CE prices are positive. The saddle-point characterization of CEs in Theorem 1 below is well-known. A proof can be established by applying the saddle-point characterization of solutions for non-linear programming problems.[6]

[5] For any positive integer q, $\Re^q_+$ denotes the non-negative orthant of the q-dimensional Euclidean space and $\Re^q_{++}$ denotes the subset of $\Re^q_+$ containing vectors in $\Re^q_+$ all with positive components.

[6] See Takayama (1985, p. 75) for the saddle-point characterization of solutions for non-linear programming problems.

Theorem 1 (Saddle-Point Characterization) *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i=1}^n$ be an exchange economy satisfying A1-A4. Then, a pair $(\bar{x}, \bar{p}) \in \Re_+^{mn} \times \Re_{++}^m$ is a CE if and only if $\bar{x}$ satisfies (2) and there exists a vector $\bar{\lambda} = (\bar{\lambda}^i)_{i=1}^n \in \Re_{++}^n$ such that for all i, the triplet $(\bar{x}^i, \bar{p}, \bar{\lambda}^i)$ satisfies*

$$u^i(x^i) + \bar{\lambda}^i \bar{p} \cdot (a^i - x^i) \leq u^i(\bar{x}^i) + \bar{\lambda}^i \bar{p} \cdot (a^i - \bar{x}^i) \leq u^i(\bar{x}^i) + \lambda^i \bar{p} \cdot (a^i - \bar{x}^i), \tag{3}$$

for all $x^i \in \Re_+^m$ and for all $\lambda^i \in \Re_+$.

Condition (3) is equivalent to $(\bar{x}^i, \bar{\lambda}^i)$ being a saddle-point of the Lagrangian function $\mathcal{L}^i(x^i, \lambda^i) = u^i(x^i) + \lambda^i p \cdot (a^i - x^i)$ for utility maximization problem (1). When a triplet $(\bar{x}, \bar{p}, \bar{\lambda})$ satisfies (2) and (3), we call it *a competitive triplet* and we call $\bar{x}$, $\bar{p}$, and $\bar{\lambda}$, respectively, a *competitive allocation*, a *competitive price vector*, and a *competitive multiplier vector*.

Two applications of Theorem 1 are relevant to the rest of the paper. Corollary 1 below shows that a competitive allocation maximizes a weighted welfare function with welfare weights equal to the reciprocals of the associated competitive multipliers. Corollary 2 shows that under some additional conditions, there is a one-to-one correspondence between CEs and competitive multiplier vectors.

Corollary 1 *Assume $\mathcal{E} = \{X^i, u^i, a^i\}_{i=1}^n$ satisfies A1-A4. If $(\bar{x}, \bar{p}, \bar{\lambda}) \in \Re_+^{mn} \times \Re_{++}^m \times \Re_{++}^n$ is a competitive triplet for $\mathcal{E}$, then $\bar{x}$ solves the weighted welfare maximization problem:*

$$\begin{array}{ll} \max \sum_{i=1}^n \frac{1}{\lambda^i} u^i(x^i) & \\ \textit{subject to} & \begin{array}{l} \sum_{i=1}^n (a^i - x^i) \geq 0, \\ x^i \in \Re_+^m, i = 1, 2, \cdots, n. \end{array} \end{array} \tag{4}$$

Corollary 1 follows easily from Theorem 1. For this reason, its proof is omitted.

Corollary 2 *Assume $\mathcal{E} = \{X^i, u^i, a^i\}_{i=1}^n$ satisfies A1 and A4. Assume further $\mathcal{E}$ satisfies*

A2′: u^i is non-satiated, continuously differentiable, and strictly concave;

A5: CE allocations are all interior allocations.[7]

Then, there is a one-to-one correspondence between CEs and competitive multiplier vectors for economy $\mathcal{E}$.

Proof. Let $(\bar{x}, \bar{p})$ be a CE. Then, by A5, $\bar{x}^i \in \Re^m_{++}$ for all i. Consequently, by A2$'$ and the Kuhn-Tucker conditions, the competitive multiplier vectors $\bar{\lambda} \in \Re^n_+$ that correspond to $(\bar{x}, \bar{p})$ satisfy [8]

$$\bigtriangledown u^i(\bar{x}^i) = \bar{\lambda}^i \bar{p}, \; i = 1, 2, \cdots, n. \tag{5}$$

The uniqueness of these competitive multiplier vectors follows from the above equation.

Conversely, by Corollary 1, competitive allocations that correspond to competitive multiplier vector $\bar{\lambda}$ solve problem (4). By the strict concavity of the utility functions, problem (4) has a unique solution given $\bar{\lambda}$. It thus follows that $\bar{\lambda}$ determines a unique competitive allocation $\bar{x}$. By *A5*, $\bar{x}^i \in \Re^m_{++}$ for all i. Thus, by the Kuhn-Tucker conditions for problem (4), there is a Lagrangian multiplier vector $\bar{p} \in \Re^m_{++}$ such that

$$\bar{p} = \sum_{i=1}^{n} \frac{1}{\bar{\lambda}^i} \bigtriangledown u^i(\bar{x}^i).$$

This shows that given $\bar{\lambda}$, the Lagrangian multiplier vector $\bar{p}$ for problem (4) is unique. Since the price vector associated with competitive allocation $\bar{x}$ is necessarily a Lagrangian multiplier vector for problem (4), it must be unique. ∎

The one-to-one correspondence in Corollary 2 implies that the products $\bar{\lambda}^i \bar{p}$, $i = 1, 2, \cdots, n$, are uniquely determined in any CE under the conditions in the corollary.

[7] A sufficient condition to guarantee the interiority of the CE allocations is for all i, $u^i(x^i) > u^i(y^i)$ whenever x^i is an interior bundle and y^i is a corner bundle. The reason is that all CE prices are strictly positive under A4 and hence the value of each trader's endowment at these prices are positive.

[8] Here $\bigtriangledown u^i(\bar{x}^i)$ denotes the gradient of u^i at $\bar{x}^i$.

3 Results

Competitive equilibrium prices are homogeneous of degree zero. Consequently, prices can be normalized without changing competitive allocations. The effects of price normalization on competitive multipliers, hence on marginal utilities of income of the traders is rarely considered in the literature. In what follows it will become clear that those effects are important in the construction of a credit mechanism for selecting a unique CE.

Suppose that traders use a credit money to buy or sell goods. A bank provides the credit money of zero interest. Before trading begins, traders exchange personal IOUs for the credit money with the bank for as much as he wants. But, after he has bought and received income from selling, trader i settles up all outstanding credit with the bank. Trader i pays a penalty of $\mu^i > 0$ in units of i's utility for each unit of debt he is unable to repay, while it is of no value to him for ending as a net creditor. Since trade is in credit money, it is natural to measure prices in terms of the money. Set $\mu = (\mu^i)_{i=1}^n$. The credit money and default penalties together transforms economy $\mathcal{E}$ into $\mathcal{E}_\mu$, in which trader i has utility function

$$U^i(x^i, p) = u^i(x^i) + \mu^i \min[p \cdot a^i - p \cdot x^i, 0]. \tag{6}$$

A CE for $\mathcal{E}_\mu$ is a pair (x^*, p^*) such that for all i, x^{*i} solves

$$\max_{x^i \in \Re_+^m} U^i(x^i, p^*) \qquad \text{(Utility Maximization)} \tag{1'}$$

and

$$\sum_{i=1}^n x^{*i} = \sum_{i=1}^n a^i. \qquad \text{(Market Clearance)} \tag{2'}$$

3.1 Price Normalization

We normalize the prices to make the wealth of total endowment $\bar{a} = \sum_{i=1}^n a^i$ of the economy constant, which we assume to be 1 without loss of any generality. That is, we consider normalized price vectors p in

$$\mathcal{P} = \left\{ p \in \Re^m_+ | p \cdot \bar{a} = 1 \right\}.$$

Definition 1 *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ be an exchange economy. We say that a CE of $\mathcal{E}$ with competitive allocation x^* and price vector $p^* \in \mathcal{P}$ is selected with default penalties μ if (x^*, p^*) is a CE for $\mathcal{E}_\mu$.*

Given default penalty vector μ, we call CEs for economy $\mathcal{E}$ that can be selected default-qualified CEs (DCEs in short) with respect to μ. In the law the default penalties are written as broad categories, not as individual penalties. In practice the litigation that often accompanies default individualizes the penalties. Thus, including a vector of individual penalties is reasonable.

The following theorem provides a connection of the DCEs with CEs.

Theorem 2 *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ be an exchange economy and $\mu = (\mu^i)_{i=1}^n$ a vector of default penalties. Assume $\mathcal{E}$ satisfies A1-A4. If (x^*, p^*) is a CE for $\mathcal{E}_\mu$, then there exists a vector $\lambda^* \in \Re^n_{++}$ with $\lambda^* \leq \mu$, such that (x^*, p^*, λ^*) is a competitive triplet for $\mathcal{E}$.*

Proof. Let (x^*, p^*) be a CE for $\mathcal{E}_\mu$. By *A4*, $p^* \in \Re^m_{++}$. From (6) and the non-satiation of u^i it follows that $p^* \cdot a^i - p^* \cdot x^{*i} \leq 0$ for all $i \in N$. Thus, (x^{*i}, x^{*i}_{m+1}) with $x^{*i}_{m+1} = p^* \cdot a^i - p^* \cdot x^i$ solves

$$\begin{array}{ll} \max u^i(x^i) + \mu^i x^i_{m+1} & \\ \text{subject to} & \begin{array}{l} p^* \cdot a^i - p^* \cdot x^i \geq x^i_{m+1}, \\ p^* \cdot a^i - p^* \cdot \bar{a} \leq x^i_{m+1} \leq 0, \ x^i \in \Re^m_+. \end{array} \end{array}$$

By the saddle-point characterization, there exists a number $\lambda^{*i} \geq 0$ such that the triplet $(x^{*i}, x^{*i}_{m+1}, \lambda^{*i})$ satisfies

$$\begin{array}{l} u^i(x^i) + \mu^i x^i_{m+1} + \lambda^{*i}(p^* \cdot a^i - p^* \cdot x^i - x^i_{m+1}) \\ \leq \\ u^i(x^{*i}) + \mu^i x^{*i}_{m+1} + \lambda^{*i}(p^* \cdot a^i - p^* \cdot x^{*i} - x^{*i}_{m+1}) \\ \leq \\ u^i(x^{*i}) + \mu^i x^{*i}_{m+1} + \lambda^i(p^* \cdot a^i - p^* \cdot x^{*i} - x^{*i}_{m+1}) \end{array} \tag{7}$$

for all $x^i \in \Re^m_+$, $p^* \cdot a^i - p^* \cdot \bar{a} \leq x^i_{m+1} \leq 0$, and for all $\lambda^i \in \Re_+$.

The non-satiation of u^i together with the first inequality in (7) implies $\lambda^{*i} > 0$. This in turn with the second inequality in (7) implies

$$p^* \cdot x^{*i} + x^{*i}_{m+1} = p^* \cdot a^i. \tag{8}$$

Since $\sum_{i \in N} x^{*i} = \sum_{i \in N} a^i$ and $x^{*i}_{m+1} \leq 0$ for all i, it follows from (8) that $x^{*i}_{m+1} = 0$ for all i.

By (7), $x^i_{m+1} = 0$ implies $(x^{*i}, p^*, \lambda^{*i})$ satisfies (3). Thus, (x^*, p^*, λ^*) is a competitive triplet for $\mathcal{E}$. Next, by taking x^i to be x^{*i} in (7), the first inequality implies

$$(\mu^i - \lambda^{*i})x^i_{m+1} \leq 0, \ \forall p^* \cdot a^i - p^* \cdot \bar{a} \leq x^i_{m+1} \leq 0.$$

By *A1* and *A3*, $p^* \cdot a^i < p^* \cdot \bar{a}$. The inequality $\lambda^{*i} \leq \mu^i$ follows automatically from the above inequality. ■

If a competitive multiplier vector $\bar{\lambda}$ associated with a CE of $\mathcal{E}$ is such that $\bar{\lambda}_i > \mu_i$ for some trader i, then the per-unit penalty on trader i is not severe enough in the sense that on the margin i gains from being in debt. When this occurs, the budget constraint will be violated. Since no one ends as a net creditor, the market for commodities will be imbalanced. Hence, the CE cannot be a DCE. A direct application of Theorem 2 implies:

Corollary 3 *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ be an exchange economy. Assume $\mathcal{E}$ satisfies A1-A4. Then, given default penalties μ, only those CEs of $\mathcal{E}$ with multiplier vectors $\bar{\lambda} \leq \mu$ are selected.*

Suppose that a CE of $\mathcal{E}$ is such that the associated competitive multiplier vector does not Pareto dominate the competitive multiplier vector of every other CE. Then, Corollary 3 implies that the CE can be uniquely selected with traders' individual per-unit default penalties equal to their associated competitive multipliers. We now present two examples with three CEs all having interior allocations such that the competitive multiplier vectors do not Pareto dominate each other. Later, we show that these are examples of economies from two classes in which the non-dominance in the Pareto sense among competitive multiplier vectors holds.

Example 1: (Shapley and Shubik 1977) There are two goods and two traders with endowments $a^1 = (40, 0)$, $a^2 = (0, 50)$, and utility functions $u^1(x^1) = x_1^1 + 100(1 - e^{-x_2^1/10})$, $u^2(x^2) = 110(1 - e^{x_1^2/10}) + x_2^2$ on $\Re_+^2$. Traders 1 and 2 are respectively named Ivan and John in Shapley and Shubik (1977); goods 1 and 2 are respectively called rubles and dollars in that paper. There are three CEs all with interior allocations in this economy. Notice the interior Pareto optimal allocations satisfy

$$x_2^2 = x_1^2 + 50 - 10 \ln 110. \tag{9}$$

Since $\bigtriangledown u^1(x^1) = (1, 10e^{-x_2^1/10})$ and $\bigtriangledown u^2(x^2) = (11e^{-x_1^2/10}, 1)$, equation (9) implies that to be Pareto optimal, trader 2's consumption of good 2 increases with his consumption of good 1. Since trader 1's marginal utility of good 1 is constant and his marginal utility of good 2 is decreasing while trader 2's marginal utility of good 2 is constant and his marginal utility of good 1 is decreasing, we have at any two interior Pareto optimal allocations $\bar{x}$ and $\tilde{x}$

$$\bigtriangledown u^i(\bar{x}^i) \cdot \bar{a} > \bigtriangledown u^i(\tilde{x}^i) \cdot \bar{a}^i \Leftrightarrow \bigtriangledown u^j(\bar{x}^j) \cdot \bar{a} > \bigtriangledown u^j(\tilde{x}^j) \cdot \bar{a}^j.$$

Thus, by (5), the competitive multiplier vectors do not dominate each other. ■

Example 2: There are two goods and two traders with endowments $a^1 \in \Re_+^2$, $a^2 \in \Re_+^2$, and utility functions $u^1(x^1) = [(x_1^1)^{-2} + \alpha(x_2^1)^{-2}]^{-\frac{1}{2}}$, $u^2(x^2) = [\beta(x_1^2)^{-2} + (x_2^2)^{-2}]^{-\frac{1}{2}}$. Assume

$$a_1^1 + a_1^2 = a_2^1 + a_2^2 > 0 \;\text{ and }\; \beta < \alpha^{-1}. \tag{10}$$

When $a^1 = (1, 0)$, $a^2 = (0, 1)$, and $\alpha = \beta = (12/37)^2$, this example coincides with exercise 15.B.6 in Mas-Colell, Whinston, and Green (1995) and has three interior CEs.

Notice at interior bundles

$$\mathrm{MRS}^1(x^1) = \alpha^{-1}(\frac{x_2^1}{x_1^1})^3 \text{ and } \mathrm{MRS}^2(x^2) = \beta(\frac{x_2^2}{x_1^2})^3.$$

It follows that interior Pareto optimal allocations must satisfy

$$\alpha^{-1}(\frac{x_2^1}{x_1^1})^3 = \beta(\frac{x_2^2}{x_1^2})^3 = K \tag{11}$$

for some $K > 0$. Since $\alpha^{-1} > \beta$, it follows from (11) that $K > \alpha^{-1}$ implies

$$x_2^1 + x_1^2 > x_1^1 + x_1^2.$$

By (10), the above inequality contradicts to the allocation being feasible. This shows $K \leq \alpha^{-1}$. A similar argument shows that $K \geq \beta$.

Notice also at interior bundles,

$$\bigtriangledown u^1(x^1) = \frac{\left[\alpha^{-1}(\frac{x_2^1}{x_1^1})^3, 1\right]}{\sqrt{\alpha}\left[1+\alpha^{-1}(\frac{x_2^1}{x_1^1})^2\right]^{\frac{3}{2}}} \text{ and } \bigtriangledown u^2(x^2) = \frac{\left[\beta(\frac{x_2^2}{x_1^2})^3, 1\right]}{\left[1+\beta(\frac{x_2^2}{x_1^2})^2\right]^{\frac{3}{2}}}.$$

It follows that,

$$\bigtriangledown u^1(x^1)\cdot\bar{a} = \frac{\left[1+\alpha^{-1}(\frac{x_2^1}{x_1^1})^3\right]\bar{a}_1}{\sqrt{\alpha}\left[1+\alpha^{-1}(\frac{x_2^1}{x_1^1})^2\right]^{\frac{3}{2}}} \text{ and } \bigtriangledown u^2(x^2)\cdot\bar{a} = \frac{\left[1+\beta(\frac{x_2^2}{x_1^2})^3\right]\bar{a}_1}{\left[1+\beta(\frac{x_2^2}{x_1^2})^2\right]^{\frac{3}{2}}}.$$

Hence, at interior Pareto optimal allocations with MRS equal to K, we have

$$\bigtriangledown u^1(x^1)\cdot\bar{a} = \frac{[1+K]\,\bar{a}_1}{\sqrt{\alpha}\left[1+\alpha^{-\frac{1}{3}}K^{\frac{2}{3}}\right]^{\frac{3}{2}}} \text{ and } \bigtriangledown u^2(x^2)\cdot\bar{a} = \frac{[1+K]\,\bar{a}_1}{\left[1+\beta^{\frac{1}{3}}K^{\frac{2}{3}}\right]^{\frac{3}{2}}}.$$

Simple calculation shows that $\bigtriangledown u^1(x^1)\cdot\bar{a}$ is decreasing while $\bigtriangledown u^2(x^2)\cdot\bar{a}$ is increasing in K over the interval $[\beta, \alpha^{-1}]$ of common marginal rates of substitution at interior Pareto optimal allocations. Thus, by (5), the non-dominance in Pareto sense among competitive multiplier vectors holds. ■

3.2 Unique Selection of a CE

We now show that the quasi-linear economy in Example 1 and CES economy in Example 2 are examples from two general classes of economies, in which the competitive multiplier vectors do not Pareto dominate each other.

Theorem 3 *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i=1}^n$ be an economy satisfying A1, A2′, and A4-A5. Then, the competitive multiplier vectors of $\mathcal{E}$ do not Pareto dominate each other under either of the following two further assumptions:*

A6*: For all i, u_i is homogenous of degree 1.*

A7*: The number of consumers is equal to the number of commodities and the commodities can be indexed such that for all i, there exist a function $v^i : \Re_+^{n-1} \longrightarrow \Re$ and constant $\theta^i > 0$ with*

$$u^i(x^i) = v^i(x_{-i}^i) + \theta^i x_i^i,$$

where x_{-i}^i is the sub-bundle obtained from x^i by excluding x_i^i.

Proof. Suppose first *A6* is satisfied in addition to *A1*, *A2′*, and *A4-A5*. By (5), at any competitive triplet (x^*, p^*, λ^*),

$$\bigtriangledown u^i(x^{*i}) \cdot x^{*i} = \lambda^{*i} p^* \cdot x^{*i}, \; i = 1, 2, \cdots, n. \tag{12}$$

Since

$$p^* \cdot x^{*i} = p^* \cdot a^i, \quad \bigtriangledown u^i(x^{*i}) \cdot x^{*i} = u^i(x^*),$$

it follows from (12) that for all i

$$u^i(x^{*i}) = \lambda^{*i} p^* \cdot a^i.$$

Hence,

$$\sum_{i=1}^n \frac{u^i(x^{*i})}{\lambda^{*i}} = p^* \cdot \bar{a}. \tag{13}$$

Now, let $(\hat{x}, \hat{p}, \hat{\lambda})$ and $(\tilde{x}, \tilde{p}, \tilde{\lambda})$ be any two competitive triplets of $\mathcal{E}$. Then, by (13) and our price normalization,

$$\sum_{i=1}^{n} \left[\frac{u^i(\hat{x}^i)}{\hat{\lambda}^i} - \frac{u^i(\tilde{x}^i)}{\tilde{\lambda}^i} \right] = 0. \tag{14}$$

On the other hand, if $\hat{\lambda}$ Pareto dominates $\tilde{\lambda}$, then

$$\sum_{i=1}^{n} \left[\frac{u^i(\hat{x}^i)}{\hat{\lambda}^i} - \frac{u^i(\tilde{x}^i)}{\tilde{\lambda}^i} \right] > \sum_{i=1}^{n} \left[\frac{u^i(\hat{x}^i)}{\hat{\lambda}^i} - \frac{u^i(\tilde{x}^i)}{\hat{\lambda}^i} \right] \geq 0$$

The first inequality holds because $u^i(\tilde{x}^i) \geq u^i(a^i) > 0$ for all i and $\hat{\lambda}$ Pareto dominates $\tilde{\lambda}$, while the last inequality directly follows from Corollary 1. This shows that the Pareto dominance of $\hat{\lambda}$ over $\tilde{\lambda}$ leads to a contradiction to (14).

Suppose now *A7* is satisfied in addition to *A1*, *A2′*, and *A4-A5*. Then, for any two competitive triplets $(\hat{x}, \hat{p}, \hat{\lambda})$ and $(\tilde{x}, \tilde{p}, \tilde{\lambda})$ of $\mathcal{E}$, from (5) it follows

$$\theta^i = \hat{\lambda}^i \hat{p}_i = \tilde{\lambda}^i \tilde{p}_i, i = 1, 2, \cdots, n. \tag{15}$$

Since

$$\hat{p} \cdot \bar{a} = \tilde{p} \cdot \bar{a},$$

(15) implies

$$\sum_{i=1}^{n} \frac{\theta^i}{\hat{\lambda}^i} = \sum_{i=1}^{n} \frac{\theta^i}{\tilde{\lambda}^i}.$$

The non-Pareto dominance between competitive multiplier vectors follows easily from the above equation. ■

Combining Theorem 3 with Corollary 3, we can now establish:

Corollary 4 *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ be an exchange economy. Assume $\mathcal{E}$ satisfies A1, A3, A2′, and A4-A5. If in addition either A6 or A7 is satisfied, then every CE of $\mathcal{E}$ can be uniquely selected with traders' per-unit default penalties equal to their associated competitive multipliers.*

We end this section with an example to demonstrate the total cash flows of CEs under our price normalization.

Example 3: Consider the 2-person economy of Shapley and Shubik (1977). If we normalize the prices by the condition $p \cdot \bar{a} = 1,000$, so that the economy's total wealth is always 1,000, then from Table 1 in Shapley and Shubik (1977, p. 875) it follows that the competitive price vectors, competitive multiplier vectors, total cash flows are as in the following table, all with a two-digit decimal rounding off:

	x^*	p^*	λ^*	TW	TCF
CE1	((32.26, 39.26), (7.74, 10.74))	(3.4, 17.27)	(0.29, 0.06)	1000	704.34
CE2	((13.17, 20.18), (26.83, 29.82))	(12.9, 9.68)	(0.08, 0.1)	1000	541.45
CE3	((3.22, 10.23), (36.78, 39.77))	(18.5, 5.19)	(0.05, 0.19)	1000	733.52

In this table, TW stands for the total wealth of the economy and TCF for the total cash flow. The cash flow required from trader i at prices p_1, p_2 and bundle x^i is given by $p_1 \max\{0, x_1^i - a_1^i\} + p_2 \max\{0, x_2^i - a_2^i\}$ and the total cash flow required in a CE is the sum of the cash flows required from both traders at their respective equilibrium bundles and the equilibrium prices. Notice that the middle CE (CE2) is the only CE with minimum cash flow. To uniquely select it, we can set the per-unit default penalties equal to the traders' competitive multipliers 0.08 and 0.1. Alternatively, we can also choose a non-discriminatory per-unit default penalty equal to 0.1. In fact, it follows from the proof of Theorem 2 that any non-discriminatory per-unit default penalties between 0.1 and the next highest maximum competitive multiplier which is equal to 0.19 would work.

4 Selection with Production

An *economy* with l goods, n traders, and household production is an array $\mathcal{E} = \{(X^i, u^i, a^i, Y^i)\}_{i \in N}$, where N is the trader set, $X^i \subseteq \Re^m$ is the consumption set of trader i, u^i is i's utility function, a^i is his endowment bundle, and $Y^i \subseteq \Re^m$ is his household production possibility set.[9] An element y^i in Y^i represents a production plan that i can carry out. As usual, inputs into production appear as negative components of y^i and outputs as positive components. For all $i \in N$, X^i and Y^i are closed and convex.

4.1 Competitive Allocations

A production plan changes a trader's initial endowment before trading. Hence, the selection of a production plan by an individual is guided by utility maximization instead of profit maximization. However, with price-taking traders, utility maximization implies profit maximization.

Definition 2 *A CE for economy $\mathcal{E} = \{(X^i, u^i, a^i, Y^i)\}_{i \in N}$ is a point*

$$((x^{*i}, y^{*i})_{i \in N}, p^*) \in (\Pi_{i \in N}(X^i \times Y^i)) \times \Re^m_+$$

such that

(i) For $i \in N$, $p^* \cdot x^{*i} = p^* \cdot a^i + p^* \cdot y^{*i}$ and $u^i(x^i) > u^i(x^{*i})$ implies $p^* \cdot x^i > p^* \cdot a^i + p^* \cdot y^i$ for all $y^i \in Y^i$;

(ii) $\sum_{i \in N} x^{*i} = \sum_{i \in N} a^i + \sum_{i \in N} y^{*i}$.

4.2 Arrow-Debreu Economy

In the Arrow-Debreu model of an economy with $m < \infty$ goods, there are a set, N, of finitely many traders with trader $i \in N$ characterized by the triplet (X^i, u^i, a^i) and a set, J, of producers with producer $j \in J$ characterized by a production possibility set Y^j. In addition, each trader i is also endowed with a relative share θ_{ij} of firm j's profit (see

[9] This model of an economy was considered in Hurwicz (1960), Rader (1964), Shapley (1973), Billera (1974), among others. Qin (1993) applies this model to study competitive outcomes in the cores of NTU market games.

Arrow and Debreu 1954, Debreu 1959). Symbolically, an *Arrow-Debreu economy* is an array $\mathcal{E} = \{\{(X^i, u^i, a^i)\}_{i\in N}, \{Y^j\}_{j\in J}, \{\theta_{ij}\}_{i\in N, j\in J}\}$. For all $i \in N$ and all $j \in J$, X^i and Y^j are closed and convex.

Definition 3 *A CE for $\mathcal{E} = \{\{(X^i, u^i, a^i)\}_{i\in N}, \{Y^j\}_{j\in J}, \{\theta_{ij}\}_{i\in N, j\in J}\}$ is a point*

$$((x^{*i})_{i\in N}, (y^{*j})_{j\in J}, p^*) \in (\Pi_{i\in N} X^i) \times (X_{j\in J} Y^j) \times \Re^m_+$$

such that

(i′a) For $i \in N$, $p^* \cdot x^{*i} = p^* \cdot a^i + \sum_{j\in J} \theta_{ij} p^* \cdot y^{*j}$ and $u^i(x^i) > u^i(x^{*i})$ implies $p^* \cdot x^i > p^* \cdot a^i + \sum_{j\in J} \theta_{ij} p^* \cdot y^{*j}$;

(i′b) For $j \in J$, $p^* \cdot y^{*j} \geq p^* \cdot y^j$, for $y^j \in Y^j$;

(ii′) $\sum_{i\in N} x^{*i} = \sum_{i\in N} a^i + \sum_{j\in J} y^{*j}$.

The relative shares θ_{ij} may be interpreted as representing private proprietorships of the production possibilities and facilities. With this interpretation, we can think of trader i as owning the technology set $\theta_{ij} Y_j$ at his disposal in firm j. Consequently, we may think of trader i as owning the following production possibility set in the Arrow-Debreu economy:

$$\tilde{Y}^i = \sum_{j\in J} \theta_{ij} Y_j. \tag{16}$$

We denote elements in $\tilde{Y}^i$ by $\tilde{y}^i = \sum_{j\in J} \theta_{ij} y^{ij}$ for some $y^{ij} \in Y^j$, $j \in J$. The reader is referred to Rader (1964, pp. 160–163) and Nikaido (1968, p. 285) for a justification of this understanding of the traders' ownership shares. With equation (16), the Arrow-Debreu economy $\mathcal{E}$ is converted into an economy with household production which we denote by $\tilde{\mathcal{E}} = \{(X^i, u^i, a^i, \tilde{Y}^i)\}_{i\in N}$.

Rader shows that an Arrow-Debreu economy $\mathcal{E}$ with convex production possibility sets is equivalent to economy $\tilde{\mathcal{E}}$, in the sense that the competitive allocations are the same across the two economies (see Rader 1964, pp. 160–163):

Theorem 4 *Let* $\mathcal{E} = \{\{(X^i, u^i, a^i)\}_{i\in N}, \{Y^j\}_{j\in J}, \{\theta_{ij}\}_{i\in N, j\in J}\}$ *be an Arrow-Debreu economy and let* $\tilde{\mathcal{E}} = \{(X^i, u^i, a^i, \tilde{Y}^i)\}_{i\in N}$ *with* $\tilde{Y}^i$ *given in (12). Then, for any CE* $((x^{*i})_{i\in N}, (y^{*j})_{j\in J}, p^*)$ *of* $\mathcal{E}$*, there are production plans* $\tilde{y}^{*i} \in \tilde{Y}^i$*,* $i \in N$*, such that* $((x^{*i}, \tilde{y}^{*i})_{i\in N}, p^*)$ *is a CE of* $\tilde{\mathcal{E}}$*. Conversely, for any CE* $((x^{*i}, \tilde{y}^{*i})_{i\in N}, p^*)$ *of* $\tilde{\mathcal{E}}$*, there are production plans* y^{*j}*,* $j \in J$*, such that* $((x^{*i})_{i\in N}, (y^{*j})_{j\in J}, p^*)$ *is a CE of* $\mathcal{E}$*.*

4.3 *Rader's Equivalence Principle*

Rader (1964) considers how to transform an economy with household production into an exchange economy using *induced preferences*. He shows that all the properties pertaining to the traders' characteristics in a production economy go over to the induced exchange economy. Furthermore, the CEs of the original economy and those of the induced exchange economy are equivalent (see Rader 1964, pp. 155-57). It follows that our credit mechanism and results in the previous sections can be extended to a production economy via its induced exchange economy.

5 Conclusion

In this paper we investigated the possibilities to enlarge the general-equilibrium structure by allowing default subject to appropriate default penalties. The enlargement of the general equilibrium structure results in a construction of a simple mechanism for a credit using society to select a unique CE under certain conditions.[10]

The implementation of the mechanism involves a bank providing a credit money that traders use as a direct and anonymous means of payment. The traders exchange personal IOUs for the credit money without limit. But, they settle up all outstanding credits with

[10] The explicit introduction of default penalties allows the traders the option to break the usual budget constraints whenever desirable. This option turns out to be essential for eliminating all but one CEs as DCE.

the bank at the end of the market. Ending as a net debtor is penalized while ending as a net creditor is worthless.

We characterized the CEs that will be selected by the mechanism. They are those CEs with traders' marginal utilities of income dominated by the corresponding per-unit default penalties. Applying this result, we showed that for two classes of economies the price normalization calling for an equal value of total endowment guarantees that any CE can be uniquely selected with default penalties equal to the associated competitive multipliers. We stress that the introduction of a default penalty is fundamentally a game theoretic addition. Viewing the process as a game of strategy, the budget constraint is no longer a pure constraint but it is adhered to as a matter of strategic choice.

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